Lab 1 The Magnetic Ball Suspension System
Preparation Report

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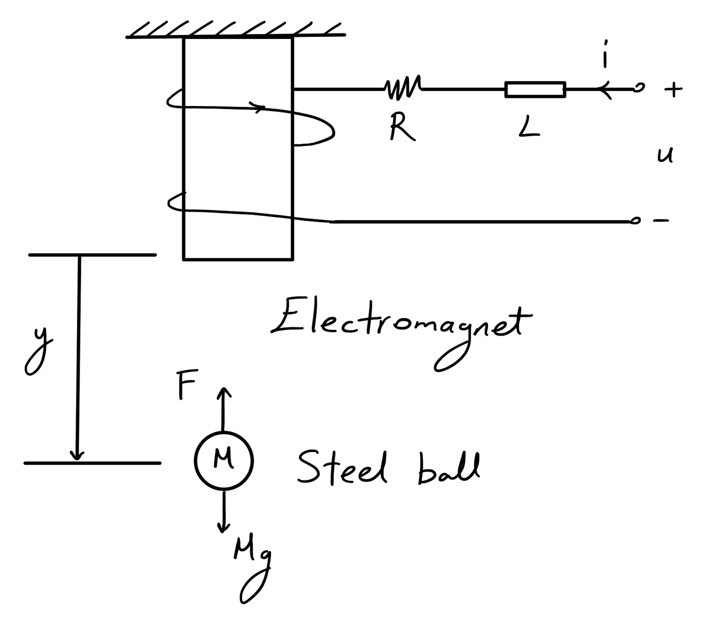

$F = \frac{i^2}{y^2}$ $g = 9.8\,m/s^2$ $M = 1\,kg$ $R = 3\,\Omega$ $L = 1\,H$

1) State model of the system with input voltage u and output position y.

$$x = \begin{bmatrix} y \\ \dot{y} \\ i \end{bmatrix} \qquad \begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}$$

Applying Newton's Second Law to the ball subsystem:

$$M\ddot{y} - Mg + \frac{i^2}{y^2} = 0 \Rightarrow \ddot{y} - 9.8 + \frac{i^2}{y^2} = 0 \quad ①$$

Applying KVL to the electromagnet subsystem:

$$u = iR + L\frac{di}{dt} \Rightarrow u = 3i + \frac{di}{dt} \quad ②$$

From ① & ②

$$x = \begin{bmatrix} y \\ \dot{y} \\ i \end{bmatrix} \Rightarrow \begin{cases} x_1 = y \\ x_2 = \dot{y} \\ x_3 = i \end{cases} \qquad \dot{x} = \begin{cases} \dot{x}_1 = \dot{y} \\ \dot{x}_2 = \ddot{y} \\ \dot{x}_3 = \dot{i} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 9.8 - \frac{(x_3)^2}{(x_1)^2} \\ \dot{x}_3 = u - 3x_3 \end{cases} \qquad y = x_1$$

$$\dot{x} = f(x,u) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ 9.8 - \frac{(x_3)^2}{(x_1)^2} \\ u - 3x_3 \end{bmatrix}$$

$$y = h(x,u) = x_1$$

2) Equilibrium conditions of the system.
$(x,u) = ([y^*, \dot{y}^*, i^*]^T, u^*)$

To get the equilibrium point set $\dot{x} = 0$

$x_2^* = 0$

$9.8 - \frac{(x_3^*)^2}{(x_1^*)^2} = 0 \Rightarrow x_3^* = 3.13y^*$

$u^* - 3x_3^* = 0 \Rightarrow u^* = 3x_3^* = 9.39y^*$

$y^* = x_1^*$

Equilibrium point
$$\boxed{(x^*, u^*) = ([y^*, 0, 3.13y^*]^T, 9.39y^*)}$$

Linearization about the equilibrium condition (x^*, u^*), to obtain a system

$\delta\dot{x} = A\delta x + B\delta u$

$\delta y = C\delta x + D\delta u$

Define $\boxed{\delta u = u - u^*}$ $\boxed{\delta x = x - x^* = [x_1 - y^*, x_2, x_3 - 3.13y^*]^T}$ $\boxed{\delta y = y - y^*}$

Compute $A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x^* \\ u=u^*}}$

$\frac{\partial f_1}{\partial x_1} = 0$ $\frac{\partial f_1}{\partial x_2} = 1$ $\frac{\partial f_1}{\partial x_3} = 0$

$\frac{\partial f_2}{\partial x_1} = 2(x_3^*)^2(x_1^*)^{-3} = 2(3.13y^*)^2(y^*)^{-3} = \frac{19.6}{y^*}$ $\frac{\partial f_2}{\partial x_2} = 0$ $\frac{\partial f_2}{\partial x_3} = -2x_3^* = -6.26y^*$

$\frac{\partial f_3}{\partial x_1} = 0$ $\frac{\partial f_3}{\partial x_2} = 0$ $\frac{\partial f_3}{\partial x_3} = -3$

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{19.6}{y^*} & 0 & -6.26 \\ 0 & 0 & -3 \end{bmatrix}}$$

Compute $B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x^* \\ u=u^*}}$

$\frac{\partial f_1}{\partial u} = 0$

$\frac{\partial f_2}{\partial u} = 0 \Rightarrow \boxed{B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

$\frac{\partial f_3}{\partial u} = 1$

Compute $C = \left.\frac{\partial h}{\partial x}\right|_{\substack{x=x^* \\ u=u^*}}$

$\frac{\partial h}{\partial x_1} = 1$ $\frac{\partial h}{\partial x_2} = 0$ $\frac{\partial h}{\partial x_3} = 0 \Rightarrow \boxed{C = [1 \ 0 \ 0]}$

Compute $D = \left.\frac{\partial h}{\partial u}\right|_{\substack{x=x^* \\ u=u^*}}$

$$\boxed{D = \frac{\partial h}{\partial u} = 0}$$

Linearized model

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{19.6}{y^*} & 0 & -6.26 \\ 0 & 0 & -3 \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\delta u$$

$$\delta y = [1 \ 0 \ 0]\delta x + 0\,\delta u$$

where $\delta u = u - u^*$ $\delta x = x - x^* = [x_1 - y^*, x_2, x_3 - 3.13y^*]^T$ $\delta y = y - y^*$

3) Set $y^* = 1$ and derive the open-loop transfer function $G(s) = \frac{\delta Y(s)}{\delta U(s)}$.

$$G(s) = C(sI - A)^{-1}B + D$$

$$= [1 \ 0 \ 0]\begin{bmatrix} s & -1 & 0 \\ -19.6 & s & 6.26 \\ 0 & 0 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{(s^2-19.6)(s+3)}[1 \ 0 \ 0]\begin{bmatrix} s(s+3) & s+3 & -6.26 \\ 19.6(s+3) & s(s+3) & -6.26s \\ 0 & 0 & s^2-19.6 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{G(s) = \frac{-6.26}{(s^2-19.6)(s+3)} = \frac{-6.26}{s^3+3s^2-19.6s+58.8}}$$

Poles and zeros of $G(s)$

$\boxed{\text{Zeros} \rightarrow \text{None}}$

Poles $\rightarrow (s^2-19.6)(s+3) = 0$

$$\boxed{\begin{aligned} P_1 &= -3 \\ P_{2,3} &= \frac{\pm\sqrt{4 \cdot 19.6}}{2} = \pm\sqrt{19.6} = \pm 4.427 \end{aligned}}$$

4) Corresponding impulse-response function $g(t) = \mathcal{L}^{-1}\{G(s)\}$ of the linearized system.

$$G(s) = \frac{r_1}{s-4.427} + \frac{r_2}{s+4.427} + \frac{r_3}{s+3}$$

$$r_1 = \left.\frac{-6.26}{(s+4.427)(s+3)}\right|_{s=4.427} = -0.1$$

$$r_2 = \left.\frac{-6.26}{(s-4.427)(s+3)}\right|_{s=-4.427} = 0.5$$

$$r_3 = \left.\frac{-6.26}{(s^2-19.6)}\right|_{s=-3} = 0.6$$

$\boxed{g(t) = (-0.1e^{4.427t} + 0.5e^{-4.427t} + 0.6e^{-3t}) \cdot u(t)}$ where $u(t)$ is the step function

Sketch of the response of $g(t)$