CPSC 513 — Assignment #3
Solutions for Question #4

In this question you were asked to prove the following.

> ***Recursion Theorem for Post-Turing Programs:*** Let $g : \mathbb{N}^{m+1} \to \mathbb{N}$ be a computable (partial or total) function.
>
> Then there exists a number $e \in \mathbb{N}$ such that for all $x_1, x_2, \dots, x_n \in \mathbb{N}$,
>
> $$\Phi_P^{(m)}(x_1, x_2, \dots, x_m, e) = g(e, x_1, x_2, \dots, x_m).$$

Solution: As the following indicates the proof of this version of the Recursion Theorem is virtually identical to the proof of the original version of the theorem: All you need to do is replace the recursively enumerable function S_m^n (respectively, the computable function $\Phi^{(m)}$) described in the original version of the Parameter Theorem with the function that is called S_m^n described in the Parameter Theorem for Post-Turing programs that was proved in the answer for Question #3 (respectively, with the function $\Phi_P^{(m)}$ that is also mentioned in the statement of this theorem).

With that noted let $g : \mathbb{N}^{m+1} \to \mathbb{N}$ be a computable (partial or total) function. Consider the partial function $h : \mathbb{N}^{m+1} \to \mathbb{N}$ such that, for all $v, x_1, x_2, \dots, x_m \in \mathbb{N}$,

$$h(x_1, x_2, \dots, x_m, v) = g(S_m^1(v, v), x_1, x_2, \dots, x_m)$$

where $S_m^1 : \mathbb{N}^2 \to \mathbb{N}$ is the primitive recursive function whose existence is proved and whose properties are described in the Parameter Theorem for Post-Turing programs. Since g is a computable partial or total function, h is a computable partial or total function as well.

It follows that h is (weakly) computed by some Post-Turing program $\mathcal{P}$. Let $z_0 = \#(\mathcal{P})$. Then

$$\begin{aligned} g(S_m^1(v, v), x_1, x_2, \dots, x_m) &= h(x_1, x_2, \dots, x_m, v) \\ &= \Phi_P^{(m+1)}(x_1, x_2, \dots, x_m, v, z_0) \\ &= \Phi_P^{(m)}(x_1, x_2, \dots, x_m, S_m^1(v, z_0)). \end{aligned}$$

Now let $v = z_0$ and let $e = S^1_m(z_0, z_0)$. Then it follows that, for all $x_1, x_2, \ldots, x_m \in \mathbb{N}$,

$$\begin{aligned} g(e, x_1, x_2, \ldots, x_m) &= g(S^1_m(z_0, z_0), x_1, x_2, \ldots, x_m) && \text{(since } e = S^1_m(z_0, z_0)\text{)} \\ &= \Phi^{(m)}_P(x_1, x_2, \ldots, x_m, S^1_m(z_0, z_0)) && \text{(as shown above)} \\ &= \Phi^{(m)}_P(x_1, x_2, \ldots, x_m, e) && \text{(since } S^1_m(z_0, z_0) = S^1_m(v, z_0) = e\text{)} \end{aligned}$$

as claimed.