

Note: the following information is not peer reviewed and mistakes may be present.

1 LOWPASS ANALOG PROTOTYPE FREQUENCY TRANSFORMATIONS

Let $H(S)$ be a lowpass filter of order N with a cutoff frequency of 1 rad/s:

$$H(S) = \frac{\sum_{i=0}^N b_i S^{N-i}}{\sum_{j=0}^N a_j S^{N-j}}$$

1.1 LOWPASS ANALOG PROTOTYPE TO LOWPASS

$H(S)$ can be transformed into a lowpass filter $H_{LPF}(S)$ of order N with cutoff frequency ω_n via:

$$H_{LPF}(S) = \frac{\sum_{i=0}^N b_i \omega_n^i S^{N-i}}{\sum_{j=0}^N a_j \omega_n^j S^{N-j}}$$

This is accomplished by using the substitution $S = \frac{S}{\omega_n}$ in $H(S)$.

1.2 LOWPASS ANALOG PROTOTYPE TO HIGHPASS

$H(S)$ can be transformed into a highpass filter $H_{HPF}(S)$ of order N with cutoff frequency ω_n via:

$$H_{HPF}(S) = \frac{\sum_{i=0}^N b_i \omega_n^{-i} S^i}{\sum_{j=0}^N a_j \omega_n^{-j} S^j}$$

This is accomplished by using the substitution $S = \frac{\omega_n}{S}$ in $H(S)$.

1.3 LOWPASS ANALOG PROTOTYPE TO BANDPASS

$H(S)$ can be transformed into a bandpass filter $H_{BPF}(S)$ of order $2N$ with geometric center frequency ω_n and quality factor Q via:

$$H_{BPF}(S) = \frac{\sum_{i=0}^N b_i [\omega_n S]^i [Q(S^2 + \omega_n^2)]^{N-i}}{\sum_{j=0}^N a_j [\omega_n S]^j [Q(S^2 + \omega_n^2)]^{N-j}}$$

This is accomplished by using the substitution $S = Q(\frac{S}{\omega_n} + \frac{\omega_n}{S})$ in $H(S)$.

$H_{BPF}(S)$ can alternatively be designed with upper and lower cutoff frequencies ω_1 and ω_2 by designing for a geometric center frequency ω_n and quality factor Q given by:

$$\omega_n = \sqrt{\omega_1 \omega_2}$$

$$Q = \frac{\omega_n}{\omega_1 - \omega_2}$$

1.3.1 Fast implementation of lowpass analog prototype to bandpass

First we will rewrite $H_{BPF}(S)$ in matrix form:

$A = [a_0 \cdots a_N]$ and is a $1 \times (N + 1)$ matrix

$B = [b_0 \cdots b_N]$ and is a $1 \times (N + 1)$ matrix

$Qq = \begin{bmatrix} Q^0 & & \\ & \ddots & \\ & & Q^{-N} \end{bmatrix}$ and is a $(N + 1) \times (N + 1)$ diagonal matrix

$P = \begin{bmatrix} p_{N,0,0} & \cdots & p_{N,0,N} \\ \vdots & \ddots & \vdots \\ p_{N,N,0} & \cdots & p_{N,N,N} \end{bmatrix}$ and is a $(N + 1) \times (N + 1)$ matrix

$Ss = \begin{bmatrix} (S/\omega_n)^N \\ \vdots \\ (S/\omega_n)^0 \end{bmatrix}$ and is a $(N + 1) \times 1$ matrix

$$A(S) = \frac{AQqPSs}{BQqPSs}$$

The matrices A, B, Qq , and Ss are self-explanatory. The matrix P holds the coefficients of the polynomial $P_{N,n} = S^n[S^2 + 1]^{N-n}$ represented as $p_{N,n,s}$ with N, n coming from $P_{N,n}$ and s meaning that this is the coefficient of the term S^{N-s} .

Each coefficient $p_{N,n,s}$ can be calculated according to:

$$p_{N,n,s} = \binom{N-n}{s-n} * 0^{(n+s)\%2} \text{ with } 0^0 = 1$$

2 ANALOG TO DIGITAL FILTER TRANSFORMATION

Let $A(S)$ be an analog filter of order N :

$$A(S) = \frac{\sum_{i=0}^N b_i S^{N-i}}{\sum_{j=0}^N a_j S^{N-j}}$$

2.1 THE BILINEAR TRANSFORM

$A(S)$ can be transformed to a digital filter $D(Z)$ of order N using the bilinear transform implemented via:

$$D(Z) = \frac{\sum_{i=0}^N b_i [2f_s]^i [Z^{-1} + 1]^i [Z^{-1} - 1]^{N-i}}{\sum_{j=0}^N a_j [2f_s]^j [Z^{-1} + 1]^j [Z^{-1} - 1]^{N-j}}$$

This is accomplished by using the substitution $S = 2f_s \frac{Z^{-1}-1}{Z^{-1}+1}$ in $A(S)$.

2.1.1 Fast implementation of the bilinear transform

We will first start by rewriting $D(Z)$ in matrix form:

$$A = [a_0 \cdots a_N] \text{ and is a } 1 \times (N+1) \text{ matrix}$$

$$B = [b_0 \cdots b_N] \text{ and is a } 1 \times (N+1) \text{ matrix}$$

$$F_s = \begin{bmatrix} (2f_s)^0 & & \\ & \ddots & \\ & & (2f_s)^N \end{bmatrix} \text{ and is a } (N+1) \times (N+1) \text{ diagonal matrix}$$

$$P = \begin{bmatrix} p_{N,0,0} & \cdots & p_{N,0,N} \\ \vdots & \ddots & \vdots \\ p_{N,N,0} & \cdots & p_{N,N,N} \end{bmatrix} \text{ and is a } (N+1) \times (N+1) \text{ matrix}$$

$$ZZ = \begin{bmatrix} Z^0 \\ \vdots \\ Z^{-N} \end{bmatrix} \text{ and is a } (N+1) \times 1 \text{ matrix}$$

$$D(Z) = \frac{AF_s P Z Z}{BF_s P Z Z}$$

The matrices A , B , F_s , and ZZ are self-explanatory. The matrix P holds the coefficients of the polynomial $P_{N,n} = [Z^{-1} + 1]^n [Z^{-1} - 1]^{N-n}$ represented as $p_{N,n,z}$ with N, n coming from $P_{N,n}$ and z meaning that this is the coefficient of the term Z^{-z} .

There is a way to recursively calculate this matrix. The first column $p_{N,n,0}$ is entirely ones, and the first row can be calculated as:

$$p_{N,0,z} = \binom{N}{z} * (-1)^z$$

An observation, valid only for $n \geq 1$ and $z \geq 1$, was made that allows for the rest of the coefficients to be calculated:

$$p_{N,n,z} = p_{N,n-1,z} + p_{N,n-1,z-1} + p_{N,n,z-1}$$

There is no formal proof of this observation, but one can use a computer to show that this is true for the first couple hundred N .

This method is quite quick, but measurements showed that it is better (for repeat transforms of the same order) to cache/precompute the results of the matrix P instead of using the recursive method every time.

2.2 FREQUENCY WARPING

For $A(S)$ to have a frequency characteristic ω_n in the digital domain, $A(S)$ must be designed to have frequency characteristic $\omega_{n,warped}$ in the analog domain where $\omega_{n,warped}$ is:

$$\omega_{n,warped} = 2f_s \tan\left(\frac{\pi\omega_n}{f_s}\right)$$

3 REFERENCES

[1] https://en.wikipedia.org/wiki/Bilinear_transform

[2] https://en.wikipedia.org/wiki/Prototype_filter