CPSC 511/611 — Assignment #1 Solutions for Problem #2

This problem can be solved by a modification of the proof of the Speedup Theorem included in Lecture #3. The online notes for that lecture includes this proof in reasonable detail.

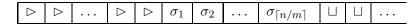
As in that proof, let m be some constant (to be selected later) such that $m \geq 2$. As in the original proof, let Γ be the tape alphabet of the Turing machine being simulated, and let us expand the tape alphabet, to get a tape alphabet $\widehat{\Gamma}$ for a new machine, such that $\widehat{\Gamma}$ includes the following sets and symbols:

- Σ (so that strings over the same input alphabet can be used as input);
- □ (because this is always required);
- Γ^m (so that the contents of m cells from the tape of the original machine can be stored on a single tape of the new one);
- \triangleright (a symbol not included in Γ that will be used as a separator).

At the beginning of a computation, an *initialization phase* will be used to change the configuration of the tape from

$$\omega_1 | \omega_2 | \ldots | \omega_n | \sqcup | \sqcup | \ldots$$

(with the tape head pointing to the leftmost cell) to the configuration



where there are n+1 copies of \triangleright at the left end of the tape before the first other symbol (either σ_1 or \sqcup) appears, and where

$$\sigma_i = \omega_{m(i-1)+1}\omega_{m(i-1)+2}\ldots\omega_{mi}$$

for $1 \le i \le \lceil n/m \rceil$, where $\omega_j = \sqcup$ for any integer j such that $n+1 \le j \le m \cdot \lceil n/m \rceil$. The tape head should be pointing to the first cell to the right of a \triangleright that contains a different symbol (either σ_1 if $n \ge 1$ or \sqcup if n = 0).

In order to carry out this initialization it is sufficient to do the following.

1. Begin by sweeping to the right, replacing the leftmost $\min(n, m)$ symbols with \triangleright and remembering the overwritten symbols using the new machine \widehat{M} 's finite control.

Continue sweeping to the right (if necessary) — without overwriting symbols — until the leftmost \sqcup is discovered; overwrite this with \triangleright , moving right. Write σ_1 (as defined above) onto the cell that is now visible, moving back to the left during this move.

If $n \leq k$ then this will case will have been detected (and remembered using \widehat{M} 's finite control). It is sufficient, in this case, to move \widehat{M} 's tape head one more position to the right (without replacing the ">" that is visible just before the move) in order to complete the initialization phase.

Otherwise one should continue as described next.

Note: At most n+3 steps of the machine \widehat{M} have been used during this first part of the initialization.

2. One should continue doing the following until a "compressed" version

$$\sigma_1 \sigma_2 \dots \sigma_{\lceil n/m \rceil}$$

of the entire input has been written to the right of the rightmost ⊳ on the tape.

- (a) Sweep back to the left past the rightmost \triangleright , and all symbols in Σ still on the tape, until another \triangleright is seen.
- (b) Once again sweep to the right, replacing $\min(\ell,m)$ symbols with \rhd , where ℓ is the number of cells, between the leftmost cells storing \rhd and the rightmost \rhd , that still store symbols in Σ . Once again \widehat{M} 's finite control should be used to remember the contents of the cells that have been overwritten.
- (c) Sweep to the right over
 - any remaining symbols in Σ ,
 - the rightmost ⊳, and
 - ullet all cells storing symbols in Γ^m

until \sqcup is discovered. Overwrite this with a symbol in Γ^m representing the contents of the cells that were overwritten with \rhd (along with additional blanks, as needed, if the number of such cells is less than m).

If a compressed version of the entire input has now been written onto the tape then this will have been noticed and remembered using \widehat{M} 's finite control.

Note that, since $m \geq 2$,

- the number of steps needed for part (a), above, is at most n, and
- the number of steps needed for parts (b) and (c) is at most n+1.

Thus each iteration of step 2 uses at most 2n+1 moves. There are $(\lceil n/m \rceil - 1) < n/m$ iterations of this part of the process, so the total number of moves needed for this part of the initialization press will be less than

$$n/m \times (2n+1) = \frac{2}{m}n^2 + \frac{1}{m}n.$$

3. All that is left to do, for the initialization phase, is reposition the tape head: One should move the tape head until \triangleright is discovered and then move it back one position to the right. The number of moves needed for this is at most $\lceil n/m \rceil + 1 \le \frac{n}{m} + 2$.

It follows that the total cost of this revised initialization phase is at most

$$(n+3) + \frac{2}{m}n^2 + \frac{1}{m}n + \frac{1}{m}n + 2 \le \frac{2}{m}n^2 + (1+\frac{2}{m})n + 5.$$

The rest of the simulation can now proceed exactly as described in the original proof: The total cost needed to simulate f(n) moves of the original machine M will be at most $6 \cdot \lceil f(n)/m \rceil \leq \frac{6}{m} f(n) + 6$.

Since it is given that M uses at most f(n) moves it follows that the total number of moves used by the new machine, \widehat{M} , on an input with length n, is at most

$$\frac{2}{m}n^2 + \left(1 + \frac{2}{m}\right)n + 5 + \frac{6}{m}f(n) + 6 \le \frac{6}{m}f(n) + \frac{2}{m}n^2 + \left(1 + \frac{2}{m}\right)n + 11.$$

Since $n^2 \in o(f)$, $f(n) \ge n^2$ for sufficiently large n. In this case the number of steps used by \widehat{M} is at most

$$\frac{8}{m}f(n) + \left(1 + \frac{2}{m}\right)\sqrt{f(n)} + 6,$$

for sufficiently large n, and this is at most $\frac{9}{m}f(n)$ for sufficiently large n as well.

It now suffices to set m to be a constant such that $\frac{9}{m} \le \epsilon$ — that is, such that $m \ge \frac{9}{\epsilon}$ — in order to establish the claim.