# CPSC 513 — Assignment #1

# Solutions for Question #2

Recall that the ***Fibonacci numbers*** $F_0, F_1, F_2, F_3, \ldots$ are defined using the following recurrence (or "recursive definition:") For every integer $n \geq 0$,

$$F_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ F_{n-2} + F_{n-1} & \text{if } n \geq 2. \end{cases}$$

You were asked to prove that the function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = F_n$, for every integer $n \geq 0$, is primitive recursive.

***Proof:*** Consider the constant

$$k = \langle 0, 1 \rangle = 2$$

and the function $g : \mathbb{N}^2 \to \mathbb{N}$ such that, for all $t, y \in \mathbb{N}$,

$$g(t, y) = \langle r(y), \ell(y) + r(y) \rangle.$$

Since addition, the pairing function, and its left and right inverses are all primitive recursive functions, and $g$ can be obtained from these using composition, it should not be hard to see that $g$ is a primitive function.

It follows that the function $h : \mathbb{N}^2 \to \mathbb{N}$, defined from the above constant function $k$ and primitive recursive function $p$, is primitive recursive too. Note that

$$h(0) = k = 2 = \langle 0, 1 \rangle = \langle F_0, F_1 \rangle$$

and, for every integer $t \geq 0$,

$$h(t+1) = g(t, h(t)) = \langle r(h(t)), \ell(h(t)) + r(h(t)) \rangle.$$

Notice that it follows that if $h(t) = \langle F_t, F_{t+1} \rangle$ then — since $t+1 \geq 1$, so that $t+2 \geq 2$ —

$$h(t+1) = \langle F_{t+1}, F_t + F_{t+1} \rangle = \langle F_{t+1}, F_{t+2} \rangle.$$

It is now trivial to prove, by induction on $n$, that

$$h(n) = \langle F_n, F_{n+1} \rangle$$

for every integer $n \geq 0$.

Now, since the left inverse of the pairing function is primitive recursive, it follows that the function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$f(n) = \ell(h(n)) = F_n$$

is primitive recursive, as claimed.