Thesis Title
Universität Leipzig

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Abstract

This is my abstract. This is my abstract. This is my abstract. This is my abstract. This is my abstract. This is my abstract. This is my abstract. This is my abstract.

Zusammenfassung

Das ist meine Zusammenfassung. Das ist meine Zusammenfassung. Das ist meine Zusammenfassung. Das ist meine Zusammenfassung. Das ist meine Zusammenfassung. Das ist meine Zusammenfassung. Das ist meine Zusammenfassung.

Danksagung

I thank everyone twice.

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1 Introduction

A brief introduction. Vivamus vehicula leo a justo. Quisque nec augue. Morbi mauris wisi, aliquet vitae, dignissim eget, sollicitudin molestie, ligula. In dictum enim sit amet risus. Curabitur vitae velit eu diam rhoncus hendrerit. Vivamus ut elit. Praesent mattis ipsum quis turpis. Curabitur rhoncus neque eu dui. Etiam vitae magna. Nam ullamcorper. Praesent interdum bibendum magna. Quisque auctor aliquam dolor. Morbi eu lorem et est porttitor fermentum. Nunc egestas arcu at tortor varius viverra. Fusce eu nulla ut nulla interdum consectetuer. Vestibulum gravida. Morbi mattis libero sed est.

1.1 The Erdös-Renyi random graph

1.2 General results on component sizes

In this section we discuss previous results on component sizes in the Erdös-Renyi random graph. We begin with the subcritical graph, that is $\mathscr{G}(n, p_n)$ with $p_n < \frac{1}{n}$.

Define $\lambda = p_n n$ and I_λ as the large deviation rate functions for Poisson random variables with mean λ, that is

$$I_\lambda = \lambda - 1 - \log(\lambda). \tag{1.1}$$

The following theorem provides an upper bound on the size of $\mathcal{C}_{\max}$, the largest component of $\mathscr{G}$. See [1, Theorem 4.4, p.125].

Theorem 1.1 (Lower bound on largest subcritical component)**.** *Fix $\lambda < 1$. Then, for all $a > 1/I_\lambda$, there exists $\delta = \delta(a, \lambda)$ such that*

$$\mathbb{P}(|\mathcal{C}_{\max}| \geq a \log n) = O\left(n^{-\delta}\right). \tag{1.2}$$

The next theorem gives an upper bound for the size of $\mathcal{C}_{\max}$. See [1, Theorem 4.5, p.125].

Theorem 1.2 (Upper bound on largest subcritical component)**.** *Fix $\lambda < 1$. Then, for all $a < 1/I_\lambda$, there exists $\delta = \delta(a, \lambda)$ such that*

$$\mathbb{P}(|\mathcal{C}_{\max}| \leq a \log n) = O\left(n^{-\delta}\right). \tag{1.3}$$

Together, these theorems imply

$$\frac{|\mathcal{C}_{\max}|}{\log n} \to_p 1/I_\lambda. \tag{1.4}$$

We have established that, for $p_n n < 1$, the expected largest component size will be of order $\log n$.

Consider the opposite case, $\lambda = p_n n > 1$. Denote by $\xi_\lambda = 1 - \eta_n \lambda$ the survival probability of a Poisson branching process with mean offspring λ. Note that $\xi_\lambda > 0$ since $\lambda > 1$. Any vertex is part of a large component with probability ξ_λ, therefore we will expect around $n\xi_\lambda$ vertices being part of large connected components. The following theorem now states that all of these vertices are, in fact, part of the same connected component, which we call the giant component. See [1, Theorem 4.8, p.131].

Theorem 1.3 (Law of large numbers for giant component)**.** *Fix $\lambda > 1$. Then, for all $\nu \in (\frac{1}{2}, 1)$, there exists $\delta = \delta(\nu, \lambda)$ such that*

$$\mathbb{P}\left(||\mathcal{C}_{\max}| - n\xi_\lambda| \geq n^\nu\right) = O\left(n^{-\delta}\right). \tag{1.5}$$

To recap: For $np_n < 1$ we expect many small clusters of size at most $\log n$, for $np_n > 1$ we expect one giant component, rapidly approaching size n. But what happens around $np_n \approx 1$? As it turns out, the emergence of the giant component occurs quite rapidly, such that shortly after $np_n = 1$ most graphs do not have any component of order between $\frac{1}{2}n^{2/3}$ and $n^{2/3}$. We switch to the language of $\mathscr{G}(n, m)$ for the following theorem.

Theorem 1.4. *test*

For an estimate on when exactly this giant component emerges, we quote [2, Theorem 6.8, p.142] for

Theorem 1.5. *Almost every graph process $\mathscr{G} = (\mathscr{G}_t)_0^n$ is such that for every $t \geq t_1 = \lfloor n/2 + 2(\log n)^{1/2} n^{2/3} \rfloor$ the graph $\mathscr{G}_t$ has a unique component of order at least $n^{2/3}$ and the other components have at most $\frac{1}{2}n^{2/3}$*

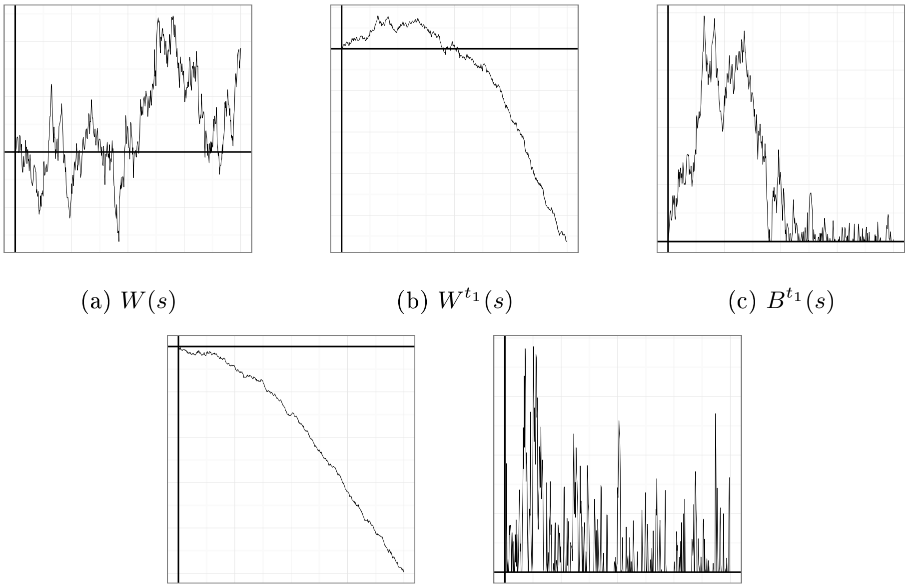
As it turns out, there is a so-called critical window in which the component sizes are not of size $\log n$ and there is no giant component yet. We call a random graph $\mathscr{G}(n, n^{-1} + tn^{-4/3})$ critical for $t \in \mathbb{R}$.

The following theorem proves that the rescaled sequence of sizes of the biggest component in $\mathscr{G}$ is tight. That is, with high probability it is contained in some compact subset of $\mathbb{R}$. See [1, Theorem 5.1, p.150]

Theorem 1.6 (Largest critical cluster)**.** *Let $\lambda = 1 + tn^{-1/3}$, with $t \in \mathbb{R}$. There exists a constant $b = b(t)$ such that for all $\omega > 1$,*

$$\mathbb{P}\left(\omega^{-1} n^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega n^{2/3}\right) \geq 1 - \frac{b}{\omega}. \tag{1.6}$$

If we rescaled the size of the biggest component by $n^{-2/3}$, it is with high probability contained in the compact interval $[\omega^{-1}, \omega]$.



(a) $W(s)$ (b) $W^{t_1}(s)$ (c) $B^{t_1}(s)$

(d) $W^{t_2}(s)$ (e) $B^{t_2}(s)$

Figure 1.1: A sample Brownian motion, Brownian motion with drift and with reflection for $t_1 > 0$ and $t_2 < 0$.

1.3 Main statements of this thesis

In this section we state the main results of this thesis, which is a refinement of Theorem 1.6. While previously the size of largest component was only estimated to be of order $n^{2/3}$, the following Theorem will give a limit distribution for all component sizes, downscaled by $n^{-2/3}$.

Denote by W the standard Brownian motion. For a fixed parameter $t \in \mathbb{R}$ we call the process W^t, defined by

$$W^t(s) := W(s) + \int_0^s (t - s) ds = W(s) + ts - \frac{1}{2}s^2, \tag{1.7}$$

the Brownian motion with drift $t - s$ at time s. The central object of our analysis will be the process W^t and its excursions above past minima. We reflect W^t at 0, defining the process B^t by

$$B^t(s) := W^t(s) - \min_{u \le s} W^t(u) \tag{1.8}$$

and calling it the reflected Brownian motion with drift.

See Figure 1.1 for an example of a Brownian motion with drift for positive and negative t and the corresponding reflected process. Note that for positive t the time intervals between zeroes of B^t, especially the first, are much longer than for negative t. We call an excursion γ of B^t a time interval $[l(\gamma), r(\gamma)]$ for which $B^t(l(\gamma)) = B^t(r(\gamma)) = 0$ and $B^t(s) > 0$ for all $l(\gamma) < s < r(\gamma)$. Denote by $|\gamma| = r(\gamma) - l(\gamma)$ the length of an excursion.

Additionally we define a Poisson counting process N^t, which equips each excursion with a number of marks, emerging with intensity $B^t(s)$ at time s. Informally speaking, the chance of encountering a mark in a time interval $[s, s + ds]$ is characterized by

$$\mathbb{P}(\text{Some mark emerges in } [s, s + ds] \mid B^t(u), u \leq s) = B^t(s)ds. \tag{1.9}$$

More formally we define N^t to be the counting process for which

$$N^t(s) - \int_0^s B^t(s)ds \tag{1.10}$$

is a martingale. Denote by $\mu(\gamma)$ the number of marks during an excursion γ.

We now state the main theorem of this thesis, which we will prove gradually in the following chapters.

Theorem 1.7 (Main theorem). *Let $\mathcal{C}_n^t(1) \geq \mathcal{C}_n^t(2) \geq ...$ be the ordered component sizes of $\mathscr{G}(n, n^{-1} + tn^{-4/3})$ and let $\sigma_n^t(j)$ be the surplus of the corresponding component. Then, as $n \to \infty$,*

$$(n^{-2/3}(\mathcal{C}_n^t(j), \sigma_n^t(j)),\ j \geq 1) \to_p ((\mathcal{C}^t(j), \sigma^t),\ j \geq 1) = (\boldsymbol{\mathcal{C}}^t, \boldsymbol{\sigma}^t), \tag{1.11}$$

where the convergence $n^{-2/3}\boldsymbol{\mathcal{C}}_n^t \to_p \boldsymbol{\mathcal{C}}^t$ holds with respect to the $l^2_\searrow$ topology.

The limit $\big((\mathcal{C}^t(j), \sigma^t(j)),\ j \geq 1\big)$ is distributed as the sequence $\big((|\gamma_j|, \mu(\gamma_j)), j \geq 1\big)$ of lengths and mark-counts of excursions of B^t.

We conclude this chapter with an overview of the remaining chapters and the structure of the proof of Theorem 1.7.

Chapter 2 will develop some preliminary theory on the function spaces C and D, convergence of probability measures and counting processes, which will prove useful in the coming chapters.

In Chapter 3 we define a way to traverse all vertices of a given graph, called the breadth-first walk Z_n^t, that reduces the graph to a one-dimensional random walk in which component sizes are decoded as excursions above past minima. We analyse its characteristics when applied to $\mathscr{G}(n, n^{-1} + tn^{-4/3})$ and discover that, after a certain rescaling, it converges in distribution to W^t.

Chapter 4 deals with the second coordinate σ_n^t in Theorem 1.7, the surplus edges. We describe a Poisson counting process N_n^t which tallies up all encountered excess edges and calculate its limit rate as B^t. The remainder of this chapter will be spent proving that this convergence of rates suffices to declare the convergence of the joint distribution of the rescaled breadth-first walk and this counting process to W^t and N^t.

It remains to be proven that not only does the rescaled random walk converge in distribution to the Brownian motion with drift, but that the convergence of component sizes to lengths of excursions of B^t in distribution follows as well. In Chapter 5 we first prove that this does indeed hold if we must not expect any large components to "wander off to infinity" as $n \to \infty$ and subsequently that, with high probability, this problem does not arise in $\mathscr{G}(n, n^{-1} + tn^{-4/3})$. This completes the proof of Theorem 1.7.

Lastly, Chapter 6 provides an overview over the remaining statements of Aldous paper, which contains a non-uniform version of Theorem 1.7 and the multiplicative coalescent, a process describing the joining of components to form larger components as the parameter t grows.

2 Preliminaries

2.1 Weak convergence of probability measures

The central notion of this thesis is the convergence in distribution of random variables in general metric spaces.

Definition 2.1 (Weak convergence, [9, p.7]). Let μ_n, μ be measures on a metric space S with associated Borel σ-algebra$\mathcal{S} = \mathscr{B}(S)$.

Denote by $C_b(S)$ the space of bounded, continuous functions $f : S \to \mathbb{R}$.

We say μ_n converges weakly to μ, $\mu_n \Rightarrow \mu$, if

$$\int_S f d\mu_n \to \int_s f d\mu \tag{2.1}$$

as $n \to \infty$ for all $f \in C_b(S)$.

We say a random variable X in $(S, \mathcal{S})$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, has distribution or law P if

$$P(A) := \mathbb{P}(X \in A) \tag{2.2}$$

for all $A \in \mathcal{S}$.

For a series of random variables in the same metric space to converge in distribution does not require them to be defined on the same probability space. Let X_n, X be random variables in a metric space $(S, \mathcal{S})$,

$$\begin{aligned} X &: (\Omega, \mathscr{F}, \mathbb{P}) \to (S, \mathcal{S}), \\ X_i &: (\Omega_i, \mathscr{F}_i, \mathbb{P}_i) \to (S, \mathcal{S}), \end{aligned} \tag{2.3}$$

with associated distributions P_n, P. Now X_n converges in distribution to X, $X_n \to_d X$, if their distribution measures converge weakly, $P_n \Rightarrow P$.

The main goal of this chapter is now to find necessary and sufficient conditions for the convergence in distribution of a given sequence of random variables or equivalently the weak convergence of their distribution measures.

A first useful reference provides the so-called Portmanteau theorem.

Theorem 2.2 (Portmanteau, [9, Theorem 2.1, p.16]). *Let P_n, P be probability measures on $(S, \mathcal{S})$. The following conditions are equivalent:*

1. $P_n \Rightarrow P$,
2. $\int_S f dP_n \to \int_S f dP$ *for all* $f \in C_b(S)$,
3. $\limsup_n P_n(F) \leq P(F)$ *for all closed* F,

4. $\liminf_n P_n(G) \geq P(G)$ *for all open* G*,*

5. $P_n(A) \to P(A)$ *for all* A *with* $P(\partial A) = 0$*.*

A powerful tool to prove weak convergence of measures is relative compactness.

Definition 2.3 (Relative compactness, [9, p.57])**.** Let Π be a family of probability measures on $(S, \mathcal{S})$. We call Π *relatively compact* if for every sequence in Π there exists a convergent subsequence. That is, for all sequences $\{P_n\} \subset \Pi$ there exists $\{P_{n_i}\} \subset \Pi$ and a probability measure Q on $(S, \mathcal{S})$, not necessarily on Π, such that $P_{n_i} \Rightarrow_i Q$.

We call a sequence of probability measures $\{P_n\}$ relatively compact if for every subsequence $\{P_{n_i}\}$ there exists a further subsequence $\{P_{n_{i_k}}\}$ and a probabilty measure Q such that $P_{n_{i_k}} \Rightarrow_k Q$.

For function spaces, knowing that a sequence of functions is relatively compact provides us with a powerful tool to prove convergence in distribution.

Definition 2.4 (Finite dimensional distributions, [7])**.** Let $X : \Omega \times \mathbb{R}_+ \to \mathbb{X}$ be a stochastic process with distribution P For $t_1, \dots, t_k \in \mathbb{R}_+$ we denote by *finite-dimensional distributions* of X the push forward measures

$$P\pi^{-1}_{t_1,\dots,t_k}(A) := \mathbb{P}\{X(t_1) \in A_1, \dots, X(t_k) \in A_k\} \tag{2.4}$$

for $A = A_1 \times \dots \times A_k \in \mathbb{X}^k$.

Now consider a relatively compact sequence of stochastic processes $\{X_n\}$, with distributions $\{P_n\}$, and a stochastic process X with distribution P, such that

$$P_n\pi^{-1}_{t_1,\dots,t_k} \Rightarrow P\pi^{-1}_{t_1,\dots,t_k} \tag{2.5}$$

for all k and $t_1, \dots, t_k \in \mathbb{R}_+$. Since $\{P_n\}$ is relatively compact, we know that every subsequence contains a further subsequence converging to some probability measure Q. It can be shown that the convergence of finite-dimensional distributions implies that all of these limit measure are in fact P, which in turn proves $P_n \Rightarrow P$. For details see [9, p.57].

If we can prove a series of probability measure to converge in finite-dimensional distributions to some limit measure we therefore only need to show relative compactness of the series for the convergence in distribution to hold.

Relative compactness is closely linked to another property of series of measures, tightness.

We call a measure μ on a metric space tight, if for all $\epsilon > 0$ there exists a compact K such that

$$\mu(K^c) < \epsilon. \tag{2.6}$$

This carries over to families of probability measures as follows:

Definition 2.5 (Tightness of families of probability measures, [9, p.59])**.** A family Π of probability measures on a metric space $(S, \mathcal{S})$ is *tight* if for every $\epsilon > 0$ there exists a compact $K \subset S$ such that $P(K) > 1 - \epsilon$ for all $P \in \Pi$.

The last main result of this section now provides a link between tightness and relative compactness and therefore an effective means of proving convergence in distribution.

Theorem 2.6 (Prohorov's theorem, [9, Theorem 5.1, p.59])**.** *Let P_n be a series of probability measures on a metric space $(S, \mathcal{S})$. If P_n is tight, it is relatively compact.*

We summarize this theorem and our considerations above in the following lemma.

Lemma 2.7. *Let X, X_n be stochastic processes with distributions P, P_n. If $\{X_n\}$ is tight and*

$$P_n \pi^{-1}_{t_1,\dots,t_k} \Rightarrow P \pi^{-1}_{t_1,\dots,t_k},$$

for all k and $t_1, \dots, t_k \in \mathbb{R}_+$, then $X_n \Rightarrow X$.

2.2 The space $D[0, T]$

The coming chapters will mainly deal with two types of stochastic processes: Brownian motion and so-called càdlàg-processes. This section will deal with the issue of proving tightness and thus convergence for the latter.

Definition 2.8 (The space $D[0, T]$, [9, p.121])**.** We call a function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}_+$, *càdlàg*, if

1. for $t \in I$, $f(t+) = \lim_{s \downarrow t} f(s)$ exists and $f(t) = f(t+)$,
2. for $t \in I$, $f(t-) = \lim_{s \uparrow t} f(s)$ exists.

These functions are right-continuous possess left limits everywhere (continue à droite, limite à gauche).

We denote by $D[0, T]$ the space of càdlàg-functions on $I = [0, T]$.

Considering topology and convergence on DT, we quickly observe that the known notion of distance known in $C(I)$, the space of continuous functions on I, where

$$f_n \to f \iff \sup_{t \in I} |f_n(t) - f(t)| \to 0, \tag{2.7}$$

is not sufficient for $D[0, T]$. Consider the functions

$$\begin{aligned} f_n(t) &= \begin{cases} 0 & t < x_n, \\ 1 & t \geq x_n, \end{cases} \\ f(t) &= \begin{cases} 0 & t < x, \\ 1 & t \geq x. \end{cases} \end{aligned} \tag{2.8}$$

For $x_n \to x$ we would expect f_n to converge to f, however $\sup_{t\in I} |f_n(t) - f(t)| = 1$ whenever $x_n \neq x$.

Thinking of the previous convergence as allowing shifts on the y-axis, we need a convergence that additionally allows shifts on the x-axis.

Definition 2.9 (Skorohod metric, [9, p.124])**.** Denote by Λ the class of strictly increasing, continuous mappings from $[0, T]$ onto itself.

For $x, y \in D[0, T]$, define the *Skorohod metric* $d(x, y)$ as the infimum of all $\epsilon > 0$ for which exists a $\lambda \in \Lambda$ such that

1. $\sup_{t\in[0,T]} |\lambda(t) - t| = \sup_{t\in[0,T]} |t - \lambda^{-1}(t)| < \epsilon$,
2. $\sup_{t\in[0,T]} |x(\lambda(t)) - y(t)| = \sup_{t\in[0,T]} |x(t) - y(\lambda^{-1}(t))| < \epsilon$.

And writing $||x||_T = \sup_{t\in[0,T]} |x(t)|$ we can provide a more compact form:

$$d(x, y) = \inf_{\lambda\in\Lambda} \{||\lambda - I||_T \vee ||x - y(\lambda)||_T\}, \tag{2.9}$$

where $I : [0, T] \to [0, T]$ is the identity map.

Definition 2.10 (Modulus of continuity, [9, p.122])**.** A set $\{t_i\}$, where $0 = t_0 < t_1 < \cdots < t_k = T$, is called δ*-sparse* if $\min_{1\leq i\leq k}(t_i - t_{i-1}) > \delta$.

For $0 < \delta < 1$ and $x \in D[0, T]$, define the *modulus of continuity* by

$$w'_x(\delta) := \inf_{\{t_i\}} \max_{1\leq i\leq k} w_x[t_{i-1}, t_i),$$

where the infimum extends over all δ-sparse sets $\{t_i\}$ and

$$w_x(I) := \sup_{s,t\in I} |x(s) - x(t)|.$$

The Arzelà-Ascoli theorem, see [9, Theorem 7.2, p.82], provides a complete characterisation of relatively compact sets in $C[0, T]$. We are now ready to present a $D[0, T]$-equivalent.

Theorem 2.11 (Arzelà-Ascoli in $D[0, T]$, [9, Theorem 12.3, p.130])**.** *A necessary and sufficient condition for a set A to be relatively compact in the Skorohod toplogy is that*

1. $\sup_{x\in A} ||x||_T < \infty$,
2. $\lim_{\delta\to 0} \sup_{x\in A} w'_x(\delta) = 0$.

2.3 Point processes

Definition 2.12 (Point process, [26, p.123]). Let S be a locally compact second countable Hausdorff space and $\mathscr{B}(S)$ its Borel σ-algebra. Let $\{x_i, i \geq 1\}$ be a collection of points of S. Let

$$\mu := \sum_{i \geq 1} \delta_{x_i}, \tag{2.10}$$

with δ_x the Dirac measure of $x \in S$, be locally compact, that is, if $C \in \mathscr{B}(S)$ is compact then $\mu(C) < \infty$. Then μ is a *point measure* on E.

Denote by $M_p(S)$ the space of all point measures on E and let $\mathscr{M}_p(S)$ be the smallest σ-algebra containing all sets of the form

$$\{m \in M_p(S) \mid m(A) \in B\}$$

for some $A \in \mathscr{B}(S)$ and $B \in \mathscr{B}(\mathbb{R}_+)$.

A *point process* N is a measurable map from a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ into $(M_p(S), \mathscr{M}_p(S))$.

For all coming observations, we will take $\mathbb{R}_+$ as th underlying Hausdorff space with its Borel σ-algebra$\mathscr{B} = \mathscr{B}(\mathbb{R}_+)$.

Definition 2.13 (Poisson point process, [26, p.130]). We call a point process N a *Poisson point process* or *Poisson process*, if

1. for all disjoint sets $A_1, A_2, \ldots, A_n \in \mathscr{B}$ the random variables $N(A_1), N(A_2), \ldots, N(A_n)$ are independent and

2. for all $A \in \mathscr{B}$, $N(A)$ has Poisson distribution $\mathrm{Poi}(\gamma)$,

   $$\mathbb{P}(N(A) = k) = \frac{\gamma^k}{k!} \exp(-\gamma)$$

   where $\gamma = \gamma(A) \in [0, \infty]$ is the *mean measure* or *intensity* of N.

The mean measure is often given in terms of a *rate* or *conditional intenstiy* λ by

$$\gamma(A) = \int_A \lambda(t) dt. \tag{2.11}$$

Definition 2.14 (Simple point process, [26, p.124]). A point process N on $\mathbb{R}_+$ is called *simple* if

$$\mathbb{P}(N(x) > 1) = 0 \tag{2.12}$$

for all $x \in \mathbb{R}_+$.

By [5, Remark 2.1, p.34] a Poisson point process is simple if and only if its mean measure γ has no discrete component, that is $\gamma(x) = 0$ for all $x \in \mathbb{R}_+$.

In order to talk reasonably about convergence of point processes, we first need to introduce a topology on the space of point measures. The *vague topology* is similar to the one generated by the weak convergence in Section 2.1. When trying to apply weak convergence to point measures we run into a problem, since a point measure $\mu = \sum_{i\geq 1} \delta_{x_i}$ may contain an infinite number of points on $\mathbb{R}_+$ and therefore

$$\int_{\mathbb{R}_+} f d\mu = \sum_{i\geq 1} f(x_i) = \infty$$

for certain $f \in C_b(\mathbb{R}_+)$, which makes a discussion of convergence in $\mathbb{R}$ of the integrals unreasonable.

To counter this, we define a new type of convergence, in which the integrals only have to converge for functions with compact support.

Definition 2.15 (Vague convergence, [26, p.140]). Let $C_K(E)$ be the space of continuous real valued functions ona Hausdorff space E with compact support, meaning there exists a compact set $K \in \mathscr{B}(E)$ such that $f(x) = 0$ for all $x \notin K$.

Let $\mu, \mu_1, \mu_2, \dots$ be point measures on E. We say μ_n *converge vaguely* to μ, $\mu_n \to_v \mu$, if

$$\int_{\mathbb{R}_+} f d\mu_n \xrightarrow{n\to\infty} \int_{\mathbb{R}_+} f d\mu \tag{2.13}$$

for all $f \in C_K(\mathbb{R}_+)$.

Theorem 2.2 gave necessary and sufficient conditions for weak convergence of measures, the following lemma may be seen as an equivalent Portmanteau Theorem in the sense of vague convergence.

Lemma 2.16 (Equivalent conditions for vague convergence, [26, Proposition 3.12, p.142]). *Let $\mu, \mu_1, \mu_2, \dots$ be point measures on a Hausdorff space E. The following are equivalent:*

1. $\mu_n \to_v \mu$,
2. $\mu_n(B) \to \mu(B)$ *for all relatively compact (i.e. with compact closure)* B *for which* $\mu(\partial(B)) = 0$.
3. $\limsup_n \mu_n(K) \leq \mu(K)$ *and* $\liminf_n \mu_n(G) \geq \mu(G)$ *for all compact* K *and all open, relatively compact* G.

Since point measures are uniquely defined by the points on their underlying space they're describing, we would expect our idea of convergence to imply some kind of convergence of points on E. In fact, as the next lemma shows, vague convergence does imply a pointwise convergence in E.

Lemma 2.17 (Pointwise convergence, [26, Proposition 3.13, p.144]). *Let $\mu, \mu_1, \mu_2, \ldots$ be point measures on E and $\mu_n \to_v \mu$. For compact K with $\mu(\partial K) = 0$ and $n \geq N(K)$ there exist a labeling of points of μ_n and μ in K such that*

$$\begin{aligned} \mu_n(\cdot \cap K) &= \sum_{i=1}^{M} \delta_{x_i^{(n)}}, \\ \mu(\cdot \cap K) &= \sum_{i=1}^{M} \delta_{x_i}, \end{aligned} \tag{2.14}$$

and in E^M

$$(x_i^{(n)}, 1 \leq i \leq M) \xrightarrow{n\to\infty} (x_i, 1 \leq i \leq M) \tag{2.15}$$

in the sense of componentwise convergence.

2.4 On Brownian motion

In the course of this thesis we will require two further results on Brownian motion. First we introduce the central limit theorem for martingales, which provides us with a means to identify convergence of a martingale to Brownian motion.

We state the theorem here, without proof, as it appears in [3, Theorem 1.4, p.339 f.], omitting one of two equivalent conditions and all references to higher dimensional processes, in order to focus on the one-dimensional case we will need for our proof.

Theorem 2.18 (Central limit theorem for martingales). *Let $\{\mathcal{F}_s^n\}$ be a filtration and M_n a $\{\mathcal{F}_s^n\}$-local martingale with sample paths in $D_{\mathbb{R}}[0,\infty)$ and $M_n(0) = 0$. Let A_n be a process with sample paths in $D_{\mathbb{R}}[0,\infty)$, increasing in s, such that $M_n^2 - A_n$ is an $\{\mathcal{F}_s^n\}$-local martingale.*

Let the following conditions hold: For each $T > 0$,

$$\lim_{n->\infty} \mathbb{E}\left[\sup_{s\leq T} |A_n(s) - A_n(s-)|\right] = 0, \tag{2.16}$$

$$\lim_{n->\infty} \mathbb{E}\left[\sup_{s\leq T} |M_n(s) - M_n(s-)|^2\right] = 0, \tag{2.17}$$

and with $c(s)$ a continuous, increasing function on $[0,\infty)$, $c(0) = 0$, let

$$A_n(s) \longrightarrow_p c(s). \tag{2.18}$$

Then $M_n \longrightarrow_d X$ where X is a process with sample paths in $C_{\mathbb{R}}[0,\infty)$ and independent Gaussian increments.

We conclude this preliminary chapter by presenting Girsanovs Theorem, which will enable us to prove certain properties of the Brownian motion with drift. We state the theorem as it appears in [4, Theorem 4.2.2, p.66].

Theorem 2.19 (Girsanov). *Let $(W(s))_{0\leq s\leq T}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\Theta(s))_{0\leq s\leq T}$ be an adapted process satisfying*

$$\int_0^T \Theta^2(u)du < \infty. \tag{2.19}$$

Define

$$\tilde{W}(s) := W(s) + \int_0^s \Theta(u)du, \tag{2.20}$$

$$X(s) := \exp\left\{-\int_0^s \Theta(u)dW(u) - \frac{1}{2}\int_0^s \Theta^2(u)du\right\}. \tag{2.21}$$

If $X(s)$ a martingale, that is $\mathbb{E}[X(s)] = \mathbb{E}[X(0)] = 1$ for all s, the measure $\mathbb{Q}$ defined by

$$\tilde{\mathbb{P}}(A) := \int_A X(\omega)d\mathbb{P}(\omega), \text{ for all } A \in \mathcal{F} \tag{2.22}$$

is a probability measure under which the process $(\tilde{W}(s))_{0\leq s\leq T}$ is a standard Brownian motion.

By [4, Remark 4.2.3, p.66], a sufficient condition for $X(t)$ to be a martingale is the so-called Novikov-condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \Theta^2(u)du\right)\right] < \infty. \tag{2.23}$$

We can apply this Theorem to the Brownian motion with drift W^t as follows: Recall the definition

$$W^t(s) = W(s) + ts - \frac{1}{2}s^2 = W(s) + \int_0^s (t-u)du. \tag{2.24}$$

For $T < \infty$, $\Theta(u) := t - u$ satisfies (2.23), therefore $X(s)$, as defined in (2.21), is a martingale and W^t is a standard Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined in (2.22). Since $X(s) > 0$ almost surely for all s, the probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ agree which events happen almost surely or almost never,

$$\begin{aligned} \mathbb{P}(A) = 0 &\iff \tilde{\mathbb{P}}(A) = 0, \\ \mathbb{P}(A) = 1 &\iff \tilde{\mathbb{P}}(A) = 1, \end{aligned} \tag{2.25}$$

for all $A \in \mathcal{F}$.

When trying to prove that certain properties hold for W^t under $\mathbb{P}$ almost surely, it suffices to prove these properties holding for a standard Brownian motion almost surely. Since W^t is a standard Brownian motion under $\tilde{\mathbb{P}}$ we can apply (2.25) to carry this realization over to W^t under $\mathbb{P}$.

3 The breadth-first walk

3.1 The breadth-first walk in discrete time

We start by describing the deterministic construction of the breadth-first walk. Consider an arbitrary graph $\mathcal{G}$ on the set of vertices $\{1, \dots, n\}$. We will define the breadth-first ordering $(v(1), \dots v(n))$ of the vertices along with an integer-valued sequence $(z(i),\ 1 \le i \le n)$ which we call the breadth-first walk on $\mathcal{G}$.

The breadth-first order follow an algorithmic construction as follows: Let $\mathcal{C}_1, \mathcal{C}_2, \dots$ be the components of $\mathcal{G}$ in order, such that $w_1, w_2, \dots,$ the vertices with the smallest label in the corresponding component, are ordered $w_1 > w_2 > \dots$. Call w_i the root of $\mathcal{C}_i$. Now order by levels (distance from the root) and within levels order by original vertex label. See Figure 3.1 for an example of the new ordering.

For a more mathematically concise definition, consider the set of vertices $\{v(1), \dots, v(i)\}$ and define the *neighbour set* $\mathcal{N}_i$ as the vertices outside of $\{v(1), \dots, v(i)\}$ that are neighbours to some vertex in $\{v(1), \dots, v(i)\}$:

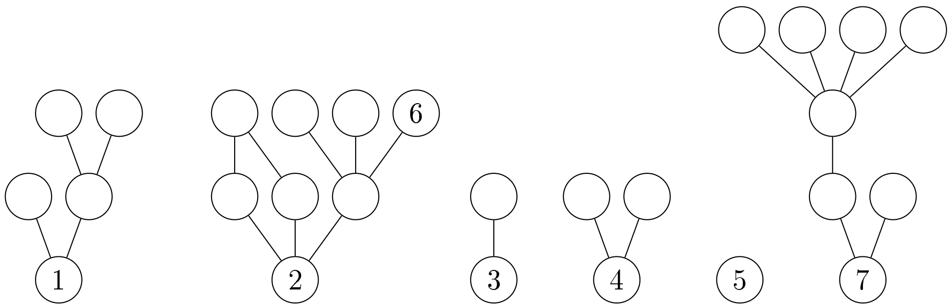
$$\mathcal{N}_i := \{v \in \{1, \dots, n\} \backslash \{v(1), \dots, v(i)\} \mid (v(j), v) \in S \text{ for some } 1 \le j \le i\} \tag{3.1}$$

This allows us to define the set of children of some vertex $v(i)$ as $\mathcal{N}_i \backslash \mathcal{N}_{i-1}$. First order the components as described above. Now consider only the first component $\mathcal{C}_1$. Define $v(1) := w_1$, the root of $\mathcal{C}_1$ and define $v(2), \dots, v(1 + |\mathcal{N}_1|)$ as the neighbours of $v(1)$, in increasing order of vertex label. Define the new label for all $i = 2, \dots, |\mathcal{C}_1|$, that is all vertices in the first component, inductively by listing all children (if any exist) of $v(i)$ in increasing order as $v(i + |\mathcal{N}_{i-1}|), \dots, v(i + |\mathcal{N}_i|)$. After labeling the last vertex in $\mathcal{C}_1$, set $v(|\mathcal{C}_1| + 1) := w_2$, the root of $\mathcal{C}_2$, and continue the construction as above. Traverse all components this way.

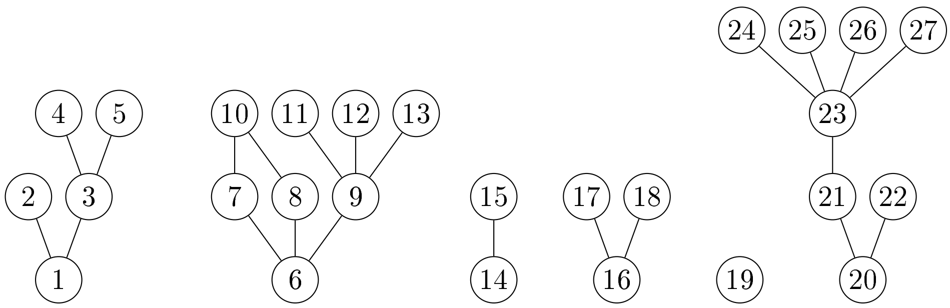
For the number of children of $v(i)$ write $c(i) = |\mathcal{N}_i \backslash \mathcal{N}_{i-1}|$. Now define the breadth-first walk $(z(i),\ 1 \le i \le n)$ by

$$\begin{aligned} z(0) &:= 0, \\ z(i) &:= z(i-1) + c(i) - 1, \quad i = 1, \dots, n. \end{aligned} \tag{3.2}$$

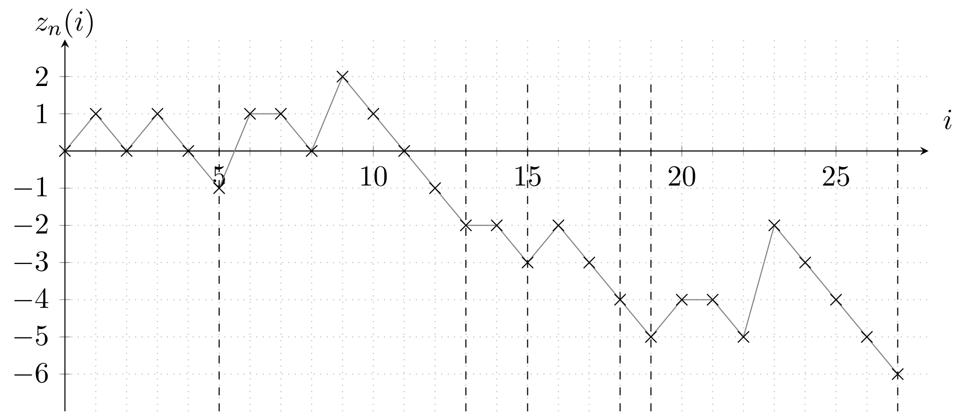
An explanation: The process divides the vertex-set into three parts: Explored, discovered and neutral vertices. Every vertex starts as neutral. At step 1, we traverse vertex $v(1)$ and mark it as explored. We search for neighbours of $v(1)$, and mark these as discovered. The next vertex to explore is the vertex already discovered with the smallest label, and each vertex switches from neutral to discovered once it gets assigned a new label. Once it's neighbours are explored, it switches to explored. Note that after traversing every vertex of one component, there are no discovered vertices left. The walk z decreases by 1 for each vertex traversed and increases by the number of new neighbours explored in each step.



(a) Original vertex labels



(b) New vertex labels



(c) Resulting breadth-first walk, dashed lines indicating the end of components

Figure 3.1: Breadth-first walk on the first components of a graph

We write

$$\zeta(j) := |\mathcal{C}_1| + \cdots + |\mathcal{C}_j|, \tag{3.3}$$

$$\zeta_n^{-1}(i) := \min\{j \mid \zeta(j) \geq i\}, \tag{3.4}$$

for the index of the last vertex in the j-th component (that's (3.3)) and (3.4), the index of the component containing $v(i)$. Now we can provide a definition of the breadth-first walk equivalent to (3.2):

$$\begin{aligned} &z^*(0) := 0, \\ &z^*(i) := |\mathcal{N}_i| - \zeta^{-1}(i), \quad i = 1, \ldots, n. \end{aligned} \tag{3.5}$$

We verify the equivalence by induction. We show that, for $i \geq 2$, increments of both functions are equal, so

$$\begin{aligned} & z(i) - z(i-1) = z^*(i) - z^*(i-1) \\ \iff\ & c(i) - 1 = |\mathcal{N}_i| - \zeta_n^{-1}(i) - |\mathcal{N}_{i-1}| + \zeta_n^{-1}(i-1) \\ \iff\ & |\mathcal{N}_i| - |\mathcal{N}_{i-1}| = c(i) + \zeta_n^{-1}(i) - \zeta_n^{-1}(i-1) \end{aligned} \tag{3.6}$$

We divide the proof into two cases. First, assume $v(i-1)$ is not the last vertex in it's component. Then $v(i)$ belongs to the same component and $\zeta_n^{-1}(i) = \zeta_n^{-1}(i-1)$. Vertex $v(i)$ has already been assigned a new label at step $i-1$, so $v(i) \in \mathcal{N}_{i-1}$. Going from $i-1$ to i, $\mathcal{N}_*$ increases by the number of new neighbours of $v(i)$ and decreases by $v(i)$ itself. So

$$|\mathcal{N}_i| - |\mathcal{N}_{i-1}| = c(i) - 1, \tag{3.7}$$

which proves the equivalence.

In the second case, if $v(i-1)$ is the last vertex of it's component, then $\zeta_n^{-1}(i) = \zeta_n^{-1}(i-1)+1$ and $|\mathcal{N}_{i-1}| = 0$. Equality (3.6) reduces to $|\mathcal{N}_i| = c(i)$, which holds since $c(i) = |\mathcal{N}_i \backslash \mathcal{N}_{i-1}| = |\mathcal{N}_i|$.

Since $|\mathcal{N}_i| = 0$ only if $v(i)$ is the last vertex in its component, (3.3) and (3.5) imply

$$z(\zeta(j)) = -j \tag{3.8}$$

and

$$z(i) \geq -j \quad \text{for all } \zeta(j) < i < \zeta(j+1). \tag{3.9}$$

So, for vertices in the j-th component, the random walk takes values greater or equal to $-(j-1)$, until the last vertex, for which z reaches a new minimum at $-j$. Knowing this we can reconstruct sizes and indices of components via

$$\zeta(j) = \min\{i \mid z(i) = -j\}, \tag{3.10}$$

$$|\mathcal{C}_j| = \zeta(j) - \zeta(j-1), \tag{3.11}$$

$$\zeta_n^{-1}(i) = 1 - \min_{j \leq i-1} z(j). \tag{3.12}$$

3.2 The breadth-first walk in continuous time

The last section defined the random walk for integer times, but to develop our theory of convergence to a Brownian motion we will have to construct $z(i)$ in continuous time.

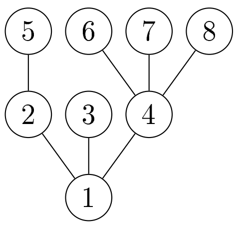
Define a series of independent random variables, uniformly distributed on $(0,1)$, $(U_{i,j}, 1 \le i \le n, 1 \le j \le c(i))$.

Then for each $i = 1, 2, \ldots$ and $0 \le u \le 1$ define the process Z by

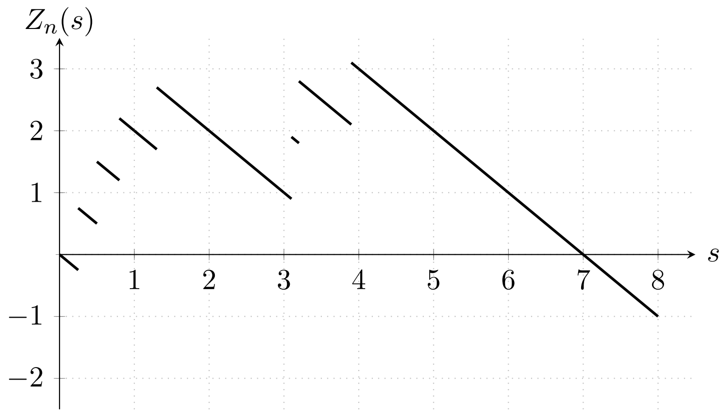
$$\begin{aligned} Z(0) &:= 0, \\ Z(i-1+u) &:= Z(i-1) - u + \sum_{1 \le j \le c(i)} \mathbb{1}_{\{U_{i,j} \le u\}}. \end{aligned} \tag{3.13}$$

So $Z(i) = Z(i-1) - 1 + c(i)$ and Z coincides with the discrete definition of the breadth-first walk at integer times.

Some more explanation on this construction: At time $i-1$, the walk has traversed vertices $(v(1), \ldots, v(i-1))$ and has encountered a list of vertices $(v(1), \ldots, v(k))$ of length $k = i - 1 + |\mathcal{N}_{i-1}|$. The discrete walk now adds the children of $v(i)$, $c(i)$, to this list at time i. The newly defined continuous walk adds those vertices at uniformly random times in $(i-1, i)$.



(a) A graph component



(b) The continuous-time breadth-first walk

Figure 3.2: Continuous breadth-first walk on a single component

For $\mathcal{G} \in \mathscr{G}(n, n^{-1} + tn^{-4/3})$, we call the continuous-time breadth-first walk on its vertices Z_n^t. We rescale Z_n^t on both axis to define

$$\bar{Z}_n^t(s) := n^{-1/3} Z_n^t(n^{2/3}s). \tag{3.14}$$

For $s \geq n^{1/3}$, let $Z_n^t(n^{2/3}s) \equiv Z_n^t(n)$. We can now state the main theorem of this chapter, the convergence of this rescaled process to a Brownian motion.

Theorem 3.1. *Let $Z_n^t(s)$, for $0 \leq s \leq n$, be the breadth-first walk associated with $\mathcal{G} \in \mathscr{G}(n, n^{-1} + tn^{-4/3})$. Rescale via*

$$\bar{Z}_n^t(s) := n^{-1/3} Z_n^t(n^{2/3}s).$$

Then $\bar{Z}_n^t(s) \to_d W^t$ as $n \to \infty$.

3.3 Decompositions of Z_n

For ease of notation, we will drop the superscript t from all random variables.

We begin with a Lemma similar to the Doob decomposition theorem, which states that any adapted stochastic process X_n with $\mathbb{E}[X_n] < \infty$ for all n may be decomposed into a martingale and an integrable predictable process. When trying to decompose Z_n, we deal with a process in continuous time for which the restriction on the expectation still holds. The following Lemma will directly provide such a decomposition.

Lemma 3.2. *The decomposition*

$$Z_n = M_n + F_n \tag{3.15}$$

holds, where M_n is a martingale and F_n is defined by

$$F_n(t) = \int_0^t a_n(s)ds - t \tag{3.16}$$

with

$$a_n(s)ds = \mathbb{P}(A \text{ new edge appears in } [s, s+ds] \mid Z_n(u),\ u \leq s). \tag{3.17}$$

Proof. We will prove that $Z_n - F_n$ is a martingale by showing that

$$\mathbb{E}[Z_n(t+u) - F_n(t+u) \mid \mathcal{F}_t] = Z_n(t) - F_n(t) \tag{3.18}$$

holds for all $u \geq 0$, where $\mathcal{F}_t$ is the natural σ-algebra generated by Z_n, $\mathcal{F}_t = \sigma(Z_n(s), s \leq t)$. This is equivalent to

$$\mathbb{E}[Z_n(t+u) - Z_n(t) \mid \mathcal{F}_t] = \mathbb{E}[F_n(t+u) - F_n(t) \mid \mathcal{F}_t]. \tag{3.19}$$

We start with the left-hand side. The change of Z_n between times t and $t+u$ is the sum of all jumps that occurred in $[t, t+u]$, minus the constant downward drift u:

$$\begin{aligned}\mathbb{E}[Z_n(t+u) - Z_n(t) \mid \mathcal{F}_t] &= \mathbb{E}[\text{Number of jumps in } [t, t+u] \mid \mathcal{F}_t] - u \\ &= \mathbb{E}[\text{Number of new edges appearing in } [t, t+u] \mid \mathcal{F}_t] - u,\end{aligned}$$

since every new edge corresponds to a jump of size 1 in Z_n.

Looking at the right-hand side, we define $\mathcal{E}_I$ as the event of a new edge appearing during the time interval I and calculate

$$\begin{aligned}\mathbb{E}[F_n t+u - F_n(t) \mid \mathcal{F}_t] &= \mathbb{E}[\int_0^{t+u} a_n(s)ds - (t+u) - \int_0^t a_n(s)ds + t \mid \mathcal{F}_t] \\ &= \mathbb{E}[\int_t^{t+u} a_n(s)ds \mid \mathcal{F}_t] - u \\ &= \int_t^{t+u} \mathbb{E}[a_n(s)ds \mid \mathcal{F}_t] - u \\ &= \int_t^{t+u} \mathbb{E}[\mathbb{P}(\mathcal{E}_{[s,s+ds]} \mid \mathcal{F}_(s)) \mid \mathcal{F}_t] - u \\ &= \int_t^{t+u} \mathbb{P}(\mathcal{E}_{[s,s+ds]} \mid \mathcal{F}_t) - u \quad \text{since } \mathcal{F}_t \subseteq \mathcal{F}_(s) \;\; \forall s \in [t, t+u] \\ &= \mathbb{E}[\text{Number of new edges appearing in } [t, t+u] \mid \mathcal{F}_t] - u.\end{aligned}$$

This proves M_n to be a martingale. □

For a better understanding of the processes involved, we need to find a concise expression for the probability used in (3.17). The next Lemma provides such an expression for $a_n(s)$ dependent on p_n, the history of the breadth-first walk up to s and the distance between s and $\lfloor (\rfloor s)$, the largest natural number smaller than s. An ensuing corollary provides some characteristics of the resulting process F_n.

Lemma 3.3. *For a_n as defined in Lemma 3.2,*

$$a_n(s) = (n - s - \zeta_n^{-1}(\lceil (\rceil s)) - Z_n(s)) \frac{p_n}{1 - (s - \lfloor s \rfloor)p_n}. \tag{3.20}$$

Proof. Consider the walk Z_n at time $s \in [i-1, i]$. Let N be the number of vertices that, at time $i-1$, were eligible to be children of vertex $v(i)$. That is, all vertices not yet explored or discovered, and excluding $v(i)$ itself. To any eligible vertex j we assign a random variable $U_{i,j}$. Let all $U_{i,j}$ be independent and identically $\mathcal{U}(0,1)$ distributed. Our understanding of the process of discovering children of $v(i)$ is as follows: At time $i - 1 + U_{i,j}$, the edge (i, j) will open with probability p_n and Z_n will make a jump of size

1. We arrive at a characterisation of our breadth-first walk, equivalent to (3.13):

$$Z_n(i-1+u) = Z_n(i-1) - u + \sum_{j=1}^{N} \mathbb{1}_{\{U_{i,j} \leq u,\ (i,j)\ \text{open}\}}. \tag{3.21}$$

We define $\mathscr{F}_s := \sigma(Z_n(t), t \leq s)$. The goal of this proof is to find an expression for $\mathbb{P}(\text{A new edge appears in } [s, s+ds] \,|\, \mathscr{F}_s)$. To do this, we condition over a finer σ-algebra and use the law of total expectation to arrive at a general statement. $\mathscr{F}_s$ tells us the history of the walk until time s. We know, how many vertices were eligible at time $i-1$ and how many open edges to $v(i)$ were found in $[\lfloor(\rfloor s), s]$. However it is unknown, exactly which vertices are now children of $v(i)$ and which vertices are still eligible.

Let

$$\begin{aligned}\mathscr{F}_s^k := \sigma(Z_n(t), t \leq s;\ & \\ & \text{There are exactly } k \text{ children of } v(i) \text{ encountered thus far,} \\ & \text{and these are } j_1, \dots, j_k).\end{aligned} \tag{3.22}$$

We know that k of the N edges eligible at time $i-1$ are already open and want to calculate the probability that one of the remaining $N-k$ edges opens in $[s, s+ds]$. The probability of some edge opening is the sum of the probabilities for single edges opening and some factor describing the, for small ds increasingly slim, chance of two or more edges opening in the interval. We denote by $\mathcal{E}_I$ the event of a new edge appearing in an interval I and write

$$\mathbb{P}(\mathcal{E}_{[s,s+ds]} \,|\, \mathscr{F}_s^k) = \sum_{j \neq j_1, \dots j_k} \mathbb{P}((i,j) \text{ opens in } [s, s+ds] \,|\, \mathscr{F}_s^k) + o\,(ds)\,. \tag{3.23}$$

For the edge (i,j), all relevant information contained in $\mathscr{F}_s^k$ is the fact that (i,j) is not yet open, the event of opening has not happened in $[\lfloor(\rfloor s), s)$. Since the opening itself, happening with probability p_n, and the uniformly distributed time of the event in $[i-1, i]$ are independent, we see that

$$\begin{aligned}&\mathbb{P}((i,j) \text{ opens in } [s, s+ds]) \\ &\quad = \mathbb{P}((i,j) \text{ opens})\mathbb{P}(\text{It happens in } [s, s+ds]) \\ &\quad = p_n ds.\end{aligned} \tag{3.24}$$

By the definition of conditional probability

$$\begin{aligned}&\mathbb{P}((i,j) \text{ opens in } [s, s+ds] \,|\, (i,j) \text{ did not open in } [\lfloor(\rfloor s), s]) \\ &\quad = \frac{\mathbb{P}((i,j) \text{ opens in } [s, s+ds])}{\mathbb{P}((i,j) \text{ did not open in } [\lfloor(\rfloor s), s])} \\ &\quad = \frac{p_n ds}{1 - p_n(s - \lfloor(\rfloor s))},\end{aligned} \tag{3.25}$$

and finally, omitting the $o\,(ds)$-term:

$$\begin{aligned}\mathbb{P}(\mathcal{E}_{[s,s+ds]} \mid \mathscr{F}_s^k) &= \sum_{j\neq j_1,\dots j_k} \frac{p_n}{1-(s-\lfloor s\rfloor)p_n} \\ &= (N-k)\frac{p_n ds}{1-p_n(s-\lfloor(\rfloor s))}.\end{aligned} \tag{3.26}$$

Seeing that $\mathscr{F}_s \subseteq \mathscr{F}_s^k$, we apply the law of total expectation:

$$\begin{aligned}\mathbb{P}(\mathcal{E}_{[s,s+ds]} \mid \mathscr{F}_s) &= \mathbb{E}[\mathbb{1}_{\mathcal{E}_{[s,s+ds]}} \mid \mathscr{F}_s] \\ &= \mathbb{E}[\mathbb{E}[\mathbb{1}_{\mathcal{E}_{[s,s+ds]}} \mid \mathscr{F}_s^k] \mid \mathscr{F}_s] \\ &= \mathbb{E}[(N-k)\frac{p_n}{1-(s-\lfloor s\rfloor)p_n} ds \mid \mathscr{F}_s].\end{aligned} \tag{3.27}$$

Conditioning on $\mathscr{F}_s$, the breadth-first walk Z_n tells us exactly how many vertices connected to vertex i until time s. We denote by $\eta_n(()s)$ the number of vertices that are at time s not eligible to be a child of $v(\lceil(\rceil s))$. Then $N-k = n-\eta_n(()s)$ at time s and

$$a_n(s)ds = \mathbb{P}(\mathcal{E}_{[s,s+ds]} \mid \mathscr{F}_s) = (n-\eta_n(()s))\frac{p_n}{1-(s-\lfloor s\rfloor)p_n}ds. \tag{3.28}$$

Finally, we find a concise expression for $\eta_n(()s)$. At time $i-1$, the ineligible vertices are the $i-1$ vertices already explored and the set $\mathcal{N}_{i-1}$ of vertices already discovered as children. If $v(i-1)$ is the last vertex of its component, $v(i)$ itself is not part of $\mathcal{N}_{i-1}$, so we need to add a term that equals 1 if $v(i-1)$ is the last vertex of its component and 0 otherwise. Together, we arrive at

$$\eta_n(i-1) = i-1+|\mathcal{N}_{i-1}| + (\zeta_n^{-1}(i) - \zeta_n^{-1}(i-1)). \tag{3.29}$$

By (3.5) this is equivalent to

$$\eta_n(i-1) = i-1+\zeta_n^{-1}(i) + Z_n(i-1). \tag{3.30}$$

At time $i-1+u$, for $0<u<1$, new vertices were discovered as children of $v(i)$ and now add to $\eta_n(i-1+u)$. The number of ineligible vertices is now

$$\begin{aligned}\eta_n(i-1+u) &= i-1+\zeta_n^{-1}(i) + Z_n(i-1) + \sum_j \mathbb{1}_{U_{i,j}\leq u} \\ &= i-1+u+\zeta_n^{-1}(i) + Z_n(i-1+u),\end{aligned} \tag{3.31}$$

by our definition (3.13) of the continuous-time breadth-first walk. So $\eta_n(()s) = s+\zeta_n^{-1}(\lceil(\rceil s)) + Z_n(s)$ which concludes the proof. □

Corollary 3.4. *The process F_n, as defined in Lemma 3.2, is a continuous process of bounded variation. Moreover, Z_n and M_n are càdlàgprocesses of bounded variation.*

Proof. Since $a_n(s) = (n - \eta_n(()s))\frac{p_n}{1-(s-\lfloor s \rfloor)p_n} \geq 0$ for all s, the integral $\int_0^t a_n(s)ds$ is a non-decreasing, continuous function in t. Therefore $F_n(t) = \int_0^t a_n(s)ds - t$ is the difference of two continuous, non-decreasing functions. By the Jordan Decomposition, see e.g. [**?**], F_n is a continuous process of bounded variation.

We remember from the definition of the continuous-time breadth-first walk in (3.13), that Z_n is the sum of the constant downward stream and jumps of size 1 for every new edge. By the Jordan Decomposition, Z_n is of bounded variation and the jumps make it a càdlàgprocess. Since $M_n = Z_n - F_n$ is the difference of two functions of bounded variation is again of bounded variation. Since F_n is continuous and Z_n is càdlàg, M_n is càdlàg. □

Having obtained a precise definition of F_n, we shift our focus to M_n, the martingale observed in Lemma 3.2. The following statement proves a similar decomposition of the squared martingale.

Lemma 3.5. *The decomposition*

$$M_n^2 = Q_n + G_n \tag{3.32}$$

holds, where Q_n is a martingale and G_n is defined by

$$G_n t = \int_0^t a_n(s)ds = F_n(t) + t \tag{3.33}$$

with a_n defined in (3.17).

Note. G_n is a continuous process.

Proof. Similar to the proof of the previous Lemma, we will show that $M_n^2 - G_n$ is a martingale.

By [**?**, Theorem 21.70, p.471], for all martingales M there exist a unique process $[M] = ([M]_t)_{t\geq 0}$ with $[M]_0 = 0$ such that $M^2 - [M]$ is a martingale. This process is called the quadratic variation of M and can be calculated as follows: Let $\Pi_n = \{t_{n,0}, \ldots, t_{n,k_n}\}$ be a sequence of partitions of $[0,t]$ with $||\Pi_n|| \to 0$ as $n \to \infty$, where $||\Pi|| := \max_{t_i, t_{i-1} \in \Pi}(t_i - t_{i-1})$ is the mesh of the partition. Then

$$[M]_t = \lim_{n\to\infty} \sum_{t_i, t_{i-1} \in \Pi_n} (M_{t_i} - M_{t_{i-1}})^2. \tag{3.34}$$

If M_t is a continuous process of bounded variation, then $[M]_t = 0$ for all t. The process M_n is right-continuous and of bounded variation. Therefore the quadratic variation vanishes on intervals where M_n is continuous, which leaves the jumps as the only discontinuities:

$$[M_n]_t = \sum_{0 \leq s \leq t} (\Delta M_n(s))^2, \tag{3.35}$$

where $\Delta M_n(s) := M_n(s) - M_n(s-)$ are the jumps of M_n. Thus

$$M_n^2 - G_n = \underbrace{(M_n^2 - [M_n])}_{\text{martingale}} + ([M_n] - G_n), \tag{3.36}$$

and to prove (3.32) it suffices to show that $[M_n] - G_n$ is a martingale. Since F_n is continuous, the jumps of M_n are exactly the jumps of Z_n. Note that the jumps $\Delta Z_n(s)$ can take one of two values: 1 if there is a jump of size 1 at time s, 0 otherwise. From this, we conclude that

$$\begin{aligned}
[M_n]_t - G_n t &= \sum_{0 \le s \le t} (\Delta M_n(s))^2 - G_n t \\
&= \sum_{0 \le s \le t} (\Delta Z_n(s))^2 - G_n t \\
&= \sum_{0 \le s \le t} \Delta Z_n(s) - G_n t \\
&= \text{Number of jumps of } Z_n \text{ in } [0,t] - G_n t \\
&= (\text{Number of jumps of } Z_n \text{ in } [0,t] - t) - F_n(t) \\
&= Z_n(t) - F_n(t),
\end{aligned}$$

which is a martingale by Lemma 3.2. □

3.4 The central limit theorem for martingales

The main goal of this section will be the use of the functional central limit theorem for martingales and subsequently proving Theorem 3.1. As stated in Theorem 2.18, with Z as X and M as N and so on, we need to make sure that our processes satisfy conditions (**??**) to (**??**).

We start by analysing the asymptotic behaviour of F_n. This will not be used directly in Girsanovs theorem, but will be helpful for later Lemmata and provide the drift of W^t.

Lemma 3.6. *For F_n defined in Lemma 3.2 and fixed $s_0 \ge 0$,*

$$n^{-1/3} \sup_{s \le n^{2/3} s_0} \left| F_n(s) + \frac{s^2}{2} n^{-1} - st n^{-1/3} \right| \longrightarrow_p 0 \tag{3.37}$$

as $n \to \infty$, where t is the fixed parameter of the random graph.

Proof. We define $a_n'(s)$ as

$$a_n'(s) := a_n(s)(1 - (s - \lfloor(\rfloor s))p_n) = (n - s - \zeta_n^{-1}(\lceil(\rceil s)) - Z_n(s))p_n. \tag{3.38}$$

First, we show that $a_n(s)$ and $a'_n(s)$ become asymptotically close, uniformly in s, for large n:

$$\begin{aligned}
|a_n(s) - a'_n(s)| &= |a_n(s)\,(1 - (1 - (s - \lfloor(\rfloor s))p_n))\,| \\
&= |\frac{(n - s - \zeta_n^{-1}(\lceil(\rceil s)) - Z_n(s))p_n}{1 - (s - \lfloor(\rfloor s))p_n}(s - \lfloor(\rfloor s))p_n)| \\
&= |\underbrace{\frac{p_n n - p_n(s + \zeta_n^{-1}(\lceil(\rceil s)) + Z_n(s))}{1 - (s - \lfloor(\rfloor s))p_n}}_{\leq 5}\underbrace{(s - \lfloor(\rfloor s))p_n}_{=O(n^{-1})}| \\
&= O\left(n^{-1}\right).
\end{aligned} \tag{3.39}$$

The last step follows from $p_n = O\left(n^{-1}\right)$ and $|Z_n(s)|, |\zeta_n^{-1}(()s)| \leq n$ for all $s \leq s_0 n^{2/3}$ and n. We substitute the definition of p_n in (3.38) to expand

$$\begin{aligned}
a'_n(s) - 1 &= \left(n - s - \zeta_n^{-1}(\lceil(\rceil s)) - Z_n(s)\right)\left(n^{-1} + tn^{-4/3}\right) - 1 \\
&= tn^{-1/3} - sn^{-1} - stn^{-4/3} \\
&\quad - \left(\zeta_n^{-1}(\lceil(\rceil s)) + Z_n(s)\right)\left(n^{-1} + tn^{-4/3}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\left|a'_n(s) - 1 + \frac{(}{s})n - \frac{t}{n^{1/3}} + \frac{st}{n^{4/3}}\right| &= \left|\frac{\zeta_n^{-1}(\lceil(\rceil s)) + Z_n(s)}{n}\left(1 + \frac{t}{n^{1/3}}\right)\right| \\
&\leq 2\left|\frac{\zeta_n^{-1}(\lceil(\rceil s)) + Z_n(s)}{n}\right| \\
&\leq 2\frac{\zeta_n^{-1}(\lceil(\rceil s)) + |Z_n(s)|}{n},
\end{aligned} \tag{3.40}$$

for $n^{1/3} \geq |t|$. Integrating the inner part of the left-hand side over s yields

$$\begin{aligned}
&\int_0^{(} s)(a'_n(u) - 1 + \frac{u}{n} - \frac{t}{n^{1/3}} + \frac{ut}{n^{4/3}})du \\
&\quad = \int_0^{(} s)(a'_n(u) - 1)du + \frac{s^2}{2n} - \frac{st}{n^{1/3}} + \frac{s^2 t}{2n^{4/3}},
\end{aligned} \tag{3.41}$$

and from (3.39) we know

$$|\int_0^{(} s)(a'_n(u) - 1)du - F_n(s)| = O\left(n^{-1}\right). \tag{3.42}$$

Using (3.12) and (3.40), the following inequalities hold for sufficiently large n:

$$
\begin{aligned}
&\left|F_n(s) + \frac{s^2}{2n} - \frac{st}{n^{1/3}} + \frac{s^2t}{2n^{4/3}}\right| \\
&\quad = \left|\int_0^(s) a_n'(u) - 1 + \frac{u}{n} - \frac{t}{n^{1/3}} + \frac{ut}{n^{4/3}} du\right| + O\left(\frac{1}{n}\right) \\
&\quad \leq \int_0^(s) \left|a_n'(u) - 1 + \frac{u}{n} - \frac{t}{n^{1/3}} + \frac{ut}{n^{4/3}}\right| du + O\left(\frac{1}{n}\right) \\
&\quad \leq \int_0^(s) 2\frac{\zeta_n^{-1}(\lceil u\rceil) + |Z_n(u)|}{n} du \qquad\qquad (3.43) \\
&\quad = \frac{2}{n}\int_0^(s)(1 - \min_{w\leq\lceil u\rceil - 1} Z_n w + |Z_n(u)|) du \\
&\quad = \frac{2}{n}\int_0^(s)(|Z_n(u)| - \min_{w\leq\lceil u\rceil - 1} Z_n w) du + O\left(\frac{(}{s})n\right) \\
&\quad \leq \frac{4s}{n}\max_{u\leq s}|Z_n(u)| + O\left(\frac{(}{s})n\right),
\end{aligned}
$$

the last inequality following from $\lceil u\rceil - 1 \leq s$ for $u \leq s$, hence $|\min_{w\leq\lceil u\rceil-1} Z_n w| \leq \max_{u\leq s}|Z_n(u)|$. The left-hand side of this inequality differs from the expression used in (3.37) by the term $\frac{s^2t}{2n^{4/3}}$. We show that this difference is negligible even after taking the supremum, by defining $A_n^*(s) := F_n(s) + \frac{s^2}{2n} - \frac{st}{n^{1/3}}$ and evaluating

$$
\begin{aligned}
&n^{-1/3}\sup_{s\leq n^{2/3}s_0}|A_n^*(s) + \frac{s^2t}{2n^{4/3}}| - n^{-1/3}\sup_{s\leq n^{2/3}s_0}|A_n^*(s)| \\
&\leq n^{-1/3}\left(\sup_{s\leq n^{2/3}s_0}|A_n^*(s)| + \sup_{s\leq n^{2/3}s_0}|\frac{s^2t}{2n^{4/3}}| - \sup_{s\leq n^{2/3}s_0}|A_n^*(s)|\right) \\
&= n^{-1/3}\sup_{s\leq n^{2/3}s_0}|\frac{s^2t}{2n^{4/3}}| \\
&= n^{-1/3}\frac{s_0^2t}{2} \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}
$$

We can now proceed to take the supremum and rescale both sides of the last inequality, arriving at

$$\begin{aligned}
&n^{-1/3} \sup_{s \leq n^{2/3} s_0} \left| F_n(s) + \frac{s^2}{2n} - \frac{st}{n^{1/3}} + \frac{s^2 t}{2n^{4/3}} \right| \\
&\leq n^{-1/3} \sup_{s \leq n^{2/3} s_0} (\frac{4s}{n} \max_{u \leq s} |Z_n(u)| + O\left(\frac{(}{s})n \right)) \\
&\leq n^{-1/3} 4 s_0 n^{-1/3} \sup_{s \leq n^{2/3} s_0} |Z_n(s)| + n^{-1/3} O\left(\frac{s_0 n^{2/3}}{n} \right) \\
&\leq 4 s_0 n^{-2/3} \sup_{s \leq n^{2/3} s_0} |Z_n(s)| + s_0 O\left(n^{-2/3} \right).
\end{aligned} \tag{3.44}$$

Since $O\left(n^{-2/3}\right) \rightarrow_p 0$ as $n \rightarrow \infty$, to establish the Lemma it is sufficient to prove

$$n^{-2/3} \sup_{s \leq n^{2/3} s_0} |Z_n(s)| \rightarrow_p 0. \tag{3.45}$$

In order to show the convergence we proceed to prove the stronger result,

$$n^{-1/3} \sup_{s \leq n^{2/3} s_0} |Z_n(s)| \text{ is stochastically bounded as } n \rightarrow \infty, \tag{3.46}$$

meaning for any $\epsilon > 0$ there is a $K > 0$ such that for n sufficiently large

$$\mathbb{P}\left(n^{-1/3} \sup_{s \leq n^{2/3} s_0} |Z_n(s)| > K \right) < \epsilon. \tag{3.47}$$

It is easily seen that this suffices to establish (3.45) by computing

$$\begin{aligned}
\mathbb{P}(n^{-2/3} \sup_{s \leq n^{2/3} s_0} |Z_n(s)| > \delta) &= \mathbb{P}(n^{-1/3} \sup_{s \leq n^{2/3} s_0} |Z_n(s)| > \delta n^{1/3}) \\
&\leq \mathbb{P}(n^{-1/3} \sup_{s \leq n^{2/3} s_0} |Z_n(s)| > K) \quad \text{for } n \geq (\delta K)^3 \\
&< \epsilon,
\end{aligned}$$

which holds for fixed $\epsilon, \delta > 0$ and sufficiently large n.

The remainder of this proof will follow a truncation argument. We define two stopping times T_n^* and T_n by

$$T_n^* := \min\{s \mid |Z_n(s)| > K n^{1/3}\}, \tag{3.48}$$

$$T_n := \min\{T_n^*, n^{2/3} s_0\}, \tag{3.49}$$

for some fixed $K > 0$ and use Markov's inequality to rewrite the left-hand side of Equation (3.47) as

$$\begin{aligned}
\mathbb{P}(\sup_{s \leq n^{2/3} s_0} |Z_n(s)| > K n^{1/3}) &= \mathbb{P}(|Z_n(T_n)| > K n^{1/3}) \\
&\leq \frac{\mathbb{E}[|Z_n(T_n)]}{K}.
\end{aligned} \tag{3.50}$$

To analyse $\mathbb{E}[|Z_n(T_n)|]$ we will use the decompositions established in the previous section. Lemma 3.2 gave $Z_n = M_n + F_n$, so

$$\mathbb{E}[|Z_n(T_n)|] \leq \mathbb{E}[|M_n(T_n)|] + \mathbb{E}[|F_n(T_n)|]. \tag{3.51}$$

By Lemma 3.5, we have $M_n^2 = Q_n + G_n$ where Q_n is a martingale. The optional sampling theorem dictates that $\mathbb{E}[Q_n(\tau)] = 0$ for all stopping times τ, hence

$$\begin{aligned}
\mathbb{E}[M_n^2(T_n)] &= \mathbb{E}[Q_n(T_n)] + \mathbb{E}[G_n T_n] \\
&= \mathbb{E}[G_n T_n] \\
&= \mathbb{E}[\int_0^{T_n} a_n(s)ds] \\
&\leq \int_0^{n^{2/3}s_0} \mathbb{E}[a_n(s)]ds.
\end{aligned}$$

By the definition of a_n in (3.17), we have

$$\begin{aligned}
a_n(s) &= (n - \nu_n(s))\frac{p_n}{1-(s-\lfloor(\rfloor s))p_n} \\
&\leq \frac{np_n}{1-(s-\lfloor(\rfloor s))p_n}
\end{aligned} \tag{3.52}$$

which is a deterministic function of s. So $\mathbb{E}[a_n(s)] \leq \frac{np_n}{1-(s-\lfloor(\rfloor s))p_n}$ and

$$\begin{aligned}
\int_0^{n^{2/3}s_0} \mathbb{E}[a_n(s)]ds &\leq \int_0^{n^{2/3}s_0} \frac{np_n}{1-(s-\lfloor(\rfloor s))p_n}ds \\
&\leq 2n^{2/3}s_0,
\end{aligned} \tag{3.53}$$

where the last inequality holds for n sufficiently large, since $\frac{np_n}{1-(s-\lfloor(\rfloor s))p_n} \to 1$ as $n \to \infty$. Now Hölder's inequality gives us

$$\mathbb{E}[|M_n(T_n)|] \leq \sqrt{\mathbb{E}[M_n^2(T_n)]} \leq (2s_0)^{1/2}n^{1/3}. \tag{3.54}$$

We proceed to the analysis of the second term in (3.51). The definition of F_n in (3.16) establishes

$$\begin{aligned}
\mathbb{E}[|F_n(T_n)|] &= \mathbb{E}[|\int_0^{T_n}(a_n(s)-1)ds|] \\
&\leq \mathbb{E}[\int_0^{T_n}|a_n(s)-1|ds] \\
&\leq \mathbb{E}[\int_0^{n^{2/3}s_0}|a_n(s)-1|ds].
\end{aligned} \tag{3.55}$$

We decompose $|a_n(s)-1|$ by

$$
\begin{aligned}
|a_n(s)-1| &= |a_n(s)+a_n'(s)-a_n'(s)+\frac{(}{s})n-\frac{t}{n^{1/3}}+\frac{st}{n^{4/3}} \\
&\quad -\frac{(}{s})n+\frac{t}{n^{1/3}}-\frac{st}{n^{4/3}}| \\
&\leq |a_n(s)-a_n'(s)|+|\frac{(}{s})n-\frac{t}{n^{1/3}}+\frac{st}{n^{4/3}}| \\
&\quad +|a_n'(s)-1+\frac{(}{s})n-\frac{t}{n^{1/3}}+\frac{st}{n^{4/3}}|
\end{aligned} \tag{3.56}
$$

to evaluate

$$
\begin{aligned}
\mathbb{E}[|F_n(T_n)|] \leq \mathbb{E}[\int_0^{n^{2/3}s_0} &|a_n(s)-a_n'(s)|ds] \\
+\mathbb{E}[\int_0^{n^{2/3}s_0} &|\frac{(}{s})n-\frac{t}{n^{1/3}}+\frac{st}{n^{4/3}}|ds] \\
+\mathbb{E}[\int_0^{n^{2/3}s_0} &|a_n'(s)-1+\frac{(}{s})n-\frac{t}{n^{1/3}}+\frac{st}{n^{4/3}}|ds].
\end{aligned} \tag{3.57}
$$

Let us look at the terms individually. By the uniform convergence $a_n \to a_n'$ we have

$$
\mathbb{E}[\int_0^{n^{2/3}s_0} |a_n(s)-a_n'(s)|ds] \leq n^{2/3}s_0 O\left(\frac{1}{n}\right), \tag{3.58}
$$

we can estimate the second term by

$$
\mathbb{E}[\int_0^{n^{2/3}s_0} |\frac{(}{s})n-\frac{t}{n^{1/3}}+\frac{st}{n^{4/3}}|ds] \leq n^{2/3}s_0 O\left(\frac{|t|}{n^{1/3}}\right), \tag{3.59}
$$

and (3.43) implies

$$
\begin{aligned}
&\mathbb{E}[\int_0^{T_n} |a_n'(s)-1+\frac{(}{s})n-\frac{t}{n^{1/3}}+\frac{st}{n^{4/3}}|ds] \\
&\leq \mathbb{E}[\frac{4T_n}{n}\max_{u\leq T_n}|Z_n(u)|+O\left(\frac{T_n}{n}\right)] \\
&\leq \frac{4n^{2/3}s_0}{n}\mathbb{E}[\max_{u\leq T_n}|Z_n(u)|]+O\left(\frac{n^{2/3}s_0}{n}\right) \\
&\leq 4s_0K+s_0O\left(n^{-1/3}\right),
\end{aligned} \tag{3.60}
$$

where the last inequality follows from the definition of T_n in (3.49), which assures $|Z_n(s)| \leq Kn^{1/3}$ for all $s \leq T_n$.

Summing these three terms we arrive at

$$\mathbb{E}[|F_n(T_n)|] \leq 4s_0K + s_0|t|O\left(n^{1/3}\right) + s_0^2O\left(n^{1/3}\right) + s_0O\left(n^{-1/3}\right) \quad (3.61)$$

We combine (3.54) and (3.61), which results in the following upper bound for large n:

$$\mathbb{E}[|Z_nT_n|] \leq \alpha n^{1/3} + 4s_0K, \quad (3.62)$$

where $\alpha = \alpha(s_0, t)$, but does not depend on n and K. Substituting $\mathbb{E}[|Z_nT_n|]$ in (3.50) we arrive at

$$\mathbb{P}(\sup_{s \leq n^{2/3}s_0} |Z_n(s)| > Kn^{1/3}) \leq \frac{\alpha}{K} + \frac{4s_0}{n^{1/3}} \quad (3.63)$$

which proves (3.47) by choosing K and n sufficiently large. □

Having established Lemma 3.6 it is now easy to deduce that G_n and M_n satisfy conditions that will make them suitable for Girsanovs theorem after a rescaling.

Lemma 3.7. *For G_n defined in Lemma 3.5 and fixed $s_0 \geq 0$,*

$$n^{-2/3}G_n n^{2/3}s_0 \longrightarrow_p s_0 \quad (3.64)$$

as $n \to \infty$.

Proof. Since $G_n(s) = F_n(s) + s$, we can rewrite (3.64) as

$$n^{-2/3}F_n(n^{2/3}s_0) \to_p 0. \quad (3.65)$$

We will show that

$$n^{-1/3}F_n(n^{-2/3}s_0) + \frac{1}{2}s_0^2 - s_0t \longrightarrow_p 0, \quad (3.66)$$

where again t denotes the probability parameter of $\mathscr{G}$. This implies

$$n^{-1/3}\left(n^{-1/3}F_n(n^{-2/3}s_0) + \frac{1}{2}s_0^2 - s_0t\right) \longrightarrow_p 0$$

which, since $\frac{1}{2}s_0^2 - s_0t$ is a constant in n, proves (3.65).

Let the function ϕ_n be defined by $\phi_n(s) := \frac{1}{2}n^{-4/3}s^2 - n^{-2/3}st$. Now $\phi_n(n^{2/3}s_0) = \frac{1}{2}s_0^2 - s_0t$ and

$$\begin{aligned}\left|n^{-1/3}F_n(n^{-2/3}s_0) + \frac{1}{2}s_0^2 - s_0t\right| &= \left|n^{-1/3}F_n(n^{-2/3}s_0) + \phi(n^{2/3}s_0)\right| \\ &\leq \sup_{s \leq n^{2/3}s_0} \left|n^{-1/3}F_n(s) + \phi(s)\right| \\ &= n^{-1/3} \sup_{s \leq n^{2/3}s_0} \left|F_n(s) + \frac{s^2}{2}n^{-1} - stn^{-1/3}\right| \\ &\longrightarrow_p 0\end{aligned}$$

by Lemma 3.6. This gives (3.66) and completes the proof. □

Lemma 3.8. *For M_n defined in Lemma 3.2 and fixed $s_0 \geq 0$,*

$$n^{-2/3}\mathbb{E}[\sup_{s \leq n^{2/3}s_0} |M_n(s) - M_n(s-)|^2] \longrightarrow 0, \tag{3.67}$$

as $n \longrightarrow \infty$.

Proof. As previously discussed in the proof of Lemma 3.5, the jumps of M_n are exactly the jumps of Z_n and therefore have size 1. The Lemma follows immediately. □

We now define the rescaled processes

$$\begin{aligned}
\bar{M}_n(s) &= n^{-1/3}M_n(n^{2/3}s), \\
\bar{F}_n(s) &= n^{-1/3}F_n(n^{2/3}s), \\
\bar{Q}_n(s) &= n^{-2/3}Q_n(n^{2/3}s), \\
\bar{G}_n(s) &= n^{-2/3}G_n(n^{2/3}s),
\end{aligned} \tag{3.68}$$

to fit the previously rescaled process $\bar{Z}_n$ in Theorem 3.1, such that

$$\begin{aligned}
\bar{Z}_n(s) &= \bar{M}_n(s) + \bar{F}_n(s), \\
\bar{M}_n^2(s) &= \bar{Q}_n(s) + \bar{G}_n(s).
\end{aligned} \tag{3.69}$$

Rescaling Lemmata 3.6, 3.7 and 3.8 gives us

$$\sup_{s \leq s_0} |\bar{F}_n(s) - \rho(s)| \longrightarrow_p 0, \tag{3.70}$$

where $\rho(s) = st - \frac{1}{2}s^2$,

$$\bar{G}_n(s) \longrightarrow_p s_0, \tag{3.71}$$

and

$$\mathbb{E}[\sup_{s \leq s_0} |\bar{M}_n(s) - \bar{M}_n(s-)|^2] \longrightarrow 0. \tag{3.72}$$

We are now ready to prove Theorem 3.1 by applying the central limit theorem for martingales, Theorem 2.18.

Proof of Theorem 3.1. Let again $\mathcal{F}_t^n = \sigma\{Z_n(s), s \leq t\}$ be the filtration generated by Z_n. The rescaling of M_n and G_n maintains martingale properties, so $\bar{M}_n$ and $\bar{Q}_n = \bar{M}_n^2 - \bar{G}_n$ are still $\{\mathcal{F}_t^n\}$-martingales. We apply Theorem 2.18 to $\bar{M}_n$ and $\bar{G}_n$. The continuity of $\bar{G}_n$ satisfies condition (2.16), condition (2.17) holds from (3.72) and (3.71) gives condition (2.18), with $c(t) = t$.

Therefore $\bar{M}_n(s) \rightarrow_d W(s)$, the standard Brownian motion, and using (3.70) we obtain

$$\bar{Z}_n(s) = \bar{M}_n(s) + \bar{F}_n(s) \rightarrow_d W(s) + \rho(s) = W^t(s). \tag{3.73}$$

□

4 Surplus edges

The goal of this chapter will be to first examine under which circumstances surplus edges can arise during the breadth-first walk, then finding an expression for the probability of encountering one and finally proving the joint convergence of $\bar{Z}_n^t$ and the surplus edge counting process to W^t and some limit process dependent on the realisation of W^t.

4.1 Counting surplus edges

We begin by describing a way to analyse the appearance of surplus edges. In Chapter 3 we defined the breadth-first walk Z_n, which counted new connections to previously not connected vertices. We remind ourselves that a surplus edge in a graph $\mathscr{G}(n, n^{-1} + tn^{-4/3})$ appears if, during the transition of vertices and components by the breadth-first walk, a vertex forms a new connection to another vertex which already has opened connections to some explored node. We associate with $\mathscr{G}$ a counting process $(N_n^t(s), 0 \leq s \leq n)$, with $N_n^t(0) = 0$, which increases by 1 at each appearance of a surplus edge.

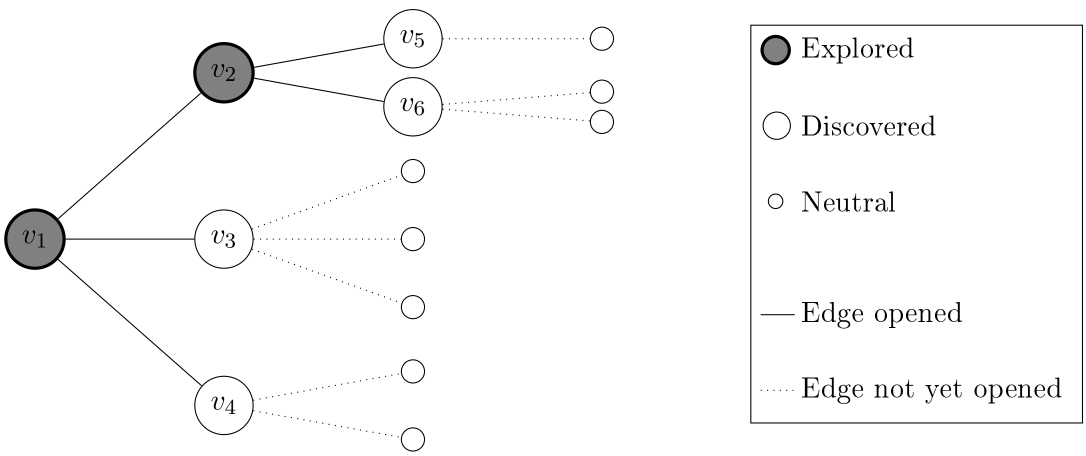


Figure 4.1: A sample component

To understand the number of vertices which can even open such excess connections, consider the breadth-first walk on the graph of Figure 4.1 at time $s = 2$. The children of v_1, v_2 to v_4, and the children of v_2, that is v_5 and v_6, are already discovered. We are interested in surplus edges to v_3. Since v_1 to v_6 are unable to form edges to become children of v_3, we have $\eta_n(2) = |\{v_1, \ldots, v_6\}|$. Of these vertices, v_1 and v_2 are already explored and every connection to neighbouring nodes is known. Vertex v_3 can not have an edge to itself, so only v_4, v_5 and v_6 are eligible to receive a surplus edge to v_3.

Let us examine what these considerations mean in terms of the breadth-first walk Z_n^t. When starting at a new component there are no vertices eligible for a surplus edge. For each new vertex found as a member of this component we have one additional eligible node and with each step taken, one more vertex is explored and thus can no longer receive a surplus edge. The number of vertices eligible for an excess edge therefore corresponds to the level of the breadth-first walk above its past minimum, which is attained at the beginning of the component. Hence we will expect the probability of encountering an excess edge at time s to be proportionate to

$$B_n^t(s) := Z_n^t(s) - \min_{u \le s} Z_n^t(u). \tag{4.1}$$

Rescaling the counting process appropriately this probability should scale to

$$\bar{B}_n^t(s) := \bar{Z}_n^t(s) - \min_{u \le s} \bar{Z}_n^t(u), \tag{4.2}$$

which converges in distribution to

$$B^t(s) := W^t(s) - \min_{u \le s} W^t(u). \tag{4.3}$$

In chapter 3 we examined a similar process, B_n, which increased by one for each appearance of a new edge to a previously not connected vertex. Lemma 3.3 established that $G_n t = \int_0^t a_n s ds$ with

$$a_n s = (n - \eta_n(s)) \frac{p_n}{1 - (s - \lfloor s \rfloor) p_n},$$

where $\eta_n(s)$ is the number of vertices ineligible to become a child of $v(\lceil s \rceil)$ at time s. In terms of counting processes, we call a_n the rate or conditional intensity of B_n. It is evident that N_n^t will have a similar rate, substituting the number of vertices eligible to become a child of $v(\lceil s \rceil)$ with the number of vertices eligible to receive a surplus edge to $v(\lceil s \rceil)$.

In general, at time $i-1$, the first i vertices are ineligible for a surplus edge to $v(i)$. The remaining $\eta_n(i-1) - i$ vertices are candidates for an excess edge opening with probability p_n. Therefore, the counting process N_n^t has rate

$$\lambda(s) = (\eta_n(\lfloor s \rfloor) - \lfloor s \rfloor) \frac{p_n}{1 - (s - \lfloor s \rfloor) p_n}. \tag{4.4}$$

Note that this rate is only exact for the chance of encountering exactly one surplus edge, but an overestimation for multiple excess edges. If we encounter a surplus edge at some time $s \in [i-1, i)$, the number of eligible vertices should decrease by one. However, (4.4) is constant for all $s \in [i-1, i)$. For ease of computation we will continue with this overestimation and later argue that the difference becomes negligible as $n \to \infty$.

Lemma 3.3 established $\eta_n(s) = s + \zeta_n^{-1}(\lceil s \rceil) + Z_n s$ and using (3.12) we can rewrite

$$\begin{aligned}\eta_n(\lfloor s \rfloor) - \lfloor s \rfloor &= \lfloor s \rfloor - \zeta_n^{-1}(\lfloor s \rfloor + 1) + Z_n^t(\lfloor s \rfloor) - \lfloor s \rfloor \\ &= 1 - \min_{u \le \lfloor s \rfloor} Z_n^t(u) + Z_n^t(\lfloor s \rfloor),\end{aligned}$$

and the conditional intensity becomes

$$\lambda(s) = (1 - \min_{u \le \lfloor s \rfloor} Z_n^t(u) + Z_n^t(\lfloor s \rfloor)) \frac{p_n}{1 - (s - \lfloor s \rfloor) p_n}. \tag{4.5}$$

We now rescale the counting process via

$$\bar{N}_n^t(s) = N_n^t(n^{2/3} s). \tag{4.6}$$

We calculate the rate of this rescaled process. Recall that the conditional intensity $\bar{\lambda}(s)$ of the process $\bar{N}_n^t(s)$ must satisfy

$$\mathbb{E}[\bar{N}_n^t(s)] = \int_0^s \bar{\lambda}(u) du. \tag{4.7}$$

Using (4.6) we evaluate the above integral in terms of $\lambda(s)$:

$$\begin{aligned}\mathbb{E}[\bar{N}_n^t(s)] &= \mathbb{E}[N_n^t(n^{2/3} s)] \\ &= \int_0^{n^{2/3} s} \lambda(u) du \\ &= \int_0^s n^{2/3} \lambda(n^{2/3} u) du.\end{aligned}$$

Comparing both integrands gives us

$$\begin{aligned}\bar{\lambda}(s) &= n^{2/3} \lambda(n^{2/3} s) \\ &= n^{2/3} \frac{1 - \min_{u \le n^{2/3} s} Z_n^t(u) + Z_n^t(n^{2/3} s)}{1 - (n^{2/3} s - \lfloor n^{2/3} s \rfloor) p_n} p_n \\ &= n^{2/3} \frac{1 - n^{1/3} \min_{u \le s} \bar{Z}_n^t(u) + n^{1/3} \bar{Z}_n^t(s)}{1 - (n^{2/3} s - \lfloor n^{2/3} s \rfloor) p_n} p_n \\ &= n p_n \frac{n^{-1/3} - \min_{u \le s} \bar{Z}_n^t(u) + \bar{Z}_n^t(s)}{1 - (n^{2/3} s - \lfloor n^{2/3} s \rfloor) p_n}\end{aligned} \tag{4.8}$$

Since $np_n \to 1$ and $|n^{2/3} s - \lfloor n^{2/3} s \rfloor| < 1$ for all s and n, this rate becomes asymptotically close to $\bar{Z}_n^t(s) - \min_{u \le s} \bar{Z}_n^t(u)$ as $n \to \infty$.

By Theorem 3.1 we have $\bar{Z}_n^t \to_d W^t$, so

$$\bar{\lambda}(s) \to_d W^t(s) - \min_{u \le s} W^t(u) = B^t(s). \tag{4.9}$$

The rate of the counting process $\bar{N}_n^t$ therefore converges in distribution to B^t.

4.2 Weak convergence of (Z_n^t, N_n^t)

Theorem 4.1. *For the previously defined processes $\bar{Z}_n^t$ and $\bar{N}_n^t$, the joint weak convergence*

$$(\bar{Z}_n^t(s), \bar{N}_n^t(s); s \geq 0) \longrightarrow_d (W^t(s), N^t(s); s \geq 0) \tag{4.10}$$

holds, where N^t is the counting process with conditional intensity B^t, i.e. the process for which

$$N^t(s) - \int_0^s B^t(u)du$$

is a martingale.

Proof. We begin with a quick overview of the proof.

We want to show that

$$(\bar{Z}_n^t, \bar{N}_n^t) \rightarrow_d (W^t, N^t), \tag{4.11}$$

meaning

$$\mathbb{E}[f(\bar{Z}_n^t, \bar{N}_n^t)] \xrightarrow{n\to\infty} \mathbb{E}[f(W^t, N^t)] \tag{4.12}$$

for all continuous, bounded functions $f : D[0,T]^2 \rightarrow \mathbb{R}$. The main idea of our proof is conditioning the expectations of some fixed realization of the Brownian motion W^t and analysing $\mathbb{E}[f(Z_n^t, N_n^t) \mid W^t]$. We first prove that for all $\epsilon > 0$ and sufficiently large n,

$$|\mathbb{E}[f(Z_n^t, N_n^t) \mid W^t] - \mathbb{E}[f(W^t, N_n^t) \mid W^t]| < \epsilon \tag{4.13}$$

holds. We then show that, conditioning on a fixed underlying Brownian motion W^t, the convergence $N_n^t \rightarrow_d N^t$ holds, which proves

$$\mathbb{E}[f(W^t, N_n^t) \mid W^t] \xrightarrow{n\to\infty} \mathbb{E}[f(W^t, N^t) \mid W^t] \tag{4.14}$$

since $f(W^t, \cdot)$ is a bounded and continuous function. The Theorem follows by averaging over all W^t using the law of total expectation.

Part 4.1.1. First, we will show that $\bar{N}_n^t$ is tight as a random process with image in $D[0,T]$. We already know that $\bar{Z}_n^t \rightarrow_d W^t$, which implies that $\bar{Z}_n^t$ is tight, so for all $\epsilon > 0$ there exists a compact $K \subset D[0,T]$ such that

$$\inf_n \mathbb{P}(\bar{Z}_n^t \in K) > 1 - \epsilon. \tag{4.15}$$

To show that $\bar{N}_n^t$ is tight, we need to prove that for all $\epsilon > 0$ there exists a $K \subset D[0,T]$ such that

$$\mathbb{P}(\bar{N}_n^t \in K) > 1 - \epsilon. \tag{4.16}$$

By the Arzelá–Ascoli theorem, it suffices to show that there exists a real number $K > 0$, such that

$$\mathbb{P}(\sup_{s \le T} \bar{N}_n^t(s) < K) > 1 - \epsilon. \tag{4.17}$$

Since $\bar{N}_n^t(s)$ is an increasing process in s, this is equivalent to

$$\mathbb{P}(\bar{N}_n^t(T) \ge K) < \epsilon. \tag{4.18}$$

We establish an analogous result for $\bar{Z}_n^t$ from (4.15). For all $\epsilon > 0$ exists $A > 0$ such that for all n

$$\mathbb{P}(\sup_{s \le T} |\bar{Z}_n^t(s)| > A) < \epsilon. \tag{4.19}$$

Define $\bar{B}_n^t(s) := \bar{Z}_n^t(s) - \min_{u \le s} \bar{Z}_n^t(u)$, the process reflecting $\bar{Z}_n^t$ at the x-axis. Since $|\bar{B}_n^t(s)| \le 2 \max_{u \le s} |\bar{Z}_n^t(u)|$, (4.19) implies

$$\mathbb{P}(\sup_{s \le T} |\bar{B}_n^t(s)| > 2A) < \epsilon. \tag{4.20}$$

Therefore, for all $\epsilon > 0$ exists an $A > 0$ such that

$$\mathbb{P}(\sup_{s \le T} |B_n^t(n^{2/3}s)| \ge A n^{1/3}) < \epsilon \tag{4.21}$$

holds for all $n \in \mathbb{N}$.

We now move to the process $\bar{N}_n^t$. Consider the unscaled process N_n^t at time $i \in [0, n^{2/3}T]$. The increment to its next step is binomially distributed on the number of vertices eligible for a surplus edge:

$$N_n^t(i) - N_n^t(i-1) \sim \mathrm{Bin}(B_n^t(i-1), p_n). \tag{4.22}$$

As previously established, for all $i \in [0, n^{2/3}T]$ we know

$$B_n^t(i) \le \sup_{j \le n^{2/3}T} B_n^t(j) \le A n^{1/3} \tag{4.23}$$

with probability greater than $1 - \epsilon$.

If we condition on the event that $B_n^t(i-1) \le A n^{1/3}$, a random variable $X_i \sim \mathrm{Bin}(B_n^t(i-1), p_n)$ will be stochastically dominated:

$$X_i \le_{\text{st.}} Y_i \sim \mathrm{Bin}(A n^{1/3}, p_n). \tag{4.24}$$

Seeing $N_n^t(T n^{2/3})$ as the sum of all its increments, we arrive at

$$N_n^t(T n^{2/3}) \le_{\text{st.}} \sum_{j=1}^{T n^{2/3}} Y_j, \tag{4.25}$$

where $Y_1, Y_2, \ldots, Y_{Tn^{2/3}} \sim \text{Bin}(An^{1/3}, p_n)$.

We denote by $\mathcal{E}_A$ the event $\sup_{j \leq n^{2/3}T} B_n^t(j) \leq An^{1/3}$ and use the law of total probability to compute

$$\begin{aligned}\mathbb{P}(\bar{N}_n^t(T) \geq K) &= \mathbb{P}(N_n^t(n^{2/3}T) \geq Kn^{1/3}) \\ &= \mathbb{P}(N_n^t(n^{2/3}T) \geq Kn^{1/3} \mid \mathcal{E}_A)\mathbb{P}(\mathcal{E}_A) \\ &\quad + \mathbb{P}(N_n^t(n^{2/3}T) \geq Kn^{1/3} \mid \neg\mathcal{E}_A)\mathbb{P}(\neg\mathcal{E}_A) \\ &\leq \mathbb{P}(N_n^t(n^{2/3}T) \geq Kn^{1/3} \mid \mathcal{E}_A) + \epsilon,\end{aligned} \tag{4.26}$$

which holds since $\mathbb{P}(\neg\mathcal{E}_A) < \epsilon$.

Since this probability is now conditioned on $\mathcal{E}_A$, the stochastic dominance (4.25) holds. Markov's inequality then gives

$$\begin{aligned}\mathbb{P}(N_n^t(n^{2/3}T) \geq Kn^{1/3} \mid \mathcal{E}_A) &\leq \mathbb{P}(\sum_{j=1}^{Tn^{2/3}} Y_j \geq K) \\ &\leq \frac{1}{K} Tn^{2/3}\mathbb{E}[Y_1] \\ &= \frac{1}{K} Tn^{2/3} p_n A n^{1/3} \\ &= \frac{1}{K} np_n TA \\ &\leq \frac{1}{K} CTA\end{aligned}$$

for some constant $C \in \mathbb{R}$, since $np_n \to 1$ as $n \to \infty$. So

$$\mathbb{P}(\bar{N}_n^t(T) \geq K) \leq \epsilon + \frac{1}{K} CTA \leq 2\epsilon \tag{4.27}$$

for large K. With high probability, $\sup_{s \leq T} \bar{N}_n^t(s)$ is bounded by some $K > 0$, hence the random variable $\bar{N}_n^t$ maps into a compact subset of $D[0,T]$ with high probability and $\bar{N}_n^t$ is tight.

Since both $\bar{Z}_n^t$ and $\bar{N}_n^t$ are tight, $(\bar{Z}_n^t, \bar{N}_n^t)$ is tight. Thus there is a compact subset $C \subset D[0,T]^2$ such that

$$\mathbb{P}((\bar{Z}_n^t, \bar{N}_n^t) \in C) > 1 - \epsilon. \tag{4.28}$$

Part 4.1.2. The next step in our proof will be showing that for all $\epsilon > 0$,

$$|\mathbb{E}[f(\bar{Z}_n^t, \bar{N}_n^t) \mid W^t] - \mathbb{E}[f(W^t, \bar{N}_n^t) \mid W^t]| < \epsilon, \tag{4.29}$$

for sufficiently large n. Recall that f is a bounded function, so there exists $M > 0$ such that $|f(x,y)| \leq M$ for all $(x,y) \in D[0,T]^2$. We denote by $\mathcal{E}_C$ the event $(Z_n^t, \bar{N}_n^t) \in C$ and use the law of total expectation to calculate

$$\begin{aligned}\mathbb{E}[f(\bar{Z}_n^t, \bar{N}_n^t) \mid W^t] &= \mathbb{E}[f(\bar{Z}_n^t, \bar{N}_n^t) \mid W^t, \mathcal{E}_C]\mathbb{P}(\mathcal{E}_C) \\ &\quad + \mathbb{E}[f(\bar{Z}_n^t, \bar{N}_n^t) \mid W^t, \neg\mathcal{E}_C]\mathbb{P}(\neg\mathcal{E}_C) \\ &\leq \mathbb{E}[f(\bar{Z}_n^t, \bar{N}_n^t) \mid W^t, \mathcal{E}_C]\mathbb{P}(\mathcal{E}_C) + \epsilon M.\end{aligned} \tag{4.30}$$

Since $\bar{Z}_n^t \to_d W^t$, we can use the Skorohod representation theorem to define random variables $\mathcal{W}^t$, $\bar{\mathcal{Z}}_1^t$, $\bar{\mathcal{Z}}_2^t$, …on the some common probability space, such that $\mathcal{W}^t \sim W^t$, $\bar{\mathcal{Z}}_i^t \sim \bar{Z}_i^t$ for all $i \in \mathbb{N}$ and $\bar{\mathcal{Z}}_n^t \to_{a.s.} \mathcal{W}^t$ as $n \to \infty$. Meaning, since $\mathcal{W}^t$ and $\bar{\mathcal{Z}}_n^t$ are random variables mapping into function spaces, we have

$$\sup_{s \leq T} |\bar{\mathcal{Z}}_n^t(s) - \mathcal{W}^t(s)| \to_{a.s.} 0, \tag{4.31}$$

which additionally implies

$$\sup_{s \leq T} |\bar{\mathcal{Z}}_n^t(s) - \mathcal{W}^t(s)| \to_p 0, \tag{4.32}$$

so for all $\epsilon > 0$:

$$\mathbb{P}(\sup_{s \leq T} |\bar{\mathcal{Z}}_n^t(s) - \mathcal{W}^t(s)| > \epsilon) \to 0. \tag{4.33}$$

Additionally, we define a process $\bar{\mathcal{N}}_n^t$ on the same probability space as $\mathcal{W}^t$ and $\bar{\mathcal{Z}}_n^t$ as a counting process with rate

$$\bar{\lambda}'(s) = np_n \frac{n^{-1/3} - \min_{u \leq s} \bar{\mathcal{Z}}_n^t(u) + \bar{\mathcal{Z}}_n^t(s)}{1 - (n^{2/3}s - \lfloor n^{2/3}s \rfloor)p_n}, \tag{4.34}$$

which makes it the equivalent of $\bar{N}_n^t$ in this new probability space, the rate of which is defined in (4.8). Since $\bar{\mathcal{Z}}_n^t \sim \bar{Z}_n^t$ we have $\bar{\mathcal{N}}_n^t \sim \bar{N}_n^t$.

Form here on out we substitute W^t, $\bar{Z}_n^t$ and $\bar{N}_n^t$ with $\mathcal{W}^t$, $\bar{\mathcal{Z}}_n^t$ and $\bar{\mathcal{N}}_n^t$ respectively and denote by $\mathcal{E}_C$ the event $(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t) \in C$ for a compact $C \subset D[0,T]^2$. By the equality in distribution, the final result on the expectation of these processes will then still hold for the original processes.

Defining the norm $||X||_T := \sup_{s \leq T} X(s)$, we denote by $\mathcal{E}_\delta$ the event $||\mathcal{W}^t - \bar{\mathcal{Z}}_n^t||_T < \delta$. By (4.33), for sufficiently large n,

$$\mathbb{P}(\mathcal{E}_\delta) > 1 - \epsilon. \tag{4.35}$$

Now an argument analogous to (4.30) gives

$$\begin{aligned} \mathbb{E}[f(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t) \mid \mathcal{W}^t] &\leq \mathbb{E}[f(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C]\mathbb{P}(\mathcal{E}_C) + \epsilon M \\ &\leq \mathbb{E}[f(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + 2\epsilon M. \end{aligned} \tag{4.36}$$

By the continuity of f, for all $\epsilon > 0$ we can choose a $\delta > 0$ such that $||\mathcal{W}^t - \bar{\mathcal{Z}}_n^t||_T < \delta$ implies that the inequality

$$\begin{aligned} f(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t) &= f(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t) + f(\mathcal{W}^t, \bar{\mathcal{N}}_n^t) - f(\mathcal{W}^t, \bar{\mathcal{N}}_n^t) \\ &\leq f(\mathcal{W}^t, \bar{\mathcal{N}}_n^t) + |f(Z_n^t, \bar{\mathcal{N}}_n^t) - f(\mathcal{W}^t, \bar{\mathcal{N}}_n^t)| \\ &\leq f(\mathcal{W}^t, \bar{\mathcal{N}}_n^t) + \epsilon \end{aligned}$$

holds. Therefore

$$\begin{aligned} \mathbb{E}[f(Z_n^t, \bar{\mathcal{N}}_n^t) \mid \mathcal{W}^t] &\leq \mathbb{E}[f(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + 2\epsilon M \\ &\leq \mathbb{E}[f(\mathcal{W}^t, \bar{\mathcal{N}}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + \epsilon + 2\epsilon M. \end{aligned} \tag{4.37}$$

Part 4.1.3. Our next objective is to establish the convergence $\bar{\mathcal{N}}_n^t \to_d N^t$. For this, we define a process M_n^t, which may be thought of as N^t in discrete time. While N^t is the continuous counting process with B^t as conditional intensity, we define M_n^t by

$$\begin{aligned} M_n^t(0) &:= 0, \\ M_n^t(k) &:= M_n^t(k-1) + \xi_k, \end{aligned} \tag{4.38}$$

where $\xi_k \sim \text{Bin}(n^{1/3}B^t(n^{-2/3}k), p_n)$, the discrete steps of M_n^t are dependent on an upscaling of the reflected Brownian motion B^t.

We will show that, if $||\bar{\mathcal{Z}}_n^t - \mathcal{W}^t|| < \delta$, then $M_n^t(k) = N_n^t(k)$ for all $k \leq n^{2/3}T$ with high probability. For this, we redefine both processes using a coupling argument. At step k, let

$$\begin{aligned} \alpha_k &:= \min(B_n^t(k), n^{1/3}B^t(n^{-2/3}k)), \\ \beta_k &:= \max(B_n^t(k), n^{1/3}B^t(n^{-2/3}k)). \end{aligned} \tag{4.39}$$

Now define random variables

$$\begin{aligned} \xi_k &\sim \text{Bin}(\alpha_k, p_n), \\ \eta_k &\sim \text{Bin}(\beta_k - \alpha_k, p_n). \end{aligned} \tag{4.40}$$

So $\xi_k + \eta_k \sim \text{Bin}(\beta_k, p_n)$.

Consider the two possibilities at time k: Either $B_n^t(k) \leq n^{1/3}B^t(n^{-2/3}k)$, then $\alpha_k = B_n^t(k)$ and $\beta_k = n^{1/3}B^t(n^{-2/3}k)$, so

$$\begin{aligned} \xi_k &=_d N_n^t(k) - N_n^t(k-1), \\ \xi_k + \eta_k &=_d M_n^t(k) - M_n^t(k-1). \end{aligned} \tag{4.41}$$

Or $B_n^t(k) > n^{1/3}B^t(n^{-2/3}k)$, then $\alpha_k = n^{1/3}B^t(n^{-2/3}k)$ and $\beta_k = B_n^t(k)$, so

$$\begin{aligned} \xi_k &=_d M_n^t(k) - M_n^t(k-1), \\ \xi_k + \eta_k &=_d N_n^t(k) - N_n^t(k-1). \end{aligned} \tag{4.42}$$

This way, we can define N_n^t and M_n^t by

$$N_n^t(k) - N_n^t(k-1) = \begin{cases} \xi_k & \text{if } B_n^t(k) \leq n^{1/3}B^t(n^{-2/3}k), \\ \xi_k + \eta_k & \text{else} \end{cases} \tag{4.43}$$

and

$$M_n^t(k) - M_n^t(k-1) = \begin{cases} \xi_k + \eta_k & \text{if } B_n^t(k) \leq n^{1/3}B^t(n^{-2/3}k), \\ \xi_k & \text{else.} \end{cases} \tag{4.44}$$

By (4.39) and (4.40), these definitions maintain

$$\begin{aligned} N_n^t(k) - N_n^t(k-1) &\sim \text{Bin}(B_n^t(k), p_n), \\ M_n^t(k) - M_n^t(k-1) &\sim \text{Bin}(n^{1/3}B^t(n^{-2/3}k), p_n). \end{aligned}$$

We see that, no matter the relation of $B_n^t(k)$ and $n^{1/3}B^t(n^{-2/3}k)$, the increments of the processes differ only by the random variable η_k.

Conditioning on $||Z_n^t - \mathcal{W}^t|| < \delta$, for $M_n^t(k) \neq N_n^t(k)$ to hold for some k, there has to have been a step in which the increments of both processes were different. We evaluate

$$\begin{aligned}
&\mathbb{P}(\exists k \leq n^{2/3}T : M_n^t(k) \neq N_n^t(k) \mid \mathcal{E}_\delta) \\
&\quad \leq \sum_{k=1}^{n^{2/3}T} \mathbb{P}(M_n^t(k) - M_n^t(k-1) \neq N_n^t(k) - N_n^t(k-1) \mid \mathcal{E}_\delta) \\
&\quad \leq \sum_{k=1}^{n^{2/3}T} \mathbb{P}(\eta_k \neq 0 \mid \mathcal{E}_\delta) \\
&\quad \leq n^{2/3}T \max_{k \leq n^{2/3}T} \mathbb{P}(\eta_k \neq 0 \mid \mathcal{E}_\delta).
\end{aligned} \tag{4.45}$$

Since $||Z_n^t - \mathcal{W}^t|| < \delta$, we know $\beta_k - \alpha_k < \delta n^{1/3}$. Therefore $\eta_k \leq_{\text{st.}} \zeta \sim \text{Bin}(\delta n^{1/3}, p_n)$ for all $k \leq n^{2/3}T$.

Using Markov's inequality gives

$$\mathbb{P}(\eta_k \neq 0 \mid \mathcal{E}_\delta) \leq \mathbb{P}(\zeta_k \geq 1) \leq \mathbb{E}[\zeta_k] = \delta n^{1/3} p_n, \tag{4.46}$$

and substituting in (4.45) we obtain

$$\begin{aligned}
&\mathbb{P}(\exists k \leq n^{2/3}T : M_n^t(k) \neq N_n^t(k) \mid \mathcal{E}_\delta) \\
&\quad \leq n^{2/3}T \max_{k \leq n^{2/3}T} \mathbb{P}(\eta_k \neq 0 \mid \mathcal{E}_\delta) \\
&\quad \leq n^{2/3}T\delta n^{1/3} p_n \\
&\quad \leq n p_n T\delta \\
&\quad \leq 2T\delta
\end{aligned} \tag{4.47}$$

for large n.

We now define $\bar{M}_n^t(s) = n^{-1/3}M_n^t(n^{2/3}s)$ and continue the estimation from (4.37), which yields

$$\begin{aligned}
\mathbb{E}[f(Z_n^t, N_n^t)] &\leq \mathbb{E}[f(\mathcal{W}^t, N_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + \epsilon + 2M\epsilon \\
&\leq \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + 2\delta TM + \epsilon + 2M\epsilon.
\end{aligned} \tag{4.48}$$

We want to drop the conditioning on $\mathcal{E}_C$ and $\mathcal{E}_\delta$ from the expectation in (4.48) and calculate

$$\begin{aligned}
\mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t] &= \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C]\mathbb{P}(\mathcal{E}_C) \\
&\quad + \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \neg\mathcal{E}_C]\mathbb{P}(\neg\mathcal{E}_C) \\
&= \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) \\
&\quad + \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \neg\mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\neg\mathcal{E}_\delta) \\
&\quad + \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \neg\mathcal{E}_C]\mathbb{P}(\neg\mathcal{E}_C).
\end{aligned}$$

As previously established, $\mathbb{P}(\mathcal{E}_C), \mathbb{P}(\mathcal{E}_\delta) > 1-\epsilon$ for sufficiently large n. The boundedness of f implies $f(\mathcal{W}^t, \bar{M}_n^t) \geq -M$, therefore

$$\begin{aligned}
&\mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) \\
&\quad = \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t] \\
&\qquad - \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \neg\mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\neg\mathcal{E}_\delta) \\
&\qquad - \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \neg\mathcal{E}_C]\mathbb{P}(\neg\mathcal{E}_C) \\
&\quad \leq \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t] + 2\epsilon M.
\end{aligned}$$

Substituting this result in (4.48) gives

$$\begin{aligned}
\mathbb{E}[f(Z_n^t, N_n^t)] &\leq \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t, \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + 2\delta TM + \epsilon + 2M\epsilon \\
&\leq \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t] + +2\delta TM + \epsilon + 4M\epsilon.
\end{aligned} \tag{4.49}$$

This inequality holds for all bounded, continuous functions f. Therefore it holds for $-f$ as well, which implies

$$\begin{aligned}
&\mathbb{E}[-f(Z_n^t, N_n^t)] \leq \mathbb{E}[-f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t] + 2\delta TM + \epsilon + 4M\epsilon \\
\iff &\mathbb{E}[f(Z_n^t, N_n^t)] \geq \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t] - 2\delta TM - \epsilon - 4M\epsilon,
\end{aligned} \tag{4.50}$$

and therefore

$$|\mathbb{E}[f(Z_n^t, N_n^t)] - \mathbb{E}[f(\mathcal{W}^t, \bar{M}_n^t) \mid \mathcal{W}^t]| \leq 2\delta TM + \epsilon + 4M\epsilon. \tag{4.51}$$

Part 4.1.4. We now show that $\bar{M}_n^t \to_d N^t$. Since N_n^t is tight and N_n^t is equal to M_n^t with high probability, M_n^t and $\bar{M}_n^t$ are tight and it suffices proving convergence in finite dimensional distributions, in this case for all $0 \leq s_1 < s_2 < \cdots < s_l \leq T$:

$$\begin{aligned}
&\mathbb{P}(\bar{M}_n^t(s_1) = k_1, \ldots, \bar{M}_n^t(s_l) = k_l) \\
&\quad \xrightarrow{n\to\infty} \mathbb{P}(N^t(s_1) = k_1, \ldots, N^t(s_l) = k_l).
\end{aligned} \tag{4.52}$$

Recall that N^t is a Poisson point process, continuous on $\mathbb{R}$ with rate B^t, thus the increments of N^t are independent and for all $a < b$:

$$N^t(b) - N^t(a) \sim \mathrm{Poi}(\int_a^b B^t(s)ds). \tag{4.53}$$

In contrast, M_n^t is a discrete process whose increments are defined by B^t at integer times, that is for all $k \leq T$:

$$M_n^t(k) - M_n^t(k-1) \sim \mathrm{Bin}(n^{1/3}B^t(n^{-2/3}k), p_n) \tag{4.54}$$

We can evaluate the distribution of the increments of $\bar{M}_n^t$ by

$$\bar{M}_n^t(k) - \bar{M}_n^t(k-1) = n^{-1/3}(M_n^t(n^{2/3}k) - M_n^t(n^{2/3}(k-1)). \tag{4.55}$$

Between times $n^{2/3}(k-1)$ and $n^{2/3}k$, there are multiple integer steps, the increment in each step as defined in (4.54). Thus

$$\bar{M}_n^t(k) - \bar{M}_n^t(k-1) \sim n^{-1/3} \sum_{i=n^{2/3}s_{j-1}+1}^{n^{2/3}s_j} \mathrm{Bin}(n^{1/3}B^t(n^{-2/3}i), p_n). \quad (4.56)$$

Since the increments are independent, we can move the sum inside the argument of the Binomial distribution. Let us define

$$R_{n,j} := \sum_{i=n^{2/3}s_{j-1}+1}^{n^{2/3}s_j} n^{1/3}B^t(n^{-2/3}i)$$

and compute the probability in (4.52) as

$$\begin{aligned}
&\mathbb{P}(\bar{M}_n^t(s_1) = k_1, \ldots, \bar{M}_n^t(s_l) = k_l) \\
&\quad = \mathbb{P}(\bar{M}_n^t(s_j) - \bar{M}_n^t(s_{j-1}) = k_j - k_{j-1}, \ \forall j = 2, \ldots, l) \\
&\quad = \prod_{j=2}^{l} \mathbb{P}(n^{1/3}\bar{M}_n^t(s_j) - n^{1/3}\bar{M}_n^t(s_{j-1}) = n^{1/3}(k_j - k_{j-1})) \qquad (4.57) \\
&\quad = \prod_{j=2}^{l} \mathbb{P}(Y_{n,j} = n^{1/3}(k_j - k_{j-1})),
\end{aligned}$$

where $Y_{n,j} \sim \mathrm{Bin}(R_{n,j}, p_n)$. Note that $Y_{n,j} =_d \sum_{k=1}^{R_{n,j}} \xi_k$, with $\xi_k \sim \mathrm{B}(p_n)$.

In the next step we use the Poisson limit theorem, see for example [**?**, Theorem 3.7, p.79], which states that for a series of binomially distributed random variables $X_k \sim \mathrm{Bin}(N_k, p_k)$ with $\mathbb{E}[X_k] = N_k p_k \to \lambda \in \mathbb{R}$ as $k \to \infty$, the convergence

$$X_k \to_d \mathrm{Poi}(\lambda) \quad (4.58)$$

holds as $k \to \infty$.

To apply this theorem, we calculate the expected value of $Y_{n,j}$:

$$\begin{aligned}
\mathbb{E}[Y_{n,j}] &= R_{n,j}p_n \\
&= \frac{1}{n}R_{n,j} + O\left(n^{-1/3}\right) \\
&= \frac{1}{n} \sum_{i=n^{2/3}s_{j-1}+1}^{n^{2/3}s_j} n^{1/3}B^t(n^{-2/3}i) + O\left(n^{-1/3}\right) \\
&= \sum_{i=n^{2/3}s_{j-1}+1}^{n^{2/3}s_j} n^{-2/3}B^t(n^{-2/3}i) + O\left(n^{-1/3}\right)
\end{aligned}$$

This sum represents a partition of the interval $[s_{j-1}, s_j]$ into $n^{2/3}(s_j - s_{j-1})$ subintervals, each of length $n^{-2/3}$. Since $n^{-2/3}i$ is an element of its correpsonding subinterval, we are dealing with a Riemann sum over the continuous function B^t. Since B^t is bounded almost surely on the compact interval $[s_{j-1}, s_j]$, the sum converges to an integral and

$$\mathbb{E}[Y_{n,j}] \longrightarrow \int_{s_{j-1}}^{s_j} B^t(u)du. \tag{4.59}$$

Now applying the Poisson limit theorem yields

$$Y_{n,j} \to_d \text{Poi}(\int_{s_{j-1}}^{s_j} B^t(u)du) =_d N^t(s_j) - N^t(s_{j-1}), \tag{4.60}$$

and, applying this convergence in (4.57), we arrive at

$$\begin{aligned}
&\mathbb{P}(\bar{M}_n^t(s_1) = k_1, \dots, \bar{M}_n^t(s_l) = k_l) \\
&= \prod_{j=2}^{l} \mathbb{P}(Y_{n,j} = n^{1/3}(k_j - k_{j-1})) \\
&\longrightarrow \prod_{j=2}^{l} \mathbb{P}(N^t(s_j) - N^t(s_{j-1}) = n^{1/3}(k_j - k_{j-1})) \\
&= \mathbb{P}(N^t(s_1) = k_1, \dots, N^t(s_l) = k_l).
\end{aligned}$$

This proves $\bar{M}_n^t \to_d N^t$ and consequently

$$\mathbb{E}[f(\mathcal{W}^t, M_n^t) \mid \mathcal{W}^t] \longrightarrow \mathbb{E}[f(\mathcal{W}^t, N^t) \mid W^t]. \tag{4.61}$$

Combining (4.51) and (4.61) yields

$$\begin{aligned}
\mathbb{E}[f(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t)] &= \mathbb{E}[\mathbb{E}[f(\bar{\mathcal{Z}}_n^t, \bar{\mathcal{N}}_n^t) \mid \mathcal{W}^t]] \\
&\xrightarrow{n \to \infty} \mathbb{E}[\mathbb{E}[f(\mathcal{W}^t, N^t) \mid \mathcal{W}^t]] = \mathbb{E}[f(\mathcal{W}^t, N^t)],
\end{aligned} \tag{4.62}$$

which proves $(\bar{Z}_n^t, \bar{N}_n^t) \to_d (W^t, N^t)$ and completes the proof.

□

We can now assure ourselves that the overestimated probability (4.4) is asymptotically negligible. Assume the chance that any vertex encounters two or more surplus edges is non-zero and does not converge to zero as $n \to \infty$. If a vertex connects by multiple excess edges, the process N_n^t makes two or more jumps during the time-interval of length 1. The rescaling (4.6) compresses the time axis until, in the limit process N^t, any distance in an interval of original length 1 will be reduced to a single point. Consequently there would be a non-zero chance that the counting process has multiple coincident points. But by definition N^t is a Poisson counting process with continuous intensity B^t, therefore it is simple with probability 1, see [5, Remark 2.1, p.34]. We conclude that the probability of a vertex having multiple surplus edges must tend to zero.

5 Convergence of component sizes

In Chapter 3 we have shown that the rescaled breadth-first walk $\bar{Z}_n^t$ on $\mathscr{G}(n, n^{-1} + tn^{-4/3})$ converges in distribution to the Brownian motion with drift B^t. Intuitively it is clear that Theorem 1.7 should follow: Component sizes are coded into the breadth-first walk as excursions above past minima, excursions of B^t are excursions of W^t above past minima. But rigorously deducing a proof of our main theorem requires a bit more work. To make sure that indeed components and excursions do match up, we describe them as two-dimensional point processes in which the first entry gives the start of an excursion or component, the second the length of an excursion or the size of a component. The following first two Lemmata prove that this sequence of Poisson point processes describing components converges to the point process describing excursions with regard to the vague topology.

For the proof of Theorem 1.7 to hold, it remains to be shown that this encompasses all relevant components and excursions. One problem that might arise is the starting point of a component of size greater than some $\delta n^{2/3}$, which needs to be considered in the convergence, to diverge to infinity as $n \to \infty$. Since we are working in the vague topology of counting processes, this would entail the component not necessarily converging in distribution to some excursion of B^t.

We start with a deterministic Lemma. Given a continuous function f with properties similar to a Brownian motion and a sequence of functions f_n converging uniformly to f, we can define a set of excursions on each f_n such that the point processes of starts and lengths of excursions converge vaguely to the point processes of starts and lengths of excursions on f.

Lemma 5.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Let $\mathcal{E}$ be the set of non-empty intervals $e = [l, r] \subset \mathbb{R}_{\geq 0}$ such that*

$$f(r) = f(l) = \min_{s \leq l} f(s), \tag{5.1}$$

$$f(l) < f(s) \quad \forall l < s < r. \tag{5.2}$$

Define $\Xi := \{(l, r - l) \mid (l, r) \in \mathcal{E}\}$.

Suppose that for intervals $(l_1, r_1), (l_2, r_2) \in \mathcal{E}$ with $l_1 < l_2$ we have

$$f(l_1) > f(l_2) \tag{5.3}$$

and the complement of $\cup_{e \in \mathcal{E}}(l, r)$ has Lebesgue measure zero,

$$\mu\left((\cup_{e \in \mathcal{E}}(l, r))^c\right) = 0. \tag{5.4}$$

Let $f_n \to f$ as $n \to \infty$ uniformly on bounded intervals. Now define $\Xi^{(n)} := \{(t_{n,i}, t_{n,i+1} - t_{n,i}) \mid i \geq 1\}$ for any sequence of sets of points $\mathcal{T}_n := (t_{n,i}, i \geq 1)$ satisfying the following conditions:

$$0 = t_{n,1} < t_{n,2} < \ldots \text{ and } \lim_{i \to \infty} t_{n,i} = \infty, \tag{5.5}$$

$$f_n(t_{n,i}) = \min_{u \leq t_{n,i}} f_n(u), \tag{5.6}$$

$$\max_{i: t_{n,i} \leq s_0} (f_n(t_{n,i}) - f_n(t_{n,i+1})) \to 0 \text{ as } n \to \infty, \text{ for all } s_0 \leq \infty. \tag{5.7}$$

Then $\Xi^{(n)} \to_v \Xi$ as $n \to \infty$.

Note. The convergence $\Xi^{(n)} \to_v \Xi$ is to be interpreted as the vague convergence of counting measures which, by Lemma 2.16, this is equivalent to the convergence

$$\Xi^{(n)}(C) \to \Xi(C) \tag{5.8}$$

for all relatively compact subsets of $[0, \infty) \times (0, \infty)$, where $\Xi(\partial C) = 0$. In this case, a relatively compact subset is a pair of intervals $[T_1, T_2] \times [d_1, d_2]$, with $T_1, T_2 \geq 0$ and $d_1, d_2 > 0$. The condition on the measure of the boundary means that the limit process must not have any excursions starting exactly at T_1 or T_2, or any excursions of length exactly d_1 or d_2. An exception is the case of $T_1 = 0$. Since the domain of f starts at 0 and the condition on the boundary is needed to prevent the case of points in $\Xi^{(n)}$ converging to some point on the boundary "from outside", we do not need to consider 0 as part of the boundary of an interval $[0, T_2] \times [d_1, d_2]$.

Proof. We begin by elaborating what the conditions (5.1) to (5.7) mean for the points in Ξ and $\Xi^{(n)}$.

To prove the Lemma, we fix some bounded subset of $[0, \infty) \times (0, \infty)$, $C := [T_1, T_2] \times [d_1, d_2]$, and show that, for sufficiently large n, $\Xi^{(n)}(C) = \Xi(C)$, that is, there are exactly as many excursions of f_n starting in (T_1, T_2), with length in (d_1, d_2), as similar excursions of the limit function f.

We will first show that every excursion of f is eventually matched by some excursion of f_n, then show that there can not be any more excursion of f_n of sufficient length.

Part 5.1.1 ($\Xi(C) \subseteq \Xi^{(n)}(C)$)**.** Let us first establish some facts about excursions of the limit function f. In the interval $[T_1, T_2]$, there can only be a finite number of excursion starting with length of at least d_1, at most T_1/d_1. We call excursions of length greater than d_1 large, all other excursions small. Let $\mathcal{E}^* := \{(l_i, r_i) \mid i = 1, \ldots, k\}$ be the set of these excursions. Consider a fix $(l, r) \in \mathcal{E}^*$. Since $l - r > d_1$, we can find an $\epsilon > 0$ such that the length of the interval $[l + \epsilon, r - \epsilon]$ is still greater than d_1.

We can also find an $\epsilon > 0$ sufficiently small, so that for every $x \in [0, l - \epsilon] \cup [l + \epsilon, r - \epsilon]$,

$$f(x) > f(l) + \delta \tag{5.9}$$

holds for some $\delta > 0$.

Assume this does not hold to the left of the ϵ-neighbourhood of l. Then there exists some $x \in [0, l - \epsilon]$ for which $f(x) \leq f(l) + \delta$ for every $\delta > 0$. This implies $f(x) \leq f(l)$, and by condition (5.1), $f(l) = \min_{u \leq l} f(u)$, so $f(x) = f(l)$. That leaves two possibilities: First, f is constant on the interval $[x, l]$. But this would be an interval of non-zero length without any excursions on it, a contradiction to (5.4). Second, there is an interval $[x', l']$, with $x \leq x' < l' \leq l$, such that $f(y) > f(x)$ for all $y \in (x', l')$. That makes (x', l') another excursion in $\mathcal{E}$, but $f(x') = f(l)$, which is a contradiction to condition (5.3).

Using the same logic, assuming (5.9) does not hold in $[l + \epsilon, r - \epsilon]$ leads to a point $x \in [l + \epsilon, r - \epsilon]$ such that $f(x) \leq f(l)$, which is a contradiction to condition (5.2).

Now consider the behaviour of f on $[r, r + \epsilon]$. As previously stated, condition (5.4) prevents f from being constant. If $f(x) \geq f(r)$ for all $x \in [r, r + \epsilon]$, there would be another excursion that contradicts (5.3). So there must exist an $r^* \in (r, r+\epsilon]$ with $f(r^*) < f(r)$. Let $\delta^* := \min\{\delta, f(r) - f(r^*)\}$, where δ is the constant used in the discussion of $[0, l - \epsilon]$ and $[l + \epsilon, r - \epsilon]$ above.

We now take a look at f_n. Fix an $x^* \in [l + \epsilon, r - \epsilon]$. We will now show that there exist points $l_n(x^*) \in [l - \epsilon, l + \epsilon]$ and $r_n(x^*) \in [r - \epsilon, r + \epsilon]$ for which condition (5.6) hold, making $(l_n(x^*), r_n(x^*) - l_n(x^*))$ a possible element of $\Xi^{(n)}$.

Define $l_n(x^*) := \min(\arg\min_{u \leq x} f_n(u))$. By the uniform convergence $f_n \to f$, we can find an $N \in \mathbb{N}$, such that $|f_n(x) - f(x)| < \delta^*/3$ for all $x \in [T_1, T_2]$ and $n \geq N$. So for every point $x \in [0, l - \epsilon] \cup [l + \epsilon, r - \epsilon]$,

$$f_n(x) > f(x) - \delta^*/3 > f(l) + \delta^* - \delta^*/3 > f(l) + \delta^*/3 > f_n(l). \tag{5.10}$$

So f_n takes its minimum over $[0, l - \epsilon] \cup [l + \epsilon, r - \epsilon]$ at l or somewhere in $[l - \epsilon, l + \epsilon]$. Therefore $l_n(x^*) \in [l - \epsilon, l + \epsilon]$ and, since we chose $l_n(x^*)$ as the smallest value for which f_n attains this minimum, $l_n(x^*) = \min_{u \leq l_n(x^*)} f_n(u)$.

Now define $r_n(x^*) := \inf\{x > x^* \mid f_n(x) = l_n(x^*)\}$. By (5.10), the function f_n can not reach its past minimum $l_n(x^*)$ before $r - \epsilon$. As previously discussed, there exists a $r^* \in (r, r + \epsilon]$ such that $f(r) - f(r^*) \geq \delta^*$. Now

$$l_n(x) > l(x) - \delta^*/3 = f(r) - \delta^*/3 \geq f(r^*) + \delta^* - \delta^*/3 > f_n(r^*), \tag{5.11}$$

which implies that $r_n(x^*)$ must be smaller than r^*, thus $r_n(x^*) \in [r - \epsilon, r + \epsilon]$. Since $l_n(x) = \min_{u \leq l_n(x^*)} f_n(u)$ and $r_n(x^*)$ is the first time this previous minimum is reached, $r_n(x^*) = \min_{u \leq l_n(x^*)} f_n(u)$.

We have shown that the only points satisfying condition (5.6) must lay near the beginning and end of excursions of f. This is not sufficient as proof of the Lemma, since the sequence $\mathcal{T}_n$ might not contain any points between two large excursions, or even not contain any points, thus skipping one or more eligible excursions. We will now show that (5.5) and (5.7) imply that any set satisfying these two conditions must contain at least one element in between two large excursions of f.

First of all, (5.5) ensures there must exist points in $\mathcal{T}_n$ and no last element of $\mathcal{T}_n$ exists.

Consider two consecutive large excursions of f, (l_1, r_1) and (l_2, r_2) and the space between r_1 and l_2. Suppose there is no element of $\mathcal{T}_n$ in $[r_1 - \epsilon, l_2 + \epsilon]$ for all $\epsilon > 0$. The latest element of $\mathcal{T}_n$ was located at or before l_1, the next will be at or around r_2. We know $f(r_1) = f(l_1) > f(l_2)$, so there is $\delta > 0$ such that for all large n, $|f_n(l^*) - f_n(r^*)| > \delta$ for all l^* and r^* in the sufficiently small ϵ-neighbourhoods around l_1 and r_2.

Any previous element of $\mathcal{T}_n$ will only yield a larger, any element after r_2 only a smaller f_n-value. This is a contradiction to (5.7). So there must be at least one point of $\mathcal{T}_n$ in between these two large excursions, no excursion can be skipped.

For any excursion (l, r) of f, there exists $l_n(x^*), r_n(x^*) \in \mathcal{T}_n$ in the respective ϵ-neighbourhoods, so the excursion of f is matched by an excursion of f_n of equal or greater length. Since $l - r + 2\epsilon > d_1$, we have found $t_{n,i} = l_n(x^*)$, $t_{n,i+1} = r_n(x^*)$, such that $(t_{n,i}, t_{n,i+1} - t_{n,i}) \in \Xi^{(n)} \cap C$ for all n greater than some $N_i = N \in \mathbb{N}$. Now let $N^* := \max\{N_1, \ldots, N_k\}$ and every excursion of f in C is matched by an excursion of f_n in C, with $n \geq N^*$. Therefore eventually $\Xi(C) \leq \Xi^{(n)}(C)$.

Part 5.1.2 ($\Xi^{(n)}(C) \subseteq \Xi(C)$)**.** Now that every excursion of f is matched, we need to show that there can not exist any additional large excursion of f_n. Considering the fact that excursions can not overlap, the only possibility for an additional large excursion is the space between two large excursions. For a pair of large excursions, (l_i, r_i) and (l_{i+1}, r_{i+1}), there is only enough space in between them if $r_i - l_{i+1} > d_1$. Consider such an interval $[r, l]$ of length greater than d_1. By condition (5.4) the space must be filled with smaller excursions of f. There is an at most countable number of these, let $\mathcal{E}^* := \{(l_i, r_i) \mid i \in \mathbb{N}\}$ be the set of such excursions starting in (r, l) with length less than or equal to d_1. We know

$$\sum_{i=1}^{\infty} r_i - l_i = l - r > d_1, \tag{5.12}$$

so we can choose a finite set of excursions $\{(l_i, r_i) \mid i = 1, \ldots, K\}$ such that, if we exclude these from the interval $[r, l]$, the space remaining is less than

d_1:

$$l - r - \sum_{i=1}^{K} r_i - l_i < d_1. \tag{5.13}$$

Let $d^* < \min\{r_i - l_i \mid i = 1, \ldots, K\}$ and apply the logic of Claim **??** to the compact set $[r, l] \times [d^*, d_1]$. For sufficiently large n, every one of these K excursions of f will be matched with an excursion of f_n, so that there will be no space left for a large excursion of f_n in $[r, l]$. Applying this logic to every one if the finitely many gaps between excursions of f, we see that there can not exist any more large excursions of f_n than those already matching f. There $\Xi^{(n)}(C) \leq \Xi(C)$ for sufficiently large n, which completes the proof.

□

The following Lemma will now link excursions of Z_n and W^t in the language of Lemma 5.1. We define a random point process containing the starts and lengths of excursions of B^t together with a sequence of random point processes describing the starts and sizes of components discovered by $\bar{Z}_n^t$. Let $\mathcal{C}_{n,i}$ be the size of the i-th component and define $\gamma_n(i) \in \{1, \ldots, n\}$ as the index for which $v(\gamma_n(i))$ is the last vertex of the $i-1$-th component encountered by the breadth-first walk Z_n. For this lemma, we need to extend the definitions of γ_n and $\mathcal{C}_{n,i}$ beyond the n-th vertex and the last component. Let i_n^* be the index of the last component and define $\gamma_n(i_n^* + 1) := \max_{1 \leq i \leq i_n^*} \gamma_n(i) + 1$ and consequently $\gamma_n(j) := \gamma_n(j-1) + 1$ for all $j > i_n^* + 1$. For component sizes, let $\mathcal{C}_{n,i} := 1$ for all $i > i^*$.

Lemma 5.2. *Let Ξ be the point process with points corresponding to excursions of B^t,*

$$\Xi := \{(l(\gamma), |\gamma|) \mid \gamma \textit{ excursion of } B^t\}. \tag{5.14}$$

Let $\Xi^{(n)}$ be the rescaled point process with points corresponding to excursions of the breadth-first walk

$$\Xi^{(n)} := \{(n^{-2/3}\gamma_n(i), n^{-2/3}\mathcal{C}_{n,i}) \mid i \geq 1\}. \tag{5.15}$$

Then $\Xi^{(n)} \to_v \Xi$ as $n \to \infty$.

Proof. First, let us give some remarks on the structure of the objects in use here and their convergence. Lemma 5.1 states that, under certain conditions, two deterministic point processes converge in the vague topology. This Lemma now introduces $\Xi^{(n)}$ and Ξ, which are random variables mapping into the space of point processes, again equipped with the vague topology.

Both $\Xi^{(n)}$ and Ξ are random in the sense that they depend on their underlying processes, $\bar{Z}_n^t$ and W^t respectively, so to clarify we define the sets

$$\begin{aligned} \Xi_{W^t} &:= \{(l(\gamma), |\gamma|) \mid \gamma \text{ excursion of } B^t\}, \\ \Xi_{Z_n^t}^{(n)} &:= \{(n^{-2/3}\gamma_n(i), n^{-2/3}\mathcal{C}_{n,i}) \mid i \geq 1\}, \\ \bar{\Xi}_{\bar{Z}_n^t}^{(n)} &:= \{(\bar{\gamma}_n(i), \bar{\mathcal{C}}_{n,i}) \mid i \geq 1\}, \end{aligned} \tag{5.16}$$

which are dependent on the specific realisation of the random processes W^t and Z_n^t, where $\bar{\gamma}_n$ and $\bar{\mathcal{C}}_{n,i}$ describe starting points and excursion lengths of $\bar{Z}_n^t$. Since $\bar{Z}_n^t(s) = n^{-1/3} Z_n^t(n^{2/3}s)$ we have $\Xi_{Z_n^t}^{(n)} = \bar{\Xi}_{\bar{Z}_n^t}^{(n)}$ and we can use $\Xi^{(n)}$ and $\bar{\Xi}^{(n)}$ interchangeably when appropriate.

Now (5.14) and (5.15) can be redefined as the random variables

$$\begin{aligned} \Xi &: \omega \mapsto \Xi_{W^t(\omega)}, \\ \Xi^{(n)} &: \omega \mapsto \Xi_{Z_n^t(\omega)}^{(n)}. \end{aligned} \tag{5.17}$$

We apply the reasoning used in the proof of Theorem 4.1 once more, and use the Skorohod representation theorem to construct random variables $\bar{\mathcal{Z}}_n^t$ and $\mathcal{W}^t$ on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, which converge almost surely. Almost sure convergence implies

$$\sup_{s \le s_0} |\bar{\mathcal{Z}}_n^t(s) - \mathcal{W}^t(s)| \to 0 \tag{5.18}$$

almost surely, so $\bar{\mathcal{Z}}_n^t(\omega') \to \mathcal{W}^t(\omega')$ uniformly for almost all $\omega' \in \Omega'$. Analogously we define

$$\mathcal{B}^t(s) := \mathcal{W}^t(s) - \min_{u \le s} \mathcal{W}^t(s). \tag{5.19}$$

On the same probability space, we now define $\Xi'^{(n)}$ and Ξ' as in (5.16) and (5.17). Since $\bar{Z}_n^t \sim \bar{\mathcal{Z}}_n^t$ and $W^t \sim \mathcal{W}^t$ we have $\Xi^{(n)} \sim \Xi'^{(n)}$ and $\Xi \sim \Xi'$.

By the definition of $\mathcal{B}^t$, Ξ' is the Ξ in Lemma 5.1 with $f = \mathcal{W}^t$. As f_n we take $\bar{Z}_n^t$ and define $t_{n,i} = n^{-2/3}\gamma(n,i)$, that is, the elements of $\mathcal{T}_n$ to be the end-points of components, rescaled to match $\bar{\mathcal{Z}}_n^t$. This way $\Xi'^{(n)}$ coincides with $\Xi^{(n)}$ in Lemma 5.1.

We still need to show that conditions (5.1) to (5.4) hold for $\mathcal{W}^t$ and conditions (5.5) to (5.7) hold for the breadth-first walk.

We start with the former. We first define the set $\mathcal{E}$ for the Brownian motion. Consider the set of positive rational numbers. For every $q \in \mathbb{Q}^+$, define

$$l(q) := \sup\{\arg\min_{s<q} \mathcal{W}^t(s)\} \tag{5.20}$$

and

$$r(q) := \inf\{s \in \mathbb{R} \mid \mathcal{W}^t(s) = l(q)\}. \tag{5.21}$$

Now every rational number belongs to one excursion $(l(q), r(q))$, while one excursion contains multiple rational numbers. We define the set of excursions

$$\mathcal{E} := \bigcup_{q \in \mathbb{Q}^+} \{(l(q), r(q))\} \tag{5.22}$$

and note that it is countable.

The following properties will be proven on a standard Brownian motion W, we later use Girsanovs theorem to apply them to $\mathcal{W}^t$.

By [6, Theorem 1.22, p.18], W is not monotonous on any interval $[a, b]$ with $0 < a < b < \infty$ almost surely.

Consider two excursions $(l_1, r_2), (l_2, r_2)$ with $l_1 < l_2$. If $W(l_2) = \min_{s \leq l_2} W(s) \geq f(l_1)$ then $W(s) \geq W(l_1)$ for all $l_1 < s < l_2$. Therefore the must exist $\epsilon > 0$ such that W is monotonously increasing on $[l_1, l_1 + \epsilon]$. For each $(l, r) \in \mathcal{E}$ this is not possible almost surely and since $\mathcal{E}$ is countable, condition (5.3) holds with probability 1.

The complement of all excursion is the set of intervals on which the Brownian motion is monotonously decreasing. By the same reasoning, almost surely there is no interval in between any two excursions on which W is monotonously decreasing, therefore condition (5.4) holds almost surely.

A standard Brownian motion W satisfies the conditions of Lemma 5.1 almost surely, therefore $\mathcal{W}^t$ does so almost surely under $\tilde{\mathbb{P}}$ and by Girsanovs theorem likewise under $\mathbb{P}$.

We now show that conditions (5.5) to (5.7) hold for the random walk Z_n and $t_{n,i} = n^{-2/3}\gamma_n(i)$.

The breadth-first walk Z_n^t is a discrete process, therefore for all $n \in \mathbb{N}$ and $i \geq 2$:

$$t_{n,i} - t_{n,i-1} = n^{-2/3}\gamma_n(i) - n^{-2/3}\gamma_n(i-1) \geq n^{-2/3} > 0. \tag{5.23}$$

By definition of $\gamma_n(i)$, $\lim_{i\to\infty} \gamma_n(i) = \infty$ for all n which establishes condition (5.5).

The breadth-first walk attains a new minimum at the end of every component, which ensures (5.6). The difference between the levels of Z_n^t at the end of two consecutive components is always 1, so

$$\max_{i:t_{n,i} \leq s_0} (\bar{\mathcal{Z}}_n^t(t_{n,i}) - \bar{\mathcal{Z}}_n^t(t_{n,i+1})) = n^{-1/3} \xrightarrow{n\to\infty} 0 \tag{5.24}$$

for all $s_0 > 0$.

For a given realisation of $\bar{\mathcal{Z}}_n^t(\omega')$ and $\mathcal{W}^t(\omega')$ the processes and sets defined meet all conditions of Lemma 5.1 and we can establish the convergence

$$\Xi'^{(n)}_{\bar{\mathcal{Z}}_n^t(\omega')} \to_v \Xi'_{\mathcal{W}^t(\omega')}. \tag{5.25}$$

This convergence holds almost surely, so

$$\Xi'^{(n)} \to_{a.s.} \Xi' \tag{5.26}$$

with regard to the vague topology. Since almost sure convergence implies convergence in distribution we have

$$\Xi'^{(n)} \to_d \Xi' \tag{5.27}$$

and therefore

$$\Xi^{(n)} \to_d \Xi \tag{5.28}$$

with regard to the vague topology, which completes the proof. □

It remains to be shown that we need not consider the problem of a large component wandering off to infinity. We prove this by analysing the behaviour of the breadth-first walk after a certain stopping time.

5.1 Late excursions of Z_n

The following Lemma proves that with high probability, all relevant excursions do in fact take place in a compact subset of $[0, \infty) \times (0, \infty)$.

Lemma 5.3. *Let $\mathbb{P}(\mathcal{E}_{\delta,C})$ be the probability that the breadth-first walk makes an excursion γ of length $|\gamma| > \delta n^{2/3}$, starting after step $Cn^{2/3}$. Then for all $\epsilon > 0$ and $\delta > 0$ exists $C > 0$ such that $\mathbb{P}(\mathcal{E}_{\delta,C}) < \epsilon$.*

Proof. By the law of total expectation,

$$\begin{aligned}\mathbb{P}(\mathcal{E}_{\delta,C}) &\leq \mathbb{E}[\text{Number of excursions } \gamma \text{ with } |\gamma| \geq Cn^{2/3} \text{ and } l(\gamma) \geq \delta n^{2/3}\ ] \\ &= \mathbb{E}[\sum_{\gamma:\, l(\gamma)\geq \delta n^{2/3}} \mathbb{1}_{\{|\gamma|\geq Cn^{2/3}\}}] \\ &\leq \mathbb{E}[\sum_{\gamma:\, l(\gamma)\geq \delta n^{2/3}} \frac{|\gamma|^2}{\delta^2 n^{4/3}} \mathbb{1}_{\{|\gamma|\geq Cn^{2/3}\}}] \\ &= \frac{1}{\delta^2 n^{4/3}} \mathbb{E}[\sum_{\gamma:\, l(\gamma)\geq \delta n^{2/3}} |\gamma|^2 \mathbb{1}_{\{|\gamma|\geq Cn^{2/3}\}}].\end{aligned} \tag{5.29}$$

Let T be the time the last excursion starting before $Cn^{2/3}$ ends. The behaviour of the breadth-first walk after T will be the same as the behaviour of a new walk on $\mathscr{G}(n-T,p)$. We write $\mathcal{C} \in \mathscr{G}$ to denote a component $\mathcal{C}$ contained in the random graph $\mathscr{G}$, and $|\mathcal{C}|$ for its size. Since the notions of excursions of the breadth-first walk and components in the underlying graph are interchangeable, we can rewrite (5.29) as

$$\begin{aligned}\mathbb{P}(\mathcal{E}_{\delta,C}) &\leq \frac{1}{\delta^2 n^{4/3}} \mathbb{E}[\sum_{\mathcal{C}\in\mathscr{G}(n-T,p)} |\mathcal{C}|^2 \mathbb{1}_{\{|\mathcal{C}|\geq \delta n^{2/3}\}}] \\ &\leq \frac{1}{\delta^2 n^{4/3}} \mathbb{E}[\sum_{\mathcal{C}\in\mathscr{G}(n-Cn^{2/3},p)} |\mathcal{C}|^2 \mathbb{1}_{\{|\mathcal{C}|\geq \delta n^{2/3}\}}] \\ &\leq \frac{1}{\delta^2 n^{4/3}} \mathbb{E}[\sum_{\mathcal{C}\in\mathscr{G}(n-Cn^{2/3},p)} |\mathcal{C}|^2].\end{aligned} \tag{5.30}$$

For ease of notation we consider the graph $\mathscr{G}(k,p)$ and calculate

$$\begin{aligned}
\mathbb{E}[\sum_{\mathcal{C}\in\mathscr{G}(k,p)} |\mathcal{C}|^2] &= \mathbb{E}[\sum_{\mathcal{C}\in\mathscr{G}(k,p)} |\mathcal{C}| \sum_{v\in\mathcal{C}} 1] \\
&= \mathbb{E}[\sum_{\mathcal{C}\in\mathscr{G}(k,p)} |\mathcal{C}| \sum_{v\in\mathscr{G}(k,p)} \mathbb{1}_{\{v\in\mathcal{C}\}}] \\
&= \sum_{v\in\mathscr{G}(k,p)} \mathbb{E}[\sum_{\mathcal{C}\in\mathscr{G}(k,p)} |\mathcal{C}|\mathbb{1}_{\{v\in\mathcal{C}\}}] \\
&= \sum_{v\in\mathscr{G}(k,p)} \mathbb{E}[|\mathcal{C}(v)|] \\
&= k\mathbb{E}[|\mathcal{C}(1)|],
\end{aligned} \tag{5.31}$$

where $\mathcal{C}(v)$ denotes the component containing the vertex v and the last inequality stems from the interchangeability of the vertex labels. We will bound the expectation of the size of this component from above by a suitable branching process $(Y_i,\ i \geq 0)$. Starting at time 0 with one vertex, we have $Y_0 = 1$. The number of children of this vertex is a $\mathrm{Bin}(k-1,p)$ distributed random variable, Y_1. In the next step, each child-vertex will itself have children, each Binomially distributed on the remaining set of vertices with probability p. We compute

$$\begin{aligned}
Y_{2,1} &\sim \mathrm{Bin}(k-1-Y_1,p), \\
Y_{2,2} &\sim \mathrm{Bin}(k-1-Y_1-Y_{2,1},p), \\
&\dots \\
Y_{2,Y_1} &\sim \mathrm{Bin}(k-1-Y_1-Y_{2,1}-\dots-Y_{2,Y_1-1},p) \\
Y_2 &= \sum_{i=1}^{Y_1} Y_{2,i}.
\end{aligned} \tag{5.32}$$

The size of the component will then be the total amount of explored vertices, which is the sum of all $Y_j,\ j \geq 0$. To provide an upper bound we consider the branching process where each amount of children is $\mathrm{Bin}(k,p)$ distributed. Define the process as follows,

$$\begin{aligned}
Z_0 &:= 1, \\
Z_j &:= \sum_{i=1}^{Z_{j-1}} Z_{j,i},
\end{aligned} \tag{5.33}$$

where $Z_{j,i} \sim \mathrm{Bin}(k,p)$ for all $i,j \geq 1$. Then the process $(Y_i,\ i \geq 0)$ is stochastically dominated by $(Z_i,\ i \geq 0)$ and

$$|\mathcal{C}(1)| \leq_{\text{st.}} Z_0 + Z_1 + Z_2 + \dots \tag{5.34}$$

which gives

$$\mathbb{E}[|\mathcal{C}(1)|] \leq \sum_{j=0}^{\infty} \mathbb{E}[Z_j]. \tag{5.35}$$

For $j \geq 0$ we calculate the expectation of Z_j by

$$\begin{aligned} \mathbb{E}[Z_j] &= \mathbb{E}[Z_{j-1}]kp \\ &\dots \\ &= \mathbb{E}[Z_0](kp)^j \\ &= (kp)^j. \end{aligned} \tag{5.36}$$

Substituting in (5.35) gives

$$\begin{aligned} \mathbb{E}[|\mathcal{C}(1)|] &\leq \sum_{j=0}^{\infty} (kp)^j \\ &= \frac{1}{1-kp}. \end{aligned} \tag{5.37}$$

We continue the calculation in (5.30) using (5.31) and (5.37), which yields

$$\begin{aligned} \mathbb{P}(\mathcal{E}_{\delta,C}) &\leq \frac{n - Cn^{2/3}}{\delta^2 n^{4/3}} \frac{1}{1-(n-Cn^{2/3})p} \\ &= \frac{n - Cn^{2/3}}{\delta^2 n^{4/3}} \frac{1}{1-(n-Cn^{2/3})(n^{-1}+tn^{-4/3})} \\ &= \frac{n - Cn^{2/3}}{\delta^2 n^{4/3}} \frac{n^{1/3}}{C-t+Ctn^{-1/3}} \\ &\leq \frac{n}{\delta^2 n^{4/3}} \frac{n^{1/3}}{C-t+Ctn^{-1/3}} \\ &= \delta^{-2} \frac{1}{C-t+Ctn^{-1/3}}. \end{aligned} \tag{5.38}$$

For fixed C, δ and t this expression converges asymptotically to $\delta^{-2}\frac{1}{C-t}$ as $n \to \infty$. Therefore for all $\epsilon > 0$ we can choose $C > 0$ such that

$$\mathbb{P}(\mathcal{E}_{\delta,C}) \leq \delta^{-2} \frac{1}{C-t+Ctn^{-1/3}} < \epsilon. \tag{5.39}$$

This completes the proof. $\square$

We have shown that with high probability there is no mass of large excursions wandering off to infinity. For all $\epsilon > 0$ we can find a $C > 0$ such that, with probability $1-\epsilon$, all excursions of size equal to or larger than $\delta n^{2/3}$ will start before step $Cn^{2/3}$. For all these excursions we can apply Lemma 5.2 to prove that $\Xi^{(n)} \to_d \Xi$. With these considerations we finish the proof of Theorem 1.7.

5.2 Graph-theory approach

In this section, we present an alternative proof of Lemma 5.3, which relies not on stochastic calculus but uses theory on random graphs.

Lemma 5.4. *Let $p(n, y, \delta)$ be the chance that $\mathscr{G}(n, n^{-1} + tn^{-4/3})$ contains a component of size greater than or equal to $\delta n^{2/3}$ which does not contain any vertex i with $1 \leq i \leq yn^{1/3}$.*

Then

$$\lim_{y\to\infty} \limsup_n p(n, y, \delta) = 0 \tag{5.40}$$

for all $\delta > 0$.

Proof. For this proof we have to change our understanding of the random graph process $\mathscr{G}(n, n^{-1}+tn^{-4/3})$ a little. Until now, we constructed a random graph by labeling all vertices $\{1, \ldots, n\}$ and then drawing Bernoulli random variables to construct edges and consequently components. Now we switch the order of actions. Start with an unlabeled empty graph with n vertices. Draw $\mathrm{B}(p)$ random variables to establish edges between vertices. Then, one by one, assign the labels 1 to n to the vertices, where the probabilities are equal for all remaining vertices. Since the construction of edges is independent of vertex labels and edge are constructed independently of each other, these two approaches are equivalent construction of the random graph.

Let $\mathscr{F}_E$ be the sigma-algebra that includes the creation of edges and components, but not the labeling of the vertices. For a component $\mathcal{C}$ of size $\alpha n^{2/3}$, let $v(\mathcal{C})$ be the label of its minimal vertex and write $\chi_n^\alpha = n^{-1/3}v(\mathcal{C})$. We show that $\chi_n^\alpha \to_d \mathrm{Exp}(\alpha)$.

Fix $\delta > 0$. Let $q(n, I)$ be the expected number of components of size greater than or equal to $\delta n^{2/3}$ with its minimal vertex label in $n^{1/3}I$. Then

$$\begin{aligned} q(n, [y, \infty)) &= \mathbb{E}\left[\mathbb{E}\left[\sum_{\mathcal{C}\colon |\mathcal{C}|\geq \delta n^{2/3}} \mathbb{1}_{\{v(\mathcal{C})>yn^{1/3}\}} \mid \mathscr{F}_E\right]\right] \\ &= \mathbb{E}\left[\sum_{\mathcal{C}\colon |\mathcal{C}|\geq \delta n^{2/3}} \mathbb{P}(v(\mathcal{C}) > yn^{1/3} \mid \mathscr{F}_E)\right] \\ &= \mathbb{E}\left[\sum_{\mathcal{C}\colon |\mathcal{C}|\geq \delta n^{2/3}} \mathbb{P}(\chi_n^\alpha > y \mid \mathscr{F}_E, |\mathcal{C}| = \alpha n^{2/3})\right] \end{aligned} \tag{5.41}$$

Since the distribution of χ_n^α converges to the exponential distribution, $\mathbb{P}(\chi_n^\alpha > y) \leq e^{-\alpha y}(1 + o(1))$, where the last factor describes an error from the convergence. Additionally, $\mathbb{P}(\chi_n^\alpha \leq 1) \leq (1 - e^{-\alpha})(1 + o(1))$, so

$$\mathbb{P}(\chi_n^\alpha > y) \leq e^{-\alpha y}(1 + o(1)) = \frac{e^{-\alpha y}}{1 - e^{-\alpha}}\mathbb{P}(\chi_n^\alpha \leq 1)(1 + o(1)). \tag{5.42}$$

Using this inequality in (5.41) leads to

$$\begin{aligned} q(n,[y,\infty)) &\le \mathbb{E}\left[\sum_{\mathcal{C}\colon |\mathcal{C}|\ge\delta n^{2/3}} \frac{e^{-|\mathcal{C}|n^{-2/3}y}}{1-e^{-|\mathcal{C}|n^{-2/3}}}\mathbb{P}(\chi_n^\alpha \le 1)(1+o\,(1))\right] \\ &\le \sup_{\alpha\ge\delta}\frac{e^{-\alpha y}}{1-e^{-\alpha}}(1+o\,(1))\mathbb{E}\left[\sum_{\mathcal{C}\colon |\mathcal{C}|\ge\delta n^{2/3}} \mathbb{P}(\chi_n^\alpha \le 1 \mid \mathscr{F}_E)\right] \\ &= \frac{e^{-\delta y}}{1-e^{-\delta}}(1+o\,(1))q(n,[0,1]), \end{aligned} \tag{5.43}$$

Dividing by $q(n,[0,1])$ gives us

$$\limsup_n \frac{q(n,[y,\infty])}{q(n,[0,1])} \le \frac{e^{-\delta y}}{1-e^{-\delta}}(1+o\,(1)) \xrightarrow{y\to\infty} 0. \tag{5.44}$$

For this convergence to hold, either $q(n,[0,1])\to\infty$ or $q(n,[y,\infty])\to 0$ as $y\to\infty$. Since the law of total expectation implies $p(n,y,\delta)\le q(n,[y,\infty])$ we only need to prove that

$$\sup_n q(n,[0,1]) < \infty. \tag{5.45}$$

Consulting existing literature on random graphs gives us $\sup_n q(n,[0,\infty]) < \infty$. Because $q(n,I_1)\le q(n,I_2)$ for $I_1\subset I_2$, this concludes the proof.

$\square$

6 Outlook

6.1 The non-uniform case

6.2 The multiplicative coalescence

6.3 Further results

Bibliography

[1] R. van der Hofstad, *Random Graphs and Complex Networks*, vol. 1 of *Cambridge series in statistical and probabilistic mathematics*. Cambridge University Press, 2016.

[2] B. Bollobás, *Random graphs*. Cambridge studies in advanced mathematics, Cambridge, UK: Cambridge Univ. Press, 2. ed., 2001.

[3] S. N. Ethier, *Markov Processes: Characterization and Convergence*. Wiley series in probability and mathematical statistics, Hoboken, NJ: Wiley-Interscience, 2005.

[4] D. Lamberton, *Introduction to Stochastic Calculus Applied to Finance*. Boca Raton, Fla.: Chapman & Hall/CRC, 2000.

[5] M. Haenggi, *Stochastic Geometry for Wireless Networks*. Cambridge, UK: Cambridge University Press, 2013.

[6] P. Mörters and Y. Peres, *Brownian Motion*. Cambridge, UK: Cambridge University Press, 2010.

[7] D. Aldous, "Brownian excursions, critical random graphs and the multiplicative coalescent," pp. 812–854, 1997.

[8] D. Aldous and V. Limic, "The entrance boundary of the multiplicative coalescent," 1998.

[9] P. Billingsley, *Convergence of probability measures*. Wiley series in probability and statistics, New York: Wiley, 2. ed., 1999.

[10] B. Bollobas, R. Kozma, and D. Miklós, *Handbook of Large-Scale Random Networks*, vol. 18 of *BOLYAI SOCIETY MATHEMATICAL STUDIES*. New York, NY: Springer, 2008.

[11] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods*. Probability and its applications, New York, NY: Springer, 2. ed., 2003.

[12] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes: Volume II: General Theory and Structure*. Probability and its applications, New York, NY: Springer, 2. ed., 2008.

[13] R. Durrett, *Stochastic Calculus: A Practical Introduction*. Probability and stochastics series, Boca Raton: CRC Press, 1996.

[14] R. Durrett, *Random graph dynamics.* Cambridge series in statistical and probabilistic mathematics, Cambridge, UK: Cambridge Univ. Press, 2007.

[15] R. Durrett, *Probability: Theory and Examples.* Cambridge, UK: Cambridge University Press, 4. ed., 2010.

[16] P. Erdős and A. Rényi, "On the evolution of random graphs," *PUBLICATION OF THE MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES*, vol. 5, pp. 17–61, 1960.

[17] J. Jacod, *Limit Theorems for Stochastic Processes.* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Berlin: Springer, 1987.

[18] S. Janson, T. Luczak, and A. Rucinski, *Random Graphs.* New York, NY: Wiley, 2000.

[19] O. Kallenberg, *Random measures.* Berlin: Akademie-Verl., 3. ed., 1983.

[20] J. F. C. Kingman, *Poisson Processes.* Oxford studies in probability, Oxford, UK: Oxford University Press, 1993.

[21] A. Klenke, *Wahrscheinlichkeitstheorie.* Berlin: Springer, 2006.

[22] T. Łuczak, B. Pittel, and J. C. Wierman, "The structure of a random graph at the point of the phase transition," *Trans. Amer. Math. Soc.*, vol. 341, pp. 721–748, 1994.

[23] A. Martin-Löf, "The final size of a nearly critical epidemic, and the first passage time of a wiener," *Journal of Applied Probability*, vol. 35, no. 3, 1998.

[24] T. Mikosch, *Non-Life Insurance Mathematics: An Introduction with Stochastic Processes.* Berlin: Springer, corrected second printing ed., 2006.

[25] P. E. Protter, *Stochastic Integration and Differential Equations.* Applications of mathematics, Berlin: Springer, 2. ed., 2004.

[26] S. I. Resnick, *Extreme Values, Regular Variation, and Point Processes.* Applied probability, New York: Springer, 2008.

[27] S. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models.* Berlin: Springer, 2004.

List of Figures

List of Symbols

$\mathbb{N}$	Natural numbers, excluding 0
$\mathbb{N}_0$	Natural numbers, including 0
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{R}_{\geq 0}$	Positive real numbers
$\mathscr{G}(n,p)$	Erdös-Renyi random graph on n vertices with edge-probability p
$\mathscr{G}(n,m)$	Erdös-Renyi random graph on n vertices with m edges
p_n	Edge-probability in $\mathscr{G}(n, n^{-1} + tn^{-4/3})$
$z(i)$	Breadth-first walk in discrete time
$Z_n^t(s)$	Breadth-first walk in continuous time
$\bar{Z}_n^t$	Rescaled breadth-first random walk
W	Standard Brownian motion
W^t	Brownian motion with drift
B^t	Reflected Brownian motion with drift
$U[a,b]$	Uniform distribution on $[a,b]$
$\mathrm{Bin}(n,p)$	Binomial distribution with parameters n, p
$\mathrm{B}(p)$	Bernoulli distribution with parameter p
$\mathrm{Exp}(\lambda)$	Exponential distribution with parameter λ
$\mathrm{Poi}(\lambda)$	Poisson distribution with parameter λ
$\mathbb{1}_x$	Indicator function of x
$\mathscr{F}$	σ-algebra
$\mathscr{F}_t$	Natural filtration generated by $(Z_n^t(s), s \leq t)$
$\mathcal{E}_X$	Event, defined by X