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Thermodynamics of Circumbinary Disks
November 21, 2014

1 Introduction

We consider the evolution of a circumbinary disks. We do not yet allow for a dead zone in the disk.

The thermal timescale is much smaller than the viscous timescale in a circumbinary disk, and so we approximate the temperature in equilibrium.

2 Thermodynamics

We allow for heating of the disk due to viscous dissipation, irradiation from the binary, and dissipation of density waves due to a tidal torque.

Assuming thermal equilibrium, we have, at low optical depths, $\tau \ll 1$,

$$\tau \sigma T_m^4 = F_{tid} + F_\nu + \tau F_{irr} \,, \tag{1}$$

and at high optical depths, $\tau \gg 1$,

$$\sigma T_m^4 = F_{tid} + \frac{3\tau}{4} F_\nu + F_{irr} \,. \tag{2}$$

Extrapolated to all optical depths,

$$\sigma T_m^4 = \left(1 + \frac{1}{\tau}\right) F_{tid} + \left(\frac{3\tau}{4} + \frac{1}{\tau}\right) F_v + F_{irr} \,, \tag{3}$$

where T_m the disk midplane temperature. The optical depth is defined as $\tau = \frac{1}{2}\kappa\Sigma$, where κ is the opacity. We will assume a power-law for the opacity,

$$\kappa = \kappa_o T^\beta \tag{4}$$

The tidal heating term is (**F_ν is derived per disk face. Is F_{tid} also per disk face or should we divide by 2?**)

$$F_{tid} = (\Omega_b - \Omega)\, \Lambda \Sigma \,, \tag{5}$$

where $\Omega_b = \sqrt{GM/a^3}$ is the binary angular velocity, with a the binary semimajor axis, $\Omega = \sqrt{GM/r^3}$ is the Keplerian velocity, M is the total mass of the binary, and Λ is the torque due to the binary. At radius $r > h(r) + a$,

$$\Lambda = f \frac{q^2 GM}{a} \left(\frac{a}{R-a}\right)^4 \,, \tag{6}$$

where q is the binary mass fraction and f is a correction factor that fixes the inner disk truncation radius. Note that Martin 2013 fix $f = 1$, which places their disk inner edge further out.

The disk height $h(r)$ is given by

$$\frac{h}{r} = \left(\frac{k\,T\,r}{GM\mu}\right)^{\frac{1}{2}} \,, \tag{7}$$

by assuming hydrostatic equilibrium for a gas-pressure dominated disk. The heating due to irradiation from the binary is

$$F_{irr} = \frac{1}{2} \frac{L_c}{4\pi r^2} \zeta \,, \tag{8}$$

where ζ is the incidence angle of radiation from the binary onto the disk at a height ηh. Far from the binary, we can approximate ζ as

$$\zeta = r\frac{d}{dr}\left(\frac{\eta h}{r}\right) . \tag{9}$$

The heating due to viscous dissipation is

$$F_\nu = \frac{3}{8\pi}\frac{F_J\Omega}{r^2} = \frac{9}{8}\Omega^2\nu\Sigma\,, \tag{10}$$

where the second equality comes from the definition of the angular momentum flux $F_J = 3\pi\nu\Sigma l$, with $l = \Omega r^2$, the specific angular momentum. (*adjust for dead zone*) The viscosity is expressed as

$$\nu = \alpha\beta^b\frac{c_s^2}{\Omega} \tag{11}$$

where the sound speed is $c_s = \sqrt{\frac{P}{\rho}}$. For a kinematic viscosity proportional to the full pressure, $b = 0$. For a kinematic pressure proportional to only the gas pressure, $b = 1$. The former case, a radiation-pressure dominated disk with $b = 0$, is thermally unstable, and we will consider only the latter case, where viscosity simplifies to

$$v = \alpha\frac{k_B T}{\mu\Omega}\,. \tag{12}$$

We can simplify the energy balance equation, Eq. **??**, by solving for the temperature in the limit of various heating processes and then interpolating our results as

$$T_m^4 = \left(T_{tid,thin}^4 + T_{tid,thick}^4\right) + \left(T_{\nu,thin}^4 + T_{\nu,thick}^4\right) + T_{irr}^4\,, \tag{13}$$

where, for example, we find T_{irr} by equating $F_{irr} = \sigma T_{irr}^4$. (**We will check for error for this approximation by solving Eq. 3 exactly for a few timesteps. Also, we can probably perform stability tests on the solution**)

In the limit of small optical depth, $\tau \ll 1$, and heating dominated by viscous dissipation, we have

$$T_{\nu,thin} = \left(\frac{9}{4}\frac{\alpha k_b}{\sigma\mu\kappa_o}\Omega\right)^{\frac{1}{3+\beta}} . \tag{14}$$

In the limit of large optical depth, $\tau \gg 1$, and heating dominated by viscous dissipation, we have

$$T_{\nu,thick} = \left(\frac{27}{64}\frac{\kappa_o\alpha k_b}{\sigma\mu}\Omega\,\Sigma^2\right)^{\frac{1}{3-\beta}} . \tag{15}$$

In the limit of small optical depth and heating dominated by tidal dissipation, we have

$$T_{tid,thin} = \left(\frac{2}{\sigma\kappa_o}(\Omega_b - \Omega)\,\Lambda\right)^{\frac{1}{4+\beta}} . \tag{16}$$

In the limit of large optical depth and heating dominated by tidal dissipation, we have

$$T_{tid,thick} = \left(\frac{1}{\sigma}(\Omega_b - \Omega)\,\Lambda\,\Sigma\right)^{\frac{1}{4}} . \tag{17}$$

For heating dominated by irradiation, we have

$$T_{irr} = \left(\left(\frac{\eta}{7}\frac{L_c}{4\pi\sigma}\right)^2\frac{k_b}{\mu GM}\right)^{\frac{1}{7}} r^{-\frac{3}{7}}\,. \tag{18}$$

3 Disk Evolution

We can now evaluate the disk evolution equation by expressing viscosity as a function of surface density Σ and radius r.

The disk evolution equation is

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[3r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\nu\Sigma\right) - \frac{2\Lambda\Sigma r^{3/2}}{(GM)^{1/2}}\right] . \tag{19}$$

(*the writeup is missing the factor of 2 in front of the external torque term*) Because we want higher time resolution at smaller times, we define a logarithmic time $\eta = \ln\left(\frac{r}{r_{in}}\right)$, where r_{in} is some arbitrary scale (possibly inner disk edge). This has the advantage of preserving a constant timestep in a finite difference numerical model.

Then, Eq. **??** becomes

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{r_{in}^2 e^{2\eta}}\frac{\partial}{\partial \eta}\left[\frac{3}{(e^\eta)^{1/2}}\frac{\partial}{\partial \eta}\left((e^\eta)^{1/2}\nu\Sigma\right) - \frac{2\Lambda\Sigma(r_{in}e^\eta)^{3/2}}{(GM)^{1/2}}\right] . \tag{20}$$

4 Conservation Equations

The equation for mass conservation is

$$M = \int 2\pi r\Sigma\, dr\,, \tag{21}$$

and for momentum conservation,

$$L = \int 2\pi r\Sigma\, r^2\Omega\, dr\,. \tag{22}$$

We express the evolution equation in a conservative form, in terms of Σ rather than F_J. This will facilitate verifying conservation every time step of the numerical procedure.

5 Initial and Boundary Conditions

We will set $\Sigma = 0$ for our inner boundary condition. Our outer boundary condition is irrelevant because our solution will never reach it, so we can likewise set $\Sigma = 0$ at our outer boundary condition.

We can derive a more general inner boundary condition with the hypothesis $\partial\dot{M}/\partial l = 0$, or equivalently,

$$\frac{\partial}{\partial l}\left(3r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\nu\Sigma\right) - \frac{2\Lambda\Sigma r^{3/2}}{(GM)^{1/2}}\right) = 0\,. \tag{23}$$

The binary torque Λ vanishes to 0 at the inner edge. If we further assume that the accretion rate $\dot{M}$ is 0, we get that Σ must be a constant, which we can fix to 0 and return our inner boundary condition above.

The initial condition is yet undetermined.

6 Numerical Method

We will use FiPy to solve Eq. **??**. The package supports partial differential equations with a nonlinear diffusion term and an advection term.