CPSC 513 — Assignment #4

Solutions for Question #3

In this question the definition of "G-computation" was extended to include cases where $G : \mathbb{N} \to \mathbb{N}$ is a partial function, instead of a total function, and the definition of the total predicate halted : $\mathbb{N}^3 \to \mathbb{N}$ was extended as well.

You were reminded that — for *total* functions $G : \mathbb{N} \to \mathbb{N}$ — the predicate

$$\text{halted}^{(n)} : \mathbb{N}^{n+2} \to \{0, 1\}$$

is always G-computable, and you were asked to prove that this is *not* the case for partial functions $G : \mathbb{N}^3 \to \mathbb{N}$ as well.

Solution: Let G be a computable partial function whose domain S is recursively enumerable but not recursive. In particular, let G be the partial function $G : \mathbb{N} \to \{0, 1\}$ such that, for all $x \in \mathbb{N}$, $G(x) = 1$ if the program with label x halts on input x, and $G(x)$ is undefined otherwise — so that the domain of this partial function is the set K that has now been discussed in several lectures.

Suppose, in order to obtain a contradiction, that the corresponding total predicate

$$\text{halted}^{(1)} : \mathbb{N}^3 \to \mathbb{N}$$

is G-computable. Then, as the Tutorial Exercise #11 indicates (and as discussed in tutorial), this function must also be "computable."

It would follow that if y is the encoding of the $\mathcal{L}$-program with an oracle

$$[A_1] \quad X_1 \leftarrow O(X_1)$$

then the predicate $f : \mathbb{N} \to \{0, 1\}$ such that, for all $x \in \mathbb{N}$,

$$f(x) = \text{halted}^{(1)}(x, y, 1)$$

is a computable total predicate as well. Note, however, that if $x \in \mathbb{N}$ then

$$\begin{aligned} f(x) = 1 &\Rightarrow \text{halted}^{(1)}(x, y, 1) = 1 \\ &\Rightarrow G(x) \text{ is defined} \\ &\Rightarrow x \in K. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(x) = 0 &\Rightarrow \text{halted}^{(1)}(x, y, 1) = 0 \\ &\Rightarrow G(x) \text{ is undefined} \\ &\Rightarrow x \notin K. \end{aligned}$$

Thus f is the characteristic function of the set K and (since f is computable) it follows that K is recursive.

Since K is a recursively enumerable set that is *not* recursive, a contradiction has been obtained.

It follows that it is *not* true that the function halted$^{(n)}$: $\mathbb{N}^{n+2} \to \{0, 1\}$, corresponding to G-computation, is *not* always a G-computable function — when G is not a total function.