Peculiar Periodicity in Press Patterns; Rectangular Lights Out Kernel Dimensions are Periodic

Daan van Berkel
Wieb Bosma

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Abstract

A lot had been said on the *Lights Out* puzzle game and it's variants. A clear connection with linear algebra exist and that connection can be used to answer various puzzle related questions, such as when a solution exist and how many solutions there are.

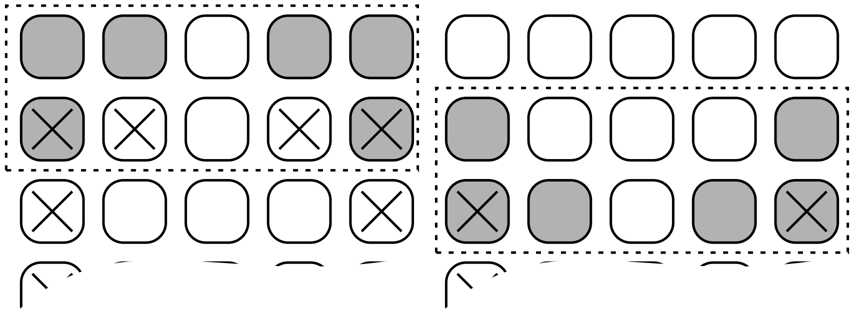
In this article we will discuss an interesting observation about the dimensions of null spaces occurring when analyzing rectangular Lights Out. We will prove that this sequence is periodic and almost palindromic.

1 Introduction

Lights Out is a handheld electronic puzzle game produced by Tiger Electronics in the 1990s. It consists of a square grid of buttons that act as lights. The object of the game is to turn off all the lights. This can be achieved by pressing a series of buttons. Each button press has the effect of changing the state of the light from off to on, and vice versa, for itself and each of it's direct neighbors.

Lights Out, and its variants and predecessors, has a long history of being studied by mathematicians. In [Fei98], and before that in [Pel87], a connection is made between Lights Out and linear algebra. Their approach amounts to solving a linear equation

$$Ap = -s$$



(a) Window of first two rows

(b) Next window

Figure 1: Chasing the lights

Where A is a 25×25 square matrix, s is the light pattern that needs to be turned off. A solution to the equation tells you which buttons to press.

Solving this matrix equation by hand is not feasible. It does give insight into when a solutions is possible and how many different solutions exist. A practical solution is given in [Mar01] where a technique known as gathering or *chasing the lights* is introduced.

A moments thought will bring the realization that the order in which we press buttons to turn the lights of is of no importance. This means that we can pick any order. We will chose to press buttons per row from left to right and the rows from top to bottom.

Now let's say that we are in the process of turning of the lights. In figure 1 we have already pressed all the buttons in the first row. There is only one way to turn of the lights that are still lit in the first row. I.e. to press each button in the second row that is directly underneath a lit button. No other buttons we still need to press effects the first row.

Once the buttons in the first row are pressed, options are forced, all the way down to the last row. This sets up a simpler linear equation. In the language of [Mar01].

$$Bp_1 = -chase(s)$$

Here B is a 5×5 matrix. p_1 are the first five components of the buttons to press, i.e. the first row. And $chase(s)$ is the effect of chasing the lights to the last row.

This allows a solution that can be committed to memory, which one of the authors has done for the standard Lights out puzzle.

In [Lea17] this method chasing the lights is extended to other rectangular board shapes. They show that analyzing a $c \times r$ Lights Out board one needs is interested in the upper left $c \times c$ sub-matrix of W^r where W describes the

	0	1	2	3	4	5	6	7	8	9	10	11
1	0	0	1	0	0	1	0	0	1	0	0	1
2	0	1	0	2	0	1	0	2	0	1	0	2
3	0	0	2	0	0	3	0	0	2	0	0	3
4	0	0	0	0	4	0	0	0	0	4	0	0

Table 1: Dimension of Kernels

effect of one step in chasing the lights.

It is in this light that the authors made their observations.

2 Observations

In discovering the theorems of [Mar01] and [Lea17] for themselves, the authors studied the table 1, and its variants, extensively. To see more entries of the table see `http://dvberkel.github.io/mathematics-articles/lights_out/periodicity.htm`

During our studies we made the following observations. For each row

1. The sequence is periodic.
2. There is a kernel of maximal dimension.
3. The period starts after the maximal dimension.
4. Before and after the maximal dimension the dimension is 0.
5. The sequence is almost palindromic.

3 Algebra

We will formalize the process of chasing down the lights and prove all of our observations. To begin we will introduce a matrix of some interest.

For all $c \in \mathbb{N}$ define the $2c \times 2c$ matrix W_c with entries in $GF(q)$ by

$$W_c := \begin{pmatrix} -E_c & I \\ -I & O \end{pmatrix}$$

where O is the zero matrix, I the identity matrix and E_c is defined as the $c \times c$ matrix with ones on the diagonal and the two main sub-diagonals, and zeroes elsewhere.

For example

$$E_4 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The reason we are looking at W_c is that its powers tell us something about the effect of chasing down the lights. In particular, if we have an $r \times c$ lights out puzzle, the $c \times c$ upper left sub-matrix of W_c^r is exactly the process of gathering the lights.

The first interesting fact is that W_c is invertible.

Lemma 1 *W_c is invertible for all $c \in \mathbb{N}$.* $\diamond$

Proof

$$\begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} O & -I \\ I & -E \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}$$

$\square$

A consequence of the invertibility of W_c is that the sequence of its powers is periodic. This also proves our observation 1, but we will have some more to say about that.

Theorem 2 *The sequence $(W_c^n)_{n\in\mathbb{N}}$ is purely periodic.* $\diamond$

Proof There are only finitely many different square matrices of size $2c$ over $GF(q)$. So the sequence $(W_c^n)_{n\in\mathbb{N}}$ must become periodic. By the preceding lemma W_c is invertible so the sequence is periodic from the start. $\square$

As mentioned in [Lea17] there is a relation between the images of chasing down the lights and Fibonacci polynomials. We find that relation in our structure lemma.

Lemma 3 (structure) *There exists a sequence of $c \times c$ matrices $(T_n)_{n\in\mathbb{N}}$ such that*

$$W_c^n = \begin{pmatrix} T_n & T_{n-1} \\ -T_{n-1} & -T_{n-2} \end{pmatrix}$$

for all $k \in \mathbb{N}$. $\diamond$

Proof Define $T_0 := I$, and for convenience $T_{-1} := O$ and $T_{n+1} := -E \cdot T_n - T_{n-1}$ for all $n \in \mathbb{N}$. So $T_1 = -E \cdot I - O = -E$.

A number $n \in \mathbb{N}$ is called strong if and only if

$$W_c^n = \begin{pmatrix} T_n & T_{n-1} \\ -T_{n-1} & -T_{n-2} \end{pmatrix}$$

Notice that $W_c^1 = \left(\begin{smallmatrix} -E & I \\ -I & O \end{smallmatrix}\right) = \left(\begin{smallmatrix} T_1 & T_0 \\ -T_0 & -T_{-1} \end{smallmatrix}\right)$ so 1 is strong.

Assume that k is strong. We will show that $k+1$ is strong as well.

$$\begin{aligned} W_c^{k+1} &= W_c \cdot W_c^k \\ &= \begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} T_k & T_{k-1} \\ -T_{k-1} & -T_{k-2} \end{pmatrix} \\ &= \begin{pmatrix} -E \cdot T_k - T_{k-1} & -E \cdot T_{k-1} - T_{k-2} \\ -T_k & -T_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} T_{k+1} & T_k \\ -T_k & -T_{k-1} \end{pmatrix} \end{aligned}$$

By mathematical induction all natural numbers are strong, finishing the proof. □

With the structure lemma under our belt we can prove our first observation, i.e. the sequence of the dimension of kernels is periodic.

Proposition 4 (Observation 1) *The sequence* $(\dim \operatorname{Ker} P_{r,c})_{r \in \mathbb{N}}$ *is periodic.* ◇

Proof By the structure lemma, for all $r \in \mathbb{N}$,

$$\dim \operatorname{Ker} P_{r,c} = \dim \operatorname{Ker} T_r$$

By the periodicity of $(W_c^r)_{r \in \mathbb{N}}$ we have the periodicity of $(\dim \operatorname{Ker}(P_{r,c})_{r \in \mathbb{N}}$. □

It could be the case that the period of $(\dim \operatorname{Ker}(P_{r,c})_{r \in \mathbb{N}}$ is a divisor of the period of $(W_c^r)_{r \in \mathbb{N}}$. In fact, these periods do not coincide. There is a relation which we will see shortly.

Next on our agenda is our observation 2. I.e. for each number of columns c there is a kernel with that dimension.

Lemma 5 (Observation 2) *There is a kernel of maximal dimension.* ◇

Proof We will be using the notation as defined by the structure lemma.

Notice that there exists $p \in \mathbb{N}$ such that $W_c^p = I$ and thus $W_c^{p-1} = W_c^{-1} \cdot W_c^p = W_c^{-1}$. Furthermore

$$\dim \operatorname{Ker} P_{p-1,c} = \dim \operatorname{Ker} T_{-1} = \dim O = c$$

$\square$

Before we will dive deeper in the question if the period of both sequences coincide, we will take a closer look at observation 4. For this we need to know the determinant of W_c^n for all $n \in \mathbb{N}$.

Lemma 6 $\det(W_c^n) = 1$ *for all* $n \in \mathbb{N}$. $\diamond$

Proof Note that $\det(W_c^0) = \det(I) = 1$ and

$$\det(W_c) = \det \begin{pmatrix} -E & I \\ -I & O \end{pmatrix} = \det(-E) \cdot \det(O) - \det(-I) \cdot \det(I) = 1$$

The lemma follows by multiplicativity of the determinant and by induction on n. $\square$

This little fact will helps us establishing the proof of observation 4.

Theorem 7 (Observation 4) *If for some* $r \in \mathbb{N}$ *we have* $\dim \operatorname{Ker} P_{r,c} = c$ *then*

- $\dim \operatorname{Ker} P_{r-1,c} = 0$
- $\dim \operatorname{Ker} P_{r+1,c} = 0$

$\diamond$

Proof Let $r \in \mathbb{N}$ be such that $\dim \operatorname{Ker} P_{r,c} = c$; then

$$W_c^r = \begin{pmatrix} O & T_{r-1} \\ -T_{r-1} & -T_{r-2} \end{pmatrix}$$

and $1 = \det W_c^r = (\det T_{r-1})^2$, so T_{r-1} is invertible.

Note that $W_c^{r-1} = \begin{pmatrix} T_{r-1} & T_{r-2} \\ -T_{r-2} & -T_{r-3} \end{pmatrix}$ and $W_c^{r+1} = \begin{pmatrix} -T_{r-1} & O \\ O & -T_{r-1} \end{pmatrix}$, hence both

- $\dim \operatorname{Ker} P_{r-1,c} = \dim \operatorname{Ker} T_{r-1} = 0$,
- $\dim \operatorname{Ker} P_{r+1,c} = \dim \operatorname{Ker} -T_{r-1} = 0$.

$\square$

Corollary 8 *If for some* $r \in \mathbb{N}$ *we have* $\dim \operatorname{Ker} P_{r,c} = c$ *then* $W_c^{r+1} = \left(\begin{smallmatrix} -T_{r-1} & O \\ O & -T_{r-1} \end{smallmatrix}\right)$ $\diamond$

In proving our observation 4 we have learned that the structure of the corresponding matrix power is particular simple. This fact will be instrumental in the relation between the period of $(W_c^n)_{n\in\mathbb{N}}$ and that of $(\dim \operatorname{Ker} P_{r,c})_{r\in\mathbb{N}}$.

But first we will see that W_c and it's inverse are conjugates.

Lemma 9 W_c *and* W_c^{-1} *are conjugates.* $\diamond$

Proof We will conjugate $W = W_c$ by $C := \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$, which is its own inverse,

$$\begin{aligned} C \cdot W \cdot C^{-1} &= \begin{pmatrix} O & I \\ I & O \end{pmatrix} \cdot \begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} O & I \\ I & O \end{pmatrix} \\ &= \begin{pmatrix} -I & O \\ -E & I \end{pmatrix} \cdot \begin{pmatrix} O & I \\ I & O \end{pmatrix} \\ &= \begin{pmatrix} O & -I \\ I & -E \end{pmatrix} \\ &= W^{-1} \end{aligned}$$

$\square$

Corollary 10 $W_c^{-n} = \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right) W_c^n \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ $\diamond$

Proof Clear. $\square$

Note that conjugating any matrix with $\left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ corresponds to rotating the four cardinal sub-matrices through 180 degrees.

This fact will be the linchpin in the proof of observation 5.

Lemma 11 *If for some* $r \in \mathbb{N}$ *we have* $W_c^r = \left(\begin{smallmatrix} Q & O \\ O & Q \end{smallmatrix}\right)$ *then for all* $i \in \mathbb{N}$.

$$W_c^{r+i} = \begin{pmatrix} QT_i & QT_{i-1} \\ -QT_{i-1} & -QT_{i-2} \end{pmatrix}$$

and

$$W_c^{r-i} = \begin{pmatrix} -QT_{i-2} & -QT_{i-1} \\ QT_{i-1} & QT_{i-2} \end{pmatrix}$$

$\diamond$

Proof By direct calculation and the structure lemma we find that

$$W_c^{r+i} = W_c^r W_c^i = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix} \begin{pmatrix} T_i & T_{i-1} \\ -T_{i-1} & -T_{i-2} \end{pmatrix} = \begin{pmatrix} QT_i & QT_{i-1} \\ -QT_{i-1} & -QT_{i-2} \end{pmatrix}$$

Furthermore, with $C = \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ we have, by the preceding lemma, $CW_c^{r-i}C^{-1} = CW_c^r C^{-1} C W^{-i} C^{-1} = W_c^r W_c^i = W_c^{r+i}$ which we set out to prove. □

We now come to our promise about the period of the dimension of kernels the kernels and the period of W. let q be the smallest number rows that has a maximal kernel dimension and let $d = q + 1$. The period of W will be p.

Lemma 12 *Either $p = d$ or $p = 2d$.* ◇

Proof If $p = d$ we are finished, so assume it is not. We will show that $p = 2d$ in that case.

By the preceding lemma we that the lower right $c \times c$ sub-matrix of W_c^{d-i} equals the upper left $c \times c$ sub-matrix of W_c^{d+i} for all i. In particular for $i = d$. The lower right sub-matrix of $W_c^{d-d} = W_c^0 = I$ is the $c \times c$ identity matrix.

Furthermore, since $W^d = \left(\begin{smallmatrix} Q & O \\ O & Q \end{smallmatrix}\right)$ for certain matrix Q, we have

$$W_c^{2d} = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix}^2 = \begin{pmatrix} Q^2 & O \\ O & Q^2 \end{pmatrix}$$

Hence, Q^2 is equal to the $c \times c$ identity matrix, therefore W^{2d} is the identity matrix, and the period of $(W_c^r)_{r \in \mathbb{N}}$ is $2d$. □

Theorem 13 (Observation 5) *The sequence* $(\dim \operatorname{Ker} P_{r,c})_{r \in \mathbb{N}}$ *is almost palindromic.* ◇

Proof Let $q \in \mathbb{N}$ such that $\dim \operatorname{Ker} P_{r,c}$ is maximal. By 8 we have $W^{q+1} = \left(\begin{smallmatrix} -T_{q-1} & O \\ O & -T_{q-1} \end{smallmatrix}\right)$. By the preceding lemma $W_c^{q+1+i} = \begin{pmatrix} -T_{q-1}T_i & -T_{q-1}T_{i-1} \\ T_{q-1}T_{i-1} & T_{q-1}T_{i-2} \end{pmatrix}$ and $W_c^{q+1-i} = \begin{pmatrix} -T_{q-1}T_{i-2} & T_{q-1}T_{i-1} \\ -T_{q-1}T_{i-1} & -T_{q-1}T_i \end{pmatrix}$

Hence $W_c^{q+1-(i+2)} = \begin{pmatrix} -T_{q-1}T_i & T_{q-1}T_{i+1} \\ -T_{q-1}T_{i+1} & T_{q-1}T_{i+2} \end{pmatrix}$

Therefore

$$\dim \operatorname{Ker} P_{q+1+i,c} = \dim \operatorname{Ker} -T_{q-1}T_i = \dim \operatorname{Ker} P_{q+1-(i+2),c} = \dim \operatorname{Ker} P_{q-1-i,c} =$$

Which shows that start and end are reflected. □

References

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