Homework 4

Graphical Models and Learning Theory

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Problem 1: Graphical Models and MCMC

Problem 1.1: A model for diagnosing illness

(a) Conditional independence structure questions

Let:

- W = weeks traveling
- S = has stomach flu
- E = has ebola
- R = rash
- A = abdominal pain

i. $vars \perp\!\!\!\perp S = \{\}$ (None)

ii. $vars \perp\!\!\!\perp S | W, A = \{\}$ (None)

iii. $vars \perp\!\!\!\perp R | E = \{A, S, W\}$

iv. $S \perp\!\!\!\perp E | A, R$? False.

(b) Computing quantities from the model:

i.

$$\begin{aligned} p(E|A=t, R=t, W=0) &= \frac{1}{Z_1} \sum_S p(E, S=s, A=t, R=t, W=0) \\ &= \frac{1}{Z_1} \sum_S p(W=0) p(S=s|W=0) p(E|W=0) p(A=t|S=s, E) p(R=t|E) \end{aligned}$$

Eliminate S: (See tables 1 and 2)

$$\begin{aligned} &\frac{1}{Z_1} \sum_S p(W=0) p(S=s|W=0) p(E|W=0) p(A=t|S=s, E) p(R=t|E) \\ &= \frac{1}{Z_1} p(W=0) p(E|W=0) p(R=t|E) \sum_S p(S=s|W=0) p(A=t|S=s, E) \end{aligned}$$

*Note $p(W=0)$ is constant so it's combined with the normalizing constant.

$$\frac{1}{Z_1}\sum_S p(W=0)p(S=s|W=0)p(E|W=0)p(A=t|S=s,E)p(R=t|E)$$
$$=\frac{1}{Z_1}p(W=0)p(E|W=0)p(R=t|E)\sum_S p(S=s|W=0)p(A=t|S=s,E)$$

$$\frac{1}{Z_1}p(W=0)p(E|W=0)p(R=t|E)\sum_S p(S=s|W=0)p(A=t|S=s,E)$$
$$=\frac{1}{Z_2}p(E|W=0)p(R=t|E)\sum_S p(S=s|W=0)p(A=t|S=s,E)$$

Table 1: $p(S=s|W=0)$

S	$p(S=s\|W=0)$
F	0.9
T	0.1

Table 2: $p(A=t|S,E)$

S	E	$p(A=t\|S,E)$
F	F	0.01
F	T	0.95
T	F	0.95
T	T	0.99

- **For E=t:**
  $p(E=t|W=0)p(R=t|E=t)*$
  $[p(S=f|W=0)p(A=t|S=f,E=t)+p(S=t|W=0)p(A=t|S=t,E=t)]$
  $=(0.001)(0.9)[(0.9)(0.95)+(0.1)(0.99)]=0.0008586$
- **For E=f:**
  $p(E=f|W=0)p(R=t|E=f)*$
  $[p(S=f|W=0)p(A=t|S=f,E=f)+p(S=t|W=0)p(A=t|S=t,E=f)]$
  $=(0.999)(0.2)[(0.9)(0.01)+(0.1)(0.95)]=0.0207792$
- **Final:**
  $p(E=t|A=t,R=t,W=0)=\frac{0.0008586}{0.0008586+0.0207792}=0.0397$

  $p(E=f|A=t,R=t,W=0)=\frac{0.0207792}{0.0008586+0.0207792}=0.9603$

ii. When $W=2$

- For E=t:
  $p(E=t|W=2)p(R=t|E=t)$
  $[p(S=f|W=2)p(A=t|S=f,E=t)+p(S=t|W=2)p(A=t|S=t,E=t)]$
  $=(0.04)(0.9)[(0.8)(0.95)+(0.2)(0.99)]=0.034488$
- For E=f:

$p(E=f|W=2)p(R=t|E=f)$
$[p(S=f|W=2)p(A=t|S=f,E=f)+p(S=t|W=2)p(A=t|S=t,E=f)]$
$=(0.96)(0.2)[(0.8)(0.01)+(0.2)(0.95)]=0.038016$

- **Final:**
  $p(E=t|A=t,R=t,W=2)=\frac{0.034488}{0.034488+0.038016}=0.4757$
  $p(E=f|A=t,R=t,W=2)=\frac{0.038016}{0.034488+0.038016}=0.5243$

iii. $p(W|R=t)=\frac{1}{Z}\sum_{A,S,E}p(W)p(S|W)p(E|W)p(A|S,E)p(R=t|E)$

Let:

- $p(W)=\Phi_W$
- $p(S|W)=\Phi_{SW}$
- $p(E|W)=\Phi_{EW}$
- $p(A|S,E)=\Phi_{ASE}$
- $p(R=t|E)=\Phi_{RE}$

a. Eliminate A:

- $P_{initial}=\frac{1}{Z}\sum_{A,S,E}\Phi_W\Phi_{SW}\Phi_{EW}\Phi_{ASE}\Phi_{RE}=\frac{1}{Z}\sum_{S,E}\Phi_W\Phi_{SW}\Phi_{EW}\Phi_{RE}\sum_A\Phi_{ASE}$
- Tables: Φ_{ASE}
- Message: $\mu_{S,E}$ (See table 3)
- $P_{final}=\frac{1}{Z}\sum_{S,E}\Phi_W\Phi_{SW}\Phi_{EW}\Phi_{RE}\,\mu_{S,E}$
- **Note: $\mu_{S,E}$ is left out of the equation after this since the marginal probability is 1.**

Table 3: $\mu_{S,E}$

S	E	$\mu_{S,E}$
F	F	1
F	T	1
T	F	1
T	T	1

b. Eliminate S:

- $P_{initial}=\frac{1}{Z}\sum_{S,E}\Phi_W\Phi_{SW}\Phi_{EW}\Phi_{RE}=\frac{1}{Z}\sum_E\Phi_W\Phi_{EW}\Phi_{RE}\sum_S\Phi_{SW}$
- Tables: Φ_{SW}
- Message: μ_{W1} (See table 4)
- $P_{final}=\frac{1}{Z}\sum_E\Phi_W\Phi_{EW}\Phi_{RE}\mu_{W1}$
- **Note: μ_{W1} is left out of the equation after this since the marginal probability is 1.**

Table 4: μ_{W1}

W	μ_{W1}
F	1
T	1

c. Eliminate E:

- $P_{initial} = \frac{1}{Z} \sum_E \Phi_W \Phi_{EW} \Phi_{RE} = \frac{1}{Z} \Phi_W \sum_E \Phi_{EW} \Phi_{RE}$
- Tables: $\Phi_{EW} \Phi_{RE}$
- Message: μ_{W2} (See tables 5, 6, 7)

  Note: We can evaluate $R = t$ here since R is observed.
- $P_{final} = \frac{1}{Z} \Phi_W \, \mu_{W2}$

Table 5: Intermediate table (1) - evaluate $p(R = t|E)$

E	$p(R = t\|E)$
F	0.2
T	0.9

Table 6: Intermediate table (2) - evaluate $\mu_{W,E}$

W	E	$\Phi_{EW} * p(R = t\|E) = \mu_{W,E}$
0	F	$0.999 * 0.2 = 0.1998$
0	T	$0.001 * 0.9 = 0.0009$
1	F	$0.980 * 0.2 = 0.196$
1	T	$0.020 * 0.9 = 0.018$
2	F	$0.960 * 0.2 = 0.192$
2	T	$0.040 * 0.9 = 0.036$

Table 7: μ_{W2}

W	μ_{W2}
0	$0.1998 + 0.009 = 0.2088$
1	$0.196 + 0.018 = 0.214$
2	$0.192 + 0.036 = 0.228$

d. **Final:**

- Combine tables: Φ_W and μ_{W2} (See table 8)
- Normalize:

$$p(W = 0|R = t) = 0.19836/0.20934 = 0.947$$
$$p(W = 1|R = t) = 0.00642/0.20934 = 0.031$$
$$p(W = 2|R = t) = 0.00456/0.20934 = 0.022$$

iv. $p(W|R = f) = \frac{1}{Z} \sum_{A,S,E} p(W)p(S|W)p(E|W)p(A|S,E)p(R = f|E)$

Let:

- $p(W) = \Phi_W$
- $p(S|W) = \Phi_{SW}$
- $p(E|W) = \Phi_{EW}$
- $p(A|S,E) = \Phi_{ASE}$

Table 8: Combine tables Φ_W and μ_{W2}

W	Φ_W	μ_{W2}	$\Phi_W * \mu_{W2}$
0	0.95	0.2088	0.1936
1	0.03	0.214	0.00642
2	0.02	0.228	0.00456

- $p(R = t|E) = \Phi_{RE}$

a. Eliminate A:

- $P_{initial} = \frac{1}{Z} \sum_{A,S,E} \Phi_W \Phi_{SW} \Phi_{EW} \Phi_{ASE} \Phi_{RE} = \frac{1}{Z} \sum_{S,E} \Phi_W \Phi_{SW} \Phi_{EW} \Phi_{RE} \sum_A \Phi_{ASE}$
- Tables: Φ_{ASE}
- Message: $\mu_{S,E}$ (See table 9)
- $P_{final} = \frac{1}{Z} \sum_{S,E} \Phi_W \Phi_{SW} \Phi_{EW} \Phi_{RE} \, \mu_{S,E}$
- **Note: $\mu_{S,E}$ is left out of the equation after this since the marginal probability is 1.**

Table 9: $\mu_{S,E}$

S	E	$\mu_{S,E}$
F	F	1
F	T	1
T	F	1
T	T	1

b. Eliminate S:

- $P_{initial} = \frac{1}{Z} \sum_{S,E} \Phi_W \Phi_{SW} \Phi_{EW} \Phi_{RE} = \frac{1}{Z} \sum_E \Phi_W \Phi_{EW} \Phi_{RE} \sum_S \Phi_{SW}$
- Tables: Φ_{SW}
- Message: μ_{W1} (See table 10)
- $P_{final} = \frac{1}{Z} \sum_E \Phi_W \Phi_{EW} \Phi_{RE} \mu_{W1}$
- **Note: μ_{W1} is left out of the equation after this since the marginal probability is 1.**

Table 10: μ_{W1}

W	μ_{W1}
F	1
T	1

c. Eliminate E:

- $P_{initial} = \frac{1}{Z} \sum_E \Phi_W \Phi_{EW} \Phi_{RE} = \frac{1}{Z} \Phi_W \sum_E \Phi_{EW} \Phi_{RE}$
- Tables: $\Phi_{EW} \Phi_{RE}$
- Message: μ_{W2} (See tables 11, 12, 13)
  Note: We can evaluate $R = t$ here since R is observed.
- $P_{final} = \frac{1}{Z} \Phi_W \, \mu_{W2}$

d. Final:

Table 11: Intermediate table (1) - evaluate $p(R=f|E)$

E	$p(R=f\|E)$
F	0.8
T	0.1

Table 12: Intermediate table (2) - evaluate $\mu_{W,E}$

W	E	$\Phi_{EW} * p(R=f\|E) = \mu_{W,E}$
0	F	$0.999 * 0.8 = 0.0.7992$
0	T	$0.001 * 0.1 = 0.0001$
1	F	$0.980 * 0.8 = 0.784$
1	T	$0.020 * 0.1 = 0.002$
2	F	$0.960 * 0.8 = 0.768$
2	T	$0.040 * 0.1 = 0.004$

- Combine tables: Φ_W and μ_{W2} (See table 14)

$$p(W=0|R=f) = 0.75933/0.79835 = 0.951$$
$$p(W=1|R=f) = 0.02358/0.79835 = 0.030$$
$$p(W=2|R=f) = 0.01544/0.79835 = 0.019$$

$$\boldsymbol{E[p(W|R=f)] = (0)(0.951) + (1)(0.030) + (2)(.019) = .068}$$

(c) $p(\theta_i|\theta_1,\ldots,\theta_{i-1},\theta_{i+1},\ldots,\theta_n) = \frac{p(\theta_1,\ldots,\theta_n)}{\theta_1,\ldots,\theta_{i-1},\theta_{i+1},\ldots,\theta_n} = \frac{1}{Z_1}\prod_{j=1}^{n} p(\theta_j|parents(\theta_j))$

We consider: (1) when θ_i is the child, (2) when θ_i is a parent, and (3) the co-parents of θ_i (included in the term $p(\theta_k|parents(\theta_k)$ shown in the equation below).

$= \frac{1}{Z_2} p(\theta_i|parents(\theta_i)) \prod_{\theta_k \epsilon\, children(\theta_i)} p(\theta_k|parents(\theta_k))$

Problem 1.2: Markov chain Monte Carlo

(d) For each problem in b, I tested the number of iterations it took for the estimates to converge within 1% of the exact quantities. Table 15 shows the average number of iterations (out of 5 tests) for each problem in b. Tables 16, 17, 18, and 19 show the number of iterations and difference for the exact value from 5 tests for each part in b. The code can be found in the code directory under script.m and mcmc_test_convergence.m.

Table 13: μ_{W2}

W	μ_{W2}
0	$0.7992 + 0.0001 = 0.7993$
1	$0.784 + 0.002 = 0.786$
2	$0.768 + 0.004 = 0.772$

Table 14: Combine tables Φ_W and μ_{W2}

W	Φ_W	μ_{W2}	$\Phi_W * \mu_{W2}$
0	0.95	0.7993	0.75933
1	0.03	0.786	0.02358
2	0.02	0.772	0.01544

Table 15: Average number of iterations

Problem	Avg. # iterations
b.i.	4694
b.ii.	1404
b.iii.	2743
b.vi	7398

Table 16: b.i. $p(\text{has ebola}|\text{abdominal pain} = T, \text{rash} = T, \text{weeks traveling}) = 0$

# iterations	error(E=f, E=t)
2311	(3.3646e-04, 8.1387e-03)
18580	(4.1329e-04, 9.9969e-03)
1198	(3.8194e-04, 9.2387e-03)
753	(1.4645e-04, 3.5425e-03)
629	(4.7514e-05, 1.1493e-03)

Table 17: b.ii. $p(\text{has ebola}|\text{abdominal pain} = T, \text{rash} = T, \text{weeks traveling} = 2)$

# iterations	error(E=f, E=t)
1930	(0.0087944, 0.0096928)
1345	(0.0087686, 0.0096645)
271	(0.0064377, 0.0070954)
3112	(0.0089614, 0.0098769)
363	(0.0069404 0.0076494)

Table 18: b.iii. $p(\text{weeks traveling}|\text{rash} = T)$

# iterations	error(W=0, W=1, W=2)
2921	(2.9391e-04, 6.0850e-03, 4.0771e-03)
1597	(2.3738e-04, 9.9582e-03, 3.8140e-03)
1789	(4.8224e-04, 8.2764e-03, 9.0960e-03)
4440	(3.9955e-04, 9.8808e-03, 3.2760e-03)
2971	(1.9086e-04, 1.0966e-03, 9.7610e-03)

Table 19: b.iv. $p(\text{weeks traveling}|\text{rash} = F)$

# iterations	error(W=0, W=1, W=2)
7296	(7.2638e-05, 8.5892e-03, 9.9261e-03)
4246	(1.3373e-05, 4.8673e-03, 8.3546e-03)
2971	(5.0293e-04, 9.7610e-03, 9.7610e-03)
1686	(2.4074e-04, 8.3037e-03, 1.0614e-03)
20793	(4.1180e-04, 6.7491e-03, 9.9553e-03)