Homework 2
Kernel SVM and Perceptron

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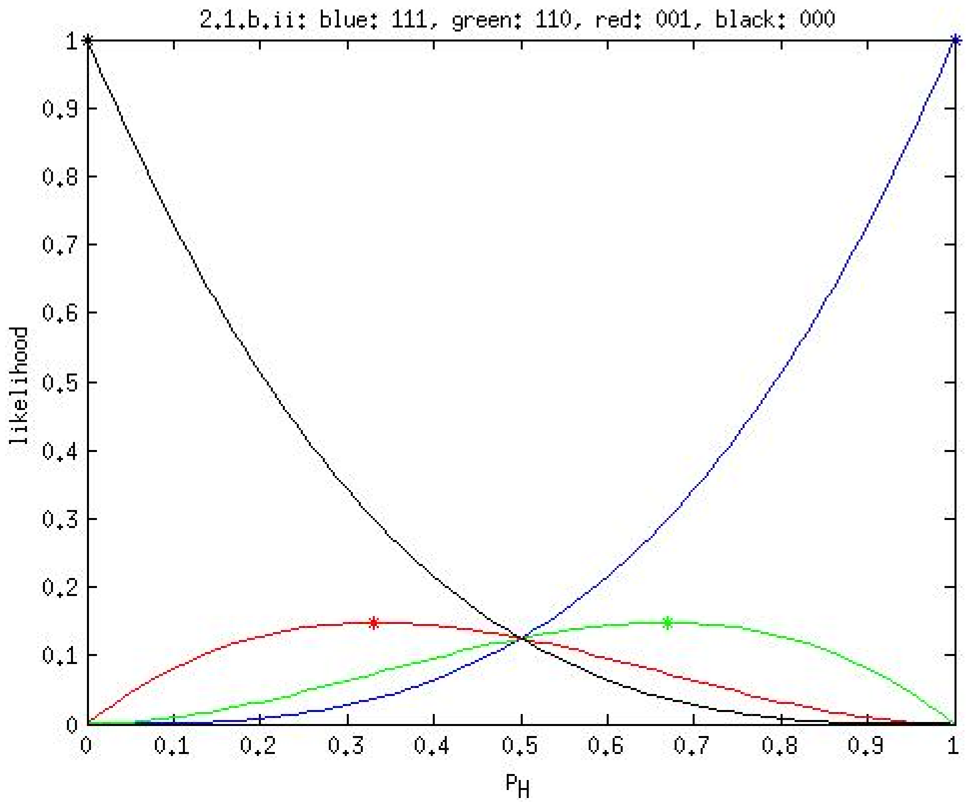
Problem 2: Understanding the Likelihood Function, Bit by Bit

1. (a) $H_i \sim Bernoulli(p_H)$, $\quad N_H \sim Binomial(N_{bits}, p_H)$.

   (b) i. $\mathrm{L}(H_1, ..., H_{Nbits}; p_H) = \prod_{i=1}^{N_{bits}} P(H_i; p_H) = \prod_{i=1}^{N_H} p_H \prod_{i=1}^{N_{bits}-N_H}(1-p_H) = p_H^{N_H}(1-p_H)^{N_{bits}-N_H}$.

   ii. Log-likelihood vs. p_H



These estimates do make sense given the data. It is obvious that the p_H that maximizes the likelihood of getting 3 heads in a row (sequence [111]) is when $p_H = 1$. The intuition is the same for getting all tails (sequence [000]). The likelihood of getting sequence [110] out of 3 coin flips is maximized when the probability of getting heads is 2/3. Similarly, the maximum likelihood of flipping 2 tails out of 3 coin flips is when the probability of getting heads is equal to 1 out of 3 flips.

iii. [000]: $\int_0^1 p_H^0(1-p_H)^3 dp_H = \int_0^1 (1-p_H)^3 dp_H = (-1)\frac{1}{4}(1-p_H)^4\Big|_0^1 = \frac{-1}{4}\left((1-1)^4-(1-0)^4\right) = \frac{1}{4}$

[111]: $\int_0^1 p_H^3(1-p_H)^0 dp_H = \int_0^1 p_H^3 dp_H = \frac{1}{4}p_H^4\Big|_0^1 = \frac{1}{4}(1^4-0^4) = \frac{1}{4}$

[110]: $\int\limits_0^1 p_H^2(1-p_H)^1 dp_H = \int\limits_0^1 (p_H^2 - p_H^3) dp_H = (\frac{1}{3}p_H^3 - \frac{1}{4}p_H^4)\big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

[001]: $\int\limits_0^1 p_H^1(1-p_H)^2 dp_H = \int\limits_0^1 p_H(1-p_H^2) dp_H = \int\limits_0^1 p_H(1-2p_H+p_H^2) dp_H$

$= \int\limits_0^1 (p_H - 2p_H^2 + p_H^3) dp_H = (\frac{1}{2}p_H^2 - \frac{2}{3}p_H^3 + \frac{1}{4}p_H^4)\big|_0^1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$

You can tell that this is not a valid probability distribution over p_H because the total sum of the area under these curves is not equal to 1. The reason that it's invalid is because these 4 sequences are only a subset of the total possible sequences, (for example, we are missing [101], [100], etc.).

(c) We choose p_H that maximizes the probability of the observed sequence. We find this value of p_H by maximizing the likelihood function.

$\hat{p}_H = \arg\max_{p_H} P(H_1, ..., H_{N_{bits}}; p_H) = \arg\max_{p_H} \big(p_H^{N_H}(1-p_H)^{N_{bits}-N_H}\big)$

There are different techniques that can be used to find the value of p_H that maximizes the likelihood function (e.g. taking the derivative, gradient descent). To find this maximizing value, we can take the derivative of this likelihood function, set it equal to 0, and solve for p_H since a closed-form solution exists for this particular likelihood function.

Instead of maximizing the likelihood function, we will instead maximize the log-likelihood function which reaches its maximum value at the same points as the original function, (this is because the logaritm function is monotonically increasing).

$\hat{p}_H = \arg\max_{p_H} log\big(P(H_1, ..., H_{N_{bits}}; p_H)\big) = \arg\max_{p_H} log\big(\big(p_H^{N_H}(1-p_H)^{N_{bits}-N_H}\big)\big)$

$= \arg\max_{p_H} log(p_H^{N_H}) + log(1-p_H)^{N_{bits}-N_H} = \arg\max_{p_H} N_H log(p_H) + (N_{bits} - N_H) log(1-p_H)$

Now take the derivative of the log-likelihood function:

$\frac{d}{dp_H}\big(N_H log(p_H) + (N_{bits} - N_H) log(1-p_H)\big) = \frac{N_H}{p_H} - \frac{N_{bits}-N_H}{1-p_H}$

Set it equal to 0 and solve for p_H:

$\frac{N_H}{p_H} - \frac{N_{bits}-N_H}{1-p_H} = 0$

$\frac{N_H}{p_H} = \frac{N_{bits}-N_H}{1-p_H}$

$\frac{N_H(1-p_H)}{p_H} = N_{bits} - N_H$

$\frac{1-p_H}{p_H} = \frac{N_{bits}-N_H}{N_H}$

$\frac{1}{p_H} - 1 = \frac{N_{bits}}{N_H} - 1$

$p_H = \frac{N_H}{N_{bits}}$

This result shows that the maximizing value of p_H is the number of heads (or 1-bits) divided by the total number of coin tosses (or bits).

2. (a) $L(O_1, ..., O_{N_{bits}}; p_H) = P(O_1, ..., O_{N_{bits}}; p_H) = \prod_{i=1}^{N_{bits}} P(O_i; p_H)$

$= \prod_{i=1}^{N_{bits}} \sum_{t=0}^{1} P(O_i, H_i = t; p_H) = \prod_{i=1}^{N_{bits}} \sum_{t=0}^{1} P(O_i | H_i = t; p_H) P(H_i = t; p_H)$

$= \prod_{i=1}^{N_{bits}} P(O_i|H_i = 0; p_H)P(H_i = 0; p_H) + P(O_i|H_i = 1; p_H)P(H_i = 1; p_H)$

$= \prod_{i=1}^{N_{bits}} P(O_i|H_i = 0; p_H)(1 - p_H) + P(O_i|H_i = 1; p_H)p_H$

Given: $O_i = H_i + \epsilon_i$, $\epsilon i \sim N(0, \sigma^2)$

$H_i = t: \quad O_i = t + \epsilon_i \qquad \text{Solve for } \epsilon_i: \qquad \epsilon_i = O_i - t$

$t = 0: \qquad N(O_i; 0, \sigma^2) \qquad t = 1: \qquad N(O_i - 1; 0, \sigma^2)$

$L(O_1, ..., O_{N_{bits}}; p_H) = \prod_{i=1}^{N_{bits}} = N(O_i; 0, \sigma^2)(1 - p_H) + N(O_i - 1; 0, \sigma^2)p_H$

Let $\alpha_i = N(O_i; 0, \sigma^2)$ and $\beta_i = N(O_i - 1; 0, \sigma^2)$

$L(O_1, ..., O_{N_{bits}}; p_H) = \prod_{i=1}^{N_{bits}} \alpha_i(1 - p_H) + \beta_i p_H$

$$L(O_1, ..., O_{N_{bits}}; p_H) = \prod_{i=1}^{N_{bits}} p_H(\beta_i - \alpha_i) + \alpha_i$$

$$LL(O_1, ..., O_{N_{bits}}; p_H) = \sum_{i=1}^{N_{bits}} log(p_H(\beta_i - \alpha_i) + \alpha_i)$$

Check concavity: check 2nd derivative

$\frac{dLL}{dp_H} = \sum_{i=1}^{N_{bits}} \frac{\beta_i - \alpha_i}{p_H(\beta_i - \alpha_i) + \alpha_i}$

$\frac{d^2LL}{dp_H^2} = \sum_{i=1}^{N_{bits}} -\left(\frac{\beta_i - \alpha_i}{p_H(\beta_i - \alpha_i) + \alpha_i}\right)^2$

The expression inside the parenthesis is always squared so it's guaranteed to be positive. The whole expression is then multiplied by -1 and we take the summation of all of these expressions. This shows that the second derivative is *always* negative for all values of p_H.

Concavity is important because it means there exists a value of some variable that minimizes/maximizes the function we are optimizing. These mins and maxs can be local or (hopefully) global.

(b) $\hat{p}_H = \arg\max_{p_H} \sum_{i=1}^{N_{bits}} log(p_H(\beta_i - \alpha_i) + \alpha_i)$

There is no closed-form solution to this equation. We can use iterative convergence methods, like gradient descent, to obtain the MLE.

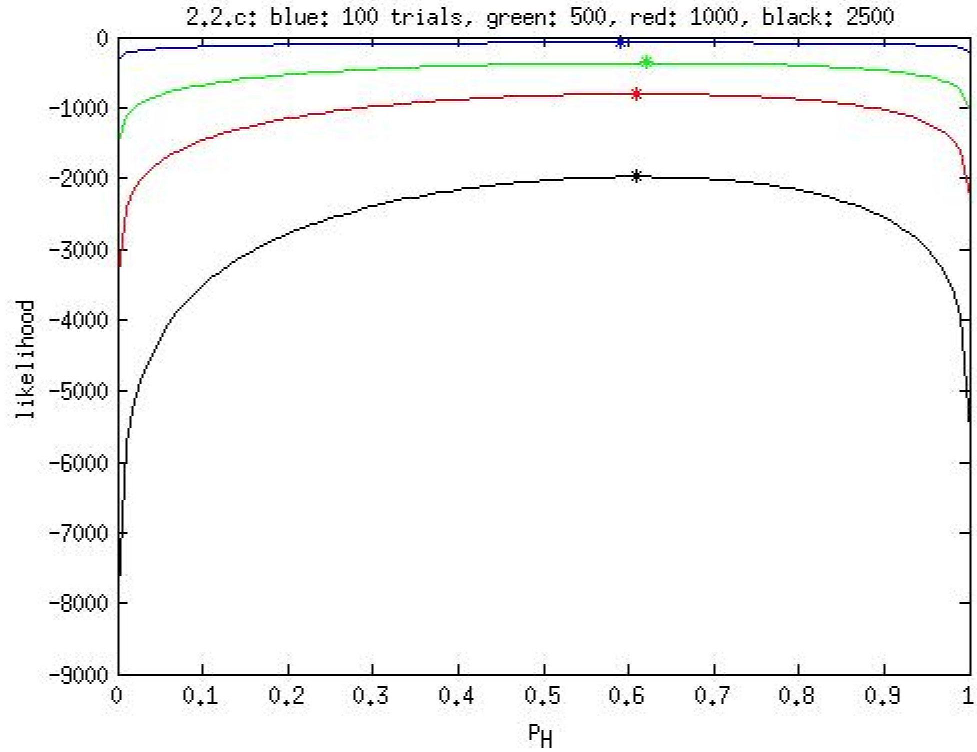
(c) Log-likelihood vs p_H
TODOTODOTODOTODOTODO!!!!!!

(d) $\hat{p}_H$, $\bar{p}_H$ vs Trial Size $\bar{p}_H$ is the observed ratio of heads to the total number of flips. It performs better than $\hat{p}_H$ in the case where our variance is very low (0.1) and constant since a low variance means that noise is going to be very close to 0. As you increase the variance, $\hat{p}_H$ does better, and converges to p_H^* as the trial size increases. TODOTODOTODOTODOTODO!!!!!!

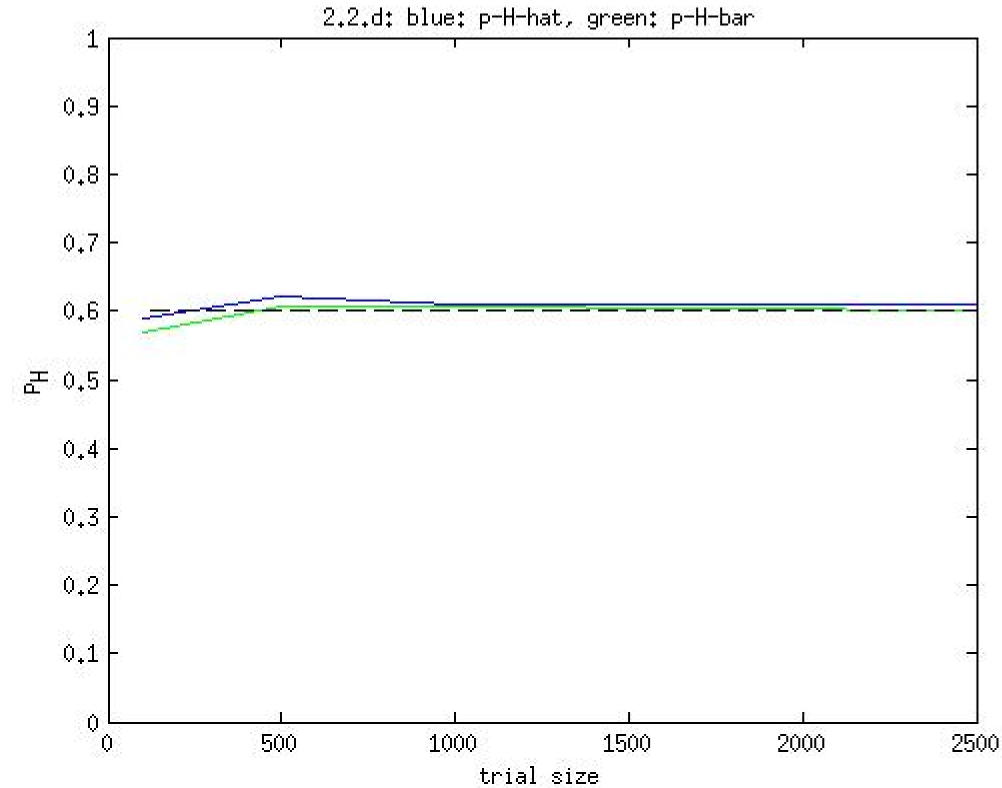
3. (a) $LL(O_1, ..., O_{N_{bits}}; p_H) = \sum_{i=1}^{N_{bits}} log(p_H(\beta_i - \alpha_i) + \alpha_i)$

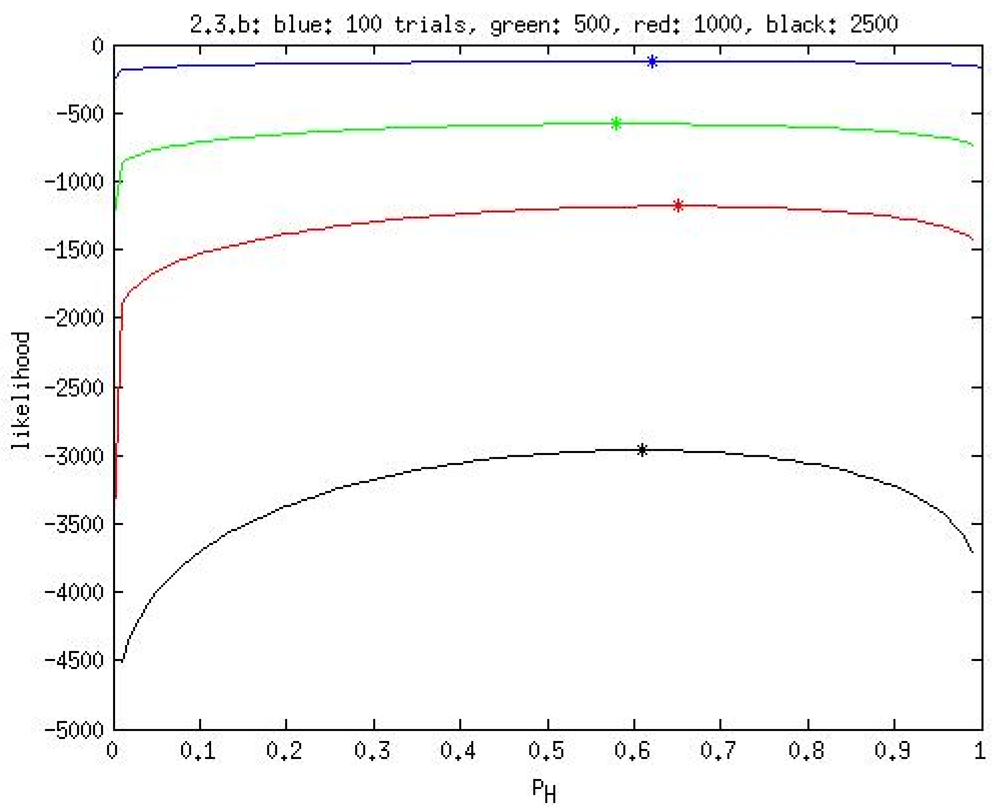
Where: $\qquad \alpha_i \sim N(O_i; 0, \sigma_i^2) \qquad \beta_i \sim N(O_i - 1; 0, \sigma_i^2)$

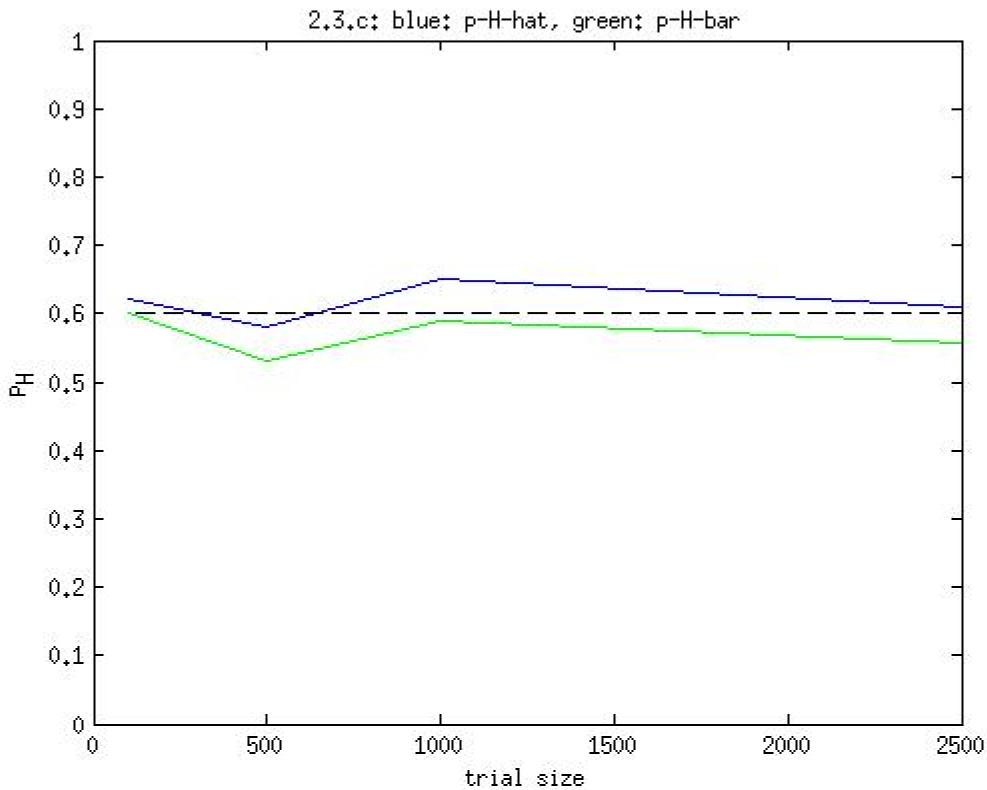
Changing the model to include a sigma that grows with consecutive bits is independent of p_H and



does not change the first or second derivative with respect to p_H. Yhe log-likelihood function is still concave for the same reason as in part 2(a).


2.2.d: blue: p-H-hat, green: p-H-bar
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
p_H
0
500
1000
1500
2000
2500
trial size


2.3.b: blue: 100 trials, green: 500, red: 1000, black: 2500
0
-500
-1000
-1500
-2000
-2500
-3000
-3500
-4000
-4500
-5000
likelihood
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
p_H


2.3.c: blue: p-H-hat, green: p-H-bar
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
p_H
0
500
1000
1500
2000
2500
trial size