Visual Group Theory

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3 september 2017

1 What is a group?

1.1 Group?

1.2 Group?

1.3 Group?

1.4 Three walls in your bedroom hold pieces of art, one hung on each wall. You are rearranging them to see which arrangement best suits your taste. You cannot use the fourth wall, because it has a window.

(a) Count the number of ways there are to rearrange the pictures, as long as only one is hung on each wall.

Solution:
There are 6 possible configurations.

(b) Consider two actions: You may swap the art on the left wall with the art on the center wall, and you may swap the art on the center wall with the art on the right wall. Can these actions alone generate all of the configurations you counted?

Solution:
Yes, consider

$$abc \to bac \to bca \to cba \to cab \to acb$$

(c) Does part (b) describe a group? If not, waht rule or rules were broken?

Solution:
It describes a group.

2 What do groups look like?

2.1 In the rectangle puzzle, what actions were the generators? What other actions are there besides the generators?

Solution:

Horizontal flip and vertical flip

Other actions are the 'do nothing' and the 'horizontal followed by the vertical flip'

2.2 In the light switch puzzle, what actions were the generators? What other actions are there besides the generators?

Solution:

See previous exercice

2.3 Can an arrow in a Cayley diagrem ever connect a node to itself?

Solution:

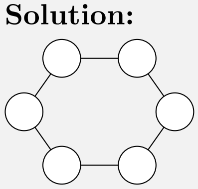
Yes, the 'do nothing' action.

2.4 Exercices 1.1 of Chapter 1 defined a group. Create its Cayley diagram using the technique from this chapter.

Solution:

2.5 Exercise 1.4 of Chapter 1 defined a group. Create its Cayley diagram using the technique from this chapter.

Solution:



2.6 Exercise 1.13 describes an infinite group which can be generated with just one generator. Can you draw an infinite Cayley diagram for it? (Just draw a portion of the diagram that makes the infinite pattern clear)

How does that Cayley diagram compare to one for the group in Exercise 1.14 part (a)?

2.7 Exercise 1.14 part (d) described a two-element group. Can you draw a Cayley diagram for it? Which arrow or arrows should you use and why?

Solution:

$1 \leftrightarrow -1$ en op beide element een pijl naar zichzelf voor de actie '$\cdot 1$'.

2.8 Section 2.2 introduced the rectangle puzzle. Imagine instead a square puzzle with its corners labeled the same way. Such a puzzle would allow a new move that was not possible with the rectangle puzzle, you could rotate a quarter-turn clockwise.

(a) Make the map of this group

Solution:
Zie D_4

(b) Why is the quarter-turn move not 'allowed' in the rectangle puzzle?

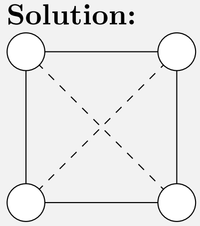
Solution:
Zie later

2.9 Most groups can be generated may different ways, and each way gives rise to a corresponding way to connect a Cayley diagram with arrows. For example, consider the group V_4, which we met in the rectangle puzzle. Let's shorten the names of its actions to n, h, v and b, meaning (respectively) no action, horizontal flip, vertical flip, and both (a horizontal flip followed by a vertical flip).

We sam that h and v together generate V_4. But it is also true that h and b together would generate V_4, or v an b together. (You can verify these facts by exploring the rectangle realm using these generators on your own numbered rectangle.)

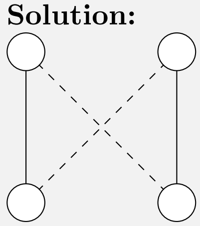
(a) Make a copy of Figure 2.9 and add to it a new type of arrow, representing the action b.

Solution:



(b) Make a copy of your answer to part (a), with the arrows representing h removed. How does your diagram show that v and b are sufficient to generate V_4?

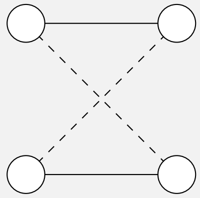
Solution:



It is still possible to reach each situation.

(c) Make a copy of your answer to part (a)n, with the arrows representing v removed. How does your diagram show that h and b are sufficient to generate V_4?

Solution:



It is still possible to reach each situation.

2.10 If you've done all the exercises to this point, you've encountered two different Cayley diagrams that have the two-node form shown here.

Can you come up with another group whose Cayley diagram has this form?

2.11 If you've done all the exercices to this point, you've encountered two different Cayley diagrams that have the four-node form shown here.

Can you come up with another group whose Cayley diagram has this form?

2.12 We have not yet seen a group whose Cayley diagram has the three-node form called C_3, shown in the top left of Figure 2.10. Can you come up with a group whose Cayley diagram has that form?

2.13 A group's generators have a special status in a Cayley diagram for the group. What is that special status?

Solution:

Represented by the arrows.

2.14 Chapter 1 required groups to satisfy Rule 1.5, which states, "There is a predefined list of actions that never changes." How does this rule impact the appearance of Cayley diagrams? (Or how would diagrams be different if this rule were not a requirement?)

Solution:

If a action would not be executed on a certain node.

2.15 Chapter 1 required groups to satisfy Rule 1.6, which states, "Every action is reversible." What constraint does this place on the arrows in a Cayley diagram? Can you draw a diagram that does not fit this constraint? (That is, draw a diagram that almost deservers the name "Cayley diagram", except for that one rule violation.)

Solution:

Every node should have an arrow departing and arriving.

2.16 Chapter 1 required groups to satisfy Rule 1.7, which states, "Every action is deterministic." What constraint does this place on the arrows in a Cayley diagram? Can you draw a diagram that does not fit this constraint? (That is, draw a diagram that almost deserves the name "Cayley diagram," except for that one rule violation.)

Solution:

An arrow departing from a node can only arrive in one node.

2.17 Chapter 1 required groups to satisfy Rule 1.8, which states, "Any sequence of consecutive actions is also an action." How do we depend upon this fact when using a Cayley diagram as a map?

Solution:

One should be able to reach every node by executing consecutive actions.

2.18 If we created a equilateral triangle puzzle, like the square puzzle in exercise 2.8, what would the valid moves be? Map the group of such a puzzle.

Solution:

Flip with respect to any symmetry axis. (Three flips) And a rotation over 60°, 120°and 180°.

This forms D_3.

It has 2 generating actions, one flip and one rotation.

2.19 A regular n-gon is a polygon with n equal sieds an n equal angles. You have already analyzed regular n-gons with $n = 3$. (equilateral triangle, Exercise 2.18) and $n = 4$ (square, Exercise 2.18).

(a) Based on what you know about the cases $n = 3$ and $n = 3$, make a conjecture about how many actions will be in the group of a regular n-gon for any $n > 2$.

Solution:

There are n axes of symmetry and n rotations of $360°$: n. This results in $2n$ symmetries.

(b) Test your conjecture by making te map of the group for a regular pentagon ($n = 5$).

Solution:

D_5 has 10 symmetries.

3 Why study groups?

4 Algebra at last

4.1 Consider the lightswitch group shown in Figure 2.8. Let L stand for the action of flipping the left switch and R stand for the action of flipping the right switch.

(a) Which of the following equations are true and which are false in this group?

Solution:

$LRRRR = RRL$

$LR = RLRLRL$

$L \neq RR$

$R^8 = R^{100}$

(b) Let N stand for the non-action (leaving the switches untouched). Which of the following equations are true?

Solution:

$(LNR)^2 \neq LNR$

$RL \neq N$

$(LNR)^3 = R^3L^3$

$NN = N$

$R^4 = N$

$LRLR = N$

(c) What is the smallest power or R that equals N?

Solution:

2

(d) What is the smallest power or L that equals N?

Solution:

2

(e) What is the smallest power or RL that equals N?

Solution:

2

(f) What is the smallest power or LR that equals N?

Solution:

2

4.2 (a) Apply the transformations in Definition 4.1 to the lightswitch Cayley diagram in Figure 2.8.

(b) Create a multiplication table for the lightswitch group

Solution:

e	L	R	LR
L	e	LR	R
R	LR	e	L
LR	R	L	e

4.3 Each part below describes a set with a binary operation on it. For each one, determine whether it is commutative and whether it is associative.

(a) the addition operation on the set of all whole numbers

Solution:

commutative and associative

(b) the subtraction operation on the set of all whole numbers

Solution:

not commutative and not associative

(c) the multiplication operation on the set of positive real numbers

Solution:

commutative and associative

(d) the division operation on the set of positive real numbers

Solution:

not commutative and not associative

(e) the exponentiation operation on the set of positive whole numbers (that is, the operation written a^b)

Solution:

not commutative and not associative

4.4 The Cayley diagrams for two groups are shown here, the cyclic group C_5 on the left and the Quaternion group Q_4 on the right.

(a) The red arrow in the diagram for C_5 represents multiplication wby what element?

Solution:

$\cdot a$

(b) What is $a^3 \cdot a$ in C_5?

Solution:

a^4

(c) What is $a^3 \cdot a \cdot a$ in C_5?

Solution:

e

(d) If 1 is the identity element, then wat do red arrows in the diagram for Q_4 represent? What do blue arrows represent?

Solution:

red $= \cdot i$

blue $= \cdot j$

(e) What is i^2? What is $j \cdot i$?

Solution:

$i^2 = -1$, $j \cdot i = k$

(f) What is $i \cdot j \cdot j$?

Solution:

$-i$

4.5 Using the Cayley diagrams from Exercise 4.4, answer the following questions.

(a) How do you use the diagram of C_5 to multiply $x \cdot a^2$ in C_5, for any element x?

Solution:

Rotate two turns.

(b) How do you use the diagram of Q_4 to multiply $x \cdot k$ in Q_4, for any element x?

Solution:

Notice how $k = j \cdot i$, then $\cdot k$ means one has to follow a blue arrow, and then a red arrow.

4.6 Create a multiplication table for each of the following Cayley diagrams.

(a) C_5, as shown on the left of Exercise 4.4. Use the template given here.

Solution:

e	a	a^2	a^3	a^4
a	a^2	a^3	a^4	e
a^2	a^3	a^4	e	a
a^3	a^4	e	a	a^2
a^4	e	a	a^2	a^3

(b) Q_4, the quaternion group with eight elements, as shown on the right of Exercise 4.5. Use the template given here.

Solution:

1	i	j	k	-1	$-i$	$-j$	$-k$
i	-1	k	$-j$	$-i$	1	$-k$	j
j	$-k$	-1	i	$-j$	k	1	$-i$
k	j	$-i$	-1	$-k$	$-j$	i	1
-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-j$	i	1	k	j	$-i$	-1

(c) A_4, the alternating group with twelve elements:

Solution:

e	a	b	c	d	a^2	b^2	c^2	d^2	x	y	z
a	a^2	c^2	d^2	b^2	e	y	x	z	b	d	c
b	d^2	b^2	a^2	c^2	y	e	z	x	a	c	d
c	b^2	d^2	c^2	a^2	x	z	e	y	d	b	a
d	c^2	a^2	b^2	d^2	z	x	y	e	c	a	b
a^2	e	x	z	y	a	d	b	c	c^2	b^2	d^2
b^2	x	e	y	z	c	b	d	a	d^2	a^2	c^2
c^2	z	y	e	x	d	a	c	b	a^2	d^2	b^2
d^2	y	z	x	e	b	c	a	d	b^2	c^2	a^2
x	c	d	a	b	b^2	a^2	d^2	c^2	e	z	y
y	b	a	d	c	d^2	c^2	b^2	a^2	z	e	x
z	d	c	b	a	c^2	d^2	a^2	b^2	y	x	e

4.7 It is possible to suggest the full multiplication table for an infinite group by showing just part of it. Fill in the following partial table for the operation of addition on the set of all whole numbers; the ellipses indicate the table continues infinitely in all directions.

4.8 Exercises 2.4 through 2.8 of Chapter 2 asked you to draw Cayley diagrams for three groups. Use the diagrams you drew to make multiplication tables for those same groups. Note that if your diagram is not yet a diagram of actions, you may need to apply the transformation in Definition 4.1.

4.9 Exercises 2.18 and 2.19 of Chapter 2 asked you to find the pattern describing the sequence of Cayley diagrams for the "n-gon puzzle." I mentioned in that exercise that the family of groups describing such puzzles are called the dihedral groups. You will study them in detail in Chapter 5, and this exercise previews some of that material.

Find the pattern decsribing the sequence of multiplication tables for those same groups. You might consider the following steps.

(a) Create multiplication tables from the Cayley diagrams for triangle, square, and regular pentagon puzzles.

Solution:

Triangle

e	r	r^2	f	rf	r^2f
r	r^2	e	rf	r^2f	f
r^2	e	r	r^2f	f	rf
f	r^2f	rf	e	r^2	r
rf	f	r^2f	r	e	r^2
r^2f	rf	f	r^2	r	e

Square

e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	e	r	r^2	r^3f	f	rf	r^2f
f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^2f	rf	f	r^3	r^2	r	e

Notice how smaller squares can be found in the large square. The Cayley-graph has an inner and outer circle.

4.10 Consider te following multiplication table that displays an binary operation.

(a) Explain succinctly why the binary operation is not associative. Can you write your answer as one equation?

Solution:

Consider $A \cdot A \cdot B$, then $A \cdot (A \cdot B) = A \cdot e = A$ while $(A \cdot A) \cdot B = e \cdot B = B$

(b) Does the operation have inverses?

Solution:

Yes, but they are not unique. (See also Exercise 4.11)

4.11 Consider the following multiplication table that displays a binary operation.

(a) Explain succinctly why the binary operation does not have inverses. Can you write your answers as one equation?

Solution:

Notice how 3 is the identity element. But notice how there is no inverse for 2 and inverse for 1.

(b) Is the operation associative?

Solution:

Yes.

4.12 Consider the following multiplication table that displays a binary operation.

(a) Does this operation have inverses? Justify your answer.

Solution:

No, x and y have no inverses.

(b) Is the operation associative? Justify your answer.

Solution:

Seems like it.

4.13 For each multiplication table below, explain why it does not depict a group.

(a) Not associative, $(4 \cdot 2) \cdot 3 = 1 \cdot 3 = 3$ while $4 \cdot (2 \cdot 3) = 4 \cdot e = 4$.

(b) a, b, c have no inverses.

(c) Not associative, $(c \cdot a) \cdot c = c \cdot c = e$ while $c \cdot (a \cdot c) = c \cdot a = c$.

(d) There is no identity.

4.14 The following multiplication table does not depict a binary operation on the set $\{e, x, y\}$. The reason is part of the definition of a binary operation; we would say that this binary operation lacks **closure**. Can you spot the problem and explain it in your own words?

Solution:

$y \cdot x = s \notin \{e, x, y\}$.

4.15 Why can the same element not appear twice in any row of a group's multiplication table? Does this restriction also apply to columns?

Solution:

If $a \cdot b = a \cdot c$ then $b = c$, since $a^{-1} \cdot a \cdot b = c \Rightarrow b = c$.

The same rule applies for columns. $a \cdot b = c \cdot b \Rightarrow a = c$.

4.16 Exercises ...

4.17 When crating a multiplication table for a group, if you try to include two different identity elements, what goes wrong? What does this lead you to conclude about groups?

Solution:

Then the multiplication table will have the same element appearing twice in a row and column. Because $e \cdot b = \tilde{e} \cdot b \Rightarrow e = \tilde{e}$.

4.18 Explain why a Cayley diagram must be connected. That is, why must there be a path from every node to every node?

Solution:

If this is not the case, then some element will appear twice in a row or column.

Different motivation, the identity element would not be connected to everything.

4.19 Complete each of the following multiplication tables so that it depicts a group. There is only one way to do so, if we require 0 to be the identity element in each table. Then search Group Explorer's group librrary to determine the names for the groups the tables represent.

Solution:

(a) C_2

0	1
1	0

(b) C_3

0	1	2
1	2	0
2	0	1

(c) $C_1 = \{0\}$

0

(d) C_4

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

(e) Looks like V_4?

0	1	2	3
1	3	0	2
2	0	3	1
3	2	1	0

4.20 The following table can be completed in more than one way, and still have the result depict a group. Find all possible such completions of the table, again using 0 as the identity element. How many did you find? Search Group Explorer's group library to determine the names for the groups each of your resulting tables represents.

4.21 From Exercise 4.19 part (a) you can conclude that there is only one pattern for a group containing two elements. This is because the only difference between the multiplication table you computed and that of any other group with two elements will be the names of those elements. So the pattern of interactions among elements (or colors if we were to color the cells of the table) would be no different.

(a) How many patterns are there for groups containing three elements?

Solution:

Only one, C_3.

(b) Containing one elements?

Solution:

Only one, $\{e\} = 1$.

(c) Containing four elements?

Solution:

Two, V_4 and C_4.

4.22 We saw earlier in this chapter that in the group V_4, the equation $RB = BR$ is true. In fact, for any two elements $a, b \in V_4$, the equation $ab = ba$ is true. That is, the order in which you combine elements does not matter. Consider each group whose multiplication table appears in Figure 4.7 (except A_5, whose details are to small to see). For which of those groups does the order of combining elements matter?

Solution:

S_3, Quasihedral group with 16 elements.

4.23 Groups in which he order of multiplication of elements does not matter are called commutative or abelian. Look through the groups in Group Explorer's group library, starting with the smallest, until you find one that is noncommutative. What is the name of the smallest noncummutative group?

Solution:

S_3 of order 6.

4.24 What visual pattern do the multiplication tables of commutative groups exhibit?

Solution:

Diagonal symmetry.

4.25 To go along with the other algebraic notation we've seen in this chapter, there is also an algebraic notation for generators. For instance, the group C_5, which appears in the first few exercises of this chapter, is generated by the element a. The standard notation for this is $C_5 = \langle a \rangle$. The $\langle a \rangle$ means "what you can generate from a," and so the equation $C_5 = \langle a \rangle$ is saying "C_5 is the group generated from a." From Figure 4.3, we can write $V_4 = \langle R, B \rangle$, saying that R and B together generate V_4.

Show your understanding of this new notation by filling in the blanks below using however many elements are necessary to generate the group. Use as few elements as possible.

(a) From the Cayley diagram in Exercise 4.4, we see that $Q_4 = \langle \rangle$

Solution:

$Q_4 = \langle i, j \rangle$

(b) From the Cayley diagram in part (c) of Exercise 4.6, we see that $A_4 = \langle \rangle$

Solution:

$A_4 = \langle a, x \rangle$

4.26 Use the multiplication tables you constructed in Exercise 4.6 to determine the inverses for each element of each of the three groups from that problem.

(a) In the cyclic group C_5, the inverses are

Solution:

$e^{-1} = e, a^{-1} = a^4, (a^2)^{-1} = a^3, \ldots$

(b) In the quaternion group Q_4, the inverses are

Solution:

$1^{-1} = 1, i^{-1} = -i, j^{-1} = -j, k^{-1} = -k$
$(-1)^{-1} = -1, (-i)^{-1} = i, \ldots$

(c) In the alternating group A_4, the inverses are

Solution:

$e^{-1} = e, a^{-1} = a^2, b^{-1} = b^2, c^{-1} = c^2, \ldots$

4.27 Inverses can be used to solve equations. In the group C_5, to solve $a^2x = a$ for x, I can proceed as in high school algebra:

Computing $(a^2)^{-1}a$ in C_5 gives $x = a^4$.

Try solving each of these equation in C_5.

(a) $a^3x = a^2$

Solution:

$\Leftrightarrow x = a^4$

(b) $a^4a^2x = a$

Solution:

$\Leftrightarrow ax = a$

$\Leftrightarrow x = e$

(c) $ax(a^3)^{-1} = e$

Solution:

$\Leftrightarrow ax = a^3$

$\Leftrightarrow x = a^2$

4.28 (a) If I have the equation $a^2x(a^2)^{-1} = a$ to solve as in the previous exercise, can I cancel the a^2 and the $(a^2)^{-1}$? Why or why not?

Solution:

Yes, as C_5 is abelian.

(b) If I have a similar equation, but in the group Q_4 from Exercise 4.6, $ixi^{-1} = j$, can I cancel the i and i^{-1}? Why or why not?

Solution:

No, as it is not abelian.
Note how $i \cdot j \cdot i^{-1} = k \cdot (-i) = -j \neq j$

4.29 Consider the equation $b^2 \cdot t \cdot a^2 = y$ in the group A_4; I want to solve for t. The previous exercise as ia warning that I cannot simply proceed as follows. What should I do instead?

Solution:

$\Leftrightarrow t \cdot a^2 = (b^2)^{-1} \cdot y = b \cdot y = c$

$\Leftrightarrow t = c \cdot (a^2)^{-1} = c \cdot a = b^2$

4.30 Solve these equations for t.

(a) In $Q_4, jitk^{-1} = -kj$

Solution:

$\Leftrightarrow -kt = i \cdot k$
$\Leftrightarrow t = (-k)^{-1} \cdot (-j)$
$\Leftrightarrow t = k \cdot (-j) = i$

(b) In $A_4, t(b^2)^2 = xyz$

Solution:

$\Leftrightarrow tb = e$
$\Leftrightarrow t = b^{-1} = b^2$

(c) In $S_3, rtf = e$

Solution:

$\Leftrightarrow rt = f$
$\Leftrightarrow t = r^2 \cdot f = fr$

4.31 Let's say you have a group G with identity element e. Take any three elements a, b, and c in G.

(a) What does the equation $ab = e$ say about the relationship between a and b?

Solution:

a is the inverse of b and vice versa.

(b) If both $ab = e$ and $ac = e$, can you use algebra to show $b = c$?

Solution:

It follows that $ab = ac \Rightarrow b = a^{-1}ac = c$.

(c) Can an element in a group have two different inverses?

Solution:

No, since $a^{-1} \cdot a = \tilde{a}^{-1}a \Rightarrow a^{-1} = \tilde{a}^{-1}$.

4.32 The set of all integers (all positive and negative whole numbers, and zero) is often written als $\mathbb{Z}$. Use Definition 4.2 to answer each of the following questions about $\mathbb{Z}$.

(a) Is it a group using ordinary addition as the operation?

Solution:

Yes.

(b) Is it a group using ordinary multiplication as the operation?

Solution:

No, there are inverses missing.

(c) The even integers are sometimes written $2\mathbb{Z}$, because they can be obtained by multiplying every integer by 2. If we think of $3\mathbb{Z}, 4\mathbb{Z}$, and in general any $n\mathbb{Z}$ in the same way, for what integers n is the set $n\mathbb{Z}$ a group using ordinary addition as the operation?

Solution:

$\forall n \in \mathbb{Z}$.

4.33 The rational numbers (often written $\mathbb{Q}$) are the set of fractions $\frac{a}{b}$, where a and b are integers (but $b \neq 0$). For example $\frac{1}{2}, \frac{-6}{11}$, and $\frac{50}{3}$ are all rational. Any integer, including zero, is rational, because you can just divide it by 1. For example, 10 is the rational number $\frac{10}{1}$.

Use Definition 4.2 to answer each of the following question about $\mathbb{Q}$.

(a) Is it a group using ordinary addition as the operation?

Solution:

Yes.

(b) Is it a group using ordinary multiplication as the operation?

Solution:

No, zero has no inverse.

(c) Call $\mathbb{Q}^+$ the positive rational numbers (only those greater than zero). Is $\mathbb{Q}^+$ a group using ordinary addition as the operation?

Solution:

No, there is no identity and no inverses.

(d) Is $\mathbb{Q}^+$ a group under ordinary multiplication?

Solution:

Yes.

(e) Call $\mathbb{Q}^*$ the nonzero rational numbers (all positive and negative ones, only leaving out zero). Is $\mathbb{Q}^*$ a group under ordinary addition?

Solution:

No, there is no identity.

(f) Is $\mathbb{Q}^*$ a group under ordinary multiplication?

Solution:

Yes.

(g) Why are groups like $\mathbb{Q}, \mathbb{Q}^+$, and $\mathbb{Q}^*$ difficult to visualize using multiplication tables and Cayley diagrams?

Solution:

Because they are difficult to list...

5 Five families

5.1 If a group is generated by just one elemnt, what kind of group is it?

Solution:

A cyclic group

5.2 (a) In the group C_5, compute $2+2$.

Solution:

$2+2=4$

(b) In the group C_5, compute $4+3$.

Solution:

$4+3=2$

(c) In the group C_{10}, compute $8+7$.

Solution:

$8+7=5$

(d) In the group C_{10}, compute $9+1$.

Solution:

$9+1=0$

(e) In the group C_3, compute $2+2+2+2+2+2$.

Solution:

$2+2+2+2+2+2=0$

(f) In the group C_{11}, compute $10-8+1-7+6+5$.

Solution:

$10-8+1-7+6+5=-4=7$

5.3 For each statement below, determine if it is true or false.

(a) Every cyclic group is abelian.

Solution:

True.

(b) Ever abelian group is cyclic.

Solution:

False, consider V_4

(c) Every dihedral group is abelian.

Solution:
False, consider D_4, then $\sigma \circ \tau_X \neq \tau_X \circ \sigma$.

(d) Some cyclic groups are dihedral.

Solution:
False, it would imply that some dihedral groups are abelian. Which is false.

(e) There is a cyclic group of order 100.

Solution:
True, C_{100}.

(f) There is a symmetric group of order 100.

Solution:
False, $|S_4| = 4! = 24$ while $|S_5| = 5! = 120$

(g) If some pair of elements in a group commute, the group is abelian.

Solution:
False, consider D_4.

(h) If every pair of elements in a group commute, the group is cyclic.

Solution:
False, consider V_4.

(i) If the pattern on the left of Figure 5.8 appears nowhere in the Cayley diagram for a group, then the group is abelian.

Solution:
True.

5.4 (a) Use the Cayley diagram of the group D_5 in Figure 5.17 to compute $r \cdot f \cdot r$ in that group.

Solution:
$r \cdot f \cdot r = f$

(b) Is the answer the same or different if you do the computation in the group D_3 instead?

Solution:
$r \cdot f \cdot r = f$

(c) Is the answer the same or different if you do the computation in the group D_n instead?

Solution:
$r \cdot f \cdot r = f$, because $r \cdot f = r^{-1} \cdot ? = f \cdot r \cdot f \cdot ?$, which results in $? = f^{-1} = f$.

5.5 Compare the strengths and weaknesses of the tree visualization techniques introduced in this book: Cayley diagrams, multiplication tables, and cycle graphs.

5.6 Sketch the following visualizations.

(a) a cycle graph for C_9

(b) a Cayley diagram for D_4

(c) a multiplication table for D_2

Solution:

0	1
1	0

5.7 Describe in words what each of the following visualizations look like for C_{999}

(a) Cayley diagram

(b) multiplication table

(c) cycle graph

5.8 Describe in words what each of the following visualizations look like for D_{999}

(a) Cayley diagram

(b) multiplication table

(c) cycle graph

5.9 What are the orders for the first ten symmetric groups, S_1 through S_{10}? What are the orders of their corresponding alternating groups, A_1 trough A_{10}? Explain your answer for the order of A_1.

Solution:

$|S_n| = n!$

Since A_1 is constructed by considering the squares of elements from S_1 one counts $|A_1| = 1$.

$|A_2| = 1$ while $|S_2| = 2$,

$|A_3| = 3$ (Figure 5.25) while $|S_3| = 3! = 6$

$|A_4| = 12$

$|A_n| = n! : 2$

5.10 The exercises for Chapter 3 asked you to create several Cayley diagrams. This chapter introduced a mehtod for telling whether a group is abelian based on its Cayley diagram. For each of the Chapter 3 exercises mentioned below, first determine whether the group belongs to any of the five families introduced in this chapter, and if so, what the group's name is (e.g. D_4, S_3, etc.). Explain how you determine each of your answers.

5.11 Explain why every cyclic group is abelian.

Solution:

Rotations around a fixed center is a commutative operation.

5.12 Qhy is it sufficient, when looking to see if a Cayley diagram represents an abelian group, to only consider the arrows? Why do we not need to examine every possible combination of paths?

Solution:

Because the path is a constructed as a sequence of arrows.

5.13 (a) Create a cycle graph for the group V_4 using the multiplication table in Figure 5.31.

Solution:
petal flower with three petals

(b) Create a cycle graph for the group A_4 using the Cayley diagram in Exercise 4.6 part (c).

Solution:
Four 2 node petals, 3 1 node petal

5.14 (a) Is there a dihedral group of order 7?

Solution:
No $|D_n| = 2n$

(b) If A_n has order 2520, what is n?

Solution:
$7, 7! = 5040$

(c) If A_n has order m, what order does S_n have?

Solution:
$2m$

5.15 For each part below, compute the orbit of the element in the group. Your answer will be alist of elements from the group that ends with the identity.

(a) The element r^2 in the group D_{10}

Solution:
$= \{r^2, r^4, r^6, r^8, e\}$

(b) The element 10 in the group C_{16}

Solution:
$= \{10, 4, 14, 8, 2, 12, 6, 0\}$

(c) The element 25 in the group C_{30}

Solution:
$= \{25, 20, 15, 10, 5, 0\}$

(d) The element 12 in the group C_{42}

Solution:

$= \{12, 24, 36, 6, 18, 30, 0\}$

(e) The element s in the group whose Cayley diagram is on the left below. (Assume the element a at the top left is the identity.)

Solution:

$= \{\ s, m, j, a\}$

(f) The element l in the group whose Cayley diagram is on the right below. (Assume the element a at the top is the identity.)

Solution:

$= \{l, e, p, a\}$

5.16 Recall the notation for generators from Exercise 4.25. Use it to fill in the blanks below with however many elements necessary to generate the group. Use as few elements as possible.

(a) $C_n = \{0, 1, \ldots, n-1\}$

Solution:

$= \langle 1 \rangle$

(b) $D_n = \{e, r, \ldots r^{n-1}, f, fr, \ldots, fr^{n-1}\}$

Solution:

$= \langle r, f \rangle$

5.17 Create multiplication tables for the smallest dihedral groups D_1, D_2, D_3 and so on, until you find the first non-abelian member of the family. Which is it an how can you tell?

Solution:

D_1

e

D_2

e	r
r	e

D_3

e	r	r^2	f	rf	r^2f
r	r^2	e	rf	r^2f	f
r^2	e	r	r^2f	f	rf
f	r^2f	rf	e	r^2	r
rf	f	r^2f	r	e	r^2
r^2f	rf	f	r^2	r	e

Is niet Abels. Zo geldt bijvoorbeeld dat $fr = r^2f$ terwijl $rf = rf$.

5.18 Repeat Exercise 5.17 for the symmetric groups S_n. Use the permutation notation from this chapter.

Solution:

S_1

e

S_2

e	(12)
(12)	e

S_3

e	(12)(3)	(1)(23)	(13)(2)	(123)	(132)
(12)(3)	e	(123)	(132)	(1)(23)	(13)(2)
(1)(23)	(132)	e	(123)	(13)(2)	(12)(3)
(13)(2)	(123)	(132)	e	(12)(3)	(1)(23)
(123)	(13)(2)	(12)(3)	(1)(23)	(132)	e
(132)	(1)(23)	(13)(2)	(12)(3)	e	(123)

Is niet Abels.

5.19 For each symmetric group whose multiplication table you created in Exercise 5.18, compute the elements of the corresponding alternating group, as in Figure 5.25. For each alternating group you compute, create

(a) a multiplication table,

Solution:

$A_1 = S_1$

$A_2 = S_1$

$A_3 = \{e, (123), (132)\}$

e	(123)	(132)
(123)	(132)	e
(132)	e	(123)

(b) a Cayley diagram, and

(c) a cycle graph

5.20 Some of the smallest meembers of the families C_n, D_n, S_n and A_n actually belong to more than one family, as long as we do not care about the names of the elements, but about the group structure. For instance, D_1 is a group with two elements, and its mulitplicationt able has the same pattern as that of C_2, as shown here.

What other groups belong to more than one of the families we studied in this chapter? (Another way to read this question is, "are there any groups of the families C_n, D_n, S_n, or A_n that are isomorphic to a group in another of those families?")

Solution:

$D_1 \cong S_1 \cong C_1 \cong A_1$

$D_2 \cong S_2 \cong C_2$

$C_3 \cong A_3$

$D_3 \cong S_3$

5.21 For each of the following questions, either exhibit a group that answers the question in the affirmative or give a clear explanation of thy the answer to the question is negative.

(a) Is here a cyclic group with exactly four generators? (Not that it takes four elements to generate the group, but that there are four different elements a, b, c, d in C_n and $C_n = \langle a \rangle = \langle b \rangle = \langle c \rangle \langle d \rangle$.) Is there more than one such group?

Solution:

Yes, C_8 with generators $\{1, 3, 5, 7\}$

C_5 with generators $\{1, 2, 3, 4\}$ complies as well.

(b) Is there a cyclic group with exactly one generator? Is there more than one?

Solution:

Both $C_1 = \langle 0 \rangle$ and $C_2 = \langle 1 \rangle$ comply.

5.22 Group Explorer

5.23 Broad

5.24 This chapter gave regular polygons as examples of objects whose symmetries are described by dihedral groups, that is, objects with both rotational and bilateral symmetry, but no other symmetries. What other objects fit in this category?

Solution:

Snowflakes, rounded polygons, ...

5.25 Analyze the symmetries of a tetrahedro using the technique from Defintion 3.1, resulting in the Cayley diagram for its symmetry group. Here are a few hints to get you started.

5.26 As you now from this chapter, S_3 and D_3 are two different names for the same group. Yet no larger dihedral group is also a symmetric group. Give an argument based on the physical featers of a n-gon for why this is so? ($n \geqslant 4$).

Solution:

$n! \geqslant 2n$ when $n \geqslant 4$. A regular n-gon has exactly n axis of symmetry, and n rotations resulting in $2n$ transformations.

5.27 Section 5.2.3 describes what the cycle graph will look like for $C_p \times C_p$ if p is a prime number. Draw the cycle graph for $C_5 \times C_5$. (It is not necessary to label the elements.)

Solution:

d

6 Subgroups

7 Products and quotients

8 The power of homomorphisms

9 Sylow theory