CPSC 511/611 — Assignment #4
Intractability

About This Assignment

This assignment can be completed by groups of up to two students in CPSC 511; students in CPSC 611 are to complete it individually.

The assignment is due by 11:59 pm on Wednesday, December 3. A printed copy of answers for written questions should be submitted by each group (for CPSC 511) or student (for CPSC 611) using the drop boxes on the second floor of the Mathematical Sciences building. If you typeset your submission then one student in your group can also submit an electronic copy (as a PDF file) using the assignment dropbox in D2L. However, a printed copy is still required if you do this!

Problems To Be Solved

1. Suppose that # is a symbol that does not belong to an alphabet Σ; let $\widehat{\Sigma} = \Sigma \cup \{\#\}$. Let

   $$\textit{pad}\colon \Sigma^\star \times \mathbb{N} \to \widehat{\Sigma}^\star$$

   be a function such that, for every string $\omega \in \Sigma^\star$ and every natural number $n \geq 0$,

   - if n is greater than or equal to the length of ω then *pad*(ω, n) is the string in $\widehat{\Sigma}^\star$ that begins with the first input string ω and that continues (and ends) with $n - |\omega|$ copies of the symbol # — so that *pad*(ω, n) has length n;
   - on the other hand, if n is less than the length of ω then *pad*(ω, n) is the first input ω, so that the length of *pad*(ω, n) is the same as the length of ω.

   Thus *pad*(ω, n) always begins with the string ω and always has length $\max(n, |\omega|)$.

   (a) Let k be an integer such that $k \geq 1$ and let Σ be any alphabet. Prove the following: If $A \subseteq \Sigma^\star$ then $A \in \mathsf{SPACE}(n^k)$ if and only if $\{\textit{pad}(\omega, |\omega|^k) \mid \omega \in A\} \in \mathsf{SPACE}(n)$.

   (b) Prove that $\mathcal{P} \neq \mathsf{SPACE}(n)$.

   Hint for (b): Consider a proof by contradiction, making use of the result of part (a) along with one or more Hierarchy Theorem.

2. Let $L \subseteq \Sigma^\star$ be an $\mathcal{NP}$-complete language.

   (a) Let
   $$\widehat{\Sigma} = \Sigma \cup \{\texttt{(},\texttt{)},\texttt{,},\texttt{T},\texttt{F}\}.$$
   and let
   $$\widehat{L} = \{\texttt{(}\omega\texttt{,T)} \mid w \in L\} \cup \{\texttt{(}\omega\texttt{,F)} \mid \omega \in \Sigma^\star \text{ and } \omega \notin L\}$$
   Give a ***short*** proof that if $\mathcal{NP} \neq$ co-$\mathcal{NP}$ than
   $$\widehat{L} \notin \mathcal{NP} \cup \text{co-}\mathcal{NP}.$$
   Hint: You will need to use that fact if L is $\mathcal{NP}$-complete then the complement of L is co-$\mathcal{NP}$-complete.

   (b) Use the above to argue that if $\mathcal{NP} \neq$ co-$\mathcal{NP}$ then
   $$\mathcal{NP} \cup \text{co-}\mathcal{NP} \subsetneq \mathcal{P}^{\textit{SAT}}.$$