CPSC 513 — Assignment #4

Solutions for Question #1

In this question you were asked to consider a language $L \subseteq \Sigma^\star$ that is the language of a non-contracting grammar

$$G = (V, \Sigma, \Pi, S).$$

You were asked to show that there is another non-contracting grammar

$$\widehat{G} = (\widehat{V}, \Sigma, \widehat{\Pi}, S)$$

such that L is also the language of $\widehat{G}$, and the only productions in Π have one of the two following forms:

(a) $\alpha \to \beta$ where $\alpha, \beta \in V^\star$ (that is, neither α nor β includes any terminals), or

(b) $A \to \sigma$ where $A \in V$ and $\sigma \in \Sigma$.

Solution: Let

$$\widehat{V} = V \cup \{V_\sigma \mid \sigma \in \Sigma\}$$

where there as many new variables as terminals — that is, if $\sigma, \tau \in \Sigma$ and $\sigma \neq \tau$ then $V_\sigma \neq V_\tau$

Consider a mapping $\varphi : V \cup \Sigma \to \widehat{V}$ such that

- $\varphi(A) = A$ for every variable $A \in V$,

and

- $\varphi_(\sigma) = V_\sigma$ for every terminal $\sigma \in \Sigma$.

This can be extended to obtain a map $\varphi : (V \cup \Sigma)^\star \to \widehat{V}^\star$ by setting $\varphi(\gamma_1\gamma_2 \dots \gamma_k)$ to be

$$\varphi(\gamma_1)\varphi(\gamma_2)\dots\varphi(\gamma_k)$$

for every integer $k \geq 0$ and for all $\gamma_1, \gamma_2, \dots, \gamma_k \in V \cup \Sigma$.

With that noted, suppose that $\widehat{\Pi}$ includes the following production:

(a) $\varphi(\alpha) \to \varphi(\beta)$ for every production $\alpha \to \beta$ in Π — so that these all have form (a), as described in the question;

(b) $V_\sigma \to \sigma$ for all $\sigma \in \Sigma$ — so that these all have form (b), as described in the question.

Since $\widehat{G}$ has the same start variable S and the same set of terminals Σ as G, this suffices to define the grammar

$$\widehat{G} = (\widehat{V}, \Sigma, \Pi, S).$$

It should not be hard to see — by inspection of the description of the rules in $\widehat{G}$ — that $\widehat{G}$ is a non-contracting grammar because G is.

Claim #1: Let $\omega \in (V \cup \Sigma)^\star$. Then $S \Rightarrow^\star_\Pi \omega$ if and only if $S \Rightarrow^\star_{\widehat{\Pi}} \varphi(\omega)$.

Method of Proof: A straightforward pair of proofs by induction on the length of the derivation of ω from S using rules in Π (or the length of the derivation of $\varphi(\omega)$ from S using rules in $\widehat{\Pi}$). Since $\varphi(S) = S$, the claim is easily established for the case that the length of the derivation is zero (as needed for the basis). Since the production $\varphi(\alpha) \to \varphi(\beta)$ is in $\widehat{\Pi}$ for every production $\alpha \in \beta$, and since it is impossible to apply productions in $\widehat{\Pi}$ to *remove* terminals from a string, the inductive step of each proof is also very easy to complete.

Claim #2: $L(G) \subseteq L(\widehat{G})$.

Proof: Let $\omega = \gamma_1\gamma_2 \dots \gamma_k \in L(G)$. Then it follows by Claim #1, above, $S \Rightarrow^\star_{\widehat{\Pi}} \varphi(\omega)$, for the string

$$\varphi(\omega) = \varphi(\gamma_1)\varphi(\gamma_2)\dots\varphi(\gamma_k) = V_{\gamma_1}V_{\gamma_2}\dots V_{\gamma_k}.$$

It suffices to note that if P_i is the production $V_{\gamma_i} \to \gamma_i$ for $1 \le i \le k$ then $P_i \in \widehat{\Pi}$ for $1 \le i \le k$, and

$$\begin{aligned}
\varphi(\omega) &= V_{\gamma_1}V_{\gamma_2}\dots V_{\gamma_k} \\
&\Rightarrow_{P_1} \gamma_1 V_{\gamma_2}V_{\gamma_3}\dots V_{\gamma_i} \\
&\Rightarrow_{P_2} \gamma_1\gamma_2 V_{\gamma_3}V_{\gamma_4}\dots V_{\gamma_k} \\
&\qquad\qquad \vdots \\
&\Rightarrow_{P_i} \gamma_1\gamma_2\dots\gamma_i V_{\gamma_{i+1}}V_{\gamma_{i+2}}\dots V_{\gamma_k} \\
&\qquad\qquad \vdots \\
&\Rightarrow_{P_{k-1}} \gamma_1\gamma_2\dots\gamma_{k-1}V_{\gamma_k} \\
&\Rightarrow_{P_k} \gamma_1\gamma_2\dots\gamma_k \\
&= \omega.
\end{aligned}$$

Note that this could easily be turned into a more formal proof by induction on i that, for $0 \le i \le k$,

$$\varphi(\omega) \Rightarrow_{\widehat{\Pi}} \gamma_1\gamma_2\dots\gamma_i V_{\gamma_{i+1}}V_{\gamma_{i+2}}\dots V_{\gamma_k}.$$

Thus

$$S \Rightarrow^{\star}_{\widehat{\Pi}} \varphi(\omega) \Rightarrow^{\star}_{\widehat{\Pi}} \omega,$$

so that $S \Rightarrow^{\star}_{\widehat{\Pi}} \omega$ and $\omega \in L(\widehat{G})$. Since ω was arbitrarily chosen from $L(G)$ it follows that $L(G) \subseteq L(\widehat{G})$, as claimed.

The converse is a little trickier to prove. One way to establish it is as follows.

Claim #3: Let $\omega \in \Sigma^{\star}$ be a string in $\Sigma^{\star}$ with length k such that

$$S \Rightarrow^{\star}_{\widehat{\Pi}} \omega.$$

Then $k \geq 1$, and *every* derivation of ω from S using productions in $\widehat{\Pi}$ includes exactly k applications of rules with the form

$$V_{\sigma} \to \sigma$$

where $\sigma \in \Sigma$.

Proof: Since $\widehat{G}$ is a non-contracting grammar the right hand side of every production in $\widehat{\Pi}$ is a nonempty string — so it is impossible to derive the empty string from S. Thus $k \geq 1$.

Every production in $\widehat{\Pi}$ has one of the forms

(a) $\varphi(\alpha) \to \varphi(\beta)$, where $\alpha \to \beta$ is a production in Π, or
(b) $V_{\sigma} \to \sigma$, where $\sigma \in \Sigma$.

In order to complete the proof it suffices to note that

- the initial string, S, does not include any symbols in Σ at all,
- each application of a production of form (a) leaves the number of terminals in the string unchanged,
- each application of a production of form (b) increases the number of terminals in the string by exactly one, and
- the final string, ω, includes exactly k (copies of) terminals.

Thus it is necessary to use exactly k applications of rules of form (b) in *any* derivation of ω from S, as claimed.

Claim #4 Let $\omega \in \Sigma^\star$. If $S \Rightarrow^\star_{\widehat{\Pi}} \omega$ then $S \Rightarrow^\star_{\widehat{\Pi}} \varphi(\omega)$.

Proof: Let $\omega = \gamma_1\gamma_2\dots\gamma_k \in \Sigma^\star$ such that $S \Rightarrow^\star_{\widehat{\Pi}} \omega$. As noted above, every derivation of ω from S using productions in $\widehat{\Pi}$ must include exactly k applications of productions with the form $V_\sigma \to \Sigma$ for $\sigma \in \Sigma$.

Notice, as well, that if a string in $\mu \in (\widehat{V} \cup \Sigma)^\star$ begins with a terminal, σ, so that $\mu = \sigma\widehat{\mu}$, P is a production in $\widehat{\Pi}$, and

$$\mu = \sigma\widehat{\mu} \Rightarrow_P \nu$$

for $\nu \in (\widehat{V} \cup \Sigma)^\star$ then — since the terminal σ does not appear on the *left* hand side of any rule in $\widehat{\Pi}$ at all — it must be the case that $\nu = \sigma\widehat{\nu}$ for some string $\widehat{\nu} \in (\widehat{V} \cup \Sigma)^\star$ such that

$$\widehat{\mu} \Rightarrow_P \widehat{\nu}$$

as well. Now, since the only production in $\widehat{\Pi}$ that can be used to include a copy of γ_1 at the beginning of a string is the production

$$V_{\gamma_1} \to \gamma_1,$$

it follows that every derivation of ω from S must have the form

$$S \Rightarrow^\star_{\widehat{\Pi}} V_{\gamma_1}\mu_1 \Rightarrow_{\widehat{\Pi}} \gamma_1\mu_1 \Rightarrow_{\widehat{\Pi}} \gamma_1\mu_2 \Rightarrow_{\widehat{\Pi}} \gamma_1\mu_3 \Rightarrow_{\widehat{\Pi}} \cdots \Rightarrow_{\widehat{\Pi}} \gamma_1\mu_\ell = \omega$$

for some integer $\ell \geq 1$ and strings $\mu_1, \mu_2, \dots, \mu_\ell \in (\widehat{V} \cup \Sigma)^\star$, such that

$$\mu_1 \Rightarrow_{\widehat{\Pi}} \mu_2 \Rightarrow_{\widehat{\Pi}} \mu_3 \Rightarrow_{widehatPi} \cdots \Rightarrow_{\widehat{\Pi}} \mu_\ell$$

as well — with the final $\ell - 1$ productions in $\widehat{\Pi}$ used in these derivations being the same.

However, *this* can be used to show that if the above application of

$$V_{\gamma_1} \to \gamma_1$$

is deleted, the above sequence of productions is applied, and then the production $V_{\gamma_1} \to \gamma_1$ is used after, then one obtains a derivation of the form

$$S \Rightarrow^\star_{\widehat{\Pi}} V_{\gamma_1}\mu_1 \Rightarrow_{\widehat{\Pi}} V_{\gamma_1}\mu_2 \Rightarrow_{\widehat{\Pi}} V_{\gamma_1}\mu_3 \Rightarrow_{\widehat{\Pi}} \cdots \Rightarrow_{\widehat{\Pi}} V_{\gamma_1}\mu_\ell \Rightarrow_{\widehat{\Pi}} \gamma_1\mu_\ell = \omega$$

instead.

Similarly — considering γ_2 now, instead of γ_1 — *this* derivation must have the form

$$S \Rightarrow^\star_{\widehat{\Pi}} V_{\gamma_1}V_{\gamma_2}\nu_1 \Rightarrow V_{\gamma_1}\gamma_2\nu_1 \Rightarrow_{\widehat{\Pi}} V_{\gamma_1}\gamma_2\nu_2 \Rightarrow_{\widehat{\Pi}} \cdots \Rightarrow_{\widehat{\Pi}} V_{\gamma_1}\gamma_2\nu_m = V_{\gamma_1}\mu_\ell \Rightarrow_{\widehat{\Pi}} \gamma_1\mu_\ell = \omega.$$

for some integer m and for strings $\mu_1, \mu_2, \dots, \mu_m \in (\widehat{V} \cup \Sigma)^\star$ such that (using the last $m-1$ applications of productions in $\widehat{\Pi}$ before the application of the rule $V_{\gamma_1} \to \gamma_1$)

$$\nu_1 \Rightarrow_{\widehat{\Pi}} \nu_2 \Rightarrow_{\widehat{\Pi}} \cdots \Rightarrow_{\widehat{\Pi}} \nu_m.$$

Again, if the application of the rule $V_{\gamma_2} \to \gamma_2$ is deleted in the middle and then included at the end, one obtains a derivation of the form

$$\begin{aligned} S \Rightarrow^{\star}_{\widehat{\Pi}} V_{\gamma_1} V_{\gamma_2} \nu_1 \Rightarrow_{\widehat{\Pi}} V_{\gamma_1} V_{\gamma_2} \nu_2 \Rightarrow_{\widehat{\Pi}} \cdots \Rightarrow_{\widehat{\Pi}} V_{\gamma_1} V_{\gamma_2} \nu_m = V_{\gamma_1} V_{\gamma_2} \gamma_3 \gamma_4 \dots \gamma_k \\ \Rightarrow_{\widehat{\Pi}} \gamma_1 V_{\gamma_2} \gamma_3 \gamma_4 \dots \gamma_k \Rightarrow_{\widehat{\Pi}} \gamma_1 \gamma_2 \gamma_3 \dots \gamma_k = \omega. \end{aligned}$$

Iterating the process another $i-2$ times (for any integer i such that $2 \leq i \leq k$) a derivation looking like

$$\begin{aligned} S &\Rightarrow^{\star}_{\widehat{\Pi}} V_{\gamma_1} V_{\gamma_2} \dots V_{\gamma_i} \tau_1 \Rightarrow_{\widehat{\Pi}} V_{\gamma_1} V_{\gamma_2} \dots V_{\gamma_i} \tau_2 \Rightarrow_{\widehat{\Pi}} \cdots \Rightarrow_{\widehat{\Pi}} V_{\gamma_1} V_{\gamma_2} \dots V_{\gamma_i} \tau_n \\ &= V_{\gamma_1} V_{\gamma_2} \dots V_{\gamma_i} \gamma_{i+1} \gamma_{i+2} \dots \gamma_k \Rightarrow_{\widehat{\Pi}} \gamma_1 V_{\gamma_2} \dots V_{\gamma_i} \gamma_{i+1}\ \gamma_{i+2} \dots \gamma_k \\ &\Rightarrow_{\widehat{\Pi}} \gamma_1 \gamma_2 V_{\gamma_3} \dots V_{\gamma_i} \gamma_{i+1} \gamma_{i+2} \dots \gamma_k \Rightarrow_{\widehat{\Pi}} \cdots \Rightarrow_{\widehat{\Pi}} \gamma_1 \gamma_2 \dots \gamma_{i-1} V_{\gamma_i} \gamma_{i+1} \gamma_{i+2} \dots \gamma_k \\ &\Rightarrow_{\widehat{\gamma}} \gamma_1 \gamma_2 \dots \gamma_k = \omega. \end{aligned}$$

Notice that this is a derivation of the form

$$S \Rightarrow^{\star}_{\widehat{\Pi}} \varphi(\gamma_1 \gamma_2 \dots \gamma_i) \gamma_{i+1} \gamma_{i+2} \dots \gamma_k \Rightarrow^{\star}_{\widehat{\Pi}} \gamma_1 \gamma_2 \dots \gamma_k = \omega.$$

In particular, when $i = k$ this is a derivation

$$S \Rightarrow^{\star}_{\widehat{\Pi}} \varphi(\omega) \Rightarrow^{\star}_{\widehat{\Pi}} \omega,$$

so that $S \Rightarrow^{\star}_{\widehat{\Pi}} \varphi(\omega)$, as claimed.

Claim #5: $L(\widehat{G}) \subseteq L(G)$.

Proof: This now follows immediately from Claim #4 and Claim #1.

It follows by Claim #2 and Claim #5 that $L(G) = L(\widehat{G})$, as needed to establish the desired result.

Note: This is certainly not the only way to prove that $L(G) = L(\widehat{G})$! A student's solution might look very different from the above but might also be correct.