Optimization:
1-Linear Programming
2- Application to industrial processes

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Objectives

- Mathematical Methods of Linear Programming:

1. Algorithms for solving Linear Systems of equations
2. Linear Programming: formulation of the optimization of a linear function (all contraints also linear) in the optimization variables.
3. The simplex algorithm
4. Lagrange Multipliers: Relation with KT problem
5. Sensitivity Analysis of the basic Simplex algorithm
6. Revised Simplex method (the reduced objective function)
7. Sensitivity analysis of the revised simplex method.
8. Duality formulation of Linear Programming
9. The interior point method

Linear Programming

Find $\mathbf{x}$ in order to

Minimize $f(\mathbf{x}) \equiv \mathbf{c}^T\mathbf{x}$

Subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

where

$\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ vector of optimization variables

$\mathbf{c} = [c_1, c_2, \ldots, c_n]^T$ vector of objective or cost coefficients

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ $m \times n$ matrix of constraint coefficients

$\mathbf{b} = [b_1, b_2, \ldots, b_m]^T \geq 0$ vector of right-hand sides of constraints

Linear Programming- Notes-1

Note that in this standard form, the problem is of minimization type. All constraints are expressed as equalities with the right-hand side greater than or equal to (≥) 0. Furthermore, all optimization variables are restricted to be positive.

Maximize $z(\mathbf{x}) = 3x_1 + 5x_2$ is the same as Minimize $f(\mathbf{x}) = -3x_1 - 5x_2$

Minimize $f(\mathbf{x}) = 3x_1 + 5x_2 + 7$

In standard LP form, it can be written as follows:

$$\begin{aligned} \text{Minimize} \quad & f(\mathbf{x}) = 3x_1 + 5x_2 + 7x_3 \\ \text{Subject to} \quad & x_3 = 1 \end{aligned}$$

Linear Programming-Notes-2

Unrestricted Variables:

The standard LP form restricts all variables to be positive. If an actual optimization variable is unrestricted in sign, it can be converted to the standard form by defining it as the difference of two new positive variables. For example, if variable x_1 is unrestricted in sign, it is replaced by two new variables y_1 and y_2 with $x_1 = y_1 - y_2$. Both the new variables are positive. After the solution is obtained, if $y_1 > y_2$, then x_1 will be positive and if $y_1 < y_2$, then x_1 will be negative.

LP-Unrestricted Variables

- Example

Maximize $z = 3x_1 + 8x_2$

$$\text{Subject to } \begin{pmatrix} 3x_1 + 4x_2 \geq -20 \\ x_1 + 3x_2 \geq 6 \\ x_1 \geq 0 \end{pmatrix}$$

Note that x_2 is unrestricted in sign. Define new variables (all ≥ 0)

$$x_1 = y_1 \quad x_2 = y_2 - y_3$$

Substituting these and multiplying the first constraint by a negative sign, the problem is as follows:

Maximize $z = 3y_1 + 8y_2 - 8y_3$

$$\text{Subject to } \begin{pmatrix} -3y_1 - 4y_2 + 4y_3 \leq 20 \\ y_1 + 3y_2 - 3y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \end{pmatrix}$$

LP-Unrestricted variables

Multiplying the objective function by a negative sign and introducing Slack/Surplus Variables in the constraints, the problem in the standard LP form is as follows:

$$\text{Minimize } f = -3y_1 - 8y_2 + 8y_3$$

$$\text{Subject to } \begin{pmatrix} -3y_1 - 4y_2 + 4y_3 + y_4 = 20 \\ y_1 + 3y_2 - 3y_3 - y_5 = 6 \\ y_1, \ldots, y_5 \geq 0 \end{pmatrix}$$

The optimum of LP program

The optimum solution of an LP problem always lies on the boundary of the feasible domain. We can easily prove this by contradiction. Suppose the solution lies inside the feasible domain; then the optimum is an unconstrained point, and hence, the necessary conditions for optimality would imply that $\partial f/\partial x_i \equiv c_i = 0, i = 1, 2, \ldots, n$, which obviously is not possible (because all $c_i = 0$ means $f \equiv \mathbf{c}^T\mathbf{x} = 0$). Thus, the solution cannot lie on the inside of the feasible domain for LP problems.

BASIC and NONBASIC variables

The solution of an LP problem reduces to solving a system of underdetermined Linear equations. From m equations, at most we can solve m variables in terms of the remained n-m variables. The variables that we choose to solve are called ***BASIC***, and the remaining variales are called ***NONBASIC***

LP Example

Minimize $f = -x_1 + x_2$

Subject to $\begin{pmatrix} x_1 - 2x_2 \geq 2 \\ x_1 + x_2 \leq 4 \\ x_1 \leq 3 \\ x_i \geq 0, i = 1, 2 \end{pmatrix}$

In the standard LP form:

Minimize $f = -x_1 + x_2$

Subject to $\begin{pmatrix} x_1 - 2x_2 - x_3 = 2 \\ x_1 + x_2 + x_4 = 4 \\ x_1 + x_5 = 3 \\ x_i \geq 0, i = 1, \ldots, 5 \end{pmatrix}$

LP Example

- In the standard LP form

$$\text{Minimize } f = -x_1 + x_2$$

$$\text{Subject to} \begin{pmatrix} x_1 - 2x_2 - x_3 = 2 \\ x_1 + x_2 + x_4 = 4 \\ x_1 + x_5 = 3 \\ x_i \geq 0, i = 1, \ldots, 5 \end{pmatrix}$$

where x_3 is a surplus variable for the first constraint, and x_4 and x_5 are slack variables for the two less-than type constraints. The total number of variables is $n = 5$, and the number of equations is $m = 3$. Thus, we can have three basic variables and two nonbasic variables. If we arbitrarily choose x_3, x_4, and x_5 as basic variables, a general solution of the constraint equations can readily be written as follows:

$$x_3 = -2 + x_1 - 2x_2 \qquad x_4 = 4 - x_1 - x_2 \qquad x_5 = 3 - x_1$$

LP Example

The general solution is valid for any values of the nonbasic variables. Since all variables are positive and we are interested in minimizing the objective function, we assign 0 values to nonbasic variables. A solution from the constraint equations obtained by setting nonbasic variables to zero is called a *basic solution*. Therefore, one possible basic solution for the above example is as follows:

$$x_3 = -2 \qquad x_4 = 4 \qquad x_5 = 3$$

Since all variables must be ≥ 0, this basic solution is infeasible because x_3 is negative.

Let's find another basic solution by choosing (again arbitrarily) x_1, x_4, and x_5 as basic variables and x_2 and x_3 as nonbasic. By setting nonbasic variables to zero, we need to solve for the basic variables from the following equations:

$$x_1 = 2 \qquad x_1 + x_4 = 4 \qquad x_1 + x_5 = 3$$

It can easily be verified that the solution is $x_1 = 2$, $x_4 = 2$, and $x_5 = 1$. Since all variables have positive values, this basic solution is feasible as well.

LP Example

Number of possible basic solutions = Binomial[n, m] $\equiv \frac{n!}{m!\,(n-m)!}$

where "!" stands for *factorial*. For the example problem where $m = 3$ and $n = 5$, therefore, the maximum number of basic solutions is

$$\frac{5!}{3!\,2!} = \frac{5 \times 4 \times 3!}{3! \times 2} = 10$$

All these basic solutions are computed from the constraint equations and are summarized in the following table. The set of basic variables for a particular solution is called a *basis* for that solution.

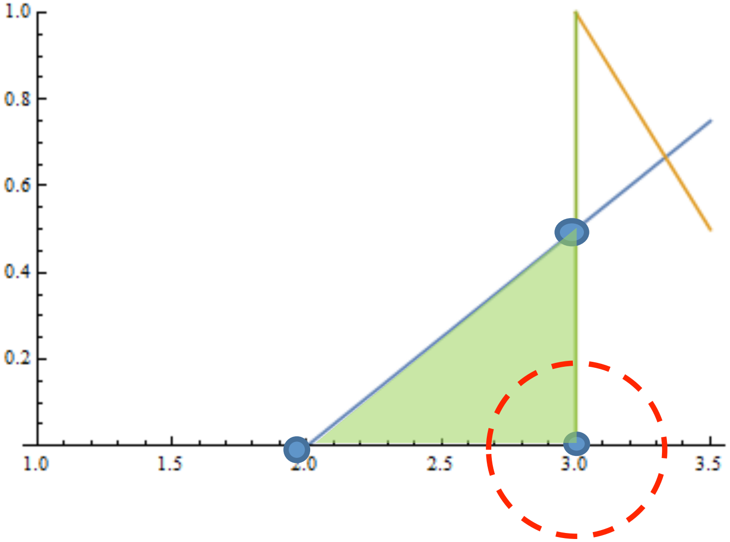
LP Example

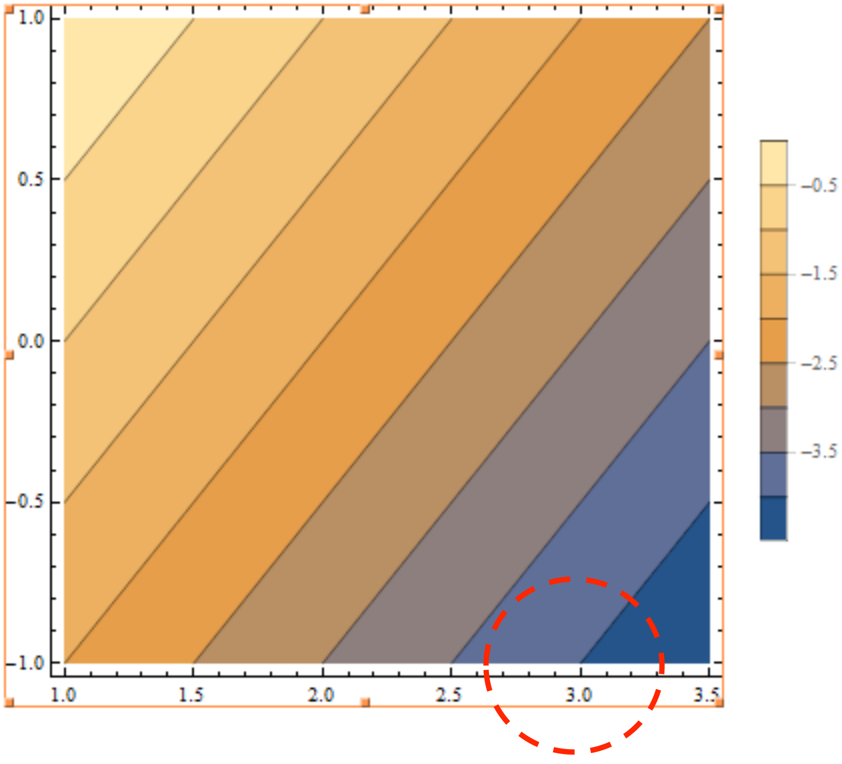
	Basis	Solution	Status	f
(1)	$\{x_1, x_2, x_3\}$	$\{3, 1, -1, 0, 0\}$	Infeasible	—
(2)	$\{x_1, x_2, x_4\}$	$\left\{3, \frac{1}{2}, 0, \frac{1}{2}, 0\right\}$	Feasible	$-\frac{5}{2}$
(3)	$\{x_1, x_2, x_5\}$	$\left\{\frac{10}{3}, \frac{2}{3}, 0, 0, -\frac{1}{3}\right\}$	Infeasible	—
(4)	$\{x_1, x_3, x_4\}$	$\{3, 0, 1, 1, 0\}$	Feasible	-3
(5)	$\{x_1, x_3, x_5\}$	$\{4, 0, 2, 0, -1\}$	Infeasible	—
(6)	$\{x_1, x_4, x_5\}$	$\{2, 0, 0, 2, 1\}$	Feasible	-2
(7)	$\{x_2, x_3, x_4\}$	$\{-\}$	NoSolution	—
(8)	$\{x_2, x_3, x_5\}$	$\{0, 4, -10, 0, 3\}$	Infeasible	—
(9)	$\{x_2, x_4, x_5\}$	$\{0, -1, 0, 5, 3\}$	Infeasible	—
(10)	$\{x_3, x_4, x_5\}$	$\{0, 0, -2, 4, 3\}$	Infeasible	—

LP Example

Optimum solution:

$$x_1^* = 3 \qquad x_2^* = 0 \qquad x_3^* = 1 \qquad x_4^* = 1 \qquad x_5^* = 0 \qquad f^* = -3$$





The Simplex Method

- The optimum solution of an LP problem corresponds to one of the basic solutions. However, the number of poissible basic solutions can be very large for practical problems. Thus, the goal is to develop a procedure that quickly finds the basic feasible solution with the lowest objectiv function value without examining all possibilities.

Simplex Method: Basic idea

- Start with a basic feasible solution
- Try to obtain a neighboring feasible solution with lower ocjective function.
- Each try one of the current basic variables is made non basic and is replaced with a variable from the non basic set.
- An optimum is reached when no other basic feasible solution can be found with lower objective function

Simplex Method

- The complete algorithm needs
  1. Finding a starting basic feasible solution
  2. Bringing a currently non basic variable into the basic set
  3. Moving a currently basic variable out of the basic set to make room for the new basic variable.

Simplex Method for LE constraints

$$\text{Minimize } f = 5x_1 - 3x_2 - 8x_3$$

$$\text{Subject to} \begin{pmatrix} 2x_1 + 5x_2 - x_3 \le 1 \\ -2x_1 - 12x_2 + 3x_3 \le 9 \\ -3x_1 - 8x_2 + 2x_3 \le 4 \\ x_i \ge 0, i = 1, \ldots, 3 \end{pmatrix}$$

Simplex Method LE constraints

For the example problem, introducing slack variables x_4, x_5, and x_6, the constraints are written in the standard LP form as follows.

$$2x_1 + 5x_2 - x_3 + x_4 = 1$$

$$-2x_1 - 12x_2 + 3x_3 + x_5 = 9$$

$$-3x_1 - 8x_2 + 2x_3 + x_6 = 4$$

Treating the slack variables as basic and the others as nonbasic, the starting basic feasible solution is

Basic: $x_4 = 1, x_5 = 9, x_6 = 4$

Nonbasic: $x_1 = x_2 = x_3 = 0$

$f = 0$

Simplex Method LE constraints

- Bringing a New Variable into the basic set

Continuing with the previous example, we have the following situation:

Basic: $x_4 = 1, x_5 = 9, x_6 = 4$

Nonbasic: $x_1 = x_2 = x_3 = 0$

$f = 0 = 5x_1 - 3x_2 - 8x_3$

The largest negative coefficient in the f equation is that of x_3. Thus, our next basic feasible solution should use x_3 as one of its basic variables.

- The variable x3 is to be brought into the basic set => it will have a value greater or equal to zero in the next feasible solution.

Simplex Method LE constraints

Constraint 1: x_4 basic: $2x_1 + 5x_2 - x_3 + x_4 = 1$

Constraint 2: x_5 basic: $-2x_1 - 12x_2 + 3x_3 + x_5 = 9$

Constraint 3: x_6 basic: $-3x_1 - 8x_2 + 2x_3 + x_6 = 4$

The new basic variable set is (x_4, x_5, x_3). We can achieve this by eliminating x_3 from the first and the second constraints. We divide the third constraint by 2 first to make the coefficient of x_3 equal to 1.

Constraint 3: x_3 basic: $-\frac{3}{2}x_1 - 4x_2 + x_3 + \frac{1}{2}x_6 = 2$

This constraint is known as the *pivot row* for computing the new basic feasible solution, and is used to eliminate x_3 from the other constraints and the objective function. Variable x_3 can be eliminated from constraint 1 by adding the pivot row to the first constraint.

Constraint 1: x_4 basic: $\frac{1}{2}x_1 + x_2 + x_4 + \frac{1}{2}x_6 = 3$

Simplex Method LE constraints

- Variable x3 is eliminated by adding (-3) time the pivot row to it

$$\text{Constraint 2:} \quad x_5 \text{ basic:} \quad \tfrac{5}{2}x_1 + x_5 - \tfrac{3}{2}x_6 = 3$$

- Adding 8 times the pivot row to the objective function we get

$$-7x_1 - 35x_2 + 4x_6 = f + 16$$

Simplex Method LE constraint

- We now have a new basic feasible solution, as follows:

Basic: $x_3 = 2, x_4 = 3, x_5 = 3$

Nonbasic: $x_1 = x_2 = x_6 = 0$

$f = -16$

Considering the objective function value, this solution is better than our starting solution.

- The series of steps are repeated until all coefficients in the objective function become positive, this indicates that we reached the lowest value possible and the current basic feasible solution represents the optimum one.

Simplex Method LE constraints

- Current situation is:

Constraint 1:	x_4 basic:	$\frac{1}{2}x_1 + x_2 + x_4 + \frac{1}{2}x_6 = 3$
Constraint 2:	x_5 basic:	$\frac{5}{2}x_1 + x_5 - \frac{3}{2}x_6 = 3$
Constraint 3:	x_3 basic:	$-\frac{3}{2}x_1 - 4x_2 + x_3 + \frac{1}{2}x_6 = 2$
Objective:		$-7x_1 - 35x_2 + 4x_6 = f + 16$

- Since x2 has the largest negative coefficient in the objective function, it is the next variable to be made basic. In the constraint expressions, possitive coefficient of x2 shows up only in the first constraint which has x4 in the basic set, thus x4 should be removed from the basic set.

Simplex Method LE constraint

- Now we must use the first equation as the pivot row (PR) as follows:

Constraint 1: x_2 basic: $\frac{1}{2}x_1 + x_2 + x_4 + \frac{1}{2}x_6 = 3$ (PR)
Constraint 2: x_5 basic: $\frac{5}{2}x_1 + x_5 - \frac{3}{2}x_6 = 3$ (no change)
Constraint 3: x_3 basic: $\frac{1}{2}x_1 + x_3 + 4x_4 + \frac{5}{2}x_6 = 14$ (Added $4 \times$ PR)
Objective: $\frac{21}{2}x_1 + 35x_4 + \frac{43}{2}x_6 = f + 121$ (Added $35 \times$ PR)

- In the objective all the coefficient of the non basic variables are possitive, thus we reached the optimum solution

Simplex Method LE constraint

Optimum solution:

Basic: $x_2 = 3, x_3 = 14, x_5 = \dot{3}$

Nonbasic: $x_1 = x_4 = x_6 = 0$

$f + 121 = 0$ giving $f^* = -121$

Simplex Tableau

Minimize $f = 5x_1 - 3x_2 - 8x_3$

$$\text{Subject to} \begin{pmatrix} 2x_1 + 5x_2 - x_3 \leq 1 \\ -2x_1 - 12x_2 + 3x_3 \leq 9 \\ -3x_1 - 8x_2 + 2x_3 \leq 4 \\ x_i \geq 0, i = 1, \ldots, 3 \end{pmatrix}$$

Introducing slack variables, the constraints are written in the standard LP form as follows:

$$2x_1 + 5x_2 - x_3 + x_4 = 1$$

$$-2x_1 - 12x_2 + 3x_3 + x_5 = 9$$

$$-3x_1 - 8x_2 + 2x_3 + x_6 = 4$$

Simplex Tableau

- This problem can be summarized in the initial tableau and the first feasible solution:

$$
\begin{pmatrix}
\text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\
x_4 & 2 & 5 & -1 & 1 & 0 & 0 & 1 \\
x_5 & -2 & -12 & 3 & 0 & 1 & 0 & 9 \\
x_6 & -3 & -8 & 2 & 0 & 0 & 1 & 4 \\
\text{Obj.} & 5 & -3 & -8 & 0 & 0 & 0 & \text{f}
\end{pmatrix}
$$

Basic: $x_4 = 1,\ x_5 = 9,\ x_6 = 4$

Nonbasic: $x_1 = x_2 = x_3 = 0$

$f = 0$

Simplex Tableau

- From the simplex tableau, we see that in the column corresponding to x3 there are two constraints rows that have possitive coefficients. Ratios of the right-hand-side and these entries are as follows:
- Ratios:
  - Second constraint : 9/3=3,
  - Third constraint: 4/2=2.
- The minimum ratio corresponds to the third constraint for which x6 is the current basic variable. Thus we should make x6 non basic.

Simplex Tableau

- The computations are as follows:

Basis	x_1	x_2	x_3	x_4	x_5	x_6	RHS		
x_4	$\frac{1}{2}$	1	0	1	0	$\frac{1}{2}$	3	$\Longleftarrow$	$\text{PR} + \text{Row1}$
x_5	$\frac{5}{2}$	0	0	0	1	$-\frac{3}{2}$	3	$\Longleftarrow$	$-3 \times \text{PR} + \text{Row2}$
x_3	$-\frac{3}{2}$	-4	1	0	0	$\frac{1}{2}$	2	$\Longleftarrow$	PR
Obj.	-7	-35	0	0	0	4	$16 + f$	$\Longleftarrow$	$8 \times \text{PR} + \text{Obj.Row}$

Basic: $x_3 = 2, x_4 = 3, x_5 = 3$ Nonbasic: $x_1 = x_2 = x_6 = 0$ $f = -16$

- x2 should be made basic and, hencefore x4 becomes non basic

Simplex Tableau

- The computations are:

Basis	x_1	x_2	x_3	x_4	x_5	x_6	RHS		
x_2	$\frac{1}{2}$	1	0	1	0	$\frac{1}{2}$	3	$\Longleftarrow$	PR
x_5	$\frac{5}{2}$	0	0	0	1	$-\frac{3}{2}$	3	$\Longleftarrow$	Row2
x_3	$\frac{1}{2}$	0	1	4	0	$\frac{5}{2}$	14	$\Longleftarrow$	$4 \times$ PR $+$ Row3
Obj.	$\frac{21}{2}$	0	0	35	0	$\frac{43}{2}$	$121 + f$	$\Longleftarrow$	$35 \times$ PR $+$ Obj.Row

Basic: $x_2 = 3, x_3 = 14, x_5 = 3$ Nonbasic: $x_1 = x_4 = x_6 = 0$ $f = -121$

- Which is the optimum.

Simplex Method GE and EQ constraints

- The starting basic feasible solution is more difficult to obtain for greater than (GE) and Equality constraints (EQ). The reason is that there is no unique positive variable associated with each constraint. A unique surplus variable is present in each GE constraint, but it is multiplied by a negative sign and thus will give an unfeasible solution if treated as a basic variable.
- we need a Phase I procedure to minimize an artificial function $\phi(\mathbf{x})$, obtained from the artificial variables associated to it.

Simplex method:Phase I

- The artificial function $\phi(\mathbf{x})$, is defined as the sum of all artificial variables needed in the problem (one per each GE constraint). Since there are no real constraints on $\phi(\mathbf{x})$, the optimum solution is reached when $\phi(\mathbf{x})$, gets zero. That is, when all artificial variables are equal to zero. This optimum solution of phase I is a basic feasible solution, and we are now in the position to start solving the original problem with the actual objective function.

Simplex Method Phase I

- Consider the following example with two GE constraints:

Minimize $f = 2x_1 + 4x_2 + 3x_3$

$$\text{Subject to} \begin{pmatrix} -x_1 + x_2 + x_3 \geq 2 \\ 2x_1 + x_2 \geq 1 \\ x_i \geq 0, i = 1, \ldots, 3 \end{pmatrix}.$$

Introducing surplus variables x_4 and x_5, the constraints are written in the standard LP form as follows:

$$-x_1 + x_2 + x_3 - x_4 = 2 \quad 2x_1 + x_2 - x_5 = 1$$

Now introducing artificial variables x_6 and x_7, the Phase I objective function and constraints are as follows:

Simplex Method Phase I

Minimize $\phi = x_6 + x_7$

$$\text{Subject to} \begin{pmatrix} -x_1 + x_2 + x_3 - x_4 + x_6 = 2 \\ 2x_1 + x_2 - x_5 + x_7 = 1 \\ x_i \geq 0, i = 1, \ldots, 7 \end{pmatrix}$$

The starting basic feasible solution for Phase I is as follows:

Basic: $x_6 = 2, x_7 = 1$

Nonbasic: $x_1 = x_2 = x_3 = x_4 = x_5 = 0$

$\phi = 3$

Before proceeding with the simplex method, the artificial objective function must be expressed in terms of nonbasic variables. It can easily be done by solving for the artificial variables from the constraint equations and substituting into the artificial objective function.

From the constraints, we have

$$x_6 = 2 + x_1 - x_2 - x_3 + x_4 \qquad x_7 = 1 - 2x_1 - x_2 + x_5$$

Simplex Method Phase I

Thus, the artificial objective function is written as

$$\phi = x_6 + x_7 = 3 - x_1 - 2x_2 - x_3 + x_4 + x_5$$

or

$$\phi - 3 = -x_1 - 2x_2 - x_3 + x_4 + x_5$$

Obviously, the actual objective function is not needed during Phase I. However, all reduction operations are performed on it as well so that at the end of Phase I, f is in the correct form (that is, it is expressed in terms on nonbasic variables only) for the simplex method. The complete solution is as follows:

Phase I: Initial Tableau

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{RHS} \\ x_6 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 2 \\ x_7 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \text{Obj.} & 2 & 4 & 3 & 0 & 0 & 0 & 0 & f \\ \text{ArtObj.} & -1 & -2 & -1 & 1 & 1 & 0 & 0 & -3+\phi \end{pmatrix}$$

New basic variable $= x_2$ (-2 is the largest negative number in the ArtObj. row)

Ratios: $\{\frac{2}{1} = 2, \frac{1}{1} = 1\}$ Minimum $= 1 \Longrightarrow x_7$ out of the basic set.

Simplex Method Phase I

Phase I: Second Tableau

$$\left(\begin{array}{lccccccccc} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{RHS} \\ x_6 & -3 & 0 & 1 & -1 & 1 & 1 & -1 & 1 \\ x_2 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \text{Obj.} & -6 & 0 & 3 & 0 & 4 & 0 & -4 & -4+f \\ \text{ArtObj.} & 3 & 0 & -1 & 1 & -1 & 0 & 2 & -1+\phi \end{array}\right) \begin{array}{l} \Longleftarrow \; -\text{PR} + \text{Row1} \\ \Longleftarrow \; \text{PR} \\ \Longleftarrow \; -4 \times \text{PR} + \text{Obj.Row} \\ \Longleftarrow \; 2 \times \text{PR} + \text{ArtObj.Row} \end{array}$$

New basic variable $= x_3$ (-1 is the first largest negative number in the ArtObj. row)

Ratios: $\{\frac{1}{1}\}$ Minimum (only choice) $\Longrightarrow x_6$ out of the basic set.

Phase I: Third Tableau

$$\left(\begin{array}{lccccccccc} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{RHS} \\ x_3 & -3 & 0 & 1 & -1 & 1 & 1 & -1 & 1 \\ x_2 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \text{Obj.} & 3 & 0 & 0 & 3 & 1 & -3 & -1 & -7+f \\ \text{ArtObj.} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \phi \end{array}\right) \begin{array}{l} \Longleftarrow \; \text{PR} \\ \Longleftarrow \; \text{Row2} \\ \Longleftarrow \; -3 \times \text{PR} + \text{Obj.Row} \\ \Longleftarrow \; \text{PR} + \text{ArtObj.Row} \end{array}$$

Simplex Method Phase I

All coefficient in the artificial objective function row are now positive, signalling that the optimum of Phase I has been reached. The solution is as follows:

Basic: $x_2 = 1 \quad x_3 = 1$

Nonbasic: $x_1 = x_4 = \cdots = x_7 = 0 \quad \phi = 0$

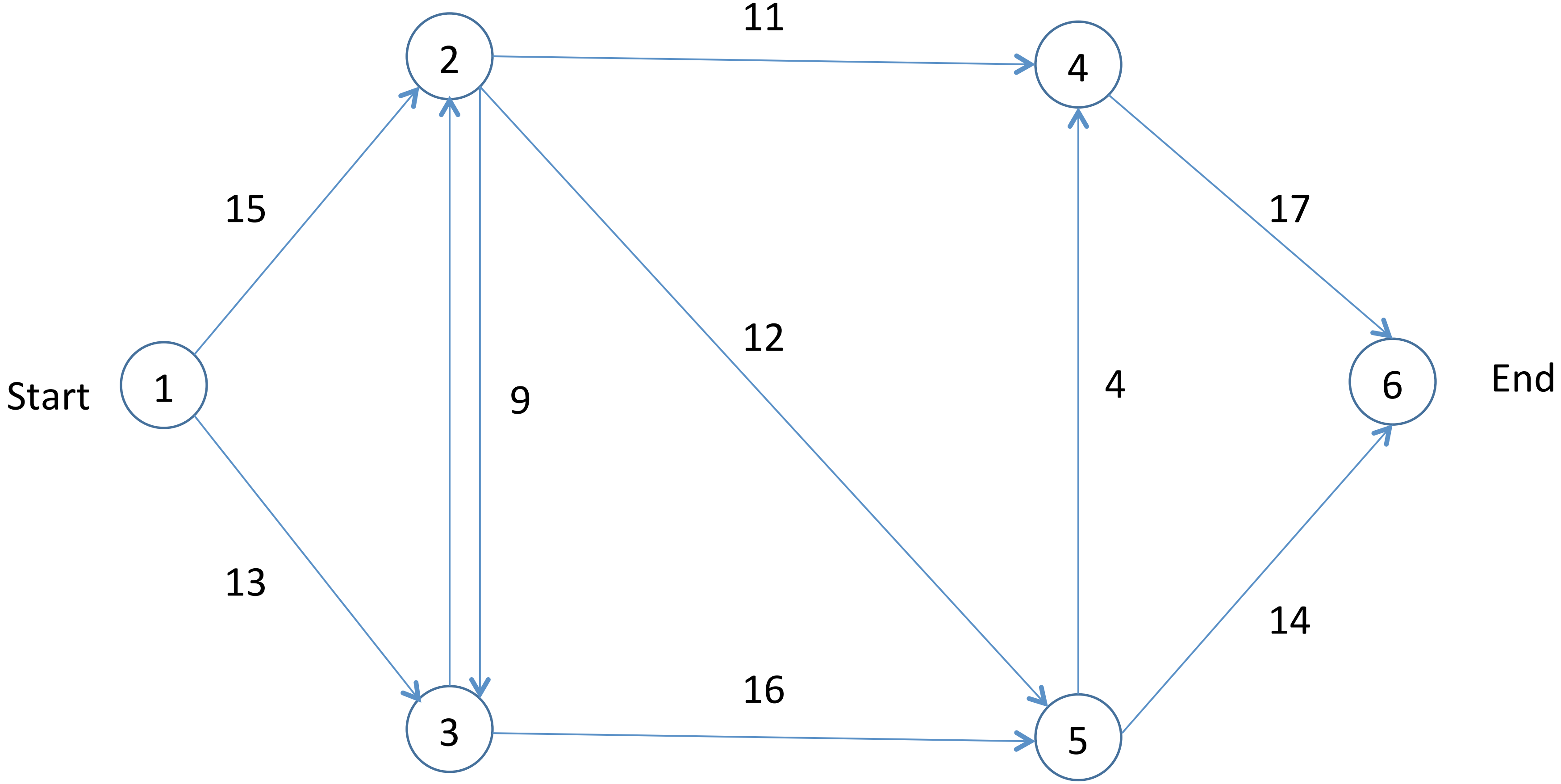
Since the artificial variables are now zero, the constraint equations now represent the original constraints, and we have a basic feasible solution for our original problem.

Phase II: Initial Tableau

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{RHS} \\ x_3 & -3 & 0 & 1 & -1 & 1 & 1 & -1 & 1 \\ x_2 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \text{Obj.} & 3 & 0 & 0 & 3 & 1 & -3 & -1 & -7+f \end{pmatrix}$$

Since all the coefficents in the objective fuction row are possitive , we have done.

Find Shortest Route



Shortest Route

- Variables= $\{x_{12}, x_{13}, x_{23}, x_{32}, x_{24}, x_{25}, x_{35}, x_{54}, x_{46}, x_{56}\}$

Minimize $f = 15x_{12} + 13x_{13} + 9x_{23} + 9x_{32} + 11x_{24} + 12x_{25} + 16x_{35} + 4x_{54} + 17x_{46} + 14x_{56}$

- The Inflow match the Outflow at every node

Node 2: $x_{12} + x_{32} = x_{24} + x_{25} + x_{23}$

Node 3: $x_{13} + x_{23} = x_{32} + x_{35}$

Node 4: $x_{24} + x_{54} = x_{46}$

Node 5: $x_{35} + x_{25} = x_{54} + x_{56}$

Node 1: $x_{12} + x_{13} = 1$

Node 6: $x_{46} + x_{56} = 1$

Post-Optimality Analysis

- We will introduce this analysis without.
- During our study let's use the example :

Minimize $f = -\frac{3}{4}x_1 + 20x_2 - \frac{1}{2}x_3 + 6x_4$

$$\text{Subject to} \begin{pmatrix} \frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \geq 1 \\ \frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 = 3 \\ x_3 + x_4 \leq 1 \\ x_i \geq 0, i = 1, \ldots, 4 \end{pmatrix}$$

To start the simplex solution the constraints are written as follows:

Constraint 1: $\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 - x_5 + x_7 = 1$ $\quad x_5 \rightarrow$ Surplus $\quad x_7 \rightarrow$ Artificial

Constraint 2: $\frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 + x_8 = 3$ $\quad x_8 \rightarrow$ Artificial

Constraint 3: $x_3 + x_4 + x_6 = 1$ $\quad x_6 \rightarrow$ Slack

After several iterations of the simplex method, the following final simplex tableau is obtained:

Basis	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
x_1	1	$-\frac{736}{33}$	0	0	$\frac{28}{33}$	$\frac{4}{11}$	$-\frac{28}{33}$	$\frac{80}{33}$	$\frac{224}{33}$
x_3	0	$\frac{8}{33}$	1	0	$\frac{4}{33}$	$\frac{10}{11}$	$-\frac{4}{33}$	$\frac{2}{33}$	$\frac{32}{33}$
x_4	0	$-\frac{8}{33}$	0	1	$-\frac{4}{33}$	$\frac{1}{11}$	$\frac{4}{33}$	$-\frac{2}{33}$	$\frac{1}{33}$
Obj.	0	$\frac{160}{33}$	0	0	$\frac{47}{33}$	$\frac{2}{11}$	$-\frac{47}{33}$	$\frac{73}{33}$	$\frac{178}{33} + f$

Post-Optimality Analysis

$$\text{Minimize } f = -\tfrac{3}{4}x_1 + 20x_2 - \tfrac{1}{2}x_3 + 6x_4$$

$$\text{Subject to } \begin{pmatrix} \tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \geq 1 \\ \tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 = 3 \\ x_3 + x_4 \leq 1 \\ x_i \geq 0, i = 1, \ldots, 4 \end{pmatrix}$$

To start the simplex solution the constraints are written as follows:

Constraint 1: $\tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 - x_5 + x_7 = 1$ $\quad x_5 \rightarrow$ Surplus $\quad x_7 \rightarrow$ Artificial

Constraint 2: $\tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 + x_8 = 3$ $\quad x_8 \rightarrow$ Artificial

Constraint 3: $x_3 + x_4 + x_6 = 1$ $\quad x_6 \rightarrow$ Slack

After several iterations of the simplex method, the following final simplex tableau is obtained:

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & \text{RHS} \\ x_1 & 1 & -\frac{736}{33} & 0 & 0 & \frac{28}{33} & \frac{4}{11} & -\frac{28}{33} & \frac{80}{33} & \frac{224}{33} \\ x_3 & 0 & \frac{8}{33} & 1 & 0 & \frac{4}{33} & \frac{10}{11} & -\frac{4}{33} & \frac{2}{33} & \frac{32}{33} \\ x_4 & 0 & -\frac{8}{33} & 0 & 1 & -\frac{4}{33} & \frac{1}{11} & \frac{4}{33} & -\frac{2}{33} & \frac{1}{33} \\ \text{Obj.} & 0 & \frac{160}{33} & 0 & 0 & \frac{47}{33} & \frac{2}{11} & -\frac{47}{33} & \frac{73}{33} & \frac{178}{33} + f \end{pmatrix}$$

The optimum solution is:

Basic: $x_1 = \frac{224}{33}, x_3 = \frac{32}{33}, x_4 = \frac{1}{33}$

Nonbasic: $x_2 = x_5 = x_6 = x_7 = x_8 = 0$ $f^* = -\frac{178}{33}$

Status of Constraints

- The status of active and inactive constraints at optimum is determined from the values of surplus/slack variables. If the associated variable is zero, then that constraint is active. In the example x5=0, x6=0. Therefore constraints 1 and 3 are active.

Lagrange Multipliers

- The Lagrange multipliers (LM) for the constraints can be read directly from the objective function row of the final simplex tableau.
- For LE the LM are the associated slack variable column.
- For GE or EQ they are the associated artificial variable.
- The sign of LM for GE should always be negative.
- The sign of LM for LE should always be positive.
- For EQ the sign of LM has no relevance.

Lagrange Multipliers

- For the example, the first constraint is GE, u1=-47/33
- The second constraint is EQ and u2=77/33
- The third constraint is LE and u3=2/11

Allowable Changes inthe RHS of constraints

- If the RHS of a constraint changes, the optimum solution may change. However, it is possible to determine the allowable changes of the RHS of constraints for which the basic variable set remains the same. This means the status of active and inactive constraints will not change and we can use the LM to obtain a new value of the objective function based on the sensitivity analysis similarly to the KT conditions.

Determination of the allowable range

- The lower limit of the allowable change of the RHS of the constraint is given by the maximum of the negative ratios, and the upper limit is obtained by the minimum of the possitive ratios.

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & \text{RHS} \\ x_1 & 1 & -\frac{736}{33} & 0 & 0 & \frac{28}{33} & \frac{4}{11} & -\frac{28}{33} & \frac{80}{33} & \frac{224}{33} \\ x_3 & 0 & \frac{8}{33} & 1 & 0 & \frac{4}{33} & \frac{10}{11} & -\frac{4}{33} & \frac{2}{33} & \frac{32}{33} \\ x_4 & 0 & -\frac{8}{33} & 0 & 1 & -\frac{4}{33} & \frac{1}{11} & \frac{4}{33} & -\frac{2}{33} & \frac{1}{33} \\ \text{Obj.} & 0 & \frac{160}{33} & 0 & 0 & \frac{47}{33} & \frac{2}{11} & -\frac{47}{33} & \frac{73}{33} & \frac{178}{33}+f \end{pmatrix}$$

- For the first contraint

the appropriate column is x7

And $\left\{-\frac{224/33}{-28/33}=8, -\frac{32/33}{-4/33}=8, -\frac{1/33}{4/33}=-\frac{1}{4}\right\}$ $\qquad \text{Max}\left[-\frac{1}{4}\right] \le \Delta b_1 \le \text{Min}[8,8] \quad \text{or} \quad -\frac{1}{4} \le \Delta b_1 \le 8$

This result implies that the basic variables at the optimum point will remain the same as long as the RHS of the first constraint is between ¾ and 9.

Sensitivity analysis: determine the New perturbed Optimum

- KT analysis: Find x that minimizes f(x) subject to h(x)=0. Here f and g ar linear functions but it's not needed in the analysis.
- The optimum is at x* and Lagrange multiplier v with objective function f*.
- The Lagrangian $L(x,v)=f(x)+vh(x)$

Suppose the RHS of the constraint is modified as follows: $h(x)-\varepsilon=0$

Sensitivity analysis

- The Lagrangian for the modified problem is as follows:

ε

$$L'(x,v,\ \mathcal{E}\ )=f(x)+v(h(x)-\mathcal{E}\ )$$

- At the optimum, the constraint is satisfied therefore

$$f(x^*(\ \mathcal{E}))=L'\ (x^*(\ \mathcal{E}\ ),v(\ \mathcal{E}\ ),\ \mathcal{E}\ ).$$

Now

$$\frac{\partial f(\mathbf{x}^*)}{\partial \epsilon} = \nabla_x \tilde{L}(\mathbf{x}^*, \upsilon)^T \frac{\partial \mathbf{x}}{\partial \epsilon} + \frac{\partial \tilde{L}}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \epsilon} + \frac{\partial \tilde{L}}{\partial \epsilon}$$

Sensitivity analysis

- Note that $\frac{\partial \tilde{L}}{\partial \epsilon} = -\upsilon$; from the optimality condition $\nabla_x \tilde{L}(\mathbf{x}^*, \upsilon) = 0$ and $\frac{\partial \tilde{L}}{\partial \upsilon} = 0$,

This taken into account, finally obtains the following relationship

$$\frac{\partial f(\mathbf{x}^*)}{\partial \epsilon} = -\upsilon$$

Sensitivity analysis

- Thus the Lagrange multiplier for a constraint represents the rate of change in the RHS of that constraint, If the Lagrange multiplier of a constraint is large, it means that changing that constraint will have a greater influence on the optimum solution.
- Using Taylor series

$$f^*_{new} \approx f(\mathbf{x}^*) + \frac{\partial f(\mathbf{x}^*)}{\partial \epsilon}\epsilon = f(\mathbf{x}^*) - \upsilon\epsilon$$

Sensitivity analyis

- In the linear case KT analysis also applies

$$\frac{\partial f}{\partial b_i} = -u_i$$

Where bi is the component of the constraint i and ui is the lagrange multiplier in the last Simplex Tableau. Therefore we evaluate f* as

$$\text{New } f^* = \text{Original } f^* - u_i\,(\text{New } b_i - \text{Original } b_i)$$

Sensitivity analysis

- Recall that this equation is based on the assumption that all inequality constraints are converted to LE form. In LP problems is better to first convert the modified original constraints and their multipliers to LE form and then use the above equation.

Sensitivity analysis Example

- The sign of the LM depends on how we write the constraint

For a constraint written as: $\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \geq 1$

the Lagrange multiplier $= -\frac{47}{33}$

For a constraint written as: $-\frac{1}{4}x_1 + 8x_2 + x_3 - 9x_4 \leq -1$

the Lagrange multiplier $= \frac{47}{33}$

The modified constraint is written as follows:

$$\frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 \geq 5 \qquad \text{or} \qquad -\frac{1}{4}x_1 + 8x_2 + x_3 - 9x_4 \leq -5$$

With the constraints in the LE form, we can get the new objective function value as follows:

$$\text{New } f^* = \text{Original } f^* - u_i\,(\text{New } b_i - \text{Original } b_i)$$
$$= -\frac{178}{33} - \left(\frac{47}{33}\right)(-5 - (-1)) = \frac{10}{33}$$

Allowable Changes in the Objective Function Coefficients

- If a coefficient in the objective is changed, without modifying the constraints, then the feasible domain will remain the same. The only thing that changes is the slope of the objective function contours. Thus as long as the changes are within certain limits, the optimum point will remain the same.

Changes in the coefficient of the objective function

- Coefficient in the non basic set

$$
\begin{pmatrix}
\text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & \text{RHS} \\
x_1 & 1 & -\frac{736}{33} & 0 & 0 & \frac{28}{33} & \frac{4}{11} & -\frac{28}{33} & \frac{80}{33} & \frac{224}{33} \\
x_3 & 0 & \frac{8}{33} & 1 & 0 & \frac{4}{33} & \frac{10}{11} & -\frac{4}{33} & \frac{2}{33} & \frac{32}{33} \\
x_4 & 0 & -\frac{8}{33} & 0 & 1 & -\frac{4}{33} & \frac{1}{11} & \frac{4}{33} & -\frac{2}{33} & \frac{1}{33} \\
\text{Obj.} & 0 & \frac{160}{33} & 0 & 0 & \frac{47}{33} & \frac{2}{11} & -\frac{47}{33} & \frac{73}{33} & \frac{178}{33}+f
\end{pmatrix}
$$

In our example x2 is a nonbasic variable in the objective function: $f = -\frac{3}{4}x_1 + 20x_2 - \frac{1}{2}x_3 + 6x_4$

$$-\frac{160}{33} \le \Delta c_2 \le \infty \qquad -\frac{160}{33} + 20 \le c_2 \le \infty + 20$$

Changes in the coefficient of the objective function

- In the basic set $f = -\frac{3}{4}x_1 + 20x_2 - \frac{1}{2}x_3 + 6x_4$

$$
\begin{pmatrix}
\text{Basis} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & \text{RHS} \\
x_1 & 1 & -\frac{736}{33} & 0 & 0 & \frac{28}{33} & \frac{4}{11} & -\frac{28}{33} & \frac{80}{33} & \frac{224}{33} \\
x_3 & 0 & \frac{8}{33} & 1 & 0 & \frac{4}{33} & \frac{10}{11} & -\frac{4}{33} & \frac{2}{33} & \frac{32}{33} \\
x_4 & 0 & -\frac{8}{33} & 0 & 1 & -\frac{4}{33} & \frac{1}{11} & \frac{4}{33} & -\frac{2}{33} & \frac{1}{33} \\
\text{Obj.} & 0 & \frac{160}{33} & 0 & 0 & \frac{47}{33} & \frac{2}{11} & -\frac{47}{33} & \frac{73}{33} & \frac{178}{33} + f
\end{pmatrix}
$$

In our example x1 is in the basic set

The allowable change in the coefficient of variable x1 is determined from the ratios of the objective funtion row and the first constraint row.

Changes in the coefficient of the objective function

Basis	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
x_1	1	$-\frac{736}{33}$	0	0	$\frac{28}{33}$	$\frac{4}{11}$	$-\frac{28}{33}$	$\frac{80}{33}$	$\frac{224}{33}$
x_3	0	$\frac{8}{33}$	1	0	$\frac{4}{33}$	$\frac{10}{11}$	$-\frac{4}{33}$	$\frac{2}{33}$	$\frac{32}{33}$
x_4	0	$-\frac{8}{33}$	0	1	$-\frac{4}{33}$	$\frac{1}{11}$	$\frac{4}{33}$	$-\frac{2}{33}$	$\frac{1}{33}$
Obj.	0	$\frac{160}{33}$	0	0	$\frac{47}{33}$	$\frac{2}{11}$	$-\frac{47}{33}$	$\frac{73}{33}$	$\frac{178}{33}+f$

The ratios are as follows:

$$\left\{\frac{160/33}{-736/33} = -\frac{5}{23}, \frac{47/33}{28/33} = \frac{47}{28}, \frac{2/11}{4/11} = \frac{1}{2}\right\}$$

Denoting the change by Δc_1, the allowable change in the coefficient of x_1 is given by

$$-\frac{5}{23} \leq \Delta c_1 \leq \text{Min}\left[\frac{47}{28}, \frac{1}{2}\right] \qquad \text{or} \qquad -\frac{5}{23} \leq \Delta c_1 \leq \frac{1}{2}$$

Adding the current value of the coefficient to both the upper and the lower limit, we can express the allowable range of the coefficient x_1 as follows:

$$-\frac{5}{23} - \frac{3}{4} \leq c_1 \leq \frac{1}{2} - \frac{3}{4} \qquad \text{or} \qquad -\frac{89}{92} \leq c_1 \leq -\frac{1}{4}$$

Matrix Form of the Simplex Method

Consider an LP problem expressed in the standard form as

Find $\mathbf{x}$ in order to Minimize $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$

where

$\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ vector of optimization variables

$\mathbf{c} = [c_1, c_2, \ldots, c_n]^T$ vector of cost coefficients

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ $m \times n$ matrix of constraint coefficients

$\mathbf{b} = [b_1, b_2, \ldots, b_m]^T \geq 0$ vector of right-hand sides of constraints

Identifying columns of the constraint coefficient matrix that multiply a given optimization variable, we write the constraint equations as follows:

$$\mathbf{A}_1 x_1 + \mathbf{A}_2 x_2 + \ldots + \mathbf{A}_n x_n = \mathbf{b}$$

Matrix Form of the Simplex Method

where $\mathbf{A}_i$ is the i^{th} column of the $\mathbf{A}$ matrix. For a basic feasible solution, denoting the vector of m basic variables by $\mathbf{x}_B$ and the $(n-m)$ vector of nonbasic variables by $\mathbf{x}_N$, the problem can be written in the partitioned form as follows:

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} \quad (\mathbf{B} \quad \mathbf{N}) \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \mathbf{b} \quad f = \begin{pmatrix} \mathbf{c}_B^T & \mathbf{c}_N^T \end{pmatrix} \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix}$$

where $m \times m$ matrix $\mathbf{B}$ consists of those columns of matrix $\mathbf{A}$ that correspond to the basic variables and $m \times (n-m)$ matrix $\mathbf{N}$ consists of those that correspond to nonbasic variables. Similarly, vector $\mathbf{c}_B$ contains cost coefficients

corresponding to basic variables, and $\mathbf{c}_N$ those that correspond to nonbasic variables.

The general solution of the constraint equations can now be written as follows:

Matrix Form of the Simplex Method

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$

giving $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$

Substituting this into the objective function, we get

$$f = \mathbf{c}_B^T\left(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N\right) + \mathbf{c}_N^T\mathbf{x}_N = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \left(\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}\right)\mathbf{x}_N$$

or

$$f = \mathbf{w}^T\mathbf{b} + \mathbf{r}^T\mathbf{x}_N$$

where $\mathbf{w}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$ and $\mathbf{r}^T = \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} = \mathbf{c}_N^T - \mathbf{w}^T\mathbf{N}$.

The vector $\mathbf{w}$ is referred to as the *simplex multipliers* vector. As will be pointed out later, these multipliers are related to the Lagrange multipliers.

From these general expressions, the basic feasible solution for the problem is obtained by setting $\mathbf{x}_N = \mathbf{0}$ and thus

Basic feasible solution: $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{0} \end{pmatrix}$ $\quad \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \quad f = \mathbf{w}^T\mathbf{b}$

Matrix Form of the Simplex Method

Also from the objective function expression, it is clear that if a new variable is brought into the basic set, the change in the objective function will be proportional to $\mathbf{r}$. Thus, this term represents the reduced objective function row of the simplex tableau.

Using this new notation, the problem can be expressed in the following form:

$$\begin{pmatrix} \text{Basic} & \text{Nonbasic} & RHS \\ \hline \mathbf{B} & \mathbf{N} & \mathbf{b} \\ \hline \mathbf{c}_B^T & \mathbf{c}_N^T & f \end{pmatrix}$$

and a basic solution can be written as follows:

$$\begin{pmatrix} \text{Basic} & \text{Nonbasic} & RHS \\ \hline \mathbf{I} & \mathbf{B}^{-1}\,\mathbf{N} & \mathbf{B}^{-1}\,\mathbf{b} \\ \hline \mathbf{0} & \mathbf{r}^T & f - \mathbf{w}^T\mathbf{b} \end{pmatrix}$$

Example

Minimize $f = -3/4x_1 + 20x_2 - 1/2x_3 + 6x_4$

$$\text{Subject to} \begin{pmatrix} 1/4x_1 - 8x_2 - x_3 + 9x_4 \leq 1 \\ 1/2x_1 - 12x_2 - 1/2x_3 + 3x_4 \leq 3 \\ x_3 + x_4 \leq 1 \\ x_i \geq 0, i = 1, \ldots, 4 \end{pmatrix}$$

Introducing slack variables, we have the standard LP form as follows:

Minimize $f = -3/4x_1 + 20x_2 - 1/2x_3 + 6x_4$

$$\text{Subject to} \begin{pmatrix} 1/4x_1 - 8x_2 - x_3 + 9x_4 + x_5 = 1 \\ 1/2x_1 - 12x_2 - 1/2x_3 + 3x_4 + x_6 = 3 \\ x_3 + x_4 + x_7 = 1 \\ x_i \geq 0, i = 1, \ldots, 7 \end{pmatrix}$$

The problem in the matrix form is as follows:

$$\mathbf{c} = \left\{-\frac{3}{4}, 20, -\frac{1}{2}, 6, 0, 0, 0\right\}^T$$

$$\mathbf{A} = \begin{pmatrix} \frac{1}{4} & -8 & -1 & 9 & 1 & 0 & 0 \\ \frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Example

the simplex tableau corresponding to $\{x_1, x_2, x_7\}$ as basic, and $\{x_3, x_4, x_5, x_6\}$ as nonbasic variables.

The partitioned matrices corresponding to selected basic and nonbasic variables are as follows:

$$\mathbf{c}_B = \left\{-\frac{3}{4}, 20, 0\right\}^T \quad \mathbf{c}_N = \left\{-\frac{1}{2}, 6, 0, 0\right\}^T$$

$$\mathbf{B} = \begin{pmatrix} \frac{1}{4} & -8 & 0 \\ \frac{1}{2} & -12 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} -1 & 9 & 1 & 0 \\ -\frac{1}{2} & 3 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Example

$$\mathbf{B}^{-1} = \begin{pmatrix} -12 & 8 & 0 \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{B}^{-1}\mathbf{N} = \begin{pmatrix} 8 & -84 & -12 & 8 \\ \frac{3}{8} & -\frac{15}{4} & -\frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 0 & 0 \end{pmatrix}. \quad \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 12 \\ \frac{1}{4} \\ 1 \end{pmatrix}$$

$$\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1} = \{-1, -1, 0\} \quad \mathbf{w}^T\mathbf{b} = -4$$

$$\mathbf{r}^T = \mathbf{c}_N^T - \mathbf{w}^T\mathbf{N} = \{-2, 18, 1, 1\}$$

The simplex tableau can now be written by simply placing these values in their appropriate places.

$$\left(\begin{array}{c|c|c} \text{Basic} & \text{Nonbasic} & RHS \\ \hline \mathbf{I} & \mathbf{B}^{-1}\,\mathbf{N} & \mathbf{B}^{-1}\,\mathbf{b} \\ \hline \mathbf{0} & \mathbf{r}^T & f - \mathbf{w}^T\mathbf{b} \end{array}\right)$$

$$\begin{pmatrix} \text{Basis} & x_1 & x_2 & x_7 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\ x_1 & 1 & 0 & 0 & 8 & -84 & -12 & 8 & 12 \\ x_2 & 0 & 1 & 0 & \frac{3}{8} & -\frac{15}{4} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ x_7 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ \text{Obj.} & 0 & 0 & 0 & -2 & 18 & 1 & 1 & 4+f \end{pmatrix}$$

Lagrangian Duality related to KT conditions

- For any optimization problem, the Lagrangian and the KT conditions can be used to define a dual optimization problem. The variables in the dual problem are the Lagrange multipliers of the original problem. These duality concept apply to any convex problem as for instance LP.

Duality

- The KT formulation of LP problems require the Lagrangian

$$M(\mathbf{u}, \mathbf{v}) = \underset{\mathbf{x}}{\text{Min}}[\mathbf{c}^T\mathbf{x} - \mathbf{u}^T\mathbf{x} + \mathbf{v}^T(-\mathbf{A}\mathbf{x} + \mathbf{b})] \quad u_i \geq 0, i = 1, \ldots, n$$

- The minimum over x can be obtained differenciating, that is

$$\underset{\mathbf{x}}{\text{Min}}[\mathbf{c}^T\mathbf{x} - \mathbf{u}^T\mathbf{x} + \mathbf{v}^T(-\mathbf{A}\mathbf{x} + \mathbf{b})] \Longrightarrow \mathbf{c} - \mathbf{u} - \mathbf{A}^T\mathbf{v} = \mathbf{0}$$

- N
  one has to get the maximum in the u’s and v’s (negative sign)

Duality

- Thus the dual LP problem is formulated as

$$\text{Maximize } \mathbf{c}^T\mathbf{x} - \mathbf{u}^T\mathbf{x} + \mathbf{v}^T(-\mathbf{A}\mathbf{x} + \mathbf{b})$$

$$\text{Subject to } \begin{pmatrix} \mathbf{c} - \mathbf{u} - \mathbf{A}^T\mathbf{v} = \mathbf{0} \\ u_i \geq 0, i = 1, \ldots, n \end{pmatrix}$$

Now, use the constraint equation to get

$$\mathbf{c}^T\mathbf{x} - \mathbf{u}^T\mathbf{x} + \mathbf{v}^T(-\mathbf{A}\mathbf{x} + \mathbf{b}) \quad \Longrightarrow \quad \mathbf{x}^T(\mathbf{c} - \mathbf{u} - \mathbf{A}^T\mathbf{v}) + \mathbf{v}^T\mathbf{b}$$

$$\text{Maximize } \mathbf{v}^T\mathbf{b}$$

$$\text{Subject to } u_i \geq 0, i = 1, \ldots, n$$

- but $\mathbf{c} - \mathbf{u} - \mathbf{A}^T\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{u} = \mathbf{c} - \mathbf{A}^T\mathbf{v}$, and we will have

$$\boxed{\begin{array}{l} \text{Maximize } \mathbf{v}^T\mathbf{b} \\ \text{Subject to } \mathbf{c} - \mathbf{A}^T\mathbf{v} \geq \mathbf{0} \end{array}}$$

Primal-Dual Interior Point Method

- The optimum of a LP problem can be obtained using (KT analysis)

$\mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$	Primal feasibility
$\mathbf{A}^T\mathbf{v} + \mathbf{u} = \mathbf{c}$	Dual feasibility
$\mathbf{X}\mathbf{U}\mathbf{e} = \mathbf{0}$	Complementary slackness conditions
$x_i \geq 0$ and $u_i \geq 0, i = 1, \ldots, n$	

- Let's t … irectly
- Using $\mu > 0$ as an indication of the error, at some iteration, the complementary slackness is written

$$\mathbf{X}\mathbf{U}\mathbf{e} = \mu\mathbf{e}$$

Newton-Raphson

- If a system of equations **F**(**x**)=**0** , **let** $\mathbf{x}^{k+1} = \mathbf{x}^k + \Delta\mathbf{x}, k = 0, 1, \ldots$

Then, $\Delta\mathbf{x}$ is determined from solving

$$\mathbf{J}(\mathbf{x}^k)\Delta\mathbf{x}^k = -\mathbf{F}(\mathbf{x}^k)$$

Where

$$\mathbf{J} = \begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{A}^T \\ \mathbf{U} & \mathbf{X} & \mathbf{0} \end{pmatrix} \begin{array}{l} \Longleftarrow \text{From primal feasibility} \\ \Longleftarrow \text{From dual feasibility} \\ \Longleftarrow \text{From complementary slackness} \end{array}$$

Denoting dx,du and dv as the changes we have

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{A}^T \\ \mathbf{U} & \mathbf{X} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{d}_x \\ \mathbf{d}_u \\ \mathbf{d}_v \end{pmatrix} = - \begin{pmatrix} \mathbf{A}\,\mathbf{x}^k - \mathbf{b} \\ \mathbf{A}^T\,\mathbf{v}^k + \mathbf{u}^k - \mathbf{c} \\ \mathbf{XUe} - \mu^k\,\mathbf{e} \end{pmatrix}$$

Interior Point method

(a) $\mathbf{A}\mathbf{d}_x = -\mathbf{A}\mathbf{x}^{\mathbf{k}} + \mathbf{b} \equiv \mathbf{r}_{\mathbf{p}}$

(b) $\mathbf{d}_u + \mathbf{A}^T\mathbf{d}_v = -\mathbf{A}^T\mathbf{v}^k - \mathbf{u}^{\mathbf{k}} + \mathbf{c} \equiv \mathbf{r}_d$

(c) $\mathbf{U}\mathbf{d}_x + \mathbf{X}\mathbf{d}_u = -\mathbf{X}\mathbf{U}\mathbf{e} + \mu^k\mathbf{e}$

Thus, $\mathbf{d}_u = -\mathbf{X}^{-1}\mathbf{U}\mathbf{d}_x - \mathbf{U}\mathbf{e} + \mu^k\mathbf{X}^{-1}\mathbf{e}$

Let's introduce the notation

$$\mathbf{r}_c = -\mathbf{U}\mathbf{e} + \mu^k\mathbf{X}^{-1}\mathbf{e}$$

To write:

(b) $\mathbf{d}_u = -\mathbf{X}^{-1}\mathbf{U}\mathbf{d}_x + \mathbf{r}_c$

Interior Point method

Multiplying both sides of equation (b) by $\mathbf{AXU}^{-1}$, we get

$$\mathbf{AXU}^{-1}\mathbf{d}_u + \mathbf{AXU}^{-1}\mathbf{A}^T\mathbf{d}_v = \mathbf{AXU}^{-1}\mathbf{r}_d$$

Substituting for $\mathbf{d}_u$, we get

$$-\mathbf{AXU}^{-1}\mathbf{X}^{-1}\mathbf{Ud}_x + \mathbf{AXU}^{-1}\mathbf{r}_c + \mathbf{AXU}^{-1}\mathbf{A}^T\mathbf{d}_v = \mathbf{AXU}^{-1}\mathbf{r}_d$$

From the form of $\mathbf{X}$ and $\mathbf{U}$ matrices, it is easy to see that

$$\mathbf{XU}^{-1}\mathbf{X}^{-1}\mathbf{U} = \mathbf{I}$$

Therefore, using equation (a), we have

$$-\mathbf{r}_p + \mathbf{AXU}^{-1}\mathbf{r}_c + \mathbf{AXU}^{-1}\mathbf{A}^T\mathbf{d}_v = \mathbf{AXU}^{-1}\mathbf{r}_d$$

or

$$\mathbf{AXU}^{-1}\mathbf{A}^T\mathbf{d}_v = \mathbf{r}_p - \mathbf{AXU}^{-1}\mathbf{r}_c + \mathbf{AXU}^{-1}\mathbf{r}_d$$

Introducing the notation $\mathbf{D} \equiv \mathbf{XU}^{-1}$,we have

$$\mathbf{ADA}^T\mathbf{d}_v = \mathbf{r}_p + \mathbf{AD}(-\mathbf{r}_c + \mathbf{r}_d)$$

Interior Point method

This system of equations can be solved for $\mathbf{d}_v$. Using this solution, the other two increments can be calculated as follows:

From (b) $\quad \mathbf{d}_u = -\mathbf{A}^T\mathbf{d}_v + \mathbf{r}_d$

From (c) $\quad \mathbf{d}_x = \mathbf{U}^{-1}\mathbf{X}(-\mathbf{d}_u + \mathbf{r}_c) \quad \text{or} \quad \mathbf{d}_x = \mathbf{D}(-\mathbf{d}_u + \mathbf{r}_c)$

Let's now introduce changes in the variables using length parameters

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha_p\mathbf{d}_x$$

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha_d\mathbf{d}_u$$

$$\mathbf{v}^{k+1} = \mathbf{v}^k + \alpha_d\mathbf{d}_v$$

We wanted to determine the optimum value of these parameters

Interior Point method

The maximum value of the step length is the one that will make one of the x_i or u_i values go to zero.

$$x_i + \alpha_p d_{xi} \geq 0 \quad \text{and} \quad u_i + \alpha_d d_{ui} \geq 0 \quad i = 1, \ldots, n$$

Variables with positive increments will obviously remain positive regardless of the step length. Also, in order to strictly maintain feasibility, the actual step length should be slightly smaller than the above maximum. The actual step length is chosen as follows:

$$\alpha = \beta \alpha_{\max} \quad \text{where } \beta = 0.999\ldots$$

Thus, the maximum step lengths are determined as follows:

$$\alpha_p = \text{Min}\left[1, -\beta x_i/d_{xi}, d_{xi} < 0\right] \quad \text{and} \quad \alpha_d = \text{Min}\left[1, -\beta u_i/d_{ui}, d_{ui} < 0\right]$$

Interior Point method

Complementary Slackness

The value of parameter μ determines how well complementary slackness conditions are satisfied. Numerical experiments suggest defining an average value of μ as follows:

$$\mu^k = \frac{(\mathbf{x}^k)^T \mathbf{u}}{n}$$

where n = number of optimization variables. This parameter should be zero at the optimum. Thus, for convergence

$$\mu \leq \epsilon_3$$

where ϵ_3 is a small positive number.

Complete Primal-Dual algorithm

Algorithm

Given: Constraint coefficient matrix $\mathbf{A}$, constraint right-hand side vector $\mathbf{b}$, objective function coefficient vector $\mathbf{c}$, step-length parameter β, and convergence tolerance parameters.

Initialization: $k = 0$, arbitrary initial values (≥ 0), say $\mathbf{x}^k = \mathbf{u}^k = \mathbf{e}$ (vector with all entries 1) and $\mathbf{v}^k = \mathbf{0}$.

The next point $\mathbf{x}^{k+1}$ is computed as follows:

1. Set $\mu^k = \left[\frac{(\mathbf{x}^k)^T\mathbf{u}}{n}\right]/(k+1)$. If $\left[\frac{\|\mathbf{A}\mathbf{x}^k-\mathbf{b}\|}{\|\mathbf{b}\|+1} \leq \epsilon_1, \frac{\|\mathbf{r}_d\|}{\|\mathbf{c}\|+1} \leq \epsilon_2, \mu^k \leq \epsilon_3\right]$, we have the optimum. Otherwise, continue.

Complete Primal-Dual algorithm

2. Form:

$$\mathbf{D} = \mathbf{X}\mathbf{U}^{-1} = \operatorname{diag}[x_i/u_i]$$

$$\mathbf{r_p} = -\mathbf{A}\mathbf{x^k} + \mathbf{b}$$

$$\mathbf{r}_d = -\mathbf{A}^T\mathbf{v}^k - \mathbf{u^k} + \mathbf{c}$$

$$\mathbf{r}_c = -\mathbf{u}^k + \mu^k\mathbf{X}^{-1}\mathbf{e}$$

3. Solve the system of linear equations for $\mathbf{d}_v$:

$$\mathbf{A}\mathbf{D}\mathbf{A}^T\mathbf{d}_v = \mathbf{r}_p + \mathbf{A}\mathbf{D}\,(-\mathbf{r}_c + \mathbf{r}_d)$$

4. Compute increments:

$$\mathbf{d}_u = -\mathbf{A}^T\mathbf{d}_v + \mathbf{r}_d$$

$$\mathbf{d}_x = \mathbf{D}\,(-\mathbf{d}_u + \mathbf{r}_c)$$

5. Check for unboundedness:

   Primal is unbounded if $\mathbf{r}_p = 0$, $\mathbf{d}_x > 0$, and $\mathbf{c}^T\mathbf{d}_x < 0$

   Dual is unbounded if $\mathbf{r}_d = 0$, $\mathbf{d}_u > 0$, and $\mathbf{b}^T\mathbf{d}_v > 0$

   If either of these conditions is true, *stop*. Otherwise, continue.

Complete Primal-Dual algorithm

6. Compute step lengths:

$$\alpha_p = \text{Min}\,[1, -\beta x_i/d_{xi}, d_{xi} < 0] \quad \text{and} \quad \alpha_d = \text{Min}\,[1, -\beta u_i/d_{ui}, d_{ui} < 0]$$

7. Compute the next point:

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha_p d_x$$

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha_d d_u$$

$$\mathbf{v}^{k+1} = \mathbf{v}^k + \alpha_d d_v$$

Primal Feasibility

The constraints must be satisfied at the optimum, i.e., $\mathbf{A}\mathbf{x}^k - \mathbf{b} = \mathbf{0}$. To use as convergence criteria, this requirement is expressed as follows:

$$\sigma_p = \frac{||\mathbf{A}\mathbf{x}^k - \mathbf{b}||}{||\mathbf{b}|| + 1} \leq \epsilon_1$$

where ϵ_1 is a small positive number. The 1 is added to the denominator to avoid division by small numbers.

Feasibility

Primal Feasibility

The constraints must be satisfied at the optimum, i.e., $\mathbf{A}\mathbf{x}^k - \mathbf{b} = \mathbf{0}$. To use as convergence criteria, this requirement is expressed as follows:

$$\sigma_p = \frac{||\mathbf{A}\mathbf{x}^k - \mathbf{b}||}{||\mathbf{b}|| + 1} \leq \epsilon_1$$

where ϵ_1 is a small positive number. The 1 is added to the denominator to avoid division by small numbers.

Dual Feasibility

We also have the requirement that

$$\mathbf{A}^T\mathbf{v}^k + \mathbf{u}^k - \mathbf{c} = \mathbf{0}$$

This gives the following convergence criteria:

$$\sigma_d = \frac{||\mathbf{r}_d||}{||\mathbf{c}|| + 1} \leq \epsilon_2$$

where ϵ_2 is a small positive number.

Application to industrial processes

- Problems:

1. Decision problems: A firm has recently adquired N facilities with cost C(i) each and the profit per year is known Profit(i). On the other hand, they need renovation at cost R(i) after some years of operation. Given the prevailing interest rate per year, which company is profitable the most?
2. Lean Manufacturing: In a plant the cost C(i) of manufacturing some good (i) decreases as the number n(i) of units produced increases. Moreover, the maintenance cost of the machines involved in producing those goods increases as a function of the total number of units delivered to the market. The whole selling price of the products drops as more goods of the type i is delivered according to p(i). Determine how many units of each product the firm should produce to maximize its profit.

Application to industrial processes

- Problems:

1. Network Logistics: A company has two Manufacturing plants of some good in the north and south of the country. In the next T period of months the known demand of this good in K different saling points is estimated as S(k). We need to satisfy this demand at the minimum transport and storage cost considering the logistics infrastructure of the firm.

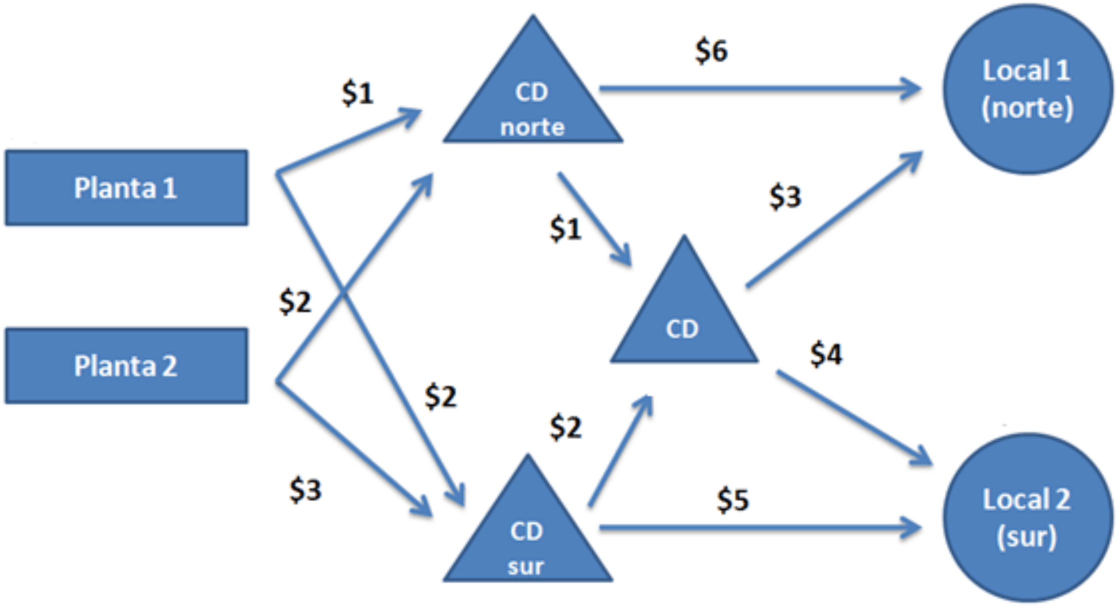
Application to industrial processes

- Problems:

1. Network problems: minimize the cost of distributing the power generated in N Energy plants to K cities when the cost C_{ij} of transport the energy produced in the plant i to the city j is known.

2. Lean Operation: schedule the available resources required to satisfy the demand in a production chain of a plant while maximizing profit of operation.

Logistic problem



Sensitivity analysis: Application to industrial processes

- Problems:

1. Network problems: Obtain the variation of the cost of distributing the power generated in N Energy plants to K cities when changes the cost Cij are taken in consideration.
2. Lean Operation: Obtain how to modify the schedule of the available resources required to satisfy the demand variations and determine how this circumstances change the profit of operation.