Peculiar Periodicity in Press Patterns; Rectangular Lights Out Kernel Dimensions are Periodic

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Abstract

A lot had been said on the *Lights Out* puzzle game and it's variants. A clear connection with linear algebra exist and that connection can be used to answer various puzzle related questions, such as when a solution exist and how many solutions there are.

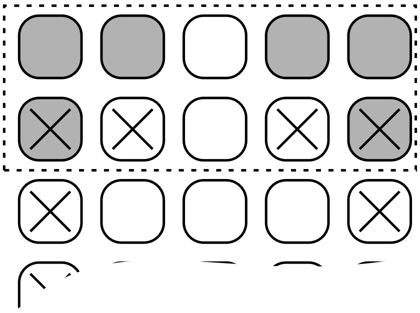
In this article we will discuss an interesting observation about the dimensions of null spaces occurring when analyzing rectangular Lights Out. We will prove that this sequence is periodic and almost palindromic.

1 Introduction

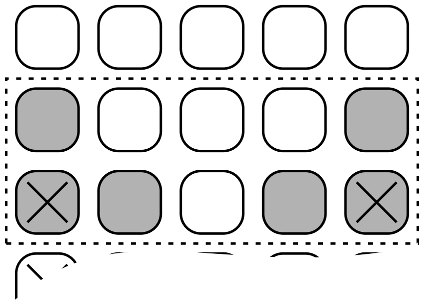
Lights Out is a handheld electronic puzzle game produced by Tiger Electronics in the 1990s. It consists of a square grid of buttons that act as lights. The object of the game is to turn off all the lights. This can be achieved by pressing a series of buttons. Each button press has the effect of changing the state of the light from off to on, and vice versa, for itself and each of it's direct neighbors.

Lights Out, and its variants and predecessors, has a long history of being studied by mathematicians. In [Fei98], and before that in [Pel87], a connection is made between Lights Out and linear algebra. Their approach amounts to solving a linear equation

$$Ap = -s$$



(a) Window of first two rows



(b) Next window

Figure 1: Chasing the lights

Where A is a 25×25 square matrix, s is the light pattern that needs to be turned off. A solution to the equation tells you which buttons to press.

Solving this matrix equation by hand is not feasible. It does give insight into when a solutions is possible and how many different solutions exist. A practical solution is given in [Mar01] where a technique known as gathering or *chasing the lights* is introduced.

A moments thought will bring the realization that the order in which we press buttons to turn the lights of is of no importance. This means that we can pick any order. We will chose to press buttons per row from left to right and the rows from top to bottom.

Now let's say that we are in the process of turning of the lights. In figure 1 we have already pressed all the buttons in the first row. There is only one way to turn of the lights that are still lit in the first row. I.e. to press each button in the second row that is directly underneath a lit button. No other buttons we still need to press effects the first row.

Once the buttons in the first row are pressed, options are forced, all the way down to the last row. This sets up a simpler linear equation. In the language of [Mar01].

$$Bp_1 = -chase(s)$$

Here B is a 5×5 matrix. p_1 are the first five components of the buttons to press, i.e. the first row. And $chase(s)$ is the effect of chasing the lights to the last row.

This allows a solution that can be committed to memory, which one of the authors has done for the standard Lights out puzzle.

In [Lea17] this method chasing the lights is extended to other rectangular board shapes. They show that analyzing a $c \times r$ Lights Out board one needs is interested in the upper left $c \times c$ sub-matrix of W^r where W describes the

	0	1	2	3	4	5	6	7	8	9	10	11
1	0	0	1	0	0	1	0	0	1	0	0	1
2	0	1	0	2	0	1	0	2	0	1	0	2
3	0	0	2	0	0	3	0	0	2	0	0	3
4	0	0	0	0	4	0	0	0	0	4	0	0

Table 1: Dimension of Kernels

effect of one step in chasing the lights.

It is in this light that the authors made their observations.

2 Observations

In discovering the theorems of [Mar01] and [Lea17] for themselves, the authors studied the table 1, and its variants, extensively. To see more entries of the table goto the website for this article[1].

During our studies we made the following observations. For each row

1. The sequence is purely periodic.

2. There is a kernel of maximal dimension.

3. The period starts after the maximal dimension.

4. The sum of two consecutive dimensions is less than or equal to the maximal dimension.

5. The sequence is almost palindromic.

We will explain each observation with the aid of the above table. Observation 1 pertains that for each number of columns, the sequence of dimensions of kernels of $c \times r$ Lights Out puzzles repeats itself eventually, and when it repeats it does so from the start.

A kernel of maximal dimension is when the kernel is the entire press space. In other words, in the sequence for the number of columns c, there is an entry of precisely c. Furthermore, the sequence repeats itself when this happens.

For observation 5, take a look at the third row of the above table. The sequence up until the value 3, i.e. $0, 0, 2, 0, 0$ is palindromic. It is the same read from left to right as from right to left.

[1] http://dvberkel.github.io/mathematics-articles/lights_out/periodicity.html

The fact that in a row consecutive numbers sum to less than the maximal dimension is observation 4.

3 Definition

In this section we will define some terms that we will use throughout the article.

In this article we are exploring rectangular Lights Out. A (n, c, r) Lights Out puzzle is a matrix M with $c \in \mathbb{N}$ columns and $r \in \mathbb{N}$ rows with entries over $\mathbb{Z}/n\mathbb{Z}$. The space of all (n, c, r) puzzles is called $\mathcal{L}_{(n,c,r)}$.

For a each $\mathcal{L}_{(n,c,r)}$ and for each (i, j) with $1 \leq i \leq r$ and $1 \leq j \leq c$ there is a basic press function $p_{(i,j)} : \mathcal{L}_{(n,c,r)} \to \mathcal{L}_{(n,c,r)}$ mapping

$$\left(p_{(i,j)}(M)\right)_{(u,v)} := \begin{cases} M_{(u,v)} + 1 & : \quad \mathrm{d}\left((i, j), (u, v)\right) \leq 1 \\ M_{(u,v)} & : \quad \text{otherwise} \end{cases}$$

where d is the *Manhattan distance*. The set of all basic presses is called B.

A press sequence is a finite sequence of basic presses. The set of all press sequences is called P. P together with concatenation of sequences makes a monoid with the empty sequence as identity element.

The effect E of a press sequence is an mapping on $\mathcal{L}_{(n,c,r)}$ that extends basic presses. I.e. the effect of the empty sequence is the identity map and for a press sequence $(q_t)_{t\in\overline{m}}$ with $\overline{m} := \{0, 1, \ldots, m - 1\}$

$$E\left((q_t)_{t\in\overline{m}})\right) = E\left((q_{t+1})_{t\in\overline{m-1}})\right) \circ q_0$$

For each press sequence $q := (q_t)_{t\in\overline{m}}$ we define a count function $N : P \to \mathbb{N}^B$ that counts the number of times that a basic press is present.

$$N_q(p_{(i,j)}) = \sum_{t\in\overline{m}} 1_{(i,j)}(q_t)$$

where $1_{(i,j)}(q)$ is 1 when $q = p_{(i,j)}$ and zero otherwise. We will write the application of N to press sequence q as N_q.

With this count we will define a fingerprint function $I : P \to (\mathbb{Z}/n\mathbb{Z})^B$ that count how many times a basic press is present in the sequence, modulo n. So for $\overline{m} := \{0, 1, \ldots, m - 1\}$ we have

$$I\left(q\right) = p_{(i,j)} \mapsto \overline{N_q(p_{(i,j)})}$$

Notice that $I(u \circ v) = I(u) + I(v)$ and $I(\epsilon) = O$. Again, we will write I_u for the application $I(u)$.

Now we will define a relation $\sim$ over P. $u \sim v$ if and only if $I(u) = I(v)$. With some thought one can see that $\sim$ is an equivalence relation. The equivalence class of $u \in P$ will be denoted by $[u]$ and wil be called a *press pattern*. The set of all press patterns will be denoted by $\mathcal{P}$.

Note that for $[s], [t] \in \mathcal{P}$ with $[s] = [t]$ we have

$$E(s)M = E(t)M$$

since

$$(E(s)M)_{(u,v)} = M_{(u,v)} + \sum_{d((i,j),(u,v)) \leq 1} I_s(p_{(i,j)}) = M_{(u,v)} + \sum_{d((i,j),(u,v)) \leq 1} I_t(p_{(i,j)}) = (E(t)M)_{(u,v)}$$

We will define the following binary operation on $\mathcal{P}$: $[u] + [v] = [u \circ v]$. Notice that for $u_0, u_1, v_0, v_1 \in \mathcal{P}$ with $[u_0] = [u_1]$ and $[v_0] = [v_1]$ we have.

$$[u_0 \circ v_0] = \{w \in P | I(w) = I(u_0 \circ v_0) = I(u_0) + I(v_0) = I(u_1) + I(v_1) = I(u_1 \circ v_1)\} = [u_1 \circ v_1]$$

which show that the addition is well defined. Since $I(u) + I(v) = I(v) + I(u)$ $(\mathcal{P}, +)$ is an abelian group.

If we define scalar multiplication with $r \in \mathbb{Z}/n/Z$ as

$$r[u] = [\underbrace{u \circ u \circ \ldots u}_{r\text{times}}]$$

We turn $\mathcal{P}$ into a free $\mathbb{Z}/n/Z$-module.

We will construct a map $\mathcal{E}$ from $\mathcal{P} \to \mathcal{L}$ with $[u] \mapsto E(u)O$. Notice that $\mathcal{E}(u \circ v) = \mathcal{E}(u)(\mathcal{E}(v))$ which makes $\mathcal{E}$ a linear transformation.

Solving a $s \in \mathcal{L}$ amounts to solving for $u \in \mathcal{P}$ the linear equation

$$\mathcal{E}(u) = -s$$

Next we consider an order $\leq$ on B, the basic presses. For $p_{(i,j)}, p_{(u,v)} \in B$ we have $p_{(i,j)} \leq p_{(u,v)}$ if and only if $j < v$ or when $j = v$ then $i \leq j$. This ordering amounts to ordering the basic pressed per row from top to bottom and for each row from left to right.

A press sequence $q := (q_t)_{t \in \overline{m}} \in P$ is called a *standard* press sequence when $q_i \leq q_j$ whenever $i \leq j$. For each press pattern $P \in \mathcal{P}$ there is a unique standard press sequence $p \in P$ such that $P = [p]$. A press sequence is *fertile* if it only contains basic presses with a row index of 0.

We will turn to chasing the light. For each light pattern $L \in \mathcal{L}$ we define a press pattern chase(L). We create a sequence of standard press patterns.

$z_0 := \epsilon$, the empty press pattern. $z_{n+1} := z_n \circ p_{(i,j+1)}$ where (i, j) is the smallest non-zero entry in $L + \mathcal{E}(z_n)$. chase(L) is the press pattern when the press sequence does not grow anymore.

Chasing amounts to finding a fertile press sequence $p \in P$ such that

$$\mathrm{chase}([p]) = -\,\mathrm{chase}(L)$$

$chase_{(n,c,r)}$ is a linear transformation.

Definition For a (n, c, r) Light Out puzzle we call $C_{(n,c,r)} := \dim \mathrm{Ker}\, \mathrm{chase}_{(n,c,r)}$ $\circ$

When n and c are clear from context we will write C_r for $C_{(n,c,r)}$.

4 Algebra

We will formalize the process of chasing down the lights and prove all of our observations. To begin we will introduce a matrix of some interest.

For all $c \in \mathbb{N}$ define the $2c \times 2c$ matrix W_c with entries in $\mathbb{Z}/n\mathbb{Z}$ by

$$W_c := \begin{pmatrix} -E_c & I \\ -I & O \end{pmatrix}$$

where O is the zero matrix, I the identity matrix and E_c is defined as the $c \times c$ matrix with ones on the diagonal and the two main sub-diagonals, and zeroes elsewhere.

For example

$$E_4 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The reason we are looking at W_c is that its powers tell us something about the effect of chasing down the lights. In particular, if we have an (n, c, r) lights out puzzle, the $c \times c$ upper left sub-matrix of W_c^r is exactly the process of gathering the lights.

The first interesting fact is that W_c is invertible.

Lemma 1 *W_c is invertible for all $c \in \mathbb{N}$.* $\diamond$

Proof

$$\begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} O & -I \\ I & -E \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}$$

$\square$

A consequence of the invertibility of W_c is that the sequence of its powers is periodic. This also proves our observation 1, but we will have some more to say about that.

Theorem 2 *The sequence $(W_c^r)_{r\in\mathbb{N}}$ is purely periodic.* $\diamond$

Proof There are only finitely many different square matrices of size $2c$ over $\mathbb{Z}/n\mathbb{Z}$. So the sequence $(W_c^r)_{r\in\mathbb{N}}$ must become periodic. By the preceding lemma W_c is invertible so the sequence is periodic from the start. $\square$

As mentioned in [Lea17] there is a relation between the images of chasing down the lights and Fibonacci polynomials. We find that relation in our structure lemma.

Lemma 3 (structure) *There exists a sequence of $c \times c$ matrices $(T_r)_{r\in\mathbb{N}}$ such that*

$$W_c^r = \begin{pmatrix} T_r & T_{r-1} \\ -T_{r-1} & -T_{r-2} \end{pmatrix}$$

for all $r \in \mathbb{N}$. $\diamond$

Proof Define $T_0 := I$, and for convenience $T_{-1} := O$ and $T_{r+1} := -E \cdot T_r - T_{r-1}$ for all $r \in \mathbb{N}$. So $T_1 = -E \cdot I - O = -E$.

A number $r \in \mathbb{N}$ is called strong if and only if

$$W_c^r = \begin{pmatrix} T_r & T_{r-1} \\ -T_{r-1} & -T_{r-2} \end{pmatrix}$$

Notice that $W_c^1 = \left(\begin{smallmatrix} -E & I \\ -I & O \end{smallmatrix}\right) = \left(\begin{smallmatrix} T_1 & T_0 \\ -T_0 & -T_{-1} \end{smallmatrix}\right)$ so 1 is strong.

Assume that k is strong. We will show that $k+1$ is strong as well.

$$\begin{aligned} W_c^{k+1} &= W_c \cdot W_c^k \\ &= \begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} T_k & T_{k-1} \\ -T_{k-1} & -T_{k-2} \end{pmatrix} \\ &= \begin{pmatrix} -E \cdot T_k - T_{k-1} & -E \cdot T_{k-1} - T_{k-2} \\ -T_k & -T_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} T_{k+1} & T_k \\ -T_k & -T_{k-1} \end{pmatrix} \end{aligned}$$

By mathematical induction all natural numbers are strong, finishing the proof. $\square$

With the structure lemma under our belt we can prove our first observation, i.e. the sequence of the dimension of kernels is periodic.

Proposition 4 (Observation 1) *The sequence* $\left(C_{(n,c,r)}\right)_{r\in\mathbb{N}}$ *is periodic.* ◇

Proof By the structure lemma, for all $r \in \mathbb{N}$,

$$C_{(n,c,r)} = \dim \operatorname{Ker} T_r$$

By the periodicity of $(W_c^r)_{r\in\mathbb{N}}$ we have the periodicity of $\left(C_{(n,c,r)}\right)_{r\in\mathbb{N}}$. □

It could be the case that the period of $\left(C_{(n,c,r)}\right)_{r\in\mathbb{N}}$ is a divisor of the period of $(W_c^r)_{r\in\mathbb{N}}$. There is a relation which we will see shortly.

Next on our agenda is our observation 2. I.e. for each number of columns c there is a kernel with that dimension.

Lemma 5 (Observation 2) *There is a kernel of maximal dimension.* ◇

Proof We will be using the notation as defined by the structure lemma.

Notice that there exists $p \in \mathbb{N}$ such that $W_c^p = I$ and thus $W_c^{p-1} = W_c^{-1} \cdot W_c^p = W_c^{-1}$. Furthermore

$$C_{(n,c,p)} = \dim \operatorname{Ker} T_{-1} = \dim \operatorname{Ker} O = c$$

□

Before we will dive deeper in the question if the period of both sequences coincide, we will take a closer look at observation 4.

Theorem 6 (Observation 4) *For all* $i \in \mathbb{N}$ *the following inequality holds*

$$C_{(n,c,i)} + C_{(n,c,i+1)} \leq c$$

◇

Proof Assume to the contrary that there is a $k \in \mathbb{N}$ such that $C_{(n,c,k)} + C_{(n,c,k+1)} > c$.

Let $U := \operatorname{Ker} \text{chase}_{(n,c,k)}$ and let $u_1, \ldots, u_m$ be a basis for U, with $m := C_{(n,c,k)}$. In similar fashion let $V := \operatorname{Ker} \text{chase}_{(n,c,k)}$ with basis $v_1, \ldots, v_n$ and $n := C_{(n,c,k+1)}$. In particular the basis for U and V are both linear independent.

Now, because each basis for a $\mathbb{Z}/n\mathbb{Z}$-module has the same rank, and the rank of $\mathcal{E}$ is c (See [Rom07]), the set of combined vector is linear dependent. I.e. there are s_i and t_i not all zero such that

$$s_1u_1 + s_2u_2 + \cdots + s_mu_m + t_1v_1 + \cdots + t_nv_n = 0$$

Because the u_i and v_i are linear independent there exist a non-zero $p \in U \cap V$. Hence

$$W^k p = \begin{pmatrix} O \\ w \end{pmatrix}$$

and

$$W^{k+1} p = \begin{pmatrix} O \\ w' \end{pmatrix}$$

but

$$\begin{pmatrix} O \\ w' \end{pmatrix} = W^{k+1}p = WW^kp = W\begin{pmatrix} O \\ w \end{pmatrix} = \begin{pmatrix} w \\ O \end{pmatrix}$$

So $W^kp = O$, which contradicts the fact that W is invertable. $\square$

This little fact will helps us establishing the proof of the following fact

Corollary 7 *If for some $r \in \mathbb{N}$ we have $C_{n,c,r} = c$ then*

- $C_{n,c,r-1} = 0$
- $C_{n,c,r+1} = 0$

$\diamond$

Proof If for some $r \in \mathbb{N}$ we have $C_{(n,c,r)} = c$, then both

$$C_{(n,c,r-1)} \leq c - C_{(n,c,r)} = c - c = 0$$

and

$$C_{(n,c,r+1)} \leq c - C_{(n,c,r)} = c - c = 0$$

$\square$

Lemma 8 *If for some $r \in \mathbb{N}$ we have $C_{(n,c,r)} = c$ then $W_c^{r+1} = \left(\begin{smallmatrix} -T_{r-1} & O \\ O & -T_{r-1} \end{smallmatrix}\right)$* ◇

Proof If for some $r \in \mathbb{N}$ we have $C_{(n,c,r)} = c$; then

$$W_c^r = \begin{pmatrix} O & T_{r-1} \\ -T_{r-1} & -T_{r-2} \end{pmatrix}$$

By direct multiplication we note that $W_c^{r+1} = \left(\begin{smallmatrix} -T_{r-1} & O \\ O & -T_{r-1} \end{smallmatrix}\right)$, □

In proving our lemma **??** we have learned that the structure of the corresponding matrix power is particular simple. This fact will be instrumental in the relation between the period of $(W_c^r)_{r\in\mathbb{N}}$ and that of $\left(C_{(n,c,r)}\right)_{r\in\mathbb{N}}$.

But first we will see that W_c and it's inverse are conjugates.

Lemma 9 *W_c and W_c^{-1} are conjugates.* ◇

Proof We will conjugate $W = W_c$ by $D := \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$, which is its own inverse,

$$\begin{aligned} D \cdot W \cdot D^{-1} &= \begin{pmatrix} O & I \\ I & O \end{pmatrix} \cdot \begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} O & I \\ I & O \end{pmatrix} \\ &= \begin{pmatrix} -I & O \\ -E & I \end{pmatrix} \cdot \begin{pmatrix} O & I \\ I & O \end{pmatrix} \\ &= \begin{pmatrix} O & -I \\ I & -E \end{pmatrix} \\ &= W^{-1} \end{aligned}$$

□

Corollary 10 $W_c^{-r} = \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right) W_c^r \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ ◇

Proof Clear. □

Note that conjugating any matrix with $\left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ corresponds to rotating the four cardinal sub-matrices through 180 degrees.

This fact will be the linchpin in the proof of observation 5.

Lemma 11 *If for some $r \in \mathbb{N}$ we have $W_c^r = \left(\begin{smallmatrix} Q & O \\ O & Q \end{smallmatrix}\right)$ then for all $i \in \mathbb{N}$.*

$$W_c^{r+i} = \begin{pmatrix} QT_i & QT_{i-1} \\ -QT_{i-1} & -QT_{i-2} \end{pmatrix}$$

and

$$W_c^{r-i} = \begin{pmatrix} -QT_{i-2} & -QT_{i-1} \\ QT_{i-1} & QT_{i-2} \end{pmatrix}$$

$\diamond$

Proof By direct calculation and the structure lemma we find that

$$W_c^{r+i} = W_c^r W_c^i = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix} \begin{pmatrix} T_i & T_{i-1} \\ -T_{i-1} & -T_{i-2} \end{pmatrix} = \begin{pmatrix} QT_i & QT_{i-1} \\ -QT_{i-1} & -QT_{i-2} \end{pmatrix}$$

Furthermore, with $C = \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ we have, by the preceding lemma, $CW_c^{r-i}C^{-1} = CW_c^r C^{-1} C W^{-i} C^{-1} = W_c^r W_c^i = W_c^{r+i}$ which we set out to prove. $\square$

We now come to our promise about the period of the dimension of kernels the kernels and the period of W. let q be the smallest number rows that has a maximal kernel dimension and let $d = q + 1$. The period of W will be p.

Lemma 12 *Either $p = d$ or $p = 2d$.* $\diamond$

Proof If $p = d$ we are finished, so assume it is not. We will show that $p = 2d$ in that case.

By the preceding lemma we that the lower right $c \times c$ sub-matrix of W_c^{d-i} equals the upper left $c \times c$ sub-matrix of W_c^{d+i} for all i. In particular for $i = d$. The lower right sub-matrix of $W_c^{d-d} = W_c^0 = I$ is the $c \times c$ identity matrix.

Furthermore, since $W^d = \left(\begin{smallmatrix} Q & O \\ O & Q \end{smallmatrix}\right)$ for certain matrix Q, we have

$$W_c^{2d} = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix}^2 = \begin{pmatrix} Q^2 & O \\ O & Q^2 \end{pmatrix}$$

Hence, Q^2 is equal to the $c \times c$ identity matrix, therefore W^{2d} is the identity matrix, and the period of $(W_c^r)_{r \in \mathbb{N}}$ is $2d$. $\square$

Theorem 13 (Observation 5) *The sequence $\left(C_{(n,c,r)}\right)_{r \in \mathbb{N}}$ is almost palindromic.*

$\diamond$

Proof let d be such that W^d has a upperleft sub matrix O.

A *press vector* is a $2 \times c$ vector with the last c components zero, an *unlit vector* is a $2 \times c$ vector with the first c components 0. Notice that for any press vector v

$$W^d v$$

is an unlit vector.

Choose $m, n \in \mathbb{N}$ such that $m + 1 + n = d$. We will show that for each press vector v for which $W^m v$ is unlit, there exist a press vector v' such that $W^n v'$ is unlit. This shows that $\dim \operatorname{Ker} T_m \leq \dim \operatorname{Ker} T_n$. Since the argument is symmetric in m and n we have $\dim \operatorname{Ker} T_m = \dim \operatorname{Ker} T_n$.

Let v be a press vector such that $W^m v$ is unlit. In particular $W^m v = u$ with $u := (0, 0, \ldots, 0, u_1, u_2, \ldots, u_c)^t$.

Define $p := Wu$. We will show that p is a press vector.

$$Wu = \begin{pmatrix} -E_c & I \\ -I & O \end{pmatrix} \begin{pmatrix} O \\ u' \end{pmatrix} = \begin{pmatrix} u' \\ O \end{pmatrix}$$

k

Since for any press vector w we have that $W^d w = O$, in particular we have

$$O = W^d v = W^n W W^m v = W^n W u = W^n p$$

which shows that for each press vector that W^m unlits, there is a press vector that W^n unlits. □

Now everything is in play to prove 3

Theorem 14 (Observation 3) *The period of $(C_{(n,c,r)})_{r \in \mathbb{N}}$ starts after the maximal dimension.* ⋄

Proof By lemma 8 the power of W has a block diagonal structure. By lemma 11 the sequence of dimensions is shifted mirror image. Together with the fact that the sequence is almost palindromic from the previous theorem, we have our proof. □

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