CPSC 511/611 — Assignment #1
Solutions for Problem #2

This problem can be solved by a modification of the proof of the Speedup Theorem included in Lecture #3. The online notes for that lecture includes this proof in reasonable detail.

As in that proof, let m be some constant (to be selected later) such that $m \geq 2$. As in the original proof, let Γ be the tape alphabet of the Turing machine being simulated, and let us expand the tape alphabet, to get a tape alphabet $\widehat{\Gamma}$ for a new machine, such that $\widehat{\Gamma}$ includes the following sets and symbols:

- Σ (so that strings over the same input alphabet can be used as input);
- $\sqcup$ (because this is always required);
- Γ^m(so that the contents of m cells from the tape of the original machine can be stored on a single tape of the new one);
- $\triangleright$ (a symbol not included in Γ that will be used as a separator).

At the beginning of a computation, an ***initialization phase*** will be used to change the configuration of the tape from

ω_1	ω_2	$\ldots$	ω_n	$\sqcup$	$\sqcup$	$\ldots$

(with the tape head pointing to the leftmost cell) to the configuration

$\triangleright$	$\triangleright$	$\ldots$	$\triangleright$	$\triangleright$	σ_1	σ_2	$\ldots$	$\sigma_{\lceil n/m \rceil}$	$\sqcup$	$\sqcup$	$\ldots$

where there are $n+1$ copies of $\triangleright$ at the left end of the tape before the first other symbol (either σ_1 or $\sqcup$) appears, and where

$$\sigma_i = \omega_{m(i-1)+1}\omega_{m(i-1)+2}\cdots\omega_{mi}$$

for $1 \leq i \leq \lceil n/m \rceil$, where $\omega_j = \sqcup$ for any integer j such that $n+1 \leq j \leq m \cdot \lceil n/m \rceil$. The tape head should be pointing to the first cell to the right of a $\triangleright$ that contains a different symbol (either σ_1 if $n \geq 1$ or $\sqcup$ if $n = 0$).

In order to carry out this initialization it is sufficient to do the following.

1. Begin by sweeping to the right, replacing the leftmost $\min(n, m)$ symbols with $\triangleright$ and remembering the overwritten symbols using the new machine $\widehat{M}$'s finite control.

   Continue sweeping to the right (if necessary) — without overwriting symbols — until the leftmost $\sqcup$ is discovered; overwrite this with $\triangleright$, moving right. Write σ_1 (as defined above) onto the cell that is now visible, moving back to the left during this move.

   If $n \leq k$ then this will case will have been detected (and remembered using $\widehat{M}$'s finite control). It is sufficient, in this case, to move $\widehat{M}$'s tape head one more position to the right (without replacing the "$\triangleright$" that is visible just before the move) in order to complete the initialization phase.

   Otherwise one should continue as described next.

   Note: At most $n + 3$ steps of the machine $\widehat{M}$ have been used during this first part of the initialization.

2. One should continue doing the following until a "compressed" version

   $$\sigma_1 \sigma_2 \ldots \sigma_{\lceil n/m \rceil}$$

   of the entire input has been written to the right of the rightmost $\triangleright$ on the tape.

   (a) Sweep back to the left — past the rightmost $\triangleright$, and all symbols in Σ still on the tape, until another $\triangleright$ is seen.

   (b) Once again sweep to the right, replacing $\min(\ell, m)$ symbols with $\triangleright$, where ℓ is the number of cells, between the leftmost cells storing $\triangleright$ and the rightmost $\triangleright$, that still store symbols in Σ. Once again $\widehat{M}$'s finite control should be used to remember the contents of the cells that have been overwritten.

   (c) Sweep to the right over

   - any remaining symbols in Σ,
   - the rightmost $\triangleright$, and
   - all cells storing symbols in Γ^m

   until $\sqcup$ is discovered. Overwrite this with a symbol in Γ^m representing the contents of the cells that were overwritten with $\triangleright$ (along with additional blanks, as needed, if the number of such cells is less than m).

   If a compressed version of the entire input has now been written onto the tape then this will have been noticed and remembered using $\widehat{M}$'s finite control.

Note that, since $m \geq 2$,

- the number of steps needed for part (a), above, is at most n, and
- the number of steps needed for parts (b) and (c) is at most $n + 1$.

Thus each iteration of step 2 uses at most $2n+1$ moves. There are $(\lceil n/m \rceil - 1) < n/m$ iterations of this part of the process, so the total number of moves needed for this part of the initialization press will be less than

$$n/m \times (2n+1) = \tfrac{2}{m}n^2 + \tfrac{1}{m}n.$$

3. All that is left to do, for the initialization phase, is reposition the tape head: One should move the tape head until $\triangleright$ is discovered and then move it back one position to the right. The number of moves needed for this is at most $\lceil n/m \rceil + 1 \leq \frac{n}{m} + 2$.

It follows that the total cost of this revised initialization phase is at most

$$(n+3) + \tfrac{2}{m}n^2 + \tfrac{1}{m}n + \tfrac{1}{m}n + 2 \leq \tfrac{2}{m}n^2 + \left(1 + \tfrac{2}{m}\right)n + 5.$$

The rest of the simulation can now proceed exactly as described in the original proof: The total cost needed to simulate $f(n)$ moves of the original machine M will be at most $6 \cdot \lceil f(n)/m \rceil \leq \frac{6}{m}f(n) + 6$.

Since it is given that M uses at most $f(n)$ moves it follows that the total number of moves used by the new machine, $\widehat{M}$, on an input with length n, is at most

$$\tfrac{2}{m}n^2 + \left(1 + \tfrac{2}{m}\right)n + 5 + \tfrac{6}{m}f(n) + 6 \leq \tfrac{6}{m}f(n) + \tfrac{2}{m}n^2 + \left(1 + \tfrac{2}{m}\right)n + 11.$$

Since $n^2 \in o(f)$, $f(n) \geq n^2$ for sufficiently large n. In this case the number of steps used by $\widehat{M}$ is at most

$$\tfrac{8}{m}f(n) + \left(1 + \tfrac{2}{m}\right)\sqrt{f(n)} + 6,$$

for sufficiently large n, and this is at most $\frac{9}{m}f(n)$ for sufficiently large n as well.

It now suffices to set m to be a constant such that $\frac{9}{m} \leq \epsilon$ — that is, such that $m \geq \frac{9}{\epsilon}$ — in order to establish the claim.