# CPSC 513 — Assignment #4

# Solutions for Question #2

In this question you were asked to continue by proving that if $L \subseteq \Sigma^\star$ is the language of a non-contracting grammar then $L$ is context-sensitive.

***Solution:*** As established when answering Question #1, it is necessary and sufficient to show that if $G = (V, \Sigma, \Pi, S)$ is a non-contracting grammar such that every production in $\Pi$ has one of the following forms

(a) $\alpha \to \beta$ where $\alpha \in V^\star$ and $\beta \in V^\star$, or

(b) $A \to \sigma$, where $A \in V$ and $\sigma \in \Sigma$,

then there exists a context-sensitive grammar $\widehat{G} = (\widehat{V}, \Sigma, \widehat{\Pi}, \widehat{S})$ such that $L(\widehat{G}) = L(G)$.

To begin, let $\widehat{V} = V$, $\widehat{\Pi} = \emptyset$, and $\widehat{S} = S$, and suppose that

$$\Pi = \{P_1, P_2, \dots, P_m\}$$

for some integer $m \geq 1$. The result is trivial if $m = 0$ since $G$ is already a context-sensitive grammar in this case (whose language is the empty set). Variables will be added to $\widehat{V}$ and productions will be added to $\widehat{\Pi}$ as each of the productions in $\Pi$ is considered (for $i$ from $1$ to $m$).

***Case:*** $P_i$ is of form $(a)$. That is, $P_i$ is a rule

$$X_1 X_2 \dots X_k \to Y_1 Y_2 \dots Y_\ell$$

where $\ell \geq k \geq 1$ and $X_1, X_2, \dots, X_k, Y_1, Y_2, \dots, Y_\ell \in V$.

In this case $k + 1$ new variables $Z_{i,1}, Z_{i,2}, \dots, Z_{i,k+1}$ should be added to $V$.

- To begin, a production

  $$P_{i,1}: \quad X_1 X_2 \dots X_k \to Z_{i,1} X_2 \dots X_k$$

  should be added to $\widehat{\Pi}$. Note that this has the form

  $$\alpha A \beta \to \alpha \gamma \beta$$

  where $\alpha$ is the empty string, $\beta$ is the string $X_2 \dots X_k$, $A$ is the variable $X_1$, and $\gamma$ is the nonempty string $Z_{i,1}$.

- For $2 \le j \le k-1$, a production

$$P_{i,j}: \quad Z_{i,1}Z_{i,2}\ldots, Z_{i,j-1}X_jX_{j+1}\ldots X_k \to Z_{i,1}Z_{i,2}\ldots Z_{i,j-1}Z_{i,j}X_{j+1}\ldots X_k$$

  should be added to $\widehat{\Pi}$. This has the form

$$\alpha A\beta \to \alpha\gamma\beta$$

  where $\alpha$ is the string $Z_{i,1}Z_{i,2}\ldots Z_{i,j-1}$, $\beta$ is the string $X_{j+1}X_{j+2}\ldots X_k$, $A$ is the variable $X_j$, and $\gamma$ is the nonempty string $Z_{i,j}$.

- A pair of productions

$$P_{i,k}: \quad Z_{i,1}Z_{i,2}\ldots Z_{i,k-1}X_k \to Z_{i,1}Z_{i,2}\ldots Z_{i,k-1}Z_{i,k}Y_{k+1}Y_{k+2}\ldots Y_\ell$$

  and

$$P_{i,k+1}: \quad Z_{i,1}Z_{i,2}\ldots Z_{i,k}Y_{k+1}Y_{k+2}\ldots Y_\ell \to Z_{i,k+1}Z_{i,2}\ldots Z_{i,k}Y_{k+1}Y_{k+2}\ldots Y_\ell$$

  should next be added to $\widehat{\Pi}$. The first has the form

$$\alpha A\beta \to \alpha\gamma\beta$$

  where $\alpha$ is the string $Z_{i,1}Z_{i,2}\ldots Z_{i,k-1}$, $\beta$ is the empty string, $A$ is the variable $X_k$, and $\gamma$ is the nonempty string $Z_{i,k}Y_{k+1}Y_{k+2}\ldots Y_\ell$. The second has this form where $\alpha$ is the empty string, $\beta$ is the string $Z_{i,2}Z_{i,3}\ldots Z_{i,k}Y_{k+1}Y_{k+2}\ldots Y_\ell$, $A$ is the variable $Z_{i,1}$, and $\gamma$ is the nonempty string $Z_{i,k+1}$.

  The second production is useful because it helps to ensure that these productions (including the ones to follow) all get applied, once each, and in order, whenever $P_{i,1}$ is applied.

- For $0 \le j \le k-2$ a production

$$P_{i,k+2+j}: \quad Z_{i,k+1}Z_{i,2}Z_{i,3}\ldots Z_{i,k-1-j}Z_{i,k-j}Y_{k+1-j}Y_{k+2-j}\ldots Y_\ell \to \\ Z_{i,k+1}Z_{i,2}Z_{i,3}\ldots Z_{i,k-1-j}Y_{k-j}Y_{k+1-j}Y_{k+2-j}\ldots Y_\ell$$

  This has the form

$$\alpha A\beta \to \alpha\gamma\beta$$

  where $\alpha$ is the string $Z_{i,k+1}Z_{i,2}Z_{i,3}\ldots Z_{i,k-1-j}$, $\beta$ is the string $Y_{k+1-j}Y_{k+2-j}\ldots Y_\ell$, $A$ is the variable $Z_{i,k-j}$, and $\gamma$ is the nonempty string $Y_{k-j}$.

  Note that these productions include the following:

$$P_{i,k+2}: \quad Z_{i,k+1}Z_{i,2}Z_{i,3}\dots Z_{i,k-1}Z_{i,k}Y_{k+1}Y_{k+2}\dots Y_\ell \to Z_{i,k+1}Z_{i,2}Z_{i,3}\dots Z_{i,k-1}Y_kY_{k+1}Y_{k+2}\dots Y_\ell$$

$$P_{i,k+3}: \quad Z_{i,k+1}Z_{i,2}Z_{i,3}\dots Z_{i,k-2}Z_{i,k-1}Y_kY_{k+1}\dots Y_\ell \to Z_{i,k+1}Z_{i,2}Z_{i,3}\dots Z_{i,k-2}Y_{k-1}Y_kY_{k+1}\dots Y_\ell$$

$$\vdots$$

$$P_{i,2k-1}: \quad Z_{i,k+1}Z_{i,2}Z_{i,3}Y_4Y_5\dots Y_\ell \to Z_{i,k+1}Z_{i,2}Y_3Y_4Y_5\dots Y_\ell$$
$$P_{i,2k}: \quad Z_{i,k+1}Z_{i,2}Y_3Y_4\dots Y_\ell \to Z_{i,k+1}Y_2Y_3\dots Y_\ell$$

- A final production

  $$P_{i,2k+1}: \quad Z_{i,k+1}Y_2Y_3\dots Y_\ell \to Y_1Y_2Y_3\dots Y_\ell$$

  should also be added to $\widehat{\Pi}$. This has the form

  $$\alpha A\beta \to \alpha\gamma\beta$$

  where $\alpha$ is the empty string, $\beta$ is the string $Y_2Y_3\dots Y_\ell$, $A$ is the variable $Z_{i,k+1}$, and $\gamma$ is the nonempty string $Y_1$.

***Claim #1:*** If $\mu, \nu \in (V \cup \Sigma)^\star$ and $\mu \Rightarrow_{P_i} \nu$ then $\mu \Rightarrow_{\widehat{\Pi}}^\star \nu$.

***Proof:*** This should be an easy exercise, now that the above productions have been added to $\widehat{\Pi}$.

***Case:*** $P_i$ is a production of form (b), that is, $A \to \sigma$ where $A \in V$ and $\sigma \in \Sigma$.

In this case a new variable $Z_{i,1}$ should be added to $\widehat{V}$ $P_i$ will be replaced by a pair of productions (which should be added to $\widehat{\Pi}$):

$$P_{i,1}: \quad A \to Z_{i,1}$$
$$P_{i,2}: \quad Z_{i,1} \to \sigma$$

Note that both of these rules are of the required form: The first has the form

$$\alpha A\beta \to \alpha\gamma\beta$$

where $\alpha$ and $\beta$ are both the empty string, $A$ is the variable $A \in V$, and $\gamma$ is the nonempty string $Z_{i,1}$. For the second production, $\alpha$ and $\beta$ are the empty string once again, $A$ is the variable $Z_{i,1}$, and $\gamma$ is the nonempty string $\sigma$.

Indeed, it would also have been permissible just to add the rule $A \to \sigma$ to $\widehat{\Pi}$. However, adding a pair of new rules involving a new variable will make a little bit easier to prove the correctness of this transformation.

***Claim #2:*** If $\mu, \nu \in (V \cup \Sigma)^\star$ and $\mu \Rightarrow_{P_i} \nu$ then $\mu \Rightarrow_{\widehat{\Pi}}^\star \nu$.

***Proof:*** This is left as another very easy exercise.

***Claim #3:*** $L(G) \subseteq L(\widehat{G})$.

***Sketch of Proof:*** Let $\omega \in \Sigma^\star$ such that $S \Rightarrow_{\Pi}^\star \omega$. Then it is necessary and sufficient to show that $S \Rightarrow_{\widehat{\Pi}} \omega$ as well.

This now follows by a straightforward proof by induction on the length of the derivation $S \Rightarrow_{\Pi}^\star \omega$, using Claims #1 and #2 when completing the inductive step.

***Claim #4:*** Suppose that $\omega \in (V \cup \Sigma)^\star$ such that $S \Rightarrow_{\widehat{\Pi}} \omega$. Then $S \Rightarrow_{\Pi}^\star \omega$ as well.

***Proof:*** Note first that this result is trivial if the derivation of $\omega$ from $S$ (using productions in $\widehat{\Pi}$) has length zero — for then $\omega = S$ and it is certainly true that $S \Rightarrow_{\Pi}^\star S$ as well.

Now let $n$ be the number of applications of rules $P_{i,1}$ (for $1 \leq i \leq m$) included in this derivation of $\omega$ from $S$. A proof that $S \Rightarrow_{\Pi}^\star \omega$ (indeed, that $S \Rightarrow_{\Pi} \omega$) is reasonably straightforward when $n \leq 1$:

- Since every rule $P_{i,j}$ such that $1 \leq i \leq m$ and $j \geq 2$ includes at least one variable in $\widehat{V} \setminus V$ on its left hand side, the first production used must be $P_{i,1}$ for some integer $i$ such that $1 \leq i \leq m$. Thus $n = 1$ if $n \leq 1$ and $\omega \neq S$.
- Suppose first that this corresponds to a production

  $$P_i : X_1 X_2 \ldots X_k \to Y_1 Y_2 \ldots Y_\ell$$

  in $\Pi$.
- Note that this application of $P_{i,1}$ must eventually be followed by applications of the productions $P_{i,2}, P_{i,3}, \ldots, P_{i,2k+1}$ in order — for the only way to eliminate the variable $Z_{i,1}$ that has now been included is to apply the production $P_{i,k+1}$ in order to replace it with $Z_{i,k+1}$ — but productions $P_{i,2}, P_{i,3}, \ldots, P_{i,k}$ must be applied, in order, before $P_{i,k+1}$ can. The production $P_{2k+1}$ must eventually be applied in order to eliminate the variable $Z_{i,k+1}$ — but $P_{i,k+2}, P_{i,k+3}, \ldots, P_{2k}$ must be applied, in order, after $P_{i,k+1}$, before $P_{i,2k+1}$ can be applied.
- Since there are no other applications of productions $P_{h,1}$ for $1 \leq h \leq m$ if $\ell = 1$, the strings derived, in the middle of the derivation either include exactly one copy of $Z_{i,1}$ and

no copies of $Z_{i,k+1}$, or vice-versa. It is not too hard to argue (by induction on the number of steps taken so far), that these are the *only* applications of productions that can be included in the derivation. Thus

$$\begin{aligned} S &= X_1 X_2 \dots X_k && \text{(so that } k = 1 \text{ and } X_1 = S\text{)} \\ &\Rightarrow_{P_{i,1}} \mu_1 \Rightarrow_{P_{i,2}} \mu_2 \cdots \Rightarrow_{P_{2k+1}} \mu_{2k+1} \\ &= Y_1 Y_2 \dots Y_\ell = \omega, \end{aligned}$$

and it should not be hard to see that $S \Rightarrow_{P_i} \omega$ as well.

- Similarly, if the first production was the production $P_{i,1}$ corresponding to a production

$$P_i : \quad A \to \sigma$$

where $A \in V$ and $\sigma \in \Sigma$, then it is not hard to argue that the derivation being considered is

$$S = A \Rightarrow_{P_{i,1}} Z_{i,1} \Rightarrow_{P_{i,2}} \sigma = \omega,$$

and the result follows since $S = A \Rightarrow_{P_i} \sigma = \omega$ as well.

If $n \geq 2$ then the proof is complicated by the fact the productions in $\widehat{\Pi}$ corresponding to different applications of productions in $\Pi$ might be interleaved. We must prove the existence of a derivation of $\omega$ from $S$ such that this is not the case.

- Suppose that some production $P_{i,1}$ is included in the derivation of $\omega$ from $S$. If this corresponds to a production

$$P_i : \quad X_1 X_2 \dots X_k \to Y_1 Y_2 \dots Y_\ell$$

then the derivation of $\omega$ from $S$ must have the form

$$S \Rightarrow^{\star}_{\widehat{\Pi}} \mu X_1 X_2 \dots X_k \nu \Rightarrow_{P_{i,1}} \mu Z_{i,1} X_2 \dots X_k \nu \Rightarrow^{\star}_{\widehat{\Pi}} \omega$$

for strings $\mu, \nu \in (\widehat{V} \cup \Sigma)^\star$.

- Once again, it is not hard to argue that the productions applied after $P_{i,1}$ must include $P_{i,2}, P_{i,3}, \dots, P_{i,2k+1}$, in order — but possibly with other productions in $\widehat{\Pi}$ in between.

- Suppose that $k \geq 1$ and one or more other productions are included between the application of $P_{i,1}$ and $P_{i,2}$. Then some of these productions modify the initial string $\mu$ — but not $\mu Z_{i,1}$, because there is no production whose left hand side ends with $Z_{i,1}$. Indeed, these productions could all have been applied (in the same relative order) if the production $P_{i,1}$ had not been applied before them.

The remaining applications of productions must modify part of $X_2X_3 \dots X_k\nu$ in between the applications of $P_{i,1}$ and $P_{i,2}$. Furthermore, if these modify the prefix $X_2X_3 \dots X_k$ then they must also restore it — because the production $P_{i,2}$ could not be applied otherwise. These productions could also have been applied, in the same relative order, if $P_{i,1}$ had not been applied before them.

It follows that the production of $P_{i,1}$ can be skipped, the applications of productions between the original application of $P_{i,1}$ and $P_{i,2}$ can still be carried out — in order — and then $P_{i,1}$ can be applied immediately *before* $P_{i,2}$. It is not hard to check that the string derived immediately after the application of $P_{i,2}$ has not been changed, so that the remaining productions can then be applied, to obtain another derivation

$$S \Rightarrow^{\star}_{\widehat{\Pi}} \omega$$

in which $P_{i,2}$ is applied immediately after $P_{i,1}$ is.

The same argument can be applied to move the application of $P_{i,j}$ so that it occurs immediately before the application of $P_{i,j+1}$ for every integer $j$ such that $1 \leq j \leq k-1$. Applying this argument repeatedly, one can eventually prove the existence of a derivation

$$S \Rightarrow^{\star}_{\widehat{\Pi}} \eta_1 \Rightarrow^{\star}_{\widehat{\Pi}} \eta_2 \Rightarrow^{\star}_{\widehat{\Pi}} \omega$$

such that

– The same productions have been applied as in the original derivation, but in a different order;
– $\eta_2$ has been derived from $\eta_1$ by applying the productions $P_{i,1}, P_{i,2}, \dots, P_{i,k}$, in order, with no other productions applied, and
– the relative order of all the *other* applications of productions has not been changed.

Note that it follows that if the applications of productions in $\widehat{\Pi}$ corresponding to some *other* application of a production in $\Pi$ already appeared in the original derivation, in order, with no other productions applied in between them, then that is also true, for this other production in $\Pi$, for the new derivation that is described above.

- Essentially the same argument can now be applied to move the applications of the productions $P_{i,k+1}, P_{i,k+2}, \dots, P_{i,2k+1}$, so that they also appear immediately *after* the application of $P_{i,k}$ — with the relative order of applications of other productions unchanged, once again (and applications of other productions that were originally between the applications of $P_{i,k}$ and $P_{i,2k+1}$ now appearing immediately after these).
- Suppose, instead, that $P_i$ is a production $A \to \sigma$ where $A \in V$ and $\sigma \in \Sigma$, so that $P_i$ has been replaced by a pair of productions $P_{i,1}$ and $P_{i,2}$. It follows by a similar (but simpler) argument that every application of $P_{i,1}$ must eventually be followed by an application of $P_{i,2}$ — and that this can be moved so that it takes place immediately after the application of $P_{i,1}$, without changing the relative order of productions being applied.

- Iterating this process another $n-1$ times (once for each of the applications of a production of $P_{i,1}$ appearing in the original derivation), we will eventually a derivation

  $$S = \zeta_0 \Rightarrow^{\star}_{\widehat{\Pi}} \zeta_1 \Rightarrow^{\star}_{\widehat{\Pi}} \zeta_2 \Rightarrow^{\star}_{\widehat{\Pi}} \cdots \Rightarrow^{\star}_{\widehat{\Pi}} \zeta_n = \omega,$$

  where, for $1 \le i \le n$, $\zeta_i$ is derived from $\zeta_{i-1}$ using all the productions in $\widehat{\Pi}$ corresponding to some production $P_{j_i} \in \Pi$, in order, and with no other productions applied.

- It is now easily shown (by induction on $h$) that $\zeta_h \in (V \cup \Sigma)^{\star}$ and, furthermore,

  $$S = \zeta_1 \Rightarrow_{P_{j_1}} \zeta_1 \Rightarrow_{P_{j_2}} \zeta_2 \Rightarrow_{P_{j_3}} \cdots \Rightarrow_{P_{j_h}} \zeta_h,$$

  so that $S \Rightarrow^{\star}_{\Pi} \zeta_h$, for every integer $h$ such $0 \le h \le n$.

- The claim now follows. since $\zeta_n = \omega$.

***Claim #5:*** $L(\widehat{G}) \subseteq L(G)$.

***Proof:*** This follows directly from Claim #4.

It follows by Claims #3 and #5 that $\widehat{G}$ is a context-free grammar such that $L(\widehat{G}) = L(G)$, as required to establish the claim given in this question.