# Peculiar Periodicity in Press Patterns; Rectangular Lights Out Kernel Dimensions are Periodic

Daan van Berkel
Wieb Bosma

November 2, 2020

**Abstract**

A lot had been said on the *Lights Out* puzzle game and it's variants. A clear connection with linear algebra exist and that connection can be used to answer various puzzle related questions, such as when a solution exist and how many solutions there are.

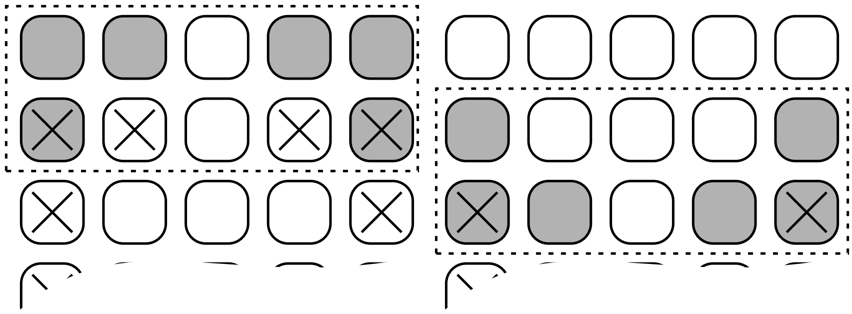
In this article we will discuss an interesting observation about the dimensions of null spaces occurring when analyzing rectangular Lights Out. We will prove that this sequence is periodic and almost palindromic.

## 1 Introduction

Lights Out is a handheld electronic puzzle game produced by Tiger Electronics in the 1990s. It consists of a square grid of buttons that act as lights. The object of the game is to turn off all the lights. This can be achieved by pressing a series of buttons. Each button press has the effect of changing the state of the light from off to on, and vice versa, for itself and each of it's direct neighbors.

Lights Out, and its variants and predecessors, has a long history of being studied by mathematicians. In [Fei98], and before that in [Pel87], a connection is made between Lights Out and linear algebra. Their approach amounts to solving a linear equation

$$Ap = -s$$



(a) Window of first two rows (b) Next window

Figure 1: Chasing the lights

Where $A$ is a $25 \times 25$ square matrix, $s$ is the light pattern that needs to be turned off. A solution to the equation tells you which buttons to press.

Solving this matrix equation by hand is not feasible. It does give insight into when a solutions is possible and how many different solutions exist. A practical solution is given in [Mar01] where a technique known as gathering or *chasing the lights* is introduced.

A moments thought will bring the realization that the order in which we press buttons to turn the lights of is of no importance. This means that we can pick any order. We will chose to press buttons per row from left to right and the rows from top to bottom.

Now let's say that we are in the process of turning of the lights. In figure 1 we have already pressed all the buttons in the first row. There is only one way to turn of the lights that are still lit in the first row. I.e. to press each button in the second row that is directly underneath a lit button. No other buttons we still need to press effects the first row.

Once the buttons in the first row are pressed, options are forced, all the way down to the last row. This sets up a simpler linear equation. In the language of [Mar01].

$$Bp_1 = -chase(s)$$

Here $B$ is a $5 \times 5$ matrix. $p_1$ are the first five components of the buttons to press, i.e. the first row. And $chase(s)$ is the effect of chasing the lights to the last row.

This allows a solution that can be committed to memory, which one of the authors has done for the standard Lights out puzzle.

In [Lea17] this method chasing the lights is extended to other rectangular board shapes. They show that analyzing a $c \times r$ Lights Out board one needs is interested in the upper left $c \times c$ sub-matrix of $W^r$ where $W$ describes the

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 2 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 2 |
| 3 | 0 | 0 | 2 | 0 | 0 | 3 | 0 | 0 | 2 | 0 | 0 | 3 |
| 4 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 4 | 0 | 0 |

Table 1: Dimension of Kernels

effect of one step in chasing the lights.

It is in this light that the authors made their observations.

# 2 Observations

In discovering the theorems of [Mar01] and [Lea17] for themselves, the authors studied the table 1, and its variants, extensively. To see more entries of the table goto the website for this article[1].

During our studies we made the following observations. For each row

1. The sequence is purely periodic.
2. There is a kernel of maximal dimension.
3. The period starts after the maximal dimension.
4. The sum of two consecutive dimensions is less than or equal to the maximal dimension.
5. The sequence is almost palindromic.

We will explain each observation with the aid of the above table. Observation 1 pertains that for each number of columns, the sequence of dimensions of kernels of $c \times r$ Lights Out puzzles repeats itself eventually, and when it repeats it does so from the start.

A kernel of maximal dimension is when the kernel is the entire press space. In other words, in the sequence for the number of columns $c$, there is an entry of precisely $c$. Furthermore, the sequence repeats itself when this happens.

For observation 5, take a look at the third row of the above table. The sequence up until the value 3, i.e. $0, 0, 2, 0, 0$ is palindromic. It is the same read from left to right as from right to left.

[1] `http://dvberkel.github.io/mathematics-articles/lights_out/periodicity.html`

The fact that in a row consecutive numbers sum to less than the maximal dimension is observation 4.

# 3 Definition

In this section we will define some terms that we will use throughout the article.

In this article we are exploring rectangular Lights Out. A $(n, c, r)$ Lights Out puzzle is a matrix $M$ with $c \in \mathbb{N}$ columns and $r \in \mathbb{N}$ rows with entries over $\mathbb{Z}/n\mathbb{Z}$. The space of all $(n, c, r)$ puzzles is called $\mathcal{L}_{(n,c,r)}$.

For a each $\mathcal{L}_{(n,c,r)}$ and for each $(i, j)$ with $1 \leq i \leq r$ and $1 \leq j \leq c$ there is a basic press function $p_{(i,j)} : \mathcal{L}_{(n,c,r)} \to \mathcal{L}_{(n,c,r)}$ mapping

$$\left(p_{(i,j)}(M)\right)_{(u,v)} := \begin{cases} M_{(u,v)} + 1 & : \quad \mathrm{d}\left((i,j),(u,v)\right) \leq 1 \\ M_{(u,v)} & : \quad \text{otherwise} \end{cases}$$

where d is the *Manhattan distance*. The set of all basic presses is called $B$.

A press sequence is a finite sequence of basic presses. The set of all press sequences is called $P$. $P$ together with concatenation of sequences makes a monoid with the empty sequence as identity element.

The effect $E$ of a press sequence is an mapping on $\mathcal{L}_{(n,c,r)}$ that extends basic presses. I.e. the effect of the empty sequence is the identity map and for a press sequence $(q_t)_{t\in\overline{m}}$ with $\overline{m} := \{0, 1, \ldots, m-1\}$

$$E\left((q_t)_{t\in\overline{m}})\right) = E\left((q_{t+1})_{t\in\overline{m-1}})\right) \circ q_0$$

For each press sequence $q := (q_t)_{t\in\overline{m}}$ we define a count function $N : P \to \mathbb{N}^B$ that counts the number of times that a basic press is present.

$$N_q(p_{(i,j)}) = \sum_{t\in\overline{m}} 1_{(i,j)}(q_t)$$

where $1_{(i,j)}(q)$ is 1 when $q = p_{(i,j)}$ and zero otherwise. We will write the application of $N$ to press sequence $q$ as $N_q$.

With this count we will define a fingerprint function $I : P \to (\mathbb{Z}/n\mathbb{Z})^B$ that count how many times a basic press is present in the sequence, modulo $n$. So for $\overline{m} := \{0, 1, \ldots, m-1\}$ we have

$$I\left(q\right) = p_{(i,j)} \mapsto \overline{N_q(p_{(i,j)})}$$

Notice that $I(u \circ v) = I(u) + I(v)$ and $I(\epsilon) = O$. Again, we will write $I_u$ for the application $I(u)$.

Now we will define a relation $\sim$ over $P$. $u \sim v$ if and only if $I(u) = I(v)$. With some thought one can see that $\sim$ is an equivalence relation. The equivalence class of $u \in P$ will be denoted by $[u]$ and wil be called a *press pattern*. The set of all press patterns will be denoted by $\mathcal{P}$.

Note that for $[s], [t] \in \mathcal{P}$ with $[s] = [t]$ we have

$$E(s)M = E(t)M$$

since

$$(E(s)M)_{(u,v)} = M_{(u,v)} + \sum_{d((i,j),(u,v)) \leq 1} I_s(p_{(i,j)}) = M_{(u,v)} + \sum_{d((i,j),(u,v)) \leq 1} I_t(p_{(i,j)}) = (E(t)M)_{(u,v)}$$

We will define the following binary operation on $\mathcal{P}$: $[u] + [v] = [u \circ v]$. Notice that for $u_0, u_1, v_0, v_1 \in \mathcal{P}$ with $[u_0] = [u_1]$ and $[v_0] = [v_1]$ we have.

$$[u_0 \circ v_0] = \{w \in P | I(w) = I(u_0 \circ v_0) = I(u_0) + I(v_0) = I(u_1) + I(v_1) = I(u_1 \circ v_1)\} = [u_1 \circ v_1]$$

which show that the addition is well defined. Since $I(u) + I(v) = I(v) + I(u)$ $(\mathcal{P}, +)$ is an abelian group.

If we define scalar multiplication with $r \in \mathbb{Z}/n/Z$ as

$$r[u] = [\underbrace{u \circ u \circ \ldots u}_{r\text{times}}]$$

We turn $\mathcal{P}$ into a free $\mathbb{Z}/n/Z$-module.

We will construct a map $\mathcal{E}$ from $\mathcal{P} \to \mathcal{L}$ with $[u] \mapsto E(u)O$. Notice that $\mathcal{E}(u \circ v) = \mathcal{E}(u)(\mathcal{E}(v))$ which makes $\mathcal{E}$ a linear transformation.

Solving a $s \in \mathcal{L}$ amounts to solving for $u \in \mathcal{P}$ the linear equation

$$\mathcal{E}(u) = -s$$

Next we consider an order $\leq$ on $B$, the basic presses. For $p_{(i,j)}, p_{(u,v)} \in B$ we have $p_{(i,j)} \leq p_{(u,v)}$ if and only if $j < v$ or when $j = v$ then $i \leq j$. This ordering amounts to ordering the basic pressed per row from top to bottom and for each row from left to right.

A press sequence $q := (q_t)_{t \in \overline{m}} \in P$ is called a *standard* press sequence when $q_i \leq q_j$ whenever $i \leq j$. For each press pattern $P \in \mathcal{P}$ there is a unique standard press sequence $p \in P$ such that $P = [p]$. A press sequence is *fertile* if it only contains basic presses with a row index of 0.

We will turn to chasing the light. For each light pattern $L \in \mathcal{L}$ we define a press pattern *chase*$(L)$. We create a sequence of standard press patterns.

$z_0 := \epsilon$, the empty press pattern. $z_{n+1} := z_n \circ p_{(i,j+1)}$ where $(i,j)$ is the smallest non-zero entry in $L + \mathcal{E}(z_n)$. $chase(L)$ is the press pattern when the press sequence does not grow anymore.

Chasing amounts to finding a fertile press sequence $p \in P$ such that

$$chase([p]) = -chase(L)$$

$chase_{(n,c,r)}$ is a linear transformation.

**Definition** For a $(n,c,r)$ Light Out puzzle we call $C_{(n,c,r)} := \dim \operatorname{Ker} chase_{(n,c,r)}$ $\circ$

When $n$ and $c$ are clear from context we will write $C_r$ for $C_{(n,c,r)}$.

# 4 Algebra

We will formalize the process of chasing down the lights and prove all of our observations. To begin we will introduce a matrix of some interest.

For all $c \in \mathbb{N}$ define the $2c \times 2c$ matrix $W_c$ with entries in $GF(q)$ by

$$W_c := \begin{pmatrix} -E_c & I \\ -I & O \end{pmatrix}$$

where $O$ is the zero matrix, $I$ the identity matrix and $E_c$ is defined as the $c \times c$ matrix with ones on the diagonal and the two main sub-diagonals, and zeroes elsewhere.

For example

$$E_4 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The reason we are looking at $W_c$ is that its powers tell us something about the effect of chasing down the lights. In particular, if we have an $r \times c$ lights out puzzle, the $c \times c$ upper left sub-matrix of $W_c^r$ is exactly the process of gathering the lights.

The first interesting fact is that $W_c$ is invertible.

**Lemma 1** *$W_c$ is invertible for all $c \in \mathbb{N}$.* $\diamond$

**Proof**

$$\begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} O & -I \\ I & -E \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}$$

$\square$

A consequence of the invertibility of $W_c$ is that the sequence of its powers is periodic. This also proves our observation 1, but we will have some more to say about that.

**Theorem 2** *The sequence $(W_c^n)_{n\in\mathbb{N}}$ is purely periodic.* $\diamond$

**Proof** There are only finitely many different square matrices of size $2c$ over $GF(q)$. So the sequence $(W_c^n)_{n\in\mathbb{N}}$ must become periodic. By the preceding lemma $W_c$ is invertible so the sequence is periodic from the start. $\square$

As mentioned in [Lea17] there is a relation between the images of chasing down the lights and Fibonacci polynomials. We find that relation in our structure lemma.

**Lemma 3** (structure) *There exists a sequence of $c \times c$ matrices $(T_n)_{n\in\mathbb{N}}$ such that*

$$W_c^n = \begin{pmatrix} T_n & T_{n-1} \\ -T_{n-1} & -T_{n-2} \end{pmatrix}$$

*for all $k \in \mathbb{N}$.* $\diamond$

**Proof** Define $T_0 := I$, and for convenience $T_{-1} := O$ and $T_{n+1} := -E \cdot T_n - T_{n-1}$ for all $n \in \mathbb{N}$. So $T_1 = -E \cdot I - O = -E$.

A number $n \in \mathbb{N}$ is called strong if and only if

$$W_c^n = \begin{pmatrix} T_n & T_{n-1} \\ -T_{n-1} & -T_{n-2} \end{pmatrix}$$

Notice that $W_c^1 = \left(\begin{smallmatrix} -E & I \\ -I & O \end{smallmatrix}\right) = \left(\begin{smallmatrix} T_1 & T_0 \\ -T_0 & -T_{-1} \end{smallmatrix}\right)$ so 1 is strong.

Assume that $k$ is strong. We will show that $k+1$ is strong as well.

$$\begin{aligned} W_c^{k+1} &= W_c \cdot W_c^k \\ &= \begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} T_k & T_{k-1} \\ -T_{k-1} & -T_{k-2} \end{pmatrix} \\ &= \begin{pmatrix} -E \cdot T_k - T_{k-1} & -E \cdot T_{k-1} - T_{k-2} \\ -T_k & -T_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} T_{k+1} & T_k \\ -T_k & -T_{k-1} \end{pmatrix} \end{aligned}$$

By mathematical induction all natural numbers are strong, finishing the proof. $\square$

With the structure lemma under our belt we can prove our first observation, i.e. the sequence of the dimension of kernels is periodic.

**Proposition 4** (Observation 1) *The sequence* $(\dim \operatorname{Ker} P_{r,c})_{r\in\mathbb{N}}$ *is periodic.* $\diamond$

**Proof** By the structure lemma, for all $r \in \mathbb{N}$,

$$\dim \operatorname{Ker} P_{r,c} = \dim \operatorname{Ker} T_r$$

By the periodicity of $(W_c^r)_{r\in\mathbb{N}}$ we have the periodicity of $(\dim \operatorname{Ker}(P_{r,c})_{r\in\mathbb{N}}$. $\square$

It could be the case that the period of $(\dim \operatorname{Ker}(P_{r,c})_{r\in\mathbb{N}}$ is a divisor of the period of $(W_c^r)_{r\in\mathbb{N}}$. In fact, these periods do not coincide. There is a relation which we will see shortly.

Next on our agenda is our observation 2. I.e. for each number of columns $c$ there is a kernel with that dimension.

**Lemma 5** (Observation 2) *There is a kernel of maximal dimension.* $\diamond$

**Proof** We will be using the notation as defined by the structure lemma.

Notice that there exists $p \in \mathbb{N}$ such that $W_c^p = I$ and thus $W_c^{p-1} = W_c^{-1} \cdot W_c^p = W_c^{-1}$. Furthermore

$$\dim \operatorname{Ker} P_{p-1,c} = \dim \operatorname{Ker} T_{-1} = \dim \operatorname{Ker} O = c$$

$\square$

Before we will dive deeper in the question if the period of both sequences coincide, we will take a closer look at observation **??**. For this we need to know the determinant of $W_c^n$ for all $n \in \mathbb{N}$.

**Lemma 6** $\det(W_c^n) = 1$ *for all* $n \in \mathbb{N}$. $\diamond$

**Proof** Note that $\det(W_c^0) = \det(I) = 1$ and

$$\det(W_c) = \det \begin{pmatrix} -E & I \\ -I & O \end{pmatrix} = \det(-E) \cdot \det(O) - \det(-I) \cdot \det(I) = 1$$

The lemma follows by multiplicativity of the determinant and by induction on $n$. $\square$

This little fact will helps us establishing the proof of observation **??**.

**Theorem 7** (Observation **??**) *If for some* $r \in \mathbb{N}$ *we have* $\dim \operatorname{Ker} P_{r,c} = c$ *then*

- $\dim \operatorname{Ker} P_{r-1,c} = 0$
- $\dim \operatorname{Ker} P_{r+1,c} = 0$

◇

**Proof** Let $r \in \mathbb{N}$ be such that $\dim \operatorname{Ker} P_{r,c} = c$; then

$$W_c^r = \begin{pmatrix} O & T_{r-1} \\ -T_{r-1} & -T_{r-2} \end{pmatrix}$$

and $1 = \det W_c^r = (\det T_{r-1})^2$, so $T_{r-1}$ is invertible.

Note that $W_c^{r-1} = \left(\begin{smallmatrix} T_{r-1} & T_{r-2} \\ -T_{r-2} & -T_{r-3} \end{smallmatrix}\right)$ and $W_c^{r+1} = \left(\begin{smallmatrix} -T_{r-1} & O \\ O & -T_{r-1} \end{smallmatrix}\right)$, hence both

- $\dim \operatorname{Ker} P_{r-1,c} = \dim \operatorname{Ker} T_{r-1} = 0$,
- $\dim \operatorname{Ker} P_{r+1,c} = \dim \operatorname{Ker} -T_{r-1} = 0$.

□

**Corollary 8** *If for some* $r \in \mathbb{N}$ *we have* $\dim \operatorname{Ker} P_{r,c} = c$ *then* $W_c^{r+1} = \left(\begin{smallmatrix} -T_{r-1} & O \\ O & -T_{r-1} \end{smallmatrix}\right)$ ◇

In proving our observation **??** we have learned that the structure of the corresponding matrix power is particular simple. This fact will be instrumental in the relation between the period of $(W_c^n)_{n\in\mathbb{N}}$ and that of $(\dim \operatorname{Ker} P_{r,c})_{r\in\mathbb{N}}$.

But first we will see that $W_c$ and it's inverse are conjugates.

**Lemma 9** $W_c$ *and* $W_c^{-1}$ *are conjugates.* ◇

**Proof** We will conjugate $W = W_c$ by $C := \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$, which is its own inverse,

$$\begin{aligned} C \cdot W \cdot C^{-1} &= \begin{pmatrix} O & I \\ I & O \end{pmatrix} \cdot \begin{pmatrix} -E & I \\ -I & O \end{pmatrix} \cdot \begin{pmatrix} O & I \\ I & O \end{pmatrix} \\ &= \begin{pmatrix} -I & O \\ -E & I \end{pmatrix} \cdot \begin{pmatrix} O & I \\ I & O \end{pmatrix} \\ &= \begin{pmatrix} O & -I \\ I & -E \end{pmatrix} \\ &= W^{-1} \end{aligned}$$

□

**Corollary 10** $W_c^{-n} = \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right) W_c^n \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ $\diamond$

**Proof** Clear. $\square$

Note that conjugating any matrix with $\left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ corresponds to rotating the four cardinal sub-matrices through 180 degrees.

This fact will be the linchpin in the proof of observation 5.

**Lemma 11** *If for some $r \in \mathbb{N}$ we have $W_c^r = \left(\begin{smallmatrix} Q & O \\ O & Q \end{smallmatrix}\right)$ then for all $i \in \mathbb{N}$.*

$$W_c^{r+i} = \begin{pmatrix} QT_i & QT_{i-1} \\ -QT_{i-1} & -QT_{i-2} \end{pmatrix}$$

*and*

$$W_c^{r-i} = \begin{pmatrix} -QT_{i-2} & -QT_{i-1} \\ QT_{i-1} & QT_{i-2} \end{pmatrix}$$

$\diamond$

**Proof** By direct calculation and the structure lemma we find that

$$W_c^{r+i} = W_c^r W_c^i = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix} \begin{pmatrix} T_i & T_{i-1} \\ -T_{i-1} & -T_{i-2} \end{pmatrix} = \begin{pmatrix} QT_i & QT_{i-1} \\ -QT_{i-1} & -QT_{i-2} \end{pmatrix}$$

Furthermore, with $C = \left(\begin{smallmatrix} O & I \\ I & O \end{smallmatrix}\right)$ we have, by the preceding lemma, $CW_c^{r-i}C^{-1} = CW_c^r C^{-1} C W^{-i} C^{-1} = W_c^r W_c^i = W_c^{r+i}$ which we set out to prove. $\square$

We now come to our promise about the period of the dimension of kernels the kernels and the period of $W$. let $q$ be the smallest number rows that has a maximal kernel dimension and let $d = q + 1$. The period of $W$ will be $p$.

**Lemma 12** *Either $p = d$ or $p = 2d$.* $\diamond$

**Proof** If $p = d$ we are finished, so assume it is not. We will show that $p = 2d$ in that case.

By the preceding lemma we that the lower right $c \times c$ sub-matrix of $W_c^{d-i}$ equals the upper left $c \times c$ sub-matrix of $W_c^{d+i}$ for all $i$. In particular for $i = d$. The lower right sub-matrix of $W_c^{d-d} = W_c^0 = I$ is the $c \times c$ identity matrix.

Furthermore, since $W^d = \left(\begin{smallmatrix} Q & O \\ O & Q \end{smallmatrix}\right)$ for certain matrix $Q$, we have

$$W_c^{2d} = \begin{pmatrix} Q & O \\ O & Q \end{pmatrix}^2 = \begin{pmatrix} Q^2 & O \\ O & Q^2 \end{pmatrix}$$

Hence, $Q^2$ is equal to the $c \times c$ identity matrix, therefore $W^{2d}$ is the identity matrix, and the period of $(W_c^r)_{r\in\mathbb{N}}$ is $2d$. $\square$

**Theorem 13** (Observation 5) *The sequence* $(\dim \operatorname{Ker} P_{r,c})_{r\in\mathbb{N}}$ *is almost palindromic.* $\diamond$

**Proof** let $d$ be such that $W^d$ has a upperleft sub matrix $O$.

A *press vector* is a $2 \times c$ vector with the last $c$ components zero, an *unlit vector* is a $2 \times c$ vector with the first $c$ components 0. Notice that for any press vector $v$

$$W^d v$$

is an unlit vector.

Choose $m, n \in \mathbb{N}$ such that $m + 1 + n = d$. We will show that for each press vector $v$ for which $W^m v$ is unlit, there exist a press vector $v'$ such that $W^n v'$ is unlit. This shows that $\dim \operatorname{Ker} T_m \leq \dim \operatorname{Ker} T_n$. Since the argument is symmetric in $m$ and $n$ we have $\dim \operatorname{Ker} T_m = \dim \operatorname{Ker} T_n$.

Let $v$ be a press vector such that $W^m v$ is unlit. In particular $W^m v = u$ with $u := (0, 0, \ldots, 0, u_1, u_2, \ldots, u_c)^t$.

Define $p := Wu$. We will show that $p$ is a press vector.

$$Wu = \begin{pmatrix} -E_c & I \\ -I & O \end{pmatrix} \begin{pmatrix} O \\ u' \end{pmatrix} = \begin{pmatrix} u' \\ O \end{pmatrix}$$

Since for any press vector $w$ we have that $W^d w = O$, in particular we have

$$O = W^d v = W^n W W^m v = W^n W u = W^m p$$

which shows that for each press vector that $W^m$ unlits, there is a press vector that $W^n$ unlits. $\square$

# References


[Fei98] Marlow Anderson & Todd Feil. Turning lights out with linear algebra. *Mathematics Magazine*, 71(4):300–303, October 1998.

[Lea17] C. David Leach. Chasing the lights in lights out. *Mathematics Magazine*, 90(2):126–133, December 2017.

[Mar01] Paraja-Flores, Cristóbal Martín-Sánchez, Óscar. Two reflected analysis of lights out. *Mathematics Magazine*, 74(4):295–304, October 2001.

[Pel87] Don Pelletier. Merlin's magic square. *The American Mathematical Monthly*, 94(2):143–150, February 1987.