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On the scaling limit of component sizes in the critical Erdős-Rényi random graph

Diplomarbeit

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Abstract

Based on a 1997 paper by David Aldous we analyse component sizes in the critical Erdős-Rényi random graph, $\mathscr{G}(n, p_n)$ with edge probability $p_n = n^{-1} + \theta n^{-4/3}$ for $n \in \mathbb{N}$ and some $\theta \in \mathbb{R}$.

In general, component sizes will be of order $n^{2/3}$. As $n \to \infty$, the ordered component sizes and corresponding counts of surplus edges (both rescaled by $n^{-2/3}$) converge in distribution to a limit.

Let $W(t)$ be a standard Brownian motion. Attach a downward drift of $\theta - t$ by $W^\theta(t) = W(t) + \int_0^t (\theta - s) ds$ and reflect this process at the x-axis by $B^\theta(t) = W^\theta(t) - \min_{s \leq t} W^\theta(s)$. The limit of component sizes and surplus edges is distributed as the ordered lengths of excursions of B^θ above zero and marks of a Poisson point process with rate B^θ.

While parts of the original paper rely on deep results on random graphs about bounds on the number of tree components, unicyclic components and complex components, we provide a self-contained proof that substantially simplifies the argument.

Danksagung

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Ein besonderer Dank gilt meinen Eltern für ihre stetige Unterstützung vor und während meines Studiums.

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List of Symbols

	Graph notation
$\mathscr{G}(n,p)$	Erdős-Rényi random graph on n vertices with edge-probability p, 2
p_n	Edge-probability in $\mathscr{G}(n, n^{-1} + \theta n^{-4/3})$, 4
$\mathcal{N}_i$	Neighbours of vertex v_i, 18
$c(i)$	Children of vertex v_i, 18
$\zeta(j)$	Index of last vertex of j-th component, 20
$\zeta^{-1}(i)$	Index of component containing i-th vertex, 20
$\eta_n(s)$	Number of vertices ineligible to connect to vertex $v_{\lceil s \rceil}$ at time s, 25
	Processes
W	Standard Brownian motion, 5
W^θ	Brownian motion with drift, 5
B^θ	Reflected Brownian motion with drift, 5
N^θ	Counting process with rate B^θ, 6
$z(i)$	Breadth-first walk in discrete time, 18
$Z(s)$	Breadth-first walk in continuous time, 21
Z_n^θ	Breadth-first walk on the critical random graph, 22
$\bar{Z}_n^\theta$	Rescaled breadth-first random walk, 22
B_n^θ	Reflected breadth-first random walk, 36
N_n^θ	Surplus edge counting process, 35
$\bar{\mathcal{Z}}_n^\theta, \mathcal{W}^\theta, \mathcal{B}^\theta, \dots$	Processes on common probability space, 9
$\mathcal{M}_n^\theta$	Discrete version of $\mathcal{N}^\theta$, 44
	Special notation used in proofs
γ	Excursion of B^θ, 6
$l(\gamma), r(\gamma)$	Left and right ends of excursion, 6
$l(q), r(q)$	Left and right ends of excursion around $q \in \mathbb{Q}$, 55
$\mu(\gamma)$	Number of marks during excursion γ, 6
$\bar{\gamma}_n$	Excursion of $\bar{Z}_n^\theta$, 60
σ_n	Number of surplus edges during excursion, 7
$\bar{\sigma}$	Rescaled number of surplus edges during excursion, 62
Y	Ordered set of excursion lengths and mark counts of B^θ, 62
Y_n	Ordered set of excursion lengths and mark counts of $\bar{Z}_n^\theta$, 62
$Y^{(k)}$	First k entries of Y, 62
$Y_n^{(k)}$	First k entries of Y_n, 62
$\Xi^{(n)}$	Two-dimensional point process for sequence function, 50
Ξ	Two-dimensional point process for limit function, 50
E	Set of excursions of continuous function f, 50

E^*	Finite set of large excursions in interval, 50
$\gamma_n(i)$	Last vertex of the $i-1$-th component, 54

General notation

$l^2_\searrow$	Space of square-summable infinite decreasing sequences in $\mathbb{R}_+$, 6
d_2	Distance on $l^2_\searrow$, 6
$D[0,T]$	Space of càdlàg functions on $[0,T]$, 11
$\lambda(t)$	Rate of counting process, 14
$\Delta X(s)$	Jump of X at time s, 27
$\|\|X\|\|_T$	Supremum norm of X on $[0,T]$, 12
w'_X	Modulus of continuity of X, 12
$\leq_{s.t.}$	Stochastic domination, 40
$\mathcal{E}$	Event, 40
$\neg\mathcal{E}$	Complementary event to $\mathcal{E}$, 40
$\mathbb{1}$	Indicator function, 21

Convergence

$x_n \to x$	Pointwise convergence in $\mathbb{R}^d$, 5
$f_n \to f$	Convergence of functions w.r.t. supremum norm, 50
$\mu_n \Rightarrow \mu$	Weak convergence of measures, 9
$N_n \to_v N$	Vague convergence of counting processes, 14
$X_n \to_p X$	Convergence in probability, 3
$X_n \to_d X$	Convergence in distribution, 9
$X_n \to_{a.s.} X$	Almost sure convergence, 9

1 Introduction

The study of randomly generated graphs started in 1959, when Paul Erdős and Alfréd Rényis paper "On random graphs" [1], and Edgar Gilberts paper "Random graphs" [2], introduced what is now known as the *Erdős-Rényi model* of random graphs.

Starting with the original papers, the examination of sizes of connected components in such graphs was of great interest. David Aldous 1997 paper "Brownian excursions, critical random graphs and the multiplicative coalescent" [3], which provides the basis for this thesis, examines component sizes in the so-called critical Erdős-Rényi random graph as the number of vertices n grows to infinity.

This thesis will provide a strict proof of one of the main results of this paper. If no other reference is given, each result (or an equivalent thereof) can be found in Chapters 1 and 2 of [3].

1.1 The Erdős-Rényi random graph

In their paper, Erdős and Rényi proposed a model of random graph that is now known as the Erdős-Rényi random graph. There are two distinct but very similar models of this name, which we both briefly introduce. We start with the model used in this first paper and continue with the model which provides the basis for Aldous' paper and consequently this thesis.

We denote by $G_{n,M}$ the set of graphs on n vertices with M edges that are

1. undirected,
2. without slings, i.e. there is no edge (v, v) for some vertex v,
3. without parallel edges, i.e. there can be at most one edge (v_1, v_2) for each pair of vertices.

A graph in $G_{n,M}$ can be constructed by choosing M out of the $\binom{n}{2}$ possible edges between the vertices, which leads to a total number of graphs

$$|G_{n,M}| = \binom{\binom{n}{2}}{M}.$$

The Erdős-Rényi random graph $\mathscr{G}(n, M)$ is now obtained by choosing one element of $G_{n,M}$ at random with equal probability for each graph, where the number of edges is usually dependent on the number of vertices, $M = M(n)$.

An equivalent definition provides the following process: Starting with a graph on n vertices with 0 edges at time $t = 1$, pick one of the $\binom{n}{2}$ possible edges at random with equal probability for each edge, label it e_1. At time $t = 2$ pick one of the remaining $\binom{n}{2} - 1$

edges, again all remaining edges being equiprobable, and denote it by e_2. Continue until M edges have been chosen at time $t = M$. The graph (V, E) on vertices $V = \{1, \dots, n\}$ with edges $E = \{e_1, \dots, e_M\}$ is the desired Erdős-Rényi random graph.

Using these definitions Erdős and Rényi prove that, when increasing the number of edges $M(n)$, the sizes of connected components undergo distinct phases leading from small sparse components and unconnected vertices to one giant component and eventually a completely connected graph.

Most current literature and the paper we are studying work with a slightly different, closely related model introduced by Gilbert. Given a set of vertices $\{1, \dots, n\}$ and an edge-probability $p_n = p(n)$ we take every edge (v_1, v_2) and add it to the set of edges E of the random graph with probability p_n independently. We call an edge $e = (v_1, v_2)$ open if $e \in E$, closed otherwise. We denote a random object constructed this way by $\mathscr{G}(n, p_n)$ and expect a realisation of the random graph to have $\binom{n}{2}p_n$ edges. Note that both $\mathscr{G}(n, M(n))$ and $\mathscr{G}(n, p)$ are called the Erdős-Rényi random graph and for $M(n)$ close to $\binom{n}{2}p_n$ the models are practically interchangeable in most analyses, see [4, p.38].

When increasing the probability p_n from 0 to 1 the resulting random graph undergoes the same so-called phase transitions discovered by Erdős and Rényi. These phase transitions include a sudden dramatic increase in average component size happening around $M(n) = \frac{n}{2}$ or $p_n = \frac{1}{n}$ which we study in this thesis. The following section will provide a more precise introduction to the different phases a random graph undergoes with increasing number of edges.

1.2 General results on component sizes

To provide an overview of the aforementioned phases which the component sizes of the Erdős-Rényi random graph pass through, we discuss relevant results that were established previous to [3].

We begin with the subcritical graph, that is $\mathscr{G}(n, p_n)$ with $p_n < \frac{1}{n}$ or $\lambda := np_n < 1$. The following theorem provides a lower bound on the size of $\mathcal{C}_{\max}$, the largest component of $\mathscr{G}$.

Theorem 1.1 (Lower bound on largest subcritical component, [5, Theorem 4.4, p.125])**.** *Fix $\lambda < 1$. Then, for all $a > 1/I_\lambda$, there exists $\delta = \delta(a, \lambda)$ such that*

$$\mathbb{P}(|\mathcal{C}_{\max}| \geq a \log n) = O\left(n^{-\delta}\right),$$

where

$$I_\lambda = \lambda - 1 - \log(\lambda)$$

is the large deviation rate function for Poisson random variables with mean λ.

The next theorem gives an upper bound for the size of $\mathcal{C}_{\max}$.

Theorem 1.2 (Upper bound on largest subcritical component, [5, Theorem 4.5, p.125])**.** *Fix $\lambda < 1$. Then, for all $a < 1/I_\lambda$, there exists $\delta = \delta(a, \lambda)$ such that*

$$\mathbb{P}(|\mathcal{C}_{\max}| \leq a \log n) = O\left(n^{-\delta}\right).$$

Together, these theorems imply

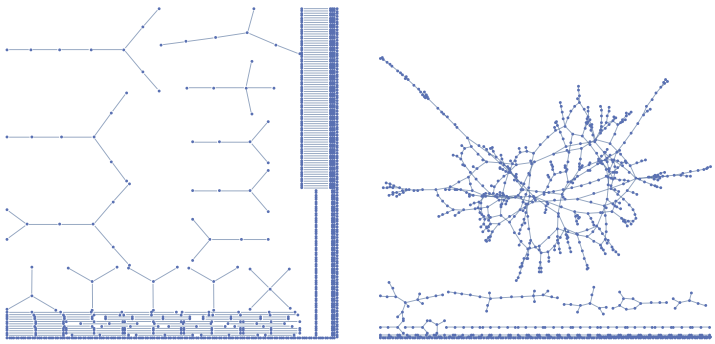
$$\frac{|\mathcal{C}_{\max}|}{\log n} \to_p 1/I_\lambda$$

and therefore the expected largest component size will be of order $\log n$.

Consider the opposite case, the supercritical graph with $\lambda > 1$. Denote by ξ_λ the survival probability of a Poisson branching process with mean offspring λ. Note that $\xi_\lambda > 0$ since $\lambda > 1$ (For an introduction into the theory of branching processes with application in random graphs, see [5, Chapter 3, p.87ff.]). Any vertex is part of a large component with probability ξ_λ, therefore we will expect around $n\xi_\lambda$ vertices being part of large connected components. The following theorem now states that all of these vertices are, in fact, part of the same connected component, which we call the giant component.

Theorem 1.3 (Law of large numbers for giant component, [5, Theorem 4.8, p.131])**.** *Fix $\lambda > 1$. Then, for all $\nu \in (\frac{1}{2}, 1)$, there exists $\delta = \delta(\nu, \lambda)$ such that*

$$\mathbb{P}\left(\left||\mathcal{C}_{\max}| - n\xi_\lambda\right| \geq n^\nu\right) = O\left(n^{-\delta}\right).$$



(a) $\lambda = 0.5$ (b) $\lambda = 1.5$

Figure 1.1: Realizations of the Erdős-Rényi random graph on $n = 1000$ vertices with varying $\lambda = np_n$.

Figure 1.1 provides two realizations of the random graph with different λ, one subcritical and one supercritical. Note how the largest connected components in the subcritical graph still consist of very few vertices (the largest component here having 9 vertices, while $\log(1000) \approx 6.9$) and how a lot of nodes are isolated. In contrast, the supercritical graph features a single large component and all other components are either drastically smaller in size or even still isolated vertices.

For $np_n < 1$ we expect many small clusters of order at most $\log n$, for $np_n > 1$ we expect one giant component, approaching size n with increasing p_n. But what happens around $np_n \approx 1$? As it turns out, the emergence of the giant component occurs quite rapidly, such that shortly after $np_n = 1$ most graphs do not have any component of order between $\frac{1}{2}n^{2/3}$ and $n^{2/3}$.

The following theorem provides an approximation of the time of emergence of the giant component, seeing the random graph on n vertices as a graph process $(\mathscr{G}_t)_0^n$, starting at $t = 0$ with 0 edges, adding one random edge at every time step. We would therefore expect the emergence starting around time $\binom{2}{n}\frac{1}{n} \approx \frac{1}{2}n$. We say a property P is shared by almost every graph if the probability of having this property approaches 1 as $n \to \infty$.

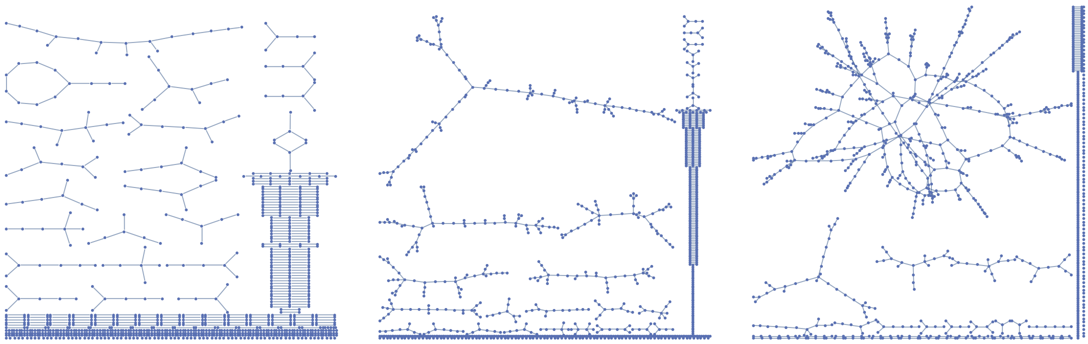
Theorem 1.4 (Emergence of the giant component, [4, Theorem 6.8, p.142])**.** *Almost every graph process $\mathscr{G} = (\mathscr{G}_t)_0^n$ is such that for every $t \geq t_1 = \lfloor n/2 + 2(\log n)^{1/2} n^{2/3} \rfloor$ the graph $\mathscr{G}_t$ has a unique component of order at least $n^{2/3}$ and the other components have at most $\frac{1}{2}n^{2/3}$.*

Subsequently, there is a so-called critical window in which the maximum component sizes are not of order $\log n$ any more but there is no single giant component yet. We call a random graph $\mathscr{G}(n, n^{-1} + \theta n^{-4/3})$ critical for $\theta \in \mathbb{R}$. The next theorem provides an approximation of the size of the largest component in a critical random graph.

Theorem 1.5 (Largest critical cluster, [5, Theorem 5.1, p.150])**.** *Let $\lambda = 1 + \theta n^{-1/3}$, with $\theta \in \mathbb{R}$. There exists a constant $b = b(\theta)$ such that for all $\omega > 1$,*

$$\mathbb{P}\left(\omega^{-1} n^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega n^{2/3}\right) \geq 1 - \frac{b}{\omega}.$$

In this critical window, the largest component of the random graph will be of order $n^{2/3}$ with high probability.



(a) $\theta = -3$ (b) $\theta = 0$ (c) $\theta = 3$

Figure 1.2: Realizations of the Erdős-Rényi random graph on $n = 1000$ vertices with varying θ.

Figure 1.2 shows realizations of the critical Erdős-Rényi random graph for different parameters θ. It is evident how the graph undergoes its transition from relatively small and simple components to the emergence of a single great component, which will later, i.e. for larger p_n, encompass more and more vertices as seen in Figure 1.1.

But is it possible to provide a similar statement not only for the largest, but for all components in a critical random graph? Aldous notes that previous to his paper the convergence of the rescaled component sizes to some limit process was generally assumed to be true, although never explicitly proven. He therefore provides the following folk theorem, which will be proven along with a more precise version in the course of this thesis.

For a connected component we define the number of surplus edges or the surplus by

$$\text{surplus} := \text{number of edges} - \text{number of vertices} + 1. \tag{1.1}$$

The surplus gives the maximum number of edges we can remove from the component so that it stays connected. A component with a surplus of zero is a tree.

Folk Theorem 1.6. *Let $\mathcal{C}_n^\theta(1) \geq \mathcal{C}_n^\theta(2) \geq \dots$ be the ordered component sizes of $\mathscr{G}(n, n^{-1} + \theta n^{-4/3})$ and let $\sigma_n^\theta(j)$ be the surplus of the corresponding component. Then*

$$(n^{-2/3}(\mathcal{C}_n^\theta(j), \sigma_n^\theta(j)),\ j \geq 1) \to_d ((\mathcal{C}^\theta(j), \sigma^\theta(j)),\ j \geq 1) = (\boldsymbol{\mathcal{C}}^\theta, \boldsymbol{\sigma}^\theta), \tag{1.2}$$

as $n \to \infty$ for some limit $(\boldsymbol{\mathcal{C}}^\theta, \boldsymbol{\sigma}^\theta)$ with $0 < \mathcal{C}^\theta(j) < \infty$ and $0 \leq \sigma^\theta(j) < \infty$ almost surely for each $j \geq 1$.

The convergence is to hold with respect to the product topology, where $\boldsymbol{x}^{(n)} \to \boldsymbol{x}$ holds for $\boldsymbol{x}^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots)$ and $\boldsymbol{x} = (x_1, x_2, \dots)$ if for all $k \in \mathbb{N}$

$$(x_1^{(n)}, \dots, x_k^{(n)}) \to (x_1, \dots, x_k)$$

by $x_i^{(n)} \to x_i$ for all $1 \leq i \leq k$ as $n \to \infty$.

Note that in our theorem the $x_i^{(n)} = (n^{-2/3}\mathcal{C}_n^\theta(i), n^{-2/3}\sigma_n^\theta(i))$ and $x_i = (\mathcal{C}^\theta(i), \sigma^\theta(i))$ are pairs of values themselves, so for $x_i^{(n)} \to x_i$ to hold we require $n^{-2/3}\mathcal{C}_n^\theta(i) \to \mathcal{C}^\theta(i)$ and $n^{-2/3}\sigma_n^\theta(i) \to \sigma^\theta(i)$ in $\mathbb{R}$ as $n \to \infty$.

1.3 Main statements of this thesis

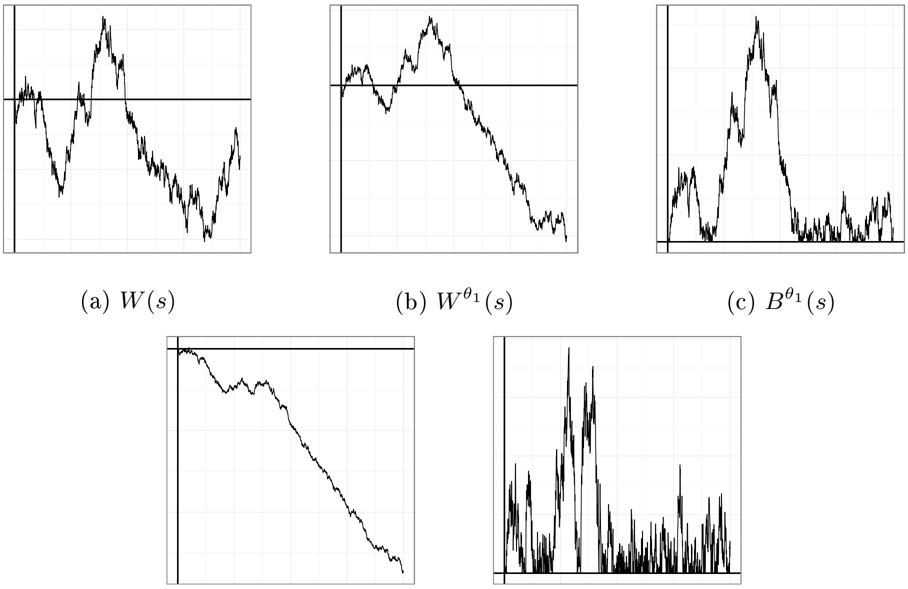
In this section we state the main results of this thesis, which is a refinement of Folk Theorem 1.6. We now identify the distribution of the limit vector precisely and state a stricter form of convergence of component sizes.

Denote by W the standard Brownian motion on $\mathbb{R}_+$. For a fixed parameter $\theta \in \mathbb{R}$ we call the process W^θ, defined by

$$W^\theta(s) := W(s) + \int_0^s (\theta - u) du = W(s) + \theta s - \frac{1}{2}s^2, \tag{1.3}$$

the Brownian motion with drift $\theta - s$ at time s. The central object of our analysis will be the process W^θ and its excursions above past minima. We reflect W^θ at the x-axis to define the process B^θ by

$$B^\theta(s) := W^\theta(s) - \min_{u \leq s} W^\theta(u). \tag{1.4}$$



(a) $W(s)$ (b) $W^{\theta_1}(s)$ (c) $B^{\theta_1}(s)$

(d) $W^{\theta_2}(s)$ (e) $B^{\theta_2}(s)$

Figure 1.3: A sample Brownian motion, Brownian motion with drift and with reflection for $\theta_1 > 0$ and $\theta_2 < 0$.

We call B^θ the reflected Brownian motion with drift and call an excursion γ of B^θ a time interval $[l(\gamma), r(\gamma)]$ for which $B^\theta(l(\gamma)) = B^\theta(r(\gamma)) = 0$ and $B^\theta(s) > 0$ for all $l(\gamma) < s < r(\gamma)$. Denote by $|\gamma| = r(\gamma) - l(\gamma)$ the length of an excursion.

See Figure 1.3 for an example of a Brownian motion with drift for positive and negative θ and the corresponding reflected process. Note that for positive θ the time intervals between zeroes of B^θ are much longer than for negative θ.

Additionally we define a Poisson counting process N^θ, which equips each excursion with a number of marks, emerging with intensity $B^\theta(s)$ at time s. Informally speaking, the chance of encountering a mark in a time interval $[s, s+ds]$ is characterized by

$$\mathbb{P}(\text{Some mark emerges in } [s, s+ds] \mid B^\theta(u), u \leq s) = B^\theta(s)ds.$$

More formally we define N^θ to be the counting process for which

$$N^\theta(s) - \int_0^s B^\theta(u)du$$

is a martingale. Denote by $\mu(\gamma)$ the number of marks during an excursion γ.

The folk theorem states a convergence of both component sizes and surplus edges in the product topology. We here provide additional information on the convergence of component sizes. Let

$$l^2_{\searrow} := \{\boldsymbol{x} = (x_1, x_2, \dots) \mid x_1 \geq x_2 \geq \dots \geq 0,\ \textstyle\sum_i x_i^2 < \infty\} \tag{1.5}$$

and define the natural metric on $l^2_\searrow$ by

$$d_2(\boldsymbol{x}, \boldsymbol{y}) := \sqrt{\sum_i (x_i - y_i)^2}. \tag{1.6}$$

For a finite sequence like $\boldsymbol{\mathcal{C}}_n^\theta$, the ordered component sizes of a graph on n vertices, we append zeroes in order to regard it as an element on $l^2_\searrow$. We now state the main theorem of this thesis, which we will prove gradually in the following chapters.

Theorem 1.7 (Main theorem). *Let $\mathcal{C}_n^\theta(1) \geq \mathcal{C}_n^\theta(2) \geq ...$ be the ordered component sizes of $\mathscr{G}(n, n^{-1}+\theta n^{-4/3})$ and let $\sigma_n^\theta(j)$ be the surplus of the corresponding component. Then, as $n \to \infty$,*

1. *the convergence*
$$(n^{-2/3}(\mathcal{C}_n^\theta(j), \sigma_n^\theta(j)),\ j \geq 1) \to_d (\boldsymbol{\mathcal{C}}^\theta, \boldsymbol{\sigma}^\theta),$$
holds with respect to the product topology,
2. *the convergence $n^{-2/3}\boldsymbol{\mathcal{C}}_n^\theta \to_d \boldsymbol{\mathcal{C}}^\theta$ holds with respect to the $l^2_\searrow$ topology and*
3. *the limit $(\boldsymbol{\mathcal{C}}^\theta, \boldsymbol{\sigma}^\theta) = \big((\mathcal{C}^\theta(j), \sigma^\theta(j)),\ j \geq 1\big)$ is distributed as the sequence of lengths and mark-counts $((|\gamma_j|, \mu(\gamma_j)), j \geq 1)$ of excursions of B^θ.*

Note. Folk Theorem 1.6 is contained in the first statement of Theorem 1.7.

We conclude this chapter with an overview of the remaining chapters and the structure of the proof of Theorem 1.7. Chapter 2 will introduce some preliminary theory on convergence of probability measures, the space of càdlàg functions $D[0, T]$, counting processes and Brownian motion.

In Chapter 3 we define a way to traverse all vertices of a given graph, called the breadth-first walk $Z(s)$, that reduces the graph to a one-dimensional random walk in which component sizes are decoded as excursions above past minima. We analyse its characteristics when applied to $\mathscr{G}(n, n^{-1} + \theta n^{-4/3})$ and discover that it converges in distribution to W^θ after rescaling.

Chapter 4 deals with the second coordinate σ_n^θ in Theorem 1.7, the surplus edges. We describe a Poisson counting process N_n^θ which tallies up all encountered excess edges and calculate its limit rate as B^θ. The remainder of this chapter will be spent proving that this convergence of rates suffices to declare the convergence of the joint distribution of the rescaled breadth-first walk and this counting process to W^θ and N^θ.

The final proof of Theorem 1.7 happens in Chapter 5. We show that the convergence in distribution of the rescaled breadth-first walk to the Brownian motion with drift indeed implies convergence of excursions of certain length on finite intervals. A problem that remains is the possibility of large excursions "wandering off to infinity", which in Aldous' paper is solved by reference to existing random graphs results. We prove that we do not need to worry about these excursions or the total weight of smaller excursions independently of literature on random graphs. We subsequently show that the convergence of component sizes and surplus edges holds in the product topology and the former additionally in the $l^2_{\searrow}$ topology.

Aldous' paper provides a similar result for a nonuniform graph model where vertices have distinct sizes and large vertices are more likely to connect to other vertices. We introduce this model and the statement of its equivalent to Theorem 1.7 briefly in Chapter 6.

Lastly, Chapter 7 provides an overview over the remaining results of Aldous' paper, which feature the multiplicative coalescent, a Markov process on $l^2_{\searrow}$ that describes the behaviour of component sizes for fixed n and variable θ.

2 Preliminaries

2.1 Weak convergence of probability measures

The central notion of this thesis is the convergence in distribution of random variables in general metric spaces. The results presented in this section and the next will be of special importance in the proof of Theorem 4.1 in Chapter 4 and of Theorem 1.7 in Chapter 5.

Definition 2.1 (Weak convergence, [6, p.7]). Let μ_n, μ be measures on a metric space S with associated Borel σ-algebra $\mathcal{S} = \mathscr{B}(S)$. Denote by $C_b(S)$ the space of bounded, continuous functions $f : S \to \mathbb{R}$. We say μ_n *converges weakly* to μ, $\mu_n \Rightarrow \mu$, if

$$\int_S f d\mu_n \to \int_S f d\mu$$

as $n \to \infty$ for all $f \in C_b(S)$.

We say a random variable X in $(S, \mathcal{S})$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, has distribution or law P if

$$P(A) = \mathbb{P}(X \in A)$$

for all $A \in \mathcal{S}$.

For a series of random variables in the same metric space to converge in distribution does not require them to be defined on the same probability space. Let X_n, X be random variables in a metric space $(S, \mathcal{S})$,

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S}),$$
$$X_i : (\Omega_i, \mathcal{F}_i, \mathbb{P}_i) \to (S, \mathcal{S}),$$

with associated distributions P_n, P. Now X_n converges in distribution to X, $X_n \to_d X$, if their distribution measures converge weakly, $P_n \Rightarrow P$.

We will make use of the following theorem to link convergence in distribution of random variables to almost sure convergence of random variables on the same probability space.

Theorem 2.2 (Skorohod, [7, Theorem 3.30, p.56]). *Let X_n, X be random elements in a metric space $(S, \mathcal{S})$ such that $X_n \to_d X$. Then, on a suitable probability space, there exist some random elements $\mathcal{X}_n \sim X_n$ and $\mathcal{X} \sim X$ such that $\mathcal{X}_n \to_{a.s.} \mathcal{X}$.*

The main goal of this chapter is now to find necessary and sufficient conditions for the convergence in distribution of a given sequence of random variables or equivalently the weak convergence of their distribution measures. A first useful reference provides the Portmanteau theorem.

Theorem 2.3 (Portmanteau, [6, Theorem 2.1, p.16]). *Let P_n, P be probability measures on $(S, \mathcal{S})$. The following conditions are equivalent:*

1. $P_n \Rightarrow P$,
2. $\int_S f dP_n \to \int_S f dP$ *for all* $f \in C_b(S)$,
3. $\limsup_{n\in\mathbb{N}} P_n(F) \leq P(F)$ *for all closed* F,
4. $\liminf_{n\in\mathbb{N}} P_n(G) \geq P(G)$ *for all open* G,
5. $P_n(A) \to P(A)$ *for all* A *with* $P(\partial A) = 0$.

A powerful tool to prove weak convergence of measures is relative compactness.

Definition 2.4 (Relative compactness, [6, p.57]). Let Π be a family of probability measures on $(S, \mathcal{S})$. We call Π *relatively compact* if for every sequence in Π there exists a convergent subsequence. That is, for all sequences $\{P_n\} \subset \Pi$ there exists $\{P_{n_i}\} \subset \Pi$ and a probability measure Q on $(S, \mathcal{S})$, not necessarily in Π, such that $P_{n_i} \Rightarrow Q$ as $i \to \infty$.

We call a sequence of probability measures $\{P_n\}$ relatively compact if for every subsequence $\{P_{n_i}\}$ there exists a further subsequence $\{P_{n_{i_k}}\}$ and a probability measure Q such that $P_{n_{i_k}} \Rightarrow Q$ as $k \to \infty$.

For function spaces we have one additional requirement to identify the limit of a relatively compact sequence of probability measures.

Definition 2.5 (Finite dimensional distributions, [6, p.11]). Let $X : \Omega \times \mathbb{R}_+ \to \mathbb{X}$ be a stochastic process with distribution P and $\mathbb{X}$ a measurable space. For $t_1, \ldots, t_k \in \mathbb{R}_+$ we denote by *finite-dimensional distributions* of X the push forward measures

$$P\pi^{-1}_{t_1,\ldots,t_k}(A) := \mathbb{P}\{X(t_1) \in A_1, \ldots, X(t_k) \in A_k\}$$

for $A = A_1 \times \cdots \times A_k \subseteq \mathbb{X}^k$.

Now consider a relatively compact sequence of stochastic processes $\{X_n\}$ with distributions $\{P_n\}$ and a stochastic process X with distribution P, such that

$$P_n\pi^{-1}_{t_1,\ldots,t_k} \Rightarrow P\pi^{-1}_{t_1,\ldots,t_k}$$

for all k and $t_1, \ldots, t_k \in \mathbb{R}_+$. Since $\{P_n\}$ is relatively compact, we know that every subsequence contains a further subsequence converging to some probability measure Q. It can be shown that the convergence of finite-dimensional distributions implies that all of these limit measures are in fact the same measure P, which in turn proves $P_n \Rightarrow P$. For details see [6, p.57f.]. If we can prove a series of probability measure to converge in finite-dimensional distributions to some limit measure we therefore only need to show relative compactness of the series for the convergence in distribution to hold.

Relative compactness is closely linked to another property of series of measures, tightness. We call a measure μ on a metric space tight, if for all $\varepsilon > 0$ there exists a compact K such that

$$\mu(K^c) < \varepsilon,$$

where K^c denotes the complement of K. This carries over to families of probability measures as follows:

Definition 2.6 (Tightness of families of probability measures, [6, p.59])**.** A family Π of probability measures on a metric space $(S, \mathcal{S})$ is *tight* if for every $\varepsilon > 0$ there exists a compact $K \subset S$ such that $P(K) > 1 - \varepsilon$ for all $P \in \Pi$.

The last main result of this section now provides a link between tightness and relative compactness and therefore an effective means of proving convergence in distribution.

Theorem 2.7 (Prohorov's theorem, [6, Theorem 5.1, p.59])**.** *Let P_n be a series of probability measures on a metric space $(S, \mathcal{S})$. If P_n is tight, it is relatively compact.*

We summarize this theorem and our considerations above in the following lemma.

Lemma 2.8. *Let X, X_n be stochastic processes with distributions P, P_n. If $\{X_n\}$ is tight and $P_n\pi^{-1}_{t_1,\ldots,t_k} \Rightarrow P\pi^{-1}_{t_1,\ldots,t_k}$, for all k and $t_1, \ldots, t_k \in \mathbb{R}_+$, then $X_n \Rightarrow X$.*

2.2 The space $D[0, T]$

The coming chapters will mainly feature two types of stochastic processes: Brownian motion and so-called càdlàg-processes. This section will deal with the issue of proving tightness and thus convergence for the latter.

Definition 2.9 (The space $D[0, T]$, [6, p.121])**.** We call a function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}_+$, *càdlàg*, if

1. for $t \in I$, $f(t+) = \lim_{s \downarrow t} f(s)$ exists and $f(t) = f(t+)$,
2. for $t \in I$, $f(t-) = \lim_{s \uparrow t} f(s)$ exists.

These functions are right-continuous and possess left limits everywhere (continue à droite, limite à gauche). We denote by $D[0, T]$ the space of càdlàg-functions on $I = [0, T]$.

Considering topology and convergence on $D[0, T]$, we quickly observe that the notion of convergence known in $C[0, T]$, the space of continuous functions on $[0, T]$ where

$$f_n \to f \iff \sup_{t \in [0,T]} |f_n(t) - f(t)| \to 0,$$

is not sufficient for $D[0, T]$. Consider the functions

$$f_n(t) = \begin{cases} 0 & t < x_n, \\ 1 & t \geq x_n, \end{cases}$$

$$f(t) = \begin{cases} 0 & t < x, \\ 1 & t \geq x. \end{cases}$$

For $x_n \to x$ we would expect f_n to converge to f, however $\sup_{t\in I}|f_n(t) - f(t)| = 1$ whenever $x_n \neq x$. Thinking of the previous convergence as allowing small shifts on the y-axis, we need a convergence that additionally allows small shifts on the x-axis.

Definition 2.10 (Skorohod metric, [6, p.124])**.** Denote by Λ the class of strictly increasing, continuous mappings from $[0,T]$ onto itself.

For $x, y \in D[0,T]$, define the *Skorohod metric* $d(x,y)$ as the infimum of all $\varepsilon > 0$ for which exists a $\lambda \in \Lambda$ such that

1. $\sup_{t\in[0,T]}|\lambda(t) - t| = \sup_{t\in[0,T]}|t - \lambda^{-1}(t)| < \varepsilon$,
2. $\sup_{t\in[0,T]}|x(\lambda(t)) - y(t)| = \sup_{t\in[0,T]}|x(t) - y(\lambda^{-1}(t))| < \varepsilon$.

And writing $||x||_T := \sup_{t\in[0,T]}|x(t)|$ we can provide a more compact form:

$$d(x,y) = \inf_{\lambda\in\Lambda}\{||\lambda - I||_T \vee ||x - y(\lambda)||_T\},$$

where $I : [0,T] \to [0,T]$ is the identity map.

Definition 2.11 (Modulus of continuity, [6, p.122])**.** We call a set $\{t_i\}$, with $0 = t_0 < t_1 < \cdots < t_k = T$, *$\delta$-sparse* if $\min_{1\leq i\leq k}(t_i - t_{i-1}) > \delta$. For $0 < \delta < 1$ and $x \in D[0,T]$, define the *modulus of continuity* by

$$w'_x(\delta) := \inf_{\{t_i\}}\max_{1\leq i\leq k} w_x([t_{i-1}, t_i)), \tag{2.1}$$

where the infimum extends over all δ-sparse sets $\{t_i\}$ and

$$w_x([t_{i-1}, t_i)) := \sup_{s,t\in[t_{i-1},t_i)}|x(s) - x(t)|. \tag{2.2}$$

The Arzelà-Ascoli theorem, see [6, Theorem 7.2, p.82], provides a complete characterisation of relatively compact sets in $C[0,T]$. We present a $D[0,T]$-equivalent.

Theorem 2.12 (Arzelà-Ascoli in $D[0,T]$, [6, Theorem 12.3, p.130])**.** *A necessary and sufficient condition for a set A to be relatively compact in the Skorohod toplogy is that*

1. $\sup_{x\in A}||x||_T < \infty$,
2. $\lim_{\delta\to 0}\sup_{x\in A} w'_x(\delta) = 0$.

Here the first condition provides a general boundedness of the set and can be found in the $C[0,T]$ Arzelà-Ascoli theorem as well, while the second condition is unique to the space $D[0,T]$. Using this result one may prove the following theorem on tightness of series of random variables in $D[0,T]$.

Theorem 2.13 (Tightness in $D[0,T]$, [6, Theorem 13.2, p.139])**.** *Let $\{X_n\}$ be a series of random variables in $D[0,T]$ with distributions $\{P_n\}$. The sequence is tight iff these two conditions hold:*

1. We have

$$\lim_{c\to\infty} \limsup_{n\in\mathbb{N}} P_n(x : ||x||_T \geq c) = 0$$

2. and for each $\varepsilon > 0$

$$\lim_{\delta\to 0} \limsup_{n\in\mathbb{N}} P_n(x : w'_x(\delta) \geq \varepsilon) = 0.$$

2.3 Point processes

In Chapter 5 we will finish the proof of Theorem 1.7 by considering component sizes and counts of surplus edges as a two-dimensional point process. This section will provide an overview of the needed theory.

Definition 2.14 (Point process, [8, p.123])**.** Let S be a locally compact second countable Hausdorff space and $\mathscr{B}(S)$ its Borel σ-algebra. Let $\{x_i, i \geq 1\}$ be a collection of points of S. Let

$$\mu := \sum_{i\geq 1} \delta_{x_i},$$

with δ_x the Dirac measure of $x \in S$, be locally compact, that is, if $C \in \mathscr{B}(S)$ is compact then $\mu(C) < \infty$. Then μ is a *point measure* on E.

Denote by $M_p(S)$ the space of all point measures on E and let $\mathscr{M}_p(S)$ be the smallest σ-algebra containing all sets of the form

$$\{m \in M_p(S) \mid m(A) \in B\}$$

for some $A \in \mathscr{B}(S)$ and $B \in \mathscr{B}(\mathbb{R}_+)$. A *point process* N is a measurable map from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into $(M_p(S), \mathscr{M}_p(S))$.

For the next observations, we will take $\mathbb{R}_+$ as the underlying Hausdorff space with its Borel σ-algebra $\mathscr{B} = \mathscr{B}(\mathbb{R}_+)$.

Definition 2.15 (Poisson point process, [8, p.127, p.130])**.** We call a point process N a *Poisson point process* or *Poisson process* if

1. for all disjoint sets $A_1, A_2, \dots, A_n \in \mathscr{B}$ the random variables $N(A_1), N(A_2), \dots, N(A_n)$ are independent and
2. for all $A \in \mathscr{B}$, $N(A)$ has Poisson distribution $\mathrm{Poi}(\gamma)$,

$$\mathbb{P}(N(A) = k) = \frac{\gamma^k}{k!}\exp(-\gamma)$$

where $\gamma = \gamma(A) \in [0, \infty]$ is the *mean measure* or *intensity* of N, defined by

$$\gamma(A) := \mathbb{E}\left[N(A)\right].$$

The mean measure is often given in terms of a *rate* or *conditional intensity* λ by

$$\gamma(A) = \int_A \lambda(t)dt. \tag{2.3}$$

We can interpret this conditional intensity as

$$\lambda(t)dt \approx \mathbb{E}\left[N(dt) \mid \mathcal{F}_t\right],$$

where $\mathcal{F}_t$ contains the history of the counting process thus far.

Definition 2.16 (Simple point process, [8, p.124])**.** A point process N on $\mathbb{R}_+$ is called *simple* if

$$\mathbb{P}(N(x) > 1) = 0$$

for all $x \in \mathbb{R}_+$.

By [9, Remark 2.1, p.34] a Poisson point process is simple if and only if its mean measure γ has no discrete component, that is $\gamma(\{x\}) = 0$ for all $x \in \mathbb{R}_+$. For more precise explanations on the conditional intensity, see [10, p.231ff.] and [11, Chapter 14].

The idea of the conditional intensity estimating the mean number of jumps of a counting process is made more clear in the following lemma.

Lemma 2.17 (Martingale decomposition of a counting process, [10, Lemma 7.2.V, p.241])**.** *Let $N(t)$, $0 \leq t < \infty$, be a counting process adapted on the history $\mathcal{F}_t$ with conditional intensity $\lambda(t)$. Let $\gamma(t) = \int_0^t \lambda(t)dt$. Then the process $M(t) = N(t) - \gamma(t)$ is an $\mathcal{F}_t$-martingale.*

In order to talk reasonably about convergence of point processes, we first need to introduce a topology on the space of point measures. The *vague topology* is similar to the one generated by the weak convergence in Section 2.1. When trying to apply weak convergence to point measures we run into a problem, since a point measure $\mu = \sum_{i\geq 1} \delta_{x_i}$ may contain an infinite number of points on $\mathbb{R}_+$ and therefore

$$\int_{\mathbb{R}_+} f d\mu = \sum_{i\geq 1} f(x_i) = \infty$$

for certain $f \in C_b(\mathbb{R}_+)$, which makes a discussion of convergence in $\mathbb{R}$ of the integrals unreasonable.

To counter this, we define a new type of convergence, in which the integrals only have to converge for functions with compact support.

Definition 2.18 (Vague convergence, [8, p.140])**.** Let $C_K(E)$ be the space of continuous real valued functions on a Hausdorff space S with compact support, meaning there exists a compact set $K \in \mathscr{B}(S)$ such that $f(x) = 0$ for all $x \notin K$.

Let $\mu, \mu_1, \mu_2, \ldots$ be point measures on S. We say μ_n *converge vaguely* to μ, $\mu_n \to_v \mu$, if

$$\int_S f d\mu_n \xrightarrow{n\to\infty} \int_S f d\mu$$

for all $f \in C_K(S)$.

Theorem 2.3 gave necessary and sufficient conditions for weak convergence of measures, the following lemma may be seen as an equivalent Portmanteau Theorem in the sense of vague convergence.

Lemma 2.19 (Equivalent conditions for vague convergence, [8, Proposition 3.12, p.142])**.** *Let $\mu, \mu_1, \mu_2, \ldots$ be point measures on a Hausdorff space S. The following are equivalent:*

1. $\mu_n \to_v \mu$,
2. $\mu_n(B) \to \mu(B)$ *for all relatively compact (i.e. with compact closure) B for which* $\mu(\partial(B)) = 0$.
3. $\limsup_{n\in\mathbb{N}} \mu_n(K) \leq \mu(K)$ *and* $\liminf_{n\in\mathbb{N}} \mu_n(G) \geq \mu(G)$ *for all compact K and all open, relatively compact G.*

Since point measures are uniquely defined by the points on their underlying space they are describing, we would expect our idea of convergence to imply some kind of convergence of points on S. In fact, as the next lemma shows, vague convergence does imply a pointwise convergence in S.

Lemma 2.20 (Pointwise convergence, [8, Proposition 3.13, p.144])**.** *Let $\mu, \mu_1, \mu_2, \ldots$ be point measures on S and $\mu_n \to_v \mu$. For compact K with $\mu(\partial K) = 0$ and $n \geq N(K)$ there exist a labelling of points of μ_n and μ in K such that*

$$\mu_n(\cdot \cap K) = \sum_{i=1}^{M} \delta_{x_i^{(n)}},$$
$$\mu(\cdot \cap K) = \sum_{i=1}^{M} \delta_{x_i},$$

and in S^M

$$(x_i^{(n)}, 1 \leq i \leq M) \xrightarrow{n\to\infty} (x_i, 1 \leq i \leq M)$$

in the sense of componentwise convergence.

2.4 Brownian motion

In the course of this thesis we will require two further results on Brownian motion. First we introduce the central limit theorem for martingales, which provides us with a means to identify convergence of a martingale to Brownian motion. It will be used in the proof of Theorem 3.1 in Chapter 3.

We state the theorem here as it appears in [12, Theorem 1.4, p.339 f.], omitting one of two equivalent conditions and all references to higher dimensional processes, in order to focus on the one-dimensional case we will need for our proof.

Theorem 2.21 (Central limit theorem for martingales). *Let $\{\mathcal{F}_s^n\}$ be a filtration and M_n a $\{\mathcal{F}_s^n\}$-local martingale with sample paths in $D[0,\infty)$ and $M_n(0) = 0$. Let A_n be a process with sample paths in $D[0,\infty)$, increasing in s, such that $M_n^2 - A_n$ is an $\{\mathcal{F}_s^n\}$-local martingale.*

Let the following conditions hold: For each $T > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{s\leq T} |A_n(s) - A_n(s-)|\right] = 0, \tag{2.4}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{s\leq T} |M_n(s) - M_n(s-)|^2\right] = 0, \tag{2.5}$$

and

$$A_n(s) \to_p c(s). \tag{2.6}$$

where $c(s)$ is a continuous, increasing function on $[0,\infty)$ with $c(0) = 0$. Then $M_n \to_d X$ where X is a martingale with sample paths in $C[0,\infty)$ and independent Gaussian increments and $X^2 - c$ is a martingale. For $c(s) = s$, the limit process X is the standard Brownian motion.

We conclude this preliminary chapter by presenting Girsanov's theorem, which will enable us to deduce certain properties of the Brownian motion with drift and prove Lemma 5.2 in Chapter 5.

Theorem 2.22 (Girsanov's theorem, [13, Theorem 4.2.2, p.66]). *Let $(W(s))_{0\leq s\leq T}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\Theta(s))_{0\leq s\leq T}$ be an adapted process satisfying*

$$\int_0^T \Theta^2(u)du < \infty.$$

Define

$$\tilde{W}(s) := W(s) + \int_0^s \Theta(u)du, \tag{2.7}$$

$$X(s) := \exp\left(-\int_0^s \Theta(u)dW(u) - \frac{1}{2}\int_0^s \Theta^2(u)du\right). \tag{2.8}$$

If $X(s)$ a martingale, particularly $\mathbb{E}[X(s)] = \mathbb{E}[X(0)] = 1$ for all s, the measure $\tilde{\mathbb{P}}$ defined by

$$\tilde{\mathbb{P}}(A) := \int_A X(\omega)d\mathbb{P}(\omega), \text{ for all } A \in \mathcal{F} \tag{2.9}$$

is a probability measure under which the process $(\tilde{W}(s))_{0\leq s\leq T}$ is a standard Brownian motion.

By [13, Remark 4.2.3, p.66], a sufficient condition for $X(t)$ to be a martingale is the so-called Novikov-condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \Theta^2(u)du\right)\right] < \infty. \tag{2.10}$$

We can apply this Theorem to the Brownian motion with drift W^t as follows: Recall that

$$W^\theta(s) = W(s) + \theta s - \frac{1}{2}s^2 = W(s) + \int_0^s (\theta - u)du.$$

For $T < \infty$, $\Theta(s) := \theta - s$ satisfies (2.10), therefore $X(s)$, as defined in (2.8), is a martingale and W^θ is a standard Brownian motion under the probability measure $\tilde{\mathbb{P}}$ defined in (2.9). Since $X(s) > 0$ almost surely for all s, the probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ agree which events happen almost surely or almost never:

$$\begin{aligned} \mathbb{P}(A) = 0 &\iff \tilde{\mathbb{P}}(A) = 0, \\ \mathbb{P}(A) = 1 &\iff \tilde{\mathbb{P}}(A) = 1 \end{aligned} \tag{2.11}$$

for all $A \in \mathcal{F}$. When trying to prove that certain properties hold for W^θ under $\mathbb{P}$ almost surely, it suffices to prove these properties holding for a standard Brownian motion almost surely. Since W^θ is a standard Brownian motion under $\tilde{\mathbb{P}}$ we can apply (2.11) to carry this realization over to W^θ under $\mathbb{P}$.

3 The breadth-first walk

This chapter introduces the breadth-first walk, a way to traverse the vertices of a given graph such that component sizes can be read from the development of a càdlàg process.

3.1 The breadth-first walk on a deterministic graph

3.1.1 In discrete time

We start by describing the deterministic construction of this process. Consider an arbitrary graph $\mathcal{G}$ on the set of vertices $V = \{1, \ldots, n\}$ with set of edges E. We will define the breadth-first ordering $(v_1, \ldots, v_n)$ of the vertices along with an integer-valued sequence $(z(i),\ 1 \leq i \leq n)$ which we call the breadth-first walk on $\mathcal{G}$.

The breadth-first order derives from an algorithmic construction as follows: Let $\mathcal{C}_1, \mathcal{C}_2, \ldots$ be the components of $\mathcal{G}$ in order, such that $w_1, w_2, \ldots,$ the vertices with the smallest label in the corresponding component, are ordered $w_1 < w_2 < \ldots$. Call w_i the root of $\mathcal{C}_i$. Now order by levels (distance from the root) and within levels order by original vertex label. See Figure 3.1 for an example of the new ordering.

For a more mathematically concise definition, consider the set of vertices $\{v_1, \ldots, v_i\}$ and define the neighbour set $\mathcal{N}_i$ as the vertices outside of $\{v_1, \ldots, v_i\}$ that are neighbours to some vertex in $\{v_1, \ldots, v_i\}$:

$$\mathcal{N}_i := \{v \in V \backslash \{v_1, \ldots, v_i\} \mid (v_j, v) \in E \text{ for some } 1 \leq j \leq i\} \tag{3.1}$$

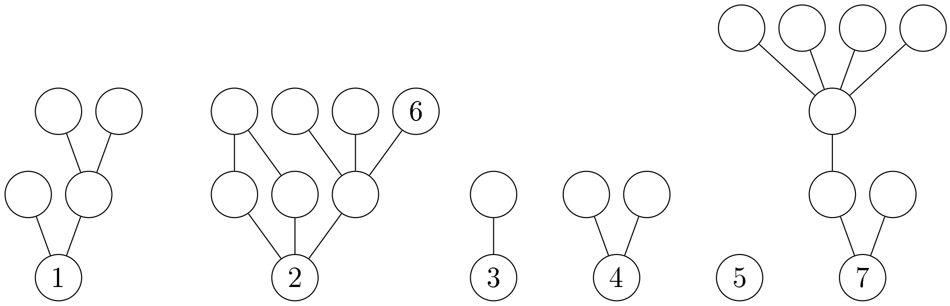
This allows us to define the set of children of some vertex v_i as $\mathcal{N}_i \backslash \mathcal{N}_{i-1}$. First order the components as described above. Now consider only the first component $\mathcal{C}_1$. Define $v_1 := w_1$, the root of $\mathcal{C}_1$ and define $v_2, \ldots, v_{|\mathcal{N}_1|+1}$ as the neighbours of v_1, in increasing order of vertex label. Define the new label for all $i = 2, \ldots, |\mathcal{C}_1|$, that is all vertices in the first component, inductively by listing all children (if any exist) of v_i in increasing order as $v_{|\mathcal{N}_{i-1}|+i}, \ldots, v_{|\mathcal{N}_i|+i}$. After labelling the last vertex in $\mathcal{C}_1$, set $v_{|\mathcal{C}_1|+1} := w_2$, the root of $\mathcal{C}_2$, and continue the construction as above. Traverse all components this way.

For the number of children of v_i write

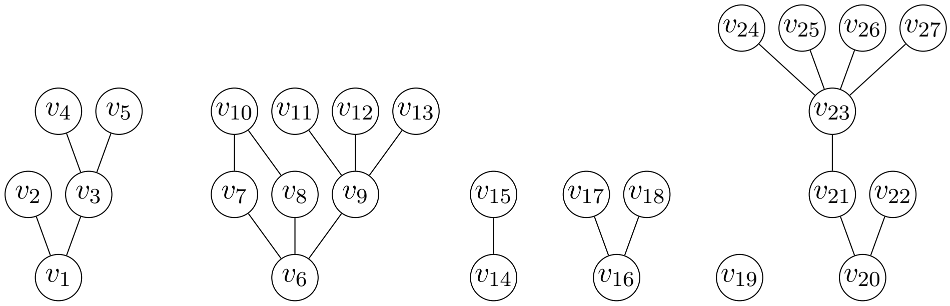
$$c(i) := |\mathcal{N}_i \backslash \mathcal{N}_{i-1}|. \tag{3.2}$$

Now define the breadth-first walk $(z(i),\ 1 \leq i \leq n)$ by

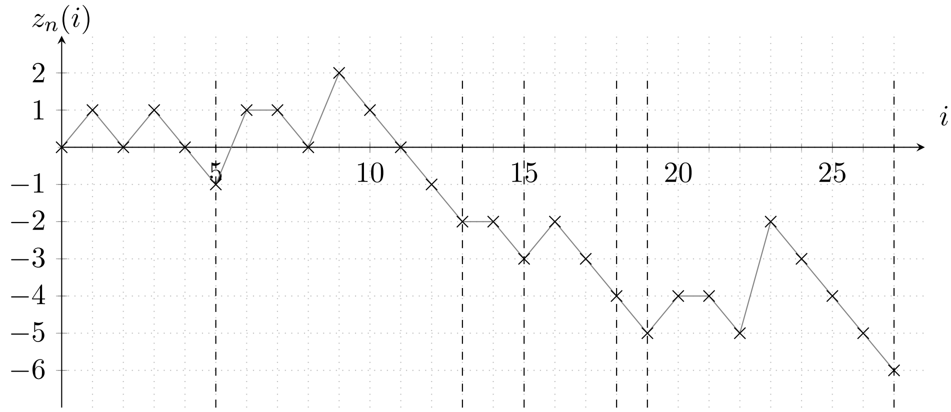
$$\begin{aligned} z(0) &:= 0, \\ z(i) &:= z(i-1) + c(i) - 1, \quad i = 1, \ldots, n. \end{aligned} \tag{3.3}$$



(a) Original vertex labels



(b) New vertex labels



(c) Resulting breadth-first walk, dashed lines indicating the end of components

Figure 3.1: Breadth-first walk on the first components of a graph

Figure 3.1 shows this process for a graph example. The root of each component is the vertex contained with the smallest original label. Most other vertex labels are omitted since they do not play a role in the assignment of new labels.

The process divides the vertex set into three parts: Explored, discovered and neutral vertices. Every vertex starts as neutral. At step 1, we traverse vertex 1 and mark it as explored with new label v_1. We search for neighbours of v_1 and assign them labels to mark as discovered. The next vertex to explore is the vertex already discovered with the smallest label and each vertex switches from neutral to discovered once it gets assigned a new label. Once we found all its neighbours, it switches to explored. After traversing every vertex of one component there are no discovered vertices left and we continue the exploration with the neutral vertex with the smallest original label.

The walk z decreases by 1 for each vertex explored and increases by the number of new neighbours discovered in each step.

Note that surplus edges, like (v_8, v_{10}) in Figure 3.1, are ignored by the breadth-first walk. We count their occurrence with another counting process in Chapter 4.

We write

$$\zeta(j) := |\mathcal{C}_1| + \cdots + |\mathcal{C}_j|, \tag{3.4}$$

$$\zeta^{-1}(i) := \min\{j \mid \zeta(j) \geq i\}, \tag{3.5}$$

for the index of the last vertex in the j-th component and the index of the component containing v_i, respectively. Now we can provide a definition of the breadth-first walk equivalent to (3.3):

$$\begin{aligned} z^*(0) &:= 0, \\ z^*(i) &:= |\mathcal{N}_i| - \zeta^{-1}(i), \quad i = 1, \ldots, n. \end{aligned} \tag{3.6}$$

We verify the equivalence by induction, showing that for $i \geq 1$ increments of both functions are equal. We have

$$\begin{aligned} & z(i) - z(i-1) = z^*(i) - z^*(i-1) \\ \iff \quad & c(i) - 1 = |\mathcal{N}_i| - \zeta^{-1}(i) - |\mathcal{N}_{i-1}| + \zeta^{-1}(i-1) \\ \iff \quad & |\mathcal{N}_i| - |\mathcal{N}_{i-1}| = c(i) + \zeta^{-1}(i) - \zeta^{-1}(i-1) - 1 \end{aligned} \tag{3.7}$$

We divide the proof into two cases. First, assume v_{i-1} is not the last vertex in its component. Then v_i belongs to the same component and $\zeta^{-1}(i) = \zeta^{-1}(i-1)$. Vertex v_i has already been assigned a new label at step $i-1$, so $v_i \in \mathcal{N}_{i-1}$. Going from $i-1$ to i, the neighbour set increases by the number of new neighbours of v_i and decreases by v_i itself. So

$$|\mathcal{N}_i| - |\mathcal{N}_{i-1}| = c(i) - 1,$$

which provides the equality.

In the second case, if v_{i-1} is the last vertex of its component, then $\zeta^{-1}(i) = \zeta^{-1}(i-1) + 1$ and $|\mathcal{N}_{i-1}| = 0$. Equality (3.7) reduces to $|\mathcal{N}_i| = c(i)$, which holds since $c(i) = |\mathcal{N}_i \backslash \mathcal{N}_{i-1}| = |\mathcal{N}_i|$. This proves the equivalence of both processes. We proceed referring to the breadth-first walk as $z(i)$.

Since $|\mathcal{N}_i| = 0$ only if v_i is the last vertex in its component, (3.4) and (3.6) imply $z(\zeta(j)) = -j$ and $z(i) \geq -j$ for all $\zeta(j) < i < \zeta(j+1)$. So, for vertices in the j-th component, the random walk takes values greater or equal to $-(j-1)$, until the last vertex, for which z reaches a new minimum at $-j$. Knowing this we can reconstruct sizes and indices of components via

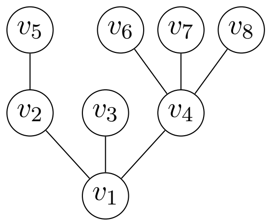
$$\zeta(j) = \min\{i \mid z(i) = -j\}, \tag{3.8}$$

$$|\mathcal{C}_j| = \zeta(j) - \zeta(j-1), \tag{3.9}$$

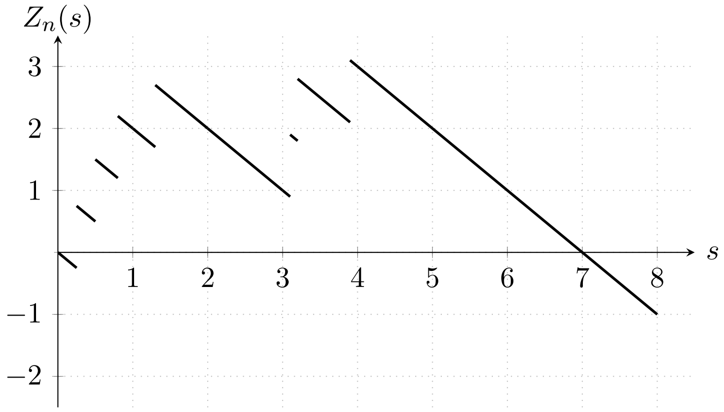
$$\zeta^{-1}(i) = 1 - \min_{j \leq i-1} z(j). \tag{3.10}$$

3.1.2 In continuous time

The last section defined the random walk for integer times, but to develop our theory of convergence to a Brownian motion we will have to construct $z(i)$ in continuous time.



(a) A graph component



(b) The continuous-time breadth-first walk

Figure 3.2: Continuous breadth-first walk on a single component

Define a sequence of independent random variables, uniformly distributed on $(0, 1)$, $(U_{i,j}, 1 \leq i \leq n, 1 \leq j \leq c(i))$, where $c(i)$ is the number of children of v_i. Then for each $i = 1, \ldots, n$ and $0 \leq u \leq 1$ define the process Z by

$$\begin{aligned} Z(0) &:= 0, \\ Z(i-1+u) &:= Z(i-1) - u + \sum_{1 \leq j \leq c(i)} \mathbb{1}_{\{U_{i,j} \leq u\}}, \end{aligned} \tag{3.11}$$

where $\mathbb{1}$ is the indicator function. So $Z(i) = Z(i-1) - 1 + c(i)$ and Z coincides with the discrete definition of the breadth-first walk at integer times.

Some more explanation on this construction: At time $i-1$, the walk has traversed vertices $(v_1, \dots, v_{i-1})$ and has discovered a list of vertices $(v_1, \dots, v_k)$ of length $k = i - 1 + |\mathcal{N}_{i-1}|$. The discrete walk now adds the children of v_i to this list at time i. The newly defined continuous time walk adds those vertices at uniformly random times in $(i-1, i)$. See Figure 3.2 for a sample component and the walk Z on its vertices.

Note that, since jumps are now happening at random times during their respective time interval, Z is a stochastic process even though it is still only defined for a given deterministic graph.

3.2 The breadth-first walk as a stochastic process

Denote by $Z_{\mathscr{G}}$ the continuous-time breadth-first walk on the graph $\mathscr{G}$ and define

$$Z_n^\theta(\omega, s) := Z_{\mathscr{G}^\theta(\omega)}(s) \tag{3.12}$$

where $\mathscr{G}^\theta(\omega)$ is a realization of the critical random graph $\mathscr{G}(n, n^{-1} + \theta n^{-4/3})$. We can now state the main theorem of this chapter, the convergence in distribution of this process to a Brownian motion with drift after rescaling.

Theorem 3.1. *Let $Z_n^\theta(s)$, for $0 \le s \le n$, be the breadth-first walk associated with $\mathscr{G}(n, n^{-1} + \theta n^{-4/3})$. Rescale via*

$$\bar{Z}_n^\theta(s) := n^{-1/3} Z_n^\theta(n^{2/3} s). \tag{3.13}$$

Then $\bar{Z}_n^\theta(s) \to_d W^\theta$ as $n \to \infty$.

3.2.1 Decompositions of Z_n^θ

In the next sections we prove Theorem 3.1. We begin by providing a suitable decomposition of Z_n^θ into a martingale and an integrable predictable process. The latter process will later provide the downwards drift of W^θ, while we go on proving the convergence of the martingale to a standard Brownian motion.

Since Z_n increases by 1 for every event of a new edge appearing in the breadth-first walk we may view the process as a Poisson point process on the positive real line with a certain constant downwards drift. This point process would then possess a conditional intensity, say $a_n(s)$, such that

$$a_n(s) ds = \mathbb{E}\left[Z_n(s + ds) - Z_n(s) + ds \mid \mathcal{F}_s\right].$$

We evaluate these considerations more elaborately in the following lemma. For ease of notation we drop the superscript θ from all random variables in this chapter.

Lemma 3.2. *The decomposition*

$$Z_n = M_n + F_n \tag{3.14}$$

holds, where M_n is a martingale and F_n is defined by

$$F_n(t) := \int_0^t a_n(s)ds - t \tag{3.15}$$

with

$$a_n(s)ds = \mathbb{P}(\textit{A new edge appears in } [s, s+ds] \mid Z_n(u), \;\; u \le s). \tag{3.16}$$

Proof: We will prove that $Z_n - F_n$ is a martingale by showing that

$$\mathbb{E}\left[Z_n(t+u) - F_n(t+u) \mid \mathcal{F}_t\right] = Z_n(t) - F_n(t)$$

holds for all $u \ge 0$, where $\mathcal{F}_t$ is the natural σ-algebra generated by Z_n, $\mathcal{F}_t = \sigma(Z_n(s), s \le t)$. This is equivalent to

$$\mathbb{E}\left[Z_n(t+u) - Z_n(t) \mid \mathcal{F}_t\right] = \mathbb{E}\left[F_n(t+u) - F_n(t) \mid \mathcal{F}_t\right].$$

We start with the left-hand side. The change of Z_n between times t and $t+u$ is the sum of all jumps that occurred in $[t, t+u]$, minus the constant downward drift u:

$$\begin{aligned}
&\mathbb{E}\left[Z_n(t+u) - Z_n(t) \mid \mathcal{F}_t\right] \\
&\quad = \mathbb{E}\left[\text{Number of jumps in } [t, t+u] \mid \mathcal{F}_t\right] - u \\
&\quad = \mathbb{E}\left[\text{Number of new edges appearing in } [t, t+u] \mid \mathcal{F}_t\right] - u,
\end{aligned}$$

since every new edge corresponds to a jump of size 1 in Z_n.

Looking at the right-hand side, we define $\mathcal{E}_I$ as the event of a new edge appearing during the time interval I and calculate

$$\begin{aligned}
&\mathbb{E}\left[F_n(t+u) - F_n(t) \mid \mathcal{F}_t\right] \\
&\quad = \mathbb{E}\left[\int_0^{t+u} a_n(s)ds - (t+u) - \int_0^t a_n(s)ds + t \mid \mathcal{F}_t\right] \\
&\quad = \mathbb{E}\left[\int_t^{t+u} a_n(s)ds \mid \mathcal{F}_t\right] - u \\
&\quad = \int_t^{t+u} \mathbb{E}\left[a_n(s)ds \mid \mathcal{F}_t\right] - u \\
&\quad = \int_t^{t+u} \mathbb{E}\left[\mathbb{P}(\mathcal{E}_{[s,s+ds]} \mid \mathcal{F}_s) \mid \mathcal{F}_t\right] - u \\
&\quad = \int_t^{t+u} \mathbb{P}(\mathcal{E}_{[s,s+ds]} \mid \mathcal{F}_t) - u \quad \text{since } \mathcal{F}_t \subseteq \mathcal{F}_s \;\; \forall s \in [t, t+u] \\
&\quad = \mathbb{E}\left[\int_t^{t+u} \mathbb{1}_{\mathcal{E}_{[s,s+ds]}} \mid \mathcal{F}_t\right] - u \\
&\quad = \mathbb{E}\left[\text{Number of new edges appearing in } [t, t+u] \mid \mathcal{F}_t\right] - u.
\end{aligned}$$

This proves M_n to be a martingale. $\square$

For a better understanding of the processes involved, we need to find a concise expression for the probability used in (3.16). An ensuing corollary provides some characteristics of the resulting process F_n. Denote by $\lfloor s \rfloor$ and $\lceil s \rceil$ the largest integer smaller than s and the smallest integer larger than s, respectively. Since the original vertex labels are no longer relevant, we use the notion of vertex v_i and vertex i interchangeably. We denote an edge between v_i and v_j by (i, j).

Lemma 3.3. *For a_n as defined in Lemma 3.2, we have*

$$a_n(s) = (n - s - \zeta_n^{-1}(\lceil s \rceil) - Z_n(s)) \frac{p_n}{1 - (s - \lfloor s \rfloor) p_n}.$$

Proof: Consider the walk Z_n at time $s \in [i-1, i]$. Let N be the number of vertices that, at time $i-1$, were eligible to be children of vertex v_i. That is the number of vertices not yet explored or discovered, which excludes v_i itself in particular. To any eligible vertex v_j we assign a random variable $U_{i,j}$. Let all $U_{i,j}$ be independent and identically $\mathcal{U}(0, 1)$ distributed. Our understanding of the process of discovering children of v_i is as follows: At time $i - 1 + U_{i,j}$, the edge (i, j) will open with probability p_n and Z_n will make a jump of size 1. We arrive at a characterisation of our breadth-first walk, equivalent to (3.11):

$$Z_n(i - 1 + u) = Z_n(i - 1) - u + \sum_{j=1}^{N} \mathbb{1}_{\{U_{i,j} \leq u,\ (i,j) \text{ open}\}}.$$

We define $\mathcal{F}_s := \sigma(Z_n(u), u \leq s)$. The goal of this proof is to find an expression for $\mathbb{P}(\text{A new edge appears in } [s, s + ds] \,|\, \mathcal{F}_s)$. To do this, we condition over a finer σ-algebra and use the law of total expectation to arrive at a general statement. $\mathcal{F}_s$ tells us the history of the walk until time s. We know how many vertices were eligible at time $i - 1$ and how many open edges to v_i were found in $[\lfloor s \rfloor, s]$. However, it is unknown exactly which vertices are now children of v_i and which vertices are still eligible. Define

$$\begin{aligned}\mathcal{F}_s^k := \sigma(Z_n(u), u \leq s;\ & \\ & \text{There are exactly } k \text{ children of } v_i \text{ encountered thus far,} \\ & \text{and these are } j_1, \ldots, j_k).\end{aligned}$$

We know that k of the N edges eligible at time $i - 1$ are already open and want to calculate the probability that one of the remaining $N - k$ edges opens in $[s, s + ds]$. The probability of some edge opening is the sum of the probabilities for single edges opening and some factor describing the, for small ds increasingly slim, chance of two or more edges opening in the interval. We denote by $\mathcal{E}_I$ the event of a new edge appearing in an interval I and write

$$\mathbb{P}\left(\mathcal{E}_{[s,s+ds]} \,|\, \mathcal{F}_s^k\right) = \sum_{j \neq j_1, \ldots j_k} \mathbb{P}\left((i, j) \text{ opens in } [s, s + ds] \,|\, \mathcal{F}_s^k\right) + o\,(ds)\,.$$

For the edge (i, j), all relevant information contained in $\mathcal{F}_s^k$ is the fact that (i, j) is not yet open, the event of opening has not happened in $[\lfloor s \rfloor, s)$. Since the opening itself, happening with probability p_n, and the uniformly distributed time of the event in $[i-1,i]$ are independent, we see that

$$\begin{aligned}&\mathbb{P}\left((i,j) \text{ opens in } [s,s+ds]\right)\\ &\quad= \mathbb{P}\left((i,j) \text{ opens}\right)\mathbb{P}\left(\text{It happens in } [s,s+ds]\right)\\ &\quad= p_n ds.\end{aligned}$$

By the definition of conditional probability

$$\begin{aligned}&\mathbb{P}\left((i,j) \text{ opens in } [s,s+ds] \mid (i,j) \text{ did not open in } [\lfloor s\rfloor, s]\right)\\ &\quad= \frac{\mathbb{P}\left((i,j) \text{ opens in } [s,s+ds]\right)}{\mathbb{P}\left((i,j) \text{ did not open in } [\lfloor s\rfloor, s]\right)}\\ &\quad= \frac{p_n ds}{1-p_n(s-\lfloor s\rfloor)},\end{aligned}$$

and finally, omitting the $o\,(ds)$-term, we have

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}_{[s,s+ds]} \mid \mathcal{F}_s^k\right) &= \sum_{j\neq j_1,\dots j_k} \frac{p_n}{1-(s-\lfloor s\rfloor)p_n}\\ &= (N-k)\frac{p_n ds}{1-p_n(s-\lfloor s\rfloor)}.\end{aligned}$$

Seeing that $\mathcal{F}_s \subseteq \mathcal{F}_s^k$, we apply the tower property:

$$\begin{aligned}\mathbb{P}\left(\mathcal{E}_{[s,s+ds]} \mid \mathcal{F}_s\right) &= \mathbb{E}\left[\mathbb{1}_{\mathcal{E}_{[s,s+ds]}} \mid \mathcal{F}_s\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\mathcal{E}_{[s,s+ds]}} \mid \mathcal{F}_s^k\right] \mid \mathcal{F}_s\right]\\ &= \mathbb{E}\left[(N-k)\frac{p_n}{1-(s-\lfloor s\rfloor)p_n} ds \mid \mathcal{F}_s\right].\end{aligned}$$

Conditioning on $\mathcal{F}_s$, the breadth-first walk Z_n tells us exactly how many vertices have connected to vertex v_i until time s. We denote by $\eta_n(s)$ the number of vertices that are at time s not eligible to be a child of $v(\lceil s\rceil)$. Then $N-k = n-\eta_n(s)$ at time s and

$$a_n(s)ds = \mathbb{P}\left(\mathcal{E}_{[s,s+ds]} \mid \mathcal{F}_s\right) = (n-\eta_n(s))\frac{p_n}{1-(s-\lfloor s\rfloor)p_n} ds.$$

Finally, we find a concise expression for $\eta_n(s)$. At time $i-1$, the ineligible vertices are the $i-1$ vertices already explored and the set $\mathcal{N}_{i-1}$ of vertices already discovered as children. If v_{i-1} is the last vertex of its component, v_i itself is not part of $\mathcal{N}_{i-1}$, so we need to add a term that equals 1 if v_{i-1} is the last vertex of its component and 0 otherwise. Together, we arrive at

$$\eta_n(i-1) = i-1+|\mathcal{N}_{i-1}| + (\zeta_n^{-1}(i) - \zeta_n^{-1}(i-1)).$$

By (3.6) this is equivalent to

$$\eta_n(i-1) = i - 1 + \zeta_n^{-1}(i) + Z_n(i-1).$$

At time $i-1+u$, for $0 < u < 1$, new vertices were discovered as children of v_i and now add to $\eta_n(i-1+u)$. The number of ineligible vertices is now

$$\begin{aligned}\eta_n(i-1+u) &= i - 1 + \zeta_n^{-1}(i) + Z_n(i-1) + \sum_j \mathbb{1}_{U_{i,j} \leq u} \\ &= i - 1 + u + \zeta_n^{-1}(i) + Z_n(i-1+u),\end{aligned}$$

by the definition of the continuous-time breadth-first walk in (3.11). So $\eta_n(s) = s + \zeta_n^{-1}(\lceil s \rceil) + Z_n(s)$ which concludes the proof. $\square$

Corollary 3.4. *The process F_n, as defined in Lemma 3.2, is a continuous process of bounded variation. Moreover, Z_n and M_n are càdlàg processes of bounded variation.*

Proof: Since $a_n(s) = (n - \eta_n(s)) \frac{p_n}{1-(s-\lfloor s \rfloor)p_n} \geq 0$ for all s, the integral $\int_0^t a_n(s)ds$ is a non-decreasing, continuous function in t. Therefore $F_n(t) = \int_0^t a_n(s)ds - t$ is the difference of two continuous, non-decreasing functions. By the Jordan Decomposition, see e.g. [15, Proposition 22, p.236], F_n is a continuous process of bounded variation.

We remember from the definition of the continuous-time breadth-first walk in (3.11) that Z_n is the sum of the constant downward stream and jumps of size 1 for every new edge. By the Jordan Decomposition, Z_n is of bounded variation and the jumps make it a càdlàg process. Since $M_n = Z_n - F_n$ is the difference of two functions of bounded variation, it is again of bounded variation. Since F_n is continuous and Z_n is càdlàg, M_n is càdlàg. $\square$

Having obtained a precise definition of F_n, we shift our focus to M_n, the martingale observed in Lemma 3.2. The following statement proves a similar decomposition of the squared martingale.

Lemma 3.5. *The decomposition*

$$M_n^2 = Q_n + G_n \tag{3.17}$$

holds, where Q_n is a martingale and G_n is defined by

$$G_n(t) := \int_0^t a_n(s)ds = F_n(t) + t \tag{3.18}$$

with a_n defined in (3.16).

Note. G_n is a continuous process.

Proof: Similar to the proof of the previous Lemma, we will show that $M_n^2 - G_n$ is a martingale. We first have to introduce the quadratic variation of M_n. Let $\Pi_n = \{t_{n,0}, \ldots, t_{n,k_n}\}$ be a sequence of partitions of the interval $[0,t]$ with $|\Pi_n| \to 0$ as $n \to \infty$, where $|\Pi| := \max_{t_i, t_{i-1} \in \Pi}(t_i - t_{i-1})$ is the mesh of the partition. Then, by [16, Theorem 21.70, p.471], we have

$$\sum_{t_i, t_{i-1} \in \Pi_n} (M_n(t_i) - M_n(t_{i-1}))^2 \to_p [M_n]_t$$

as $n \to \infty$, where $[M_n]_t$ is the so-called quadratic variation of M_n and $M_n^2 - [M_n]$ is a martingale. By [16, Remark 21.59, p.468], for any continuous process of bounded variation $X(t)$ we have $[X]_t = 0$ for all t.

The process M_n is right-continuous and of bounded variation. Therefore the quadratic variation vanishes on intervals where M_n is continuous, which leaves the jumps as the only discontinuities:

$$[M_n]_t = \sum_{0 \le s \le t} (\Delta M_n(s))^2,$$

where $\Delta M_n(s) := M_n(s) - M_n(s-)$ are the jumps of M_n. Therefore

$$M_n^2 - G_n = \underbrace{(M_n^2 - [M_n])}_{\text{martingale}} + ([M_n] - G_n),$$

and to prove (3.17) it suffices to show that $[M_n] - G_n$ is a martingale. Since F_n is continuous, the jumps of M_n are exactly the jumps of Z_n. Note that the jumps $\Delta Z_n(s)$ can take one of two values: 1 if there is a jump of size 1 at time s, 0 otherwise. From this, we conclude that

$$\begin{aligned}
[M_n]_t - G_n(t) &= \sum_{0 \le s \le t} (\Delta M_n(s))^2 - G_n(t) \\
&= \sum_{0 \le s \le t} (\Delta Z_n(s))^2 - G_n(t) \\
&= \sum_{0 \le s \le t} \Delta Z_n(s) - G_n(t) \\
&= \text{Number of jumps of } Z_n \text{ in } [0,t] - G_n(t) \\
&= (\text{Number of jumps of } Z_n \text{ in } [0,t] - t) - F_n(t) \\
&= Z_n(t) - F_n(t),
\end{aligned}$$

which is a martingale by Lemma 3.2. □

3.2.2 Convergence of rescaled processes

We state three technical lemmas describing the asymptotic behaviour of the processes established in the previous section. For proofs of these lemmas see Section 3.2.3.

Lemma 3.6. *For F_n defined in Lemma 3.2 and fixed $s_0 \geq 0$, we have*

$$n^{-1/3} \sup_{s \leq n^{2/3} s_0} \left| F_n(s) + \frac{s^2}{2} n^{-1} - s\theta n^{-1/3} \right| \to_p 0 \tag{3.19}$$

as $n \to \infty$.

Lemma 3.7. *For G_n defined in Lemma 3.5 and fixed $s_0 \geq 0$, we have*

$$n^{-2/3} G_n(n^{2/3} s_0) \to_p s_0 \tag{3.20}$$

as $n \to \infty$.

Lemma 3.8. *For M_n defined in Lemma 3.2 and fixed $s_0 \geq 0$, we have*

$$n^{-2/3} \mathbb{E} \left[\sup_{s \leq n^{2/3} s_0} |M_n(s) - M_n(s-)|^2 \right] \to 0, \tag{3.21}$$

as $n \to \infty$.

We now define the rescaled processes

$$\begin{aligned} \bar{M}_n(s) &:= n^{-1/3} M_n(n^{2/3} s), \\ \bar{F}_n(s) &:= n^{-1/3} F_n(n^{2/3} s), \\ \bar{Q}_n(s) &:= n^{-2/3} Q_n(n^{2/3} s), \\ \bar{G}_n(s) &:= n^{-2/3} G_n(n^{2/3} s), \end{aligned} \tag{3.22}$$

to fit the previously rescaled process $\bar{Z}_n$ in Theorem 3.1, such that

$$\begin{aligned} \bar{Z}_n(s) &= \bar{M}_n(s) + \bar{F}_n(s), \\ \bar{M}_n^2(s) &= \bar{Q}_n(s) + \bar{G}_n(s). \end{aligned} \tag{3.23}$$

Rescaling Lemmas 3.6, 3.7 and 3.8 gives us

$$\sup_{s \leq s_0} |\bar{F}_n(s) - \rho(s)| \to_p 0, \tag{3.24}$$

where $\rho(s) = st - \frac{1}{2}s^2$,

$$\bar{G}_n(s) \to_p s_0, \tag{3.25}$$

and

$$\mathbb{E} \left[\sup_{s \leq s_0} |\bar{M}_n(s) - \bar{M}_n(s-)|^2 \right] \to 0. \tag{3.26}$$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1: Let again $\mathcal{F}_t^n = \sigma\{Z_n(s), s \leq t\}$ be the filtration generated by Z_n. The rescaling of M_n and G_n maintains martingale properties, so $\bar{M}_n$ and $\bar{Q}_n = \bar{M}_n^2 - \bar{G}_n$ are still $\{\mathcal{F}_t^n\}$-martingales. We apply Theorem 2.21 to $\bar{M}_n$ and $\bar{G}_n$. The continuity of $\bar{G}_n$ satisfies condition (2.4), condition (2.5) holds from (3.26) and (3.25) gives condition (2.6), with $c(s) = s$.

Therefore $\bar{M}_n(s) \to_d W(s)$, the standard Brownian motion, and using (3.24) we obtain

$$\bar{Z}_n(s) = \bar{M}_n(s) + \bar{F}_n(s) \to_d W(s) + \rho(s) = W^\theta(s).$$

□

3.2.3 Proofs of technical lemmas

In this section we prove the previously stated Lemmas 3.6, 3.7 and 3.8. We begin with two further auxiliary lemmas.

Lemma 3.9. *For all $\sigma_0 \in \mathbb{R}_+$ and F_n defined in Lemma 3.2 we have*

$$\left| F_n(s) + \frac{s^2}{2n} - \frac{s\theta}{n^{1/3}} + \frac{s^2\theta}{2n^{4/3}} \right| \leq \frac{4s}{n} \max_{u \leq s} |Z_n(u)| + O\left(\frac{s}{n}\right)$$

for all $s \leq s_0 n^{2/3}$ for sufficiently large n.

Proof: Defining $a_n'(s)$ as

$$\begin{aligned} a_n'(s) &:= a_n(s)\left(1 - (s - \lfloor s \rfloor)p_n\right) \\ &= \left(n - s - \zeta_n^{-1}(\lceil s \rceil) - Z_n(s)\right) p_n, \end{aligned} \tag{3.27}$$

we show that $a_n(s)$ and $a_n'(s)$ become asymptotically close, uniformly in s, for large n:

$$\begin{aligned} |a_n(s) - a_n'(s)| &= |a_n(s)\left(1 - (1 - (s - \lfloor s \rfloor)p_n)\right)| \\ &= \left| \frac{(n - s - \zeta_n^{-1}(\lceil s \rceil) - Z_n(s))p_n}{1 - (s - \lfloor s \rfloor)p_n} (s - \lfloor s \rfloor)p_n \right| \\ &= \left| \underbrace{\frac{p_n n - p_n(s + \zeta_n^{-1}(\lceil s \rceil) + Z_n(s))}{1 - (s - \lfloor s \rfloor)p_n}}_{\leq 5} \underbrace{(s - \lfloor s \rfloor)p_n}_{=O(n^{-1})} \right| \\ &= O\left(n^{-1}\right). \end{aligned} \tag{3.28}$$

The last step follows from $p_n = O\left(n^{-1}\right)$ and $|Z_n(s)|, |\zeta_n^{-1}(s)| \leq n$ for all $s \leq s_0 n^{2/3}$ and n. We substitute the definition of p_n in (3.27) to expand

$$\begin{aligned} a_n'(s) - 1 &= \left(n - s - \zeta_n^{-1}(\lceil s \rceil) - Z_n(s)\right)\left(n^{-1} + \theta n^{-4/3}\right) - 1 \\ &= tn^{-1/3} - sn^{-1} - s\theta n^{-4/3} \\ &\quad - \left(\zeta_n^{-1}(\lceil s \rceil) + Z_n(s)\right)\left(n^{-1} + \theta n^{-4/3}\right). \end{aligned}$$

Therefore

$$\begin{aligned}
\left|a_n'(s) - 1 + \frac{s}{n} - \frac{\theta}{n^{1/3}} + \frac{s\theta}{n^{4/3}}\right| &= \left|\frac{\zeta_n^{-1}(\lceil s\rceil) + Z_n(s)}{n}\left(1 + \frac{\theta}{n^{1/3}}\right)\right| \\
&\leq 2\left|\frac{\zeta_n^{-1}(\lceil s\rceil) + Z_n(s)}{n}\right| \\
&\leq 2\frac{\zeta_n^{-1}(\lceil s\rceil) + |Z_n(s)|}{n},
\end{aligned} \tag{3.29}$$

for $n^{1/3} \geq |\theta|$. Integrating the inner part of the left-hand side over s yields

$$\int_0^s \left(a_n'(u) - 1 + \frac{u}{n} - \frac{\theta}{n^{1/3}} + \frac{u\theta}{n^{4/3}}\right) du = \int_0^s \left(a_n'(u) - 1\right) du + \frac{s^2}{2n} - \frac{s\theta}{n^{1/3}} + \frac{s^2\theta}{2n^{4/3}},$$

and from (3.28) we know

$$\left|\int_0^s (a_n'(u) - 1) du - F_n(s)\right| = O\left(\frac{s}{n}\right).$$

Using (3.10) and (3.29), the following inequalities hold for sufficiently large n:

$$\begin{aligned}
&\left|F_n(s) + \frac{s^2}{2n} - \frac{s\theta}{n^{1/3}} + \frac{s^2\theta}{2n^{4/3}}\right| \\
&= \left|\int_0^s a_n'(u) - 1 + \frac{u}{n} - \frac{\theta}{n^{1/3}} + \frac{u\theta}{n^{4/3}} du\right| + O\left(\frac{s}{n}\right) \\
&\leq \int_0^s \left|a_n'(u) - 1 + \frac{u}{n} - \frac{\theta}{n^{1/3}} + \frac{u\theta}{n^{4/3}}\right| du + O\left(\frac{s}{n}\right) \\
&\leq \int_0^s 2\frac{\zeta_n^{-1}(\lceil u\rceil) + |Z_n(u)|}{n} du + O\left(\frac{s}{n}\right) \\
&= \frac{2}{n}\int_0^s \left(1 - \min_{w\leq\lceil u\rceil - 1} Z_n(w) + |Z_n(u)|\right) du + O\left(\frac{s}{n}\right) \\
&= \frac{2}{n}\int_0^s \left(|Z_n(u)| - \min_{w\leq\lceil u\rceil - 1} Z_n(w)\right) du + \frac{2s}{n} + O\left(\frac{s}{n}\right) \\
&\leq \frac{4s}{n}\max_{u\leq s}|Z_n(u)| + O\left(\frac{s}{n}\right),
\end{aligned} \tag{3.30}$$

the last inequality holds since $|\min_{w\leq\lceil u\rceil-1} Z_n(w)| \leq \max_{u\leq s}|Z_n(u)|$ for $u \leq s$. $\square$

Lemma 3.10. *$n^{-1/3}\sup_{s\leq n^{2/3}s_0}|Z_n(s)|$ is stochastically bounded as $n \to \infty$, meaning for all $\varepsilon > 0$ exist $K > 0$ such that*

$$\mathbb{P}\left(n^{-1/3}\sup_{s\leq n^{2/3}s_0}|Z_n(s)| > K\right) < \varepsilon \tag{3.31}$$

for all $n \in \mathbb{N}$.

Proof: This proof will follow a truncation argument. We define two stopping times T_n^* and T_n by

$$T_n^* := \min\{s \mid |Z_n(s)| > Kn^{1/3}\}, \tag{3.32}$$

$$T_n := \min\{T_n^*, n^{2/3}s_0\}, \tag{3.33}$$

for some fixed $K > 0$ and use Markov's inequality to rewrite the left-hand side of (3.31) as

$$\begin{aligned} \mathbb{P}\left(\sup_{s\le n^{2/3}s_0} |Z_n(s)| > Kn^{1/3}\right) &= \mathbb{P}\left(|Z_n(T_n)| > Kn^{1/3}\right) \\ &\le \frac{\mathbb{E}\left[|Z_n(T_n)|\right]}{Kn^{1/3}}. \end{aligned} \tag{3.34}$$

To analyse $\mathbb{E}\left[|Z_n(T_n)|\right]$ we will use the decompositions established in the previous section. Lemma 3.2 gave $Z_n = M_n + F_n$, so

$$\mathbb{E}\left[|Z_n(T_n)|\right] \le \mathbb{E}\left[|M_n(T_n)|\right] + \mathbb{E}\left[|F_n(T_n)|\right]. \tag{3.35}$$

By Lemma 3.5, we have $M_n^2 = Q_n + G_n$ where Q_n is a martingale. The optional sampling theorem (see [16, Theorem 10.11, p.203]) dictates that $\mathbb{E}\left[Q_n(\tau)\right] = 0$ for all stopping times τ, hence

$$\begin{aligned} \mathbb{E}\left[M_n^2(T_n)\right] &= \mathbb{E}\left[Q_n(T_n)\right] + \mathbb{E}\left[G_n(T_n)\right] \\ &= \mathbb{E}\left[G_n(T_n)\right] \\ &= \mathbb{E}\left[\int_0^{T_n} a_n(s)ds\right] \\ &\le \int_0^{n^{2/3}s_0} \mathbb{E}\left[a_n(s)\right] ds. \end{aligned}$$

By the definition of a_n in (3.16), we have

$$\begin{aligned} a_n(s) &= (n - \nu_n(s))\frac{p_n}{1-(s-\lfloor s \rfloor)p_n} \\ &\le \frac{np_n}{1-(s-\lfloor s \rfloor)p_n} \end{aligned}$$

which is a deterministic function of s. So $\mathbb{E}\left[a_n(s)\right] \le \frac{np_n}{1-(s-\lfloor s \rfloor)p_n}$ and

$$\begin{aligned} \int_0^{n^{2/3}s_0} \mathbb{E}\left[a_n(s)\right] ds &\le \int_0^{n^{2/3}s_0} \frac{np_n}{1-(s-\lfloor s \rfloor)p_n} ds \\ &\le 2n^{2/3}s_0, \end{aligned}$$

where the last inequality holds for n sufficiently large, since $\frac{np_n}{1-(s-\lfloor s \rfloor)p_n} \to 1$ as $n \to \infty$. Now Hölders inequality (see [15, p.113]) gives us

$$\mathbb{E}\left[|M_n(T_n)|\right] \le \sqrt{\mathbb{E}\left[M_n^2(T_n)\right]} \le (2s_0)^{1/2} n^{1/3}. \tag{3.36}$$

We proceed to the analysis of the second term in (3.35). The definition of F_n in (3.15) establishes

$$\begin{aligned}\mathbb{E}\left[|F_n(T_n)|\right] &= \mathbb{E}\left[\left|\int_0^{T_n}(a_n(s)-1)ds\right|\right]\\ &\leq \mathbb{E}\left[\int_0^{T_n}|a_n(s)-1|ds\right]\\ &\leq \mathbb{E}\left[\int_0^{n^{2/3}s_0}|a_n(s)-1|ds\right].\end{aligned}$$

We decompose $|a_n(s)-1|$ by

$$\begin{aligned}|a_n(s)-1| &= \Bigg|a_n(s)+a_n'(s)-a_n'(s)+\frac{s}{n}-\frac{\theta}{n^{1/3}}+\frac{s\theta}{n^{4/3}}\\ &\qquad -\frac{s}{n}+\frac{\theta}{n^{1/3}}-\frac{s\theta}{n^{4/3}}\Bigg|\\ &\leq \left|a_n(s)-a_n'(s)\right|+\left|\frac{s}{n}-\frac{\theta}{n^{1/3}}+\frac{s\theta}{n^{4/3}}\right|\\ &\qquad +\left|a_n'(s)-1+\frac{s}{n}-\frac{\theta}{n^{1/3}}+\frac{s\theta}{n^{4/3}}\right|\end{aligned}$$

to evaluate

$$\begin{aligned}\mathbb{E}\left[|F_n(T_n)|\right] \leq{}& \mathbb{E}\left[\int_0^{n^{2/3}s_0}\left|a_n(s)-a_n'(s)\right|ds\right]\\ &+\mathbb{E}\left[\int_0^{n^{2/3}s_0}\left|\frac{s}{n}-\frac{\theta}{n^{1/3}}+\frac{s\theta}{n^{4/3}}\right|ds\right]\\ &+\mathbb{E}\left[\int_0^{n^{2/3}s_0}\left|a_n'(s)-1+\frac{s}{n}-\frac{\theta}{n^{1/3}}+\frac{s\theta}{n^{4/3}}\right|ds\right].\end{aligned}$$

Let us look at the terms individually. By (3.28) we have

$$\begin{aligned}\mathbb{E}\left[\int_0^{n^{2/3}s_0}\left|a_n(s)-a_n'(s)\right|ds\right] &\leq n^{2/3}s_0 O\left(n^{-1}\right)\\ &= s_0 O\left(n^{-1/3}\right),\end{aligned}$$

we can estimate the second term by

$$\mathbb{E}\left[\int_0^{n^{2/3}s_0}\left|\frac{s}{n}-\frac{\theta}{n^{1/3}}+\frac{s\theta}{n^{4/3}}\right|ds\right] \leq \frac{1}{2}s_0^2 n^{1/3}+s_0|\theta|n^{1/3}+\frac{1}{2}s_0^2|\theta|$$

and (3.30) implies

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T_n}\left|a_n'(s)-1+\frac{s}{n}-\frac{\theta}{n^{1/3}}+\frac{s\theta}{n^{4/3}}\right|ds\right]\\
&\le \mathbb{E}\left[\frac{4T_n}{n}\max_{u\le T_n}|Z_n(u)|+O\left(\frac{T_n}{n}\right)\right]\\
&\le \frac{4n^{2/3}s_0}{n}\mathbb{E}\left[\max_{u\le T_n}|Z_n(u)|\right]+O\left(\frac{s_0n^{2/3}}{n}\right)\\
&\le 4s_0K+s_0O\left(n^{-1/3}\right),
\end{aligned}
\tag{3.37}
$$

where the last inequality follows from the definition of T_n in (3.33), which assures $|Z_n(s)|\le Kn^{1/3}$ for all $s\le T_n$.

Summing these three terms we arrive at

$$
\begin{aligned}
\mathbb{E}\left[|F_n(T_n)|\right]\le 4s_0K+s_0|\theta|O\left(n^{1/3}\right)+\frac{1}{2}s_0^2n^{1/3}+s_0|\theta|n^{1/3}\\
+\frac{1}{2}s_0^2|\theta|+s_0O\left(n^{-1/3}\right).
\end{aligned}
\tag{3.38}
$$

We combine (3.36) and (3.38), which results in the following upper bound for large n:

$$
\mathbb{E}\left[|Z_n(T_n)|\right]\le \alpha n^{1/3}+4s_0K,
$$

where $\alpha=\alpha(s_0,\theta)$ does not depend on n and K. Substituting $\mathbb{E}\left[|Z_n(T_n)|\right]$ in (3.34) we arrive at

$$
\mathbb{P}(\sup_{s\le n^{2/3}s_0}|Z_n(s)|>Kn^{1/3})\le\frac{\alpha}{K}+\frac{4s_0}{n^{1/3}}
$$

which proves (3.31) by choosing K and n sufficiently large. □

Proof of Lemma 3.6: We can now proceed to estimate (3.19) using Lemmas 3.9 and 3.10:

$$
\begin{aligned}
&n^{-1/3}\sup_{s\le n^{2/3}s_0}\left|F_n(s)+\frac{s^2}{2}n^{-1}+s\theta n^{-1/3}\right|\\
&\le n^{-1/3}\sup_{s\le n^{2/3}s_0}\left|F_n(s)+\frac{s^2}{2n}-\frac{s\theta}{n^{1/3}}+\frac{s^2\theta}{2n^{4/3}}\right|+\frac{\theta s_0^2}{2n^{1/3}}\\
&\le n^{-1/3}\sup_{s\le n^{2/3}s_0}\left(\frac{4s}{n}\max_{u\le s}|Z_n(u)|\right)+O\left(n^{-1/3}\right)\\
&\le n^{-1/3}4s_0n^{-1/3}\sup_{s\le n^{2/3}s_0}|Z_n(s)|+O\left(n^{-1/3}\right)\\
&\le 4s_0n^{-2/3}\sup_{s\le n^{2/3}s_0}|Z_n(s)|+O\left(n^{-1/3}\right).
\end{aligned}
$$

To establish the Lemma it is now sufficient to prove

$$n^{-2/3} \sup_{s \leq n^{2/3}s_0} |Z_n(s)| \to_p 0. \tag{3.39}$$

By Lemma 3.10 we know that for any $\varepsilon > 0$ there is a $K > 0$ such that for n sufficiently large

$$\mathbb{P}\left(n^{-1/3} \sup_{s \leq n^{2/3}s_0} |Z_n(s)| > K\right) < \varepsilon.$$

It is easily seen that this suffices to establish (3.39) by computing

$$\begin{aligned}\mathbb{P}\left(n^{-2/3} \sup_{s \leq n^{2/3}s_0} |Z_n(s)| > \delta\right) &= \mathbb{P}\left(n^{-1/3} \sup_{s \leq n^{2/3}s_0} |Z_n(s)| > \delta n^{1/3}\right) \\ &\leq \mathbb{P}\left(n^{-1/3} \sup_{s \leq n^{2/3}s_0} |Z_n(s)| > K\right) \\ &< \varepsilon,\end{aligned}$$

which holds for fixed $\varepsilon, \delta > 0$ and $n \geq (\delta K)^3$. □

Proof of Lemma 3.7: Since $G_n(s) = F_n(s) + s$, we can rewrite (3.20) as

$$n^{-2/3} F_n(n^{2/3}s_0) \to_p 0. \tag{3.40}$$

We will show that

$$n^{-1/3} F_n(n^{-2/3}s_0) + \frac{1}{2}s_0^2 - s_0\theta \to_p 0. \tag{3.41}$$

This implies

$$n^{-1/3}\left(n^{-1/3} F_n(n^{-2/3}s_0) + \frac{1}{2}s_0^2 - s_0\theta\right) \to_p 0$$

which, since $\frac{1}{2}s_0^2 - s_0\theta$ is a constant in n, proves (3.40).

Define $\phi_n(s) := \frac{1}{2}n^{-4/3}s^2 - n^{-2/3}s\theta$. Now $\phi_n(n^{2/3}s_0) = \frac{1}{2}s_0^2 - s_0\theta$ and

$$\begin{aligned}\left|n^{-1/3} F_n(n^{-2/3}s_0) + \frac{1}{2}s_0^2 - s_0\theta\right| &= \left|n^{-1/3} F_n(n^{-2/3}s_0) + \phi(n^{2/3}s_0)\right| \\ &\leq \sup_{s \leq n^{2/3}s_0} \left|n^{-1/3} F_n(s) + \phi(s)\right| \\ &= n^{-1/3} \sup_{s \leq n^{2/3}s_0} \left|F_n(s) + \frac{s^2}{2}n^{-1} - s\theta n^{-1/3}\right| \\ &\to_p 0\end{aligned}$$

by Lemma 3.6. This gives (3.41) and completes the proof. □

Proof of Lemma 3.8: As previously discussed in the proof of Lemma 3.5, the jumps of M_n are exactly the jumps of Z_n and therefore have size 1. The Lemma follows immediately. □

4 Surplus edges

The first goal of this chapter will be to examine under which circumstances surplus edges can arise during the breadth-first walk and consequently finding an expression for the probability of encountering one. We then prove the joint convergence of $\bar{Z}_n^\theta$ and the surplus edge counting process to W^θ and some limit process dependent on the realisation of W^θ.

4.1 Counting surplus edges

We begin by describing a way to analyse the appearance of surplus edges. In Chapter 3 we defined the breadth-first walk Z_n, which counted new connections to previously not connected vertices. We remind ourselves that a surplus edge in a graph $\mathscr{G}(n, n^{-1} + \theta n^{-4/3})$ appears if a vertex forms a new connection to another vertex which already has opened connections to some explored node. We associate with $\mathscr{G}$ a counting process $(N_n^\theta(s), 0 \leq s \leq n)$, with $N_n^\theta(0) = 0$, which increases by 1 at each appearance of a surplus edge.

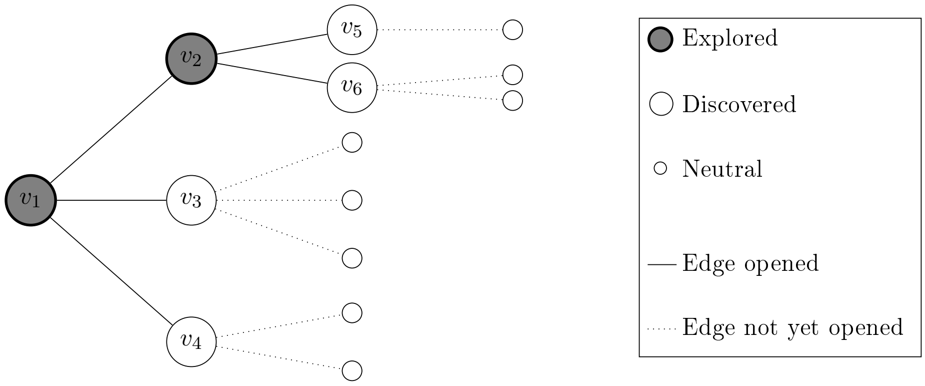


Figure 4.1: A sample component

To find an expression for the number of vertices which are able to open such excess connections, consider the breadth-first walk on the graph of Figure 4.1 at time $s = 2$. The children of v_1 (v_2 to v_4) and the children of v_2 (v_5 and v_6) are already discovered. We are interested in surplus edges to v_3. Since v_1 to v_6 are unable to form edges to become children of v_3, fr the number of ineligible vertices we have $\eta(2) = |\{v_1, \ldots, v_6\}| = 6$. Of these vertices, v_1 and v_2 are already explored and every connection to neighbouring nodes is known. Vertex v_3 can not have an edge to itself, so only v_4, v_5 and v_6 are eligible to receive a surplus edge to v_3.

Let us examine what these considerations mean in terms of the breadth-first walk Z_n^θ. When starting at a new component there are no vertices eligible for a surplus edge. For each new vertex found as a member of this component we have one additional eligible node and with each step taken, one more vertex is explored and thus can no longer receive a surplus edge. The number of vertices eligible for an excess edge therefore corresponds to the level of the breadth-first walk above its past minimum, which is attained at the beginning of the component. Hence we will expect the probability of encountering an excess edge at time s to be proportionate to

$$B_n^\theta(s) := Z_n^\theta(s) - \min_{u \le s} Z_n^\theta(u). \tag{4.1}$$

Rescaling the counting process appropriately this probability should scale to

$$\bar{B}_n^\theta(s) := \bar{Z}_n^\theta(s) - \min_{u \le s} \bar{Z}_n^\theta(u), \tag{4.2}$$

which converges in distribution to

$$B^\theta(s) = W^\theta(s) - \min_{u \le s} W^\theta(u).$$

In chapter 3 we examined a similar process, G_n, which increased by one for each appearance of a new edge to a previously not connected vertex. Lemma 3.3 established that $G_n(s) = \int_0^s a_n(u)du$ with

$$a_n(s) = (n - \eta_n(s)) \frac{p_n}{1 - (s - \lfloor s \rfloor)p_n},$$

where $\eta_n(s)$ is the number of vertices ineligible to become a child of $v_{\lceil s \rceil}$ at time s. In terms of counting processes, we call a_n the rate or conditional intensity of G_n. It is evident that N_n^θ will have a similar rate, substituting the number of vertices eligible to become a child of $v_{\lceil s \rceil}$ with the number of vertices eligible to receive a surplus edge to $v_{\lceil s \rceil}$.

In general, at time $i-1$, the first i vertices are ineligible for a surplus edge to v_i. The remaining $\eta_n(i-1)-i$ vertices are candidates for an excess edge opening with probability p_n. Therefore, the counting process N_n^θ has rate

$$\lambda(s) = (\eta_n(\lfloor s \rfloor) - \lfloor s \rfloor) \frac{p_n}{1 - (s - \lfloor s \rfloor)p_n}. \tag{4.3}$$

Note that this rate is only exact for the chance of encountering exactly one surplus edge, but an overestimation for multiple excess edges. If we encounter a surplus edge at some time $s \in [i-1, i)$, the number of eligible vertices should decrease by one. However, the number of ineligible vertices in (4.3) is constant for all $s \in [i-1, i)$. For ease of computation we will continue with this overestimation and later argue that the difference becomes negligible as $n \to \infty$.

Lemma 3.3 established $\eta_n(s) = s + \zeta_n^{-1}(\lceil s \rceil) + Z_n(s)$ and using (3.10) we can rewrite

$$\begin{aligned} \eta_n(\lfloor s \rfloor) - \lfloor s \rfloor &= \lfloor s \rfloor - \zeta_n^{-1}(\lfloor s \rfloor + 1) + Z_n^\theta(\lfloor s \rfloor) - \lfloor s \rfloor \\ &= 1 - \min_{u \le \lfloor s \rfloor} Z_n^\theta(u) + Z_n^\theta(\lfloor s \rfloor), \end{aligned}$$

and the conditional intensity becomes

$$\lambda(s) = \left(1 - \min_{u \le \lfloor s \rfloor} Z_n^\theta(u) + Z_n^\theta(\lfloor s \rfloor)\right) \frac{p_n}{1 - (s - \lfloor s \rfloor)p_n}.$$

We now rescale the counting process via

$$\bar{N}_n^\theta(s) = N_n^\theta(n^{2/3}s). \tag{4.4}$$

We calculate the rate of this rescaled process. Recall that the conditional intensity $\bar{\lambda}(s)$ of the process $\bar{N}_n^\theta(s)$ must satisfy

$$\mathbb{E}\left[\bar{N}_n^\theta(s)\right] = \int_0^s \bar{\lambda}(u)du.$$

Using (4.4) we evaluate the integral above in terms of $\lambda(s)$:

$$\begin{aligned}
\mathbb{E}\left[\bar{N}_n^\theta(s)\right] &= \mathbb{E}\left[N_n^\theta(n^{2/3}s)\right] \\
&= \int_0^{n^{2/3}s} \lambda(u)du \\
&= \int_0^s n^{2/3}\lambda(n^{2/3}u)du.
\end{aligned}$$

Comparing both integrands gives us

$$\begin{aligned}
\bar{\lambda}(s) &= n^{2/3}\lambda(n^{2/3}s) \\
&= n^{2/3}\frac{1 - \min_{u \le n^{2/3}s} Z_n^\theta(u) + Z_n^\theta(n^{2/3}s)}{1 - (n^{2/3}s - \lfloor n^{2/3}s \rfloor)p_n} p_n \\
&= n^{2/3}\frac{1 - n^{1/3}\min_{u \le s} \bar{Z}_n^\theta(u) + n^{1/3}\bar{Z}_n^\theta(s)}{1 - (n^{2/3}s - \lfloor n^{2/3}s \rfloor)p_n} p_n \\
&= np_n \frac{n^{-1/3} - \min_{u \le s} \bar{Z}_n^\theta(u) + \bar{Z}_n^\theta(s)}{1 - (n^{2/3}s - \lfloor n^{2/3}s \rfloor)p_n}.
\end{aligned} \tag{4.5}$$

Since $np_n \to 1$ and $\left|n^{2/3}s - \lfloor n^{2/3}s \rfloor\right| < 1$ for all s and n, this rate becomes asymptotically close to $\bar{Z}_n^\theta(s) - \min_{u \le s} \bar{Z}_n^\theta(u)$ as $n \to \infty$.

By Theorem 3.1 we have $\bar{Z}_n^\theta \to_d W^\theta$, so

$$\bar{\lambda}(s) \to_d W^\theta(s) - \min_{u \le s} W^\theta(u) = B^\theta(s). \tag{4.6}$$

The rate of the counting process $\bar{N}_n^\theta$ therefore converges in distribution to B^θ.

Figure 4.2 shows a realization of B^θ and a counting process, say N, with rate B^θ, represented as marks on the x-axis. For any interval I the number of points N encounters in I, here $N(I) = 3$, has Poisson distribution with parameter $\int_I B^\theta(s)ds$.

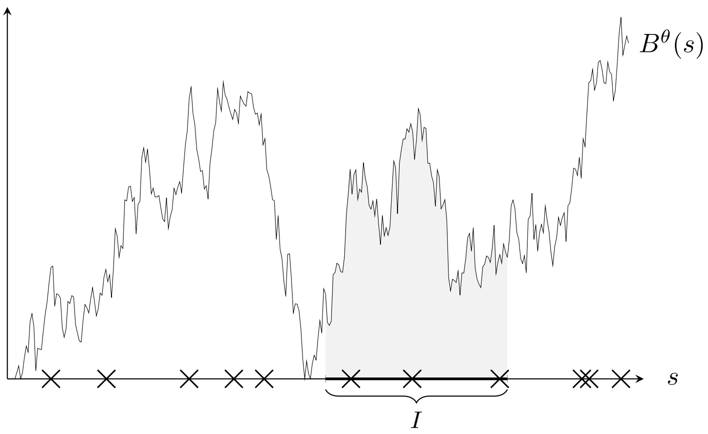


Figure 4.2: A counting process with conditional intensity B^θ.

4.2 Joint convergence of $(\bar{Z}_n^\theta, \bar{N}_n^\theta)$

We know that $\bar{Z}_n^\theta \to_d W^\theta$ and that the rate of $\bar{N}_n^\theta$ converges to B^θ. But is that enough to deduce the convergence in distribution of the highly correlated pair $(\bar{Z}_n^\theta, \bar{N}_n^\theta)$ to (W^θ, N^θ)? The remainder of this chapter is dedicated to proving this convergence.

Theorem 4.1. *For the previously defined processes $\bar{Z}_n^\theta$ and $\bar{N}_n^\theta$, the joint weak convergence*

$$(\bar{Z}_n^\theta(s), \bar{N}_n^\theta(s); s \geq 0) \to_d (W^\theta(s), N^\theta(s); s \geq 0)$$

holds, where N^θ is the counting process with conditional intensity B^θ, i.e. the process for which

$$N^\theta(s) - \int_0^s B^\theta(u)du$$

is a martingale.

Proof: We want to show that

$$(\bar{Z}_n^\theta, \bar{N}_n^\theta) \to_d (W^\theta, N^\theta),$$

meaning

$$\mathbb{E}\left[f(\bar{Z}_n^\theta, \bar{N}_n^\theta)\right] \to \mathbb{E}\left[f(W^\theta, N^\theta)\right]$$

for all continuous, bounded functions $f : D[0,T]^2 \to \mathbb{R}$ as $n \to \infty$.

We begin with a quick overview of the proof.

1. We show that the process $(\bar{Z}_n^\theta, \bar{N}_n^\theta)$ is tight, which allows us to restrict the expectation to $\mathbb{E}\left[f(\bar{Z}_n^\theta, \bar{N}_n^\theta) \mid \mathcal{E}_C\right]$ where $\mathcal{E}_C = \{(\bar{Z}_n^\theta, \bar{N}_n^\theta) \in C\}$ for some compact $C \subseteq D[0,T]^2$.

2. Since $\bar{Z}_n^\theta \to_d W^\theta$ we can define them on the same probability space such that $||\bar{Z}_n^\theta(s) - W^\theta(s)||_T < \delta$. Calling this event $\mathcal{E}_\delta$, it suffices to look at $\mathbb{E}\left[f(W^\theta, \bar{N}_n^\theta) \mid \mathcal{E}_C, \mathcal{E}_\delta\right]$.

3. If $||\bar{Z}_n^\theta(s) - W^\theta(s)||_T < \delta$ then with high probability $N_n^\theta = \mathcal{M}_n^\theta$, where $\mathcal{M}_n^\theta$ is a discrete process with binomially distributed increments, dependent on an upscaling of B^θ. It then suffices to prove convergence of $\mathbb{E}\left[f(W^\theta, \mathcal{M}_n^\theta) \mid \mathcal{E}_C, \mathcal{E}_\delta\right]$.

4. Finally we prove $\mathbb{E}\left[f(W^\theta, \mathcal{M}_n^\theta)\right] \to \mathbb{E}\left[f(W^\theta, N^\theta)\right]$ by showing that for almost all realizations of W^θ we have $\mathcal{M}_n^\theta \to_d N^\theta$.

Part 1: First, we will show that $\bar{N}_n^\theta$ is tight as a random process with image in $D[0,T]$ by using Theorem 2.13. We first take a closer look at $\bar{Z}_n^\theta$. We already know that $\bar{Z}_n^\theta \to_d W^\theta$, which implies that $\bar{Z}_n^\theta$ is tight, so for all $\varepsilon > 0$ there exists a compact $K \subset D[0,T]$ such that

$$\inf_{n\in\mathbb{N}} \mathbb{P}\left(\bar{Z}_n^\theta \in K\right) > 1 - \varepsilon.$$

Therefore for all $\varepsilon > 0$ exists $A > 0$ such that for all n

$$\mathbb{P}\left(\sup_{s\leq T} |\bar{Z}_n^\theta(s)| > A\right) < \varepsilon. \tag{4.7}$$

As previously we define

$$B_n^\theta(s) := Z_n^\theta(s) - \min_{u\leq s} Z_n^\theta(u),$$

the process reflecting Z_n^θ at the x-axis and its rescaling

$$\bar{B}_n^\theta(s) := n^{-1/3} B_n^\theta(n^{2/3}s) = \bar{Z}_n^\theta(s) - \min_{u\leq s} \bar{Z}_n^\theta(u).$$

Since $|\bar{B}_n^\theta(s)| \leq 2\max_{u\leq s} |\bar{Z}_n^\theta(u)|$, (4.7) implies

$$\mathbb{P}\left(\sup_{s\leq T} |\bar{B}_n^\theta(s)| > 2A\right) < \varepsilon.$$

Therefore, for all $\varepsilon > 0$ exists an $A > 0$ such that

$$\mathbb{P}\left(\sup_{s\leq T} |B_n^\theta(n^{2/3}s)| \geq An^{1/3}\right) < \varepsilon$$

holds for all $n \in \mathbb{N}$.

We now move to the process $\bar{N}_n^\theta$. Consider the unscaled process N_n^θ at time $i \in [0, n^{2/3}T]$. The increment to its next step is binomially distributed on the number of vertices eligible for a surplus edge:

$$N_n^\theta(i) - N_n^\theta(i-1) \sim \mathrm{Bin}(B_n^\theta(i-1), p_n).$$

As previously established, for all $i \in [0, n^{2/3}T]$ we know

$$B_n^\theta(i) \le \sup_{j \le n^{2/3}T} B_n^\theta(j) \le An^{1/3}$$

with probability greater than $1 - \varepsilon$.

If we condition on the event that $B_n^\theta(i-1) \le An^{1/3}$, a random variable $X_i \sim \mathrm{Bin}(B_n^\theta(i-1), p_n)$ will be stochastically dominated:

$$X_i \le_{\text{st.}} Y_i \sim \mathrm{Bin}(An^{1/3}, p_n). \tag{4.8}$$

Seeing $N_n^\theta(Tn^{2/3})$ as the sum of all its increments, we arrive at

$$N_n^\theta(Tn^{2/3}) \le_{\text{st.}} \sum_{j=1}^{Tn^{2/3}} Y_j, \tag{4.9}$$

where $Y_1, Y_2, \ldots, Y_{Tn^{2/3}} \sim \mathrm{Bin}(An^{1/3}, p_n)$.

We denote by $\mathcal{E}_A$ the event $\sup_{j \le n^{2/3}T} B_n^\theta(j) \le An^{1/3}$ and use the law of total probability to compute

$$\begin{aligned}
\mathbb{P}\left(\bar{N}_n^\theta(T) \ge K\right) &= \mathbb{P}\left(N_n^\theta(n^{2/3}T) \ge Kn^{1/3}\right) \\
&= \mathbb{P}\left(N_n^\theta(n^{2/3}T) \ge Kn^{1/3} \mid \mathcal{E}_A\right) \mathbb{P}(\mathcal{E}_A) \\
&\quad + \mathbb{P}\left(N_n^\theta(n^{2/3}T) \ge Kn^{1/3} \mid \neg\mathcal{E}_A\right) \mathbb{P}(\neg\mathcal{E}_A) \\
&\le \mathbb{P}\left(N_n^\theta(n^{2/3}T) \ge Kn^{1/3} \mid \mathcal{E}_A\right) + \varepsilon,
\end{aligned} \tag{4.10}$$

which holds since $\mathbb{P}(\neg\mathcal{E}_A) < \varepsilon$.

Since this probability is now conditioned on $\mathcal{E}_A$, the stochastic dominance (4.9) holds. Markov's inequality then gives

$$\begin{aligned}
\mathbb{P}\left(N_n^\theta(n^{2/3}T) \ge Kn^{1/3} \mid \mathcal{E}_A\right) &\le \mathbb{P}\left(\sum_{j=1}^{Tn^{2/3}} Y_j \ge K\right) \\
&\le \frac{1}{K} Tn^{2/3} \mathbb{E}\left[Y_1\right] \\
&= \frac{1}{K} Tn^{2/3} p_n An^{1/3} \\
&= \frac{1}{K} np_n TA \\
&\le \frac{1}{K} CTA
\end{aligned}$$

for some constant $C \in \mathbb{R}$, since $np_n \to 1$ as $n \to \infty$. So

$$\mathbb{P}\left(\bar{N}_n^\theta(T) \ge K\right) \le \varepsilon + \frac{1}{K} CTA \le 2\varepsilon$$

for large K, which satisfies the first condition of Theorem 2.13.

To show that $\bar{N}_n^\theta$ satisfies the second condition,

$$\lim_{\delta\to 0}\limsup_{n\in\mathbb{N}} \mathbb{P}(w'_{\bar{N}_n^\theta}(\delta) \geq \varepsilon) = 0, \tag{4.11}$$

let us analyse $w'_{\bar{N}_n^\theta}(\delta)$. For a realization $N = \bar{N}_n^\theta(\omega)$ we have

$$w'_N(\delta) = \inf_{\{t_i\}} \max_{1\leq i\leq k} \sup_{s,t\in[t_{i-1},t_i)} |N(s) - N(t)|.$$

For an interval $[t_{i-1}, t_i)$, $\sup_{s,t\in[t_{i-1},t_i)} |N(s)-N(t)| > 0$ only if there is a jump in $[t_{i-1}, t_i)$. For a set $\{t_i\}$, $\max_{1\leq i\leq k} \sup_{s,t\in[t_{i-1},t_i)} |N(s)-N(t)| = 0$ if there is no jump in any interval $[t_{i-1}, t_i)$. Therefore

$$\inf_{\{t_i\}} \max_{1\leq i\leq k} \sup_{s,t\in[t_{i-1},t_i)} |N(s) - N(t)| = 0$$

if there exists a δ-sparse set $\{t_i\}$ such that all jumps of N happen on a t_i, that is if all jumps of N have at least a distance of δ from each other.

We again use the fact that we can stochastically dominate N_n^θ as in (4.9). For fixed δ, in the interval $[0, Tn^{2/3}]$ we consider all T/δ intervals of length $\delta n^{2/3}$. On these, N_n^θ is stochastically dominated by $\sum_{j=1}^{\delta n^{2/3}} Y_j$ where $Y_j \sim \text{Bin}(An^{1/3}, p_n)$. Therefore

$$\begin{aligned}
&\mathbb{P}\left(\exists \text{ interval with } \geq 2 \text{ points}\right) \\
&\quad \leq \mathbb{P}\left(\exists \text{ interval with } \sum_{j=1}^{\delta n^{2/3}} Y_j \geq 2\right) \\
&\quad = 1 - \mathbb{P}\left(\sum_{j=1}^{\delta n^{2/3}} Y_j \leq 1 \text{ for every interval}\right) \\
&\quad = 1 - \prod_{i=1}^{T/\delta} \mathbb{P}\left(\sum_{j=1}^{\delta n^{2/3}} Y_j \leq 1\right) \\
&\quad = 1 - \mathbb{P}\left(Z \leq 1\right)^{T/\delta}
\end{aligned}$$

where $Z \sim \text{Bin}(An^{1/3}\delta n^{2/3}, p_n) = \text{Bin}(A\delta n, p_n)$.

To analyse (4.11) we substitute $w'_{\bar{N}_n^\theta}(\delta)$ using the variables above and have

$$\begin{aligned}
&\limsup_{n\in\mathbb{N}} \mathbb{P}\left(w'_{\bar{N}_n^\theta}(\delta) \geq \varepsilon\right) \\
&\quad \leq \limsup_{n\in\mathbb{N}} \left(1 - \mathbb{P}\left(Z \leq 1\right)^{T/\delta}\right) \\
&\quad = 1 - \limsup_{n\in\mathbb{N}} \left(\mathbb{P}(Z=0) + \mathbb{P}(Z=1)\right)^{T/\delta} \\
&\quad = 1 - \limsup_{n\in\mathbb{N}} \left((1-p_n)^{A\delta n} + A\delta n p_n (1-p_n)^{A\delta n - 1}\right)^{T/\delta} \\
&\quad = 1 - \left(\exp(-A\delta) + A\delta \exp(-A\delta)\right)^{T/\delta} \\
&\quad = 1 - \exp(-A\delta)^{T/\delta}(1+A\delta)^{T/\delta} \\
&\quad = 1 - \exp(-TA)(1+A\delta)^{T/\delta}.
\end{aligned}$$

Since T and A are constant and $\lim_{\delta\to 0}(1+A\delta)^{T/\delta} = \exp(TA)$ we have

$$\lim_{\delta\to 0} \limsup_{n\in\mathbb{N}} \mathbb{P}(w'_{\bar{N}_n^\theta}(\delta) \geq \varepsilon) = \lim_{\delta\to 0}(1 - \exp(-TA)(1+A\delta)^{T/\delta}) = 0$$

which proves $\bar{N}_n^\theta$ to satisfy the second condition of Theorem 2.13 and therefore $\bar{N}_n^\theta$ to be a tight process in $D[0,T]$. Since both $\bar{Z}_n^\theta$ and $\bar{N}_n^\theta$ are tight, $(\bar{Z}_n^\theta, \bar{N}_n^\theta)$ is tight. Therefore there exists a compact subset $C \subset D[0,T]^2$ such that $\mathbb{P}\left((\bar{Z}_n^\theta, \bar{N}_n^\theta) \in C\right) > 1 - \varepsilon$.

Part 2: In the next step of our proof we will be showing that for all $\varepsilon > 0$,

$$\left|\mathbb{E}[f(\bar{Z}_n^\theta, \bar{N}_n^\theta)] - \mathbb{E}[f(\mathcal{W}^\theta, \bar{\mathcal{N}}_n^\theta)]\right| < \varepsilon$$

for sufficiently large n, where $(\mathcal{W}^\theta, \bar{\mathcal{N}}_n^\theta)$ is a coupling of W^θ and $\bar{N}_n^\theta$ constructed below. Recall that f is a bounded function, so there exists $M > 0$ such that $|f(x,y)| \leq M$ for all $(x,y) \in D[0,T]^2$. We denote by $\mathcal{E}_C$ the event $(Z_n^\theta, \bar{N}_n^\theta) \in C$ and use the law of total expectation to calculate

$$\begin{aligned}
\mathbb{E}[f(\bar{Z}_n^\theta, \bar{N}_n^\theta)] &= \mathbb{E}[f(\bar{Z}_n^\theta, \bar{N}_n^\theta) \mid \mathcal{E}_C]\mathbb{P}(\mathcal{E}_C) \\
&\quad + \mathbb{E}[f(\bar{Z}_n^\theta, \bar{N}_n^\theta) \mid \neg\mathcal{E}_C]\mathbb{P}(\neg\mathcal{E}_C) \\
&\leq \mathbb{E}[f(\bar{Z}_n^\theta, \bar{N}_n^\theta) \mid \mathcal{E}_C]\mathbb{P}(\mathcal{E}_C) + \varepsilon M.
\end{aligned} \tag{4.12}$$

Since $\bar{Z}_n^\theta \to_d W^\theta$, we can use the Skorohod representation theorem (Theorem 2.2) to define random variables $\mathcal{W}^\theta$, $\bar{\mathcal{Z}}_1^\theta$, $\bar{\mathcal{Z}}_2^\theta$, ... on the same probability space, such that $\mathcal{W}^\theta \sim W^\theta$, $\bar{\mathcal{Z}}_i^\theta \sim \bar{Z}_i^\theta$ for all $i \in \mathbb{N}$ and $\bar{\mathcal{Z}}_n^\theta \to_{a.s.} \mathcal{W}^\theta$ as $n \to \infty$. Meaning, since $\mathcal{W}^\theta$ and $\bar{\mathcal{Z}}_n^\theta$ are random variables mapping into function spaces, we have

$$\sup_{s\leq T} |\bar{\mathcal{Z}}_n^\theta(s) - \mathcal{W}^\theta(s)| \to_{a.s.} 0,$$

which implies

$$\sup_{s\le T} |\bar{\mathcal{Z}}_n^\theta(s) - \mathcal{W}^\theta(s)| \to_p 0,$$

so for all $\varepsilon > 0$:

$$\mathbb{P}\left(\sup_{s\le T} |\bar{\mathcal{Z}}_n^\theta(s) - \mathcal{W}^\theta(s)| > \varepsilon\right) \to 0. \tag{4.13}$$

Additionally we define

$$\begin{aligned}
\mathcal{B}^\theta(t) &:= \mathcal{W}^\theta(t) - \min_{s\le t} \mathcal{W}^\theta(s),\\
\bar{\mathcal{B}}_n^\theta(t) &:= \bar{\mathcal{Z}}_n^\theta(t) - \min_{s\le t} \bar{\mathcal{Z}}_n^\theta(s),\\
\mathcal{B}_n^\theta(t) &:= n^{1/3}\bar{\mathcal{B}}_n^\theta(n^{-2/3}t)
\end{aligned} \tag{4.14}$$

as the equivalents of B^θ, $\bar{B}_n^\theta$ and B_n^θ on the same probability space as $\bar{\mathcal{Z}}_n^\theta$ and $\mathcal{W}^\theta$, as well as $\mathcal{N}^\theta$ which is the counting process with rate $\mathcal{B}^\theta$. We define a process $\bar{\mathcal{N}}_n^\theta$ as a counting process with rate

$$\bar{\lambda}'(s) = np_n \frac{n^{-1/3} - \min_{u\le s} \bar{\mathcal{Z}}_n^\theta(u) + \bar{\mathcal{Z}}_n^\theta(s)}{1 - (n^{2/3}s - \lfloor n^{2/3}s \rfloor)p_n},$$

which makes it the equivalent of $\bar{N}_n^\theta$ in this new probability space, the rate of which is defined in (4.5). Define the equivalent to N_n^θ by

$$\mathcal{N}_n^\theta(s) = \bar{\mathcal{N}}_n^\theta(n^{-2/3}s). \tag{4.15}$$

Since $\bar{\mathcal{Z}}_n^\theta \sim \bar{Z}_n^\theta$ we have $\mathcal{N}^\theta \sim N^\theta$, $\bar{\mathcal{N}}_n^\theta \sim \bar{N}_n^\theta$ and $\mathcal{N}_n^\theta \sim N_n^\theta$.

Form here on out we substitute $\mathcal{W}^\theta$, $\bar{\mathcal{Z}}_n^\theta$ and the process defined above for W^θ, $\bar{Z}_n^\theta$ and the original processes and denote by $\mathcal{E}_C$ the event $(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta) \in C$ for a compact $C \subset D[0,T]^2$. By the equality in distribution, the final result on the expectation of these processes will then still hold for the original processes.

We denote by $\mathcal{E}_\delta$ the event $||\mathcal{W}^\theta - \bar{\mathcal{Z}}_n^\theta||_T < \delta$. By (4.13), for sufficiently large n,

$$\mathbb{P}(\mathcal{E}_\delta) > 1 - \varepsilon.$$

Now an argument analogous to (4.12) gives

$$\begin{aligned}
\mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta)] &\le \mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta) \mid \mathcal{E}_C]\mathbb{P}(\mathcal{E}_C) + \varepsilon M\\
&\le \mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta) \mid \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + 2\varepsilon M.
\end{aligned}$$

By the uniform continuity of f on the compact set C, for all $\varepsilon > 0$ we can choose a $\delta > 0$ such that the inequality

$$\begin{aligned}
f(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta) &= f(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta) + f(\mathcal{W}^\theta, \bar{\mathcal{N}}_n^\theta) - f(\mathcal{W}^\theta, \bar{\mathcal{N}}_n^\theta)\\
&\le f(\mathcal{W}^\theta, \bar{\mathcal{N}}_n^\theta) + |f(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta) - f(\mathcal{W}^\theta, \bar{\mathcal{N}}_n^\theta)|\\
&\le f(\mathcal{W}^\theta, \bar{\mathcal{N}}_n^\theta) + \varepsilon
\end{aligned}$$

holds for $||\mathcal{W}^\theta - \bar{\mathcal{Z}}_n^\theta||_T < \delta$. Therefore

$$\begin{aligned}\mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta)] &\leq \mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \bar{\mathcal{N}}_n^\theta) \mid \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + 2\varepsilon M \\ &\leq \mathbb{E}[f(\mathcal{W}^\theta, \bar{\mathcal{N}}_n^\theta) \mid \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + \varepsilon + 2\varepsilon M.\end{aligned} \tag{4.16}$$

Part 3: Our next objective is to establish the convergence $\bar{\mathcal{N}}_n^\theta \to_d \mathcal{N}^\theta$. For this, we describe a process $\mathcal{M}_n^\theta$, which may be thought of as $\mathcal{N}^\theta$ in discrete time. We define $\mathcal{M}_n^\theta$ by

$$\begin{aligned}\mathcal{M}_n^\theta(0) &:= 0, \\ \mathcal{M}_n^\theta(k) &:= \mathcal{M}_n^\theta(k-1) + \xi_k,\end{aligned} \tag{4.17}$$

where $\xi_k \sim \mathrm{Bin}(n^{1/3}\mathcal{B}^\theta(n^{-2/3}k), p_n)$, the discrete steps of $\mathcal{M}_n^\theta$ are dependent on an upscaling of the reflected Brownian motion $\mathcal{B}^\theta$.

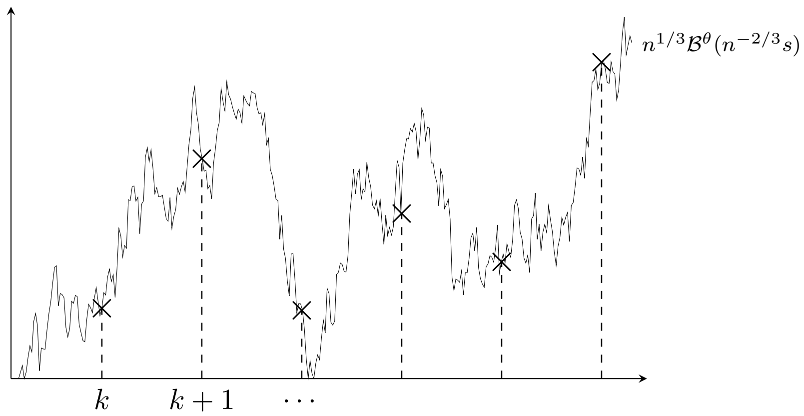


Figure 4.3: The increments of $\mathcal{M}_n^\theta$ are dependent on discrete steps of $\mathcal{B}^\theta$.

We will show that, if $||\bar{\mathcal{Z}}_n^\theta - \mathcal{W}^\theta|| < \delta$, then $\mathcal{M}_n^\theta(k) = \mathcal{N}_n^\theta(k)$ for all $k \leq n^{2/3}T$ with high probability. For this, we redefine both processes using a coupling argument. At step k, let

$$\begin{aligned}\alpha_k &:= \min\{\mathcal{B}_n^\theta(k), n^{1/3}\mathcal{B}^\theta(n^{-2/3}k)\}, \\ \beta_k &:= \max\{\mathcal{B}_n^\theta(k), n^{1/3}\mathcal{B}^\theta(n^{-2/3}k)\},\end{aligned} \tag{4.18}$$

Now define random variables

$$\begin{aligned}\xi_k &\sim \mathrm{Bin}(\alpha_k, p_n), \\ \eta_k &\sim \mathrm{Bin}(\beta_k - \alpha_k, p_n).\end{aligned} \tag{4.19}$$

So $\xi_k + \eta_k \sim \mathrm{Bin}(\beta_k, p_n)$.

Consider the two possibilities at time k: Either $\mathcal{B}_n^\theta(k) \leq n^{1/3}\mathcal{B}^\theta(n^{-2/3}k)$, then $\alpha_k = \mathcal{B}_n^\theta(k)$ and $\beta_k = n^{1/3}\mathcal{B}^\theta(n^{-2/3}k)$, so

$$\begin{aligned}\xi_k &=_d \mathcal{N}_n^\theta(k) - \mathcal{N}_n^\theta(k-1), \\ \xi_k + \eta_k &=_d \mathcal{M}_n^\theta(k) - \mathcal{M}_n^\theta(k-1).\end{aligned}$$

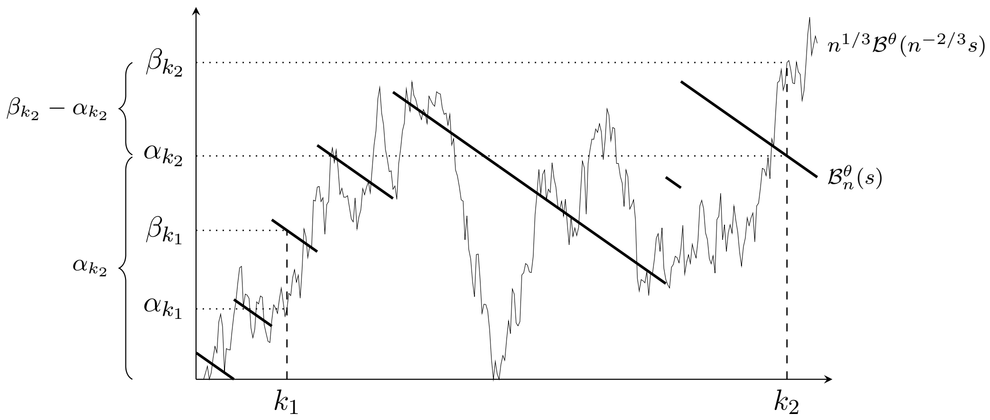


Figure 4.4: The upscaled reflected Brownian motion and the reflected breadth-first walk define α_k and β_k.

Or $\mathcal{B}_n^\theta(k) > n^{1/3}\mathcal{B}^\theta(n^{-2/3}k)$, then $\alpha_k = n^{1/3}\mathcal{B}^\theta(n^{-2/3}k)$ and $\beta_k = \mathcal{B}_n^\theta(k)$, so

$$\xi_k =_d \mathcal{M}_n^\theta(k) - \mathcal{M}_n^\theta(k-1),$$
$$\xi_k + \eta_k =_d \mathcal{N}_n^\theta(k) - \mathcal{N}_n^\theta(k-1).$$

This way, we can define $\mathcal{N}_n^\theta$ and $\mathcal{M}_n^\theta$ by

$$\mathcal{N}_n^\theta(k) - \mathcal{N}_n^\theta(k-1) = \begin{cases} \xi_k & \text{if } \mathcal{B}_n^\theta(k) \leq n^{1/3}\mathcal{B}^\theta(n^{-2/3}k), \\ \xi_k + \eta_k & \text{else} \end{cases}$$

and

$$\mathcal{M}_n^\theta(k) - \mathcal{M}_n^\theta(k-1) = \begin{cases} \xi_k + \eta_k & \text{if } \mathcal{B}_n^\theta(k) \leq n^{1/3}\mathcal{B}^\theta(n^{-2/3}k), \\ \xi_k & \text{else.} \end{cases}$$

By (4.18) and (4.19), these definitions maintain

$$\mathcal{N}_n^\theta(k) - \mathcal{N}_n^\theta(k-1) \sim \mathrm{Bin}(\mathcal{B}_n^\theta(k), p_n),$$
$$\mathcal{M}_n^\theta(k) - \mathcal{M}_n^\theta(k-1) \sim \mathrm{Bin}(n^{1/3}\mathcal{B}^\theta(n^{-2/3}k), p_n).$$

We see that, no matter the relation of $\mathcal{B}_n^\theta(k)$ and $n^{1/3}\mathcal{B}^\theta(n^{-2/3}k)$, the increments of the processes differ only by the random variable η_k.

Conditioning on $||\bar{\mathcal{Z}}_n^\theta - \mathcal{W}^\theta|| < \delta$, for $\mathcal{M}_n^\theta(k) \neq \mathcal{N}_n^\theta(k)$ to hold for some k, there has to have been a step in which the increments of both processes were different. We evaluate

$$\begin{aligned}
&\mathbb{P}\left(\exists k \leq n^{2/3}T : \mathcal{M}_n^\theta(k) \neq \mathcal{N}_n^\theta(k) \mid \mathcal{E}_\delta\right) \\
&\leq \sum_{k=1}^{n^{2/3}T} \mathbb{P}\left(\mathcal{M}_n^\theta(k) - \mathcal{M}_n^\theta(k-1) \neq \mathcal{N}_n^\theta(k) - \mathcal{N}_n^\theta(k-1) \mid \mathcal{E}_\delta\right) \\
&\leq \sum_{k=1}^{n^{2/3}T} \mathbb{P}\left(\eta_k \neq 0 \mid \mathcal{E}_\delta\right) \\
&\leq n^{2/3}T \max_{k \leq n^{2/3}T} \mathbb{P}\left(\eta_k \neq 0 \mid \mathcal{E}_\delta\right).
\end{aligned} \tag{4.20}$$

Since $||Z_n^\theta - \mathcal{W}^\theta|| < \delta$, we know $\beta_k - \alpha_k < \delta n^{1/3}$. Therefore $\eta_k \leq_{\text{st.}} \zeta \sim \text{Bin}(\delta n^{1/3}, p_n)$ for all $k \leq n^{2/3}T$. Using Markov's inequality gives

$$\mathbb{P}(\eta_k \neq 0 \mid \mathcal{E}_\delta) \leq \mathbb{P}(\zeta_k \geq 1) \leq \mathbb{E}\left[\zeta_k\right] = \delta n^{1/3} p_n,$$

and substituting in (4.20) we obtain

$$\begin{aligned}
&\mathbb{P}\left(\exists k \leq n^{2/3}T : \mathcal{M}_n^\theta(k) \neq \mathcal{N}_n^\theta(k) \mid \mathcal{E}_\delta\right) \\
&\leq n^{2/3}T \max_{k \leq n^{2/3}T} \mathbb{P}(\eta_k \neq 0 \mid \mathcal{E}_\delta) \\
&\leq n^{2/3}T\delta n^{1/3} p_n \\
&\leq n p_n T\delta \\
&\leq 2T\delta
\end{aligned}$$

for large n.

We now define $\bar{\mathcal{M}}_n^\theta(s) = n^{-1/3}\mathcal{M}_n^\theta(n^{2/3}s)$ and continue the estimation from (4.16), which yields

$$\begin{aligned}
\mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \mathcal{N}_n^\theta)] &\leq \mathbb{E}[f(\mathcal{W}^\theta, \mathcal{N}_n^\theta) \mid \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + \varepsilon + 2M\varepsilon \\
&\leq \mathbb{E}[f(\mathcal{W}^\theta, \bar{\mathcal{M}}_n^\theta) \mid \mathcal{E}_C, \mathcal{E}_\delta]\mathbb{P}(\mathcal{E}_C)\mathbb{P}(\mathcal{E}_\delta) + 2\delta TM + \varepsilon + 2M\varepsilon \\
&\leq \mathbb{E}[f(\mathcal{W}^\theta, \bar{\mathcal{M}}_n^\theta)] + 2\delta TM + \varepsilon + 2M\varepsilon.
\end{aligned}$$

This inequality holds for all bounded, continuous functions f. Therefore it holds for $-f$ as well, which implies

$$\begin{aligned}
&\mathbb{E}[-f(\bar{\mathcal{Z}}_n^\theta, \mathcal{N}_n^\theta)] \leq \mathbb{E}[-f(\mathcal{W}^\theta, \bar{\mathcal{M}}_n^\theta)] + 2\delta TM + \varepsilon + 2M\varepsilon \\
\iff \quad &\mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \mathcal{N}_n^\theta)] \geq \mathbb{E}[f(\mathcal{W}^\theta, \bar{\mathcal{M}}_n^\theta)] - 2\delta TM - \varepsilon - 2M\varepsilon,
\end{aligned}$$

and therefore

$$\left|\mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \mathcal{N}_n^\theta)] - \mathbb{E}[f(\mathcal{W}^\theta, \bar{\mathcal{M}}_n^\theta)]\right| \leq 2\delta TM + \varepsilon + 2M\varepsilon. \tag{4.21}$$

Part 4: We now show that for any realization of $\mathcal{W}^\theta$ we have $\bar{\mathcal{M}}_n^\theta \to_d \mathcal{N}^\theta$. Fix $\mathcal{W}^\theta = \mathcal{W}^\theta(\omega)$ and therefore $\mathcal{B}^\theta = \mathcal{B}^\theta(\omega)$. Since $\mathcal{N}_n^\theta$ is tight and $\mathcal{N}_n^\theta$ is equal to $\mathcal{M}_n^\theta$ with high probability, $\mathcal{M}_n^\theta$ and $\bar{\mathcal{M}}_n^\theta$ are tight and it suffices proving convergence in finite dimensional distributions, in this case for all $0 \le s_1 < s_2 < \cdots < s_l \le T$:

$$\begin{aligned}&\mathbb{P}(\bar{\mathcal{M}}_n^\theta(s_1) = k_1, \dots, \bar{\mathcal{M}}_n^\theta(s_l) = k_l) \\ &\quad \to \mathbb{P}(\mathcal{N}^\theta(s_1) = k_1, \dots, \mathcal{N}^\theta(s_l) = k_l)\end{aligned} \tag{4.22}$$

as $n \to \infty$.

Recall that $\mathcal{N}^\theta$ is a Poisson point process, continuous on $\mathbb{R}$ with rate $\mathcal{B}^\theta$, therefore the increments of $\mathcal{N}^\theta$ are independent and for all $a < b$:

$$\mathcal{N}^\theta(b) - \mathcal{N}^\theta(a) \sim \text{Poi}\left(\int_a^b \mathcal{B}^\theta(s)ds\right).$$

In contrast, $\mathcal{M}_n^\theta$ is a discrete process whose increments are defined by $\mathcal{B}^\theta$ at integer times, that is for all $k \le n^{2/3}T$:

$$\mathcal{M}_n^\theta(k) - \mathcal{M}_n^\theta(k-1) \sim \text{Bin}\left(n^{1/3}\mathcal{B}^\theta(n^{-2/3}k), p_n\right). \tag{4.23}$$

We can evaluate the distribution of the increments of $\bar{\mathcal{M}}_n^\theta$ by

$$\bar{\mathcal{M}}_n^\theta(s) - \bar{\mathcal{M}}_n^\theta(s-1) = n^{-1/3}\left(\mathcal{M}_n^\theta(n^{2/3}s) - \mathcal{M}_n^\theta(n^{2/3}(s-1))\right)$$

for $s \le T$. Between times $n^{2/3}(s-1)$ and $n^{2/3}s$, there are multiple integer steps, the increment in each step as defined in (4.23). Therefore

$$\bar{\mathcal{M}}_n^\theta(s_j) - \bar{\mathcal{M}}_n^\theta(s_{j-1}) \sim n^{-1/3} \sum_{i=n^{2/3}s_{j-1}+1}^{n^{2/3}s_j} \text{Bin}\left(n^{1/3}\mathcal{B}^\theta(n^{-2/3}i), p_n\right).$$

Since the increments are independent, we can move the sum inside the argument of the Binomial distribution. Let us define

$$R_{n,j} := \sum_{i=n^{2/3}s_{j-1}+1}^{n^{2/3}s_j} n^{1/3}\mathcal{B}^\theta(n^{-2/3}i)$$

and compute the probability in (4.22) as

$$\begin{aligned}&\mathbb{P}\left(\bar{\mathcal{M}}_n^\theta(s_1) = k_1, \dots, \bar{\mathcal{M}}_n^\theta(s_l) = k_l\right) \\ &\quad = \mathbb{P}\left(\bar{\mathcal{M}}_n^\theta(s_j) - \bar{\mathcal{M}}_n^\theta(s_{j-1}) = k_j - k_{j-1},\ \forall j = 2, \dots, l\right) \\ &\quad = \prod_{j=2}^{l} \mathbb{P}\left(n^{1/3}\bar{\mathcal{M}}_n^\theta(s_j) - n^{1/3}\bar{\mathcal{M}}_n^\theta(s_{j-1}) = n^{1/3}(k_j - k_{j-1})\right) \\ &\quad = \prod_{j=2}^{l} \mathbb{P}\left(Y_{n,j} = n^{1/3}(k_j - k_{j-1})\right),\end{aligned} \tag{4.24}$$

where $Y_{n,j} \sim \mathrm{Bin}(R_{n,j}, p_n)$. Note that $Y_{n,j} =_d \sum_{k=1}^{R_{n,j}} \xi_k$, with $\xi_k \sim \mathrm{B}(p_n)$ Bernoulli distributed random variables.

In the next step we use the Poisson limit theorem, see [16, Theorem 3.7, p.79], which states that for a series of binomially distributed random variables $X_k \sim \mathrm{Bin}(N_k, p_k)$ with $\mathbb{E}[X_k] = N_k p_k \to \lambda \in \mathbb{R}$ as $k \to \infty$, the convergence

$$X_k \to_d \mathrm{Poi}(\lambda) \tag{4.25}$$

holds as $k \to \infty$. To apply this theorem, we calculate the expected value of $Y_{n,j}$:

$$\begin{aligned}
\mathbb{E}[Y_{n,j}] &= R_{n,j} p_n \\
&= \frac{1}{n} R_{n,j} + O\left(n^{-1/3}\right) \\
&= \frac{1}{n} \sum_{i=n^{2/3}s_{j-1}+1}^{n^{2/3}s_j} n^{1/3} \mathcal{B}^\theta(n^{-2/3} i) + O\left(n^{-1/3}\right) \\
&= \sum_{i=n^{2/3}s_{j-1}+1}^{n^{2/3}s_j} n^{-2/3} \mathcal{B}^\theta(n^{-2/3} i) + O\left(n^{-1/3}\right).
\end{aligned}$$

This sum represents a partition of the interval $[s_{j-1}, s_j]$ into $n^{2/3}(s_j - s_{j-1})$ subintervals, each of length $n^{-2/3}$. Since $n^{-2/3} i$ is an element of its corresponding subinterval, we are dealing with a Riemann sum over the continuous function $\mathcal{B}^\theta$. Since $\mathcal{B}^\theta$ is bounded almost surely on the compact interval $[s_{j-1}, s_j]$, the sum converges to an integral and

$$\mathbb{E}[Y_{n,j}] \to \int_{s_{j-1}}^{s_j} \mathcal{B}^\theta(u) du$$

as $n \to \infty$. Now applying the Poisson limit theorem yields

$$Y_{n,j} \to_d \mathrm{Poi}\left(\int_{s_{j-1}}^{s_j} \mathcal{B}^\theta(u) du\right) =_d \mathcal{N}^\theta(s_j) - \mathcal{N}^\theta(s_{j-1}), \tag{4.26}$$

and, applying this convergence in (4.24), we arrive at

$$\begin{aligned}
&\mathbb{P}\left(\bar{\mathcal{M}}_n^\theta(s_1) = k_1, \ldots, \bar{\mathcal{M}}_n^\theta(s_l) = k_l\right) \\
&\quad = \prod_{j=2}^{l} \mathbb{P}\left(Y_{n,j} = n^{1/3}(k_j - k_{j-1})\right)) \\
&\quad \to \prod_{j=2}^{l} \mathbb{P}\left(\mathcal{N}^\theta(s_j) - \mathcal{N}^\theta(s_{j-1}) = n^{1/3}(k_j - k_{j-1})\right) \\
&\quad = \mathbb{P}\left(\mathcal{N}^\theta(s_1) = k_1, \ldots, \mathcal{N}^\theta(s_l) = k_l\right).
\end{aligned}$$

For any fixed continuous $\mathcal{W}^\theta(\omega)$ we therefore have $\bar{\mathcal{M}}_n^\theta \to_d \mathcal{N}^\theta$. Since $f(\mathcal{W}^\theta, \cdot)$ is a bounded continuous function this implies

$$\mathbb{E}[f(\mathcal{W}^\theta(\omega), \bar{\mathcal{M}}_n^\theta) \mid \mathcal{W}^\theta(\omega)] \to \mathbb{E}[f(\mathcal{W}^\theta(\omega), \mathcal{N}^\theta) \mid \mathcal{W}^\theta(\omega)].$$

By the boundedness of f we can apply the dominated convergence theorem, see [16, Corollary 6.26, p.135], which gives

$$\begin{aligned}
\mathbb{E}[f(\mathcal{W}^\theta, \mathcal{N}^\theta)] &= \mathbb{E}\left[\mathbb{E}[f(\mathcal{W}^\theta(\omega), \mathcal{N}^\theta) \mid \mathcal{W}^\theta(\omega)]\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} \mathbb{E}[f(\mathcal{W}^\theta(\omega), \bar{\mathcal{M}}_n^\theta) \mid \mathcal{W}^\theta(\omega)]\right] \\
&\overset{\text{d.c.}}{=} \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}[f(\mathcal{W}^\theta(\omega), \bar{\mathcal{M}}_n^\theta) \mid \mathcal{W}^\theta(\omega)]\right] \\
&= \lim_{n\to\infty} \mathbb{E}[f(\mathcal{W}^\theta, \bar{\mathcal{M}}_n^\theta)].
\end{aligned}$$

Finally, the closeness of $\mathbb{E}[f(\mathcal{W}^\theta, \bar{\mathcal{M}}_n^\theta)]$ and $\mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \mathcal{N}_n^\theta)]$ in (4.21) implies

$$\mathbb{E}[f(\bar{\mathcal{Z}}_n^\theta, \mathcal{N}_n^\theta)] \to \mathbb{E}[f(\mathcal{W}^\theta, \mathcal{N}^\theta)]$$

which completes the proof. □

We can now assure ourselves that the overestimated probability (4.3) is asymptotically negligible. Assume the chance that any vertex encounters two or more surplus edges is non-zero and does not converge to zero as $n \to \infty$. If a vertex connects by multiple excess edges, the process N_n^θ makes two or more jumps during the time-interval of length 1. The rescaling (4.4) compresses the time axis until, in the limit process N^θ, any distance in an interval of original length 1 will be reduced to a single point. Consequently there would be a non-zero chance that the counting process has multiple coincident points. But since $\int_{\{x\}} B^\theta(s)ds = 0$ for all $x \in \mathbb{R}_+$, N^θ is simple and the probability of multiple coincident points occurring is 0. We conclude that the probability of a vertex having multiple surplus edges must tend to zero.

5 Convergence of component sizes and surplus edges

In Chapter 3 we have shown that the rescaled breadth-first walk $\bar{Z}_n^\theta$ on $\mathscr{G}(n, n^{-1}+\theta n^{-4/3})$ converges in distribution to B^θ, the Brownian motion with drift. Intuitively it is clear that Theorem 1.7 should follow: Component sizes are coded into the breadth-first walk as excursions above past minima, excursions of B^θ are excursions of W^θ above past minima. But rigorously deducing a proof of our main theorem requires a bit more work. To make sure that indeed components and excursions do match up, we describe them as two-dimensional point processes in which the first entry gives the start of an excursion or component, the second the length of an excursion or the size of a component. The following two Lemmas in Section 5.1 prove that this sequence of point processes describing components converges to the point process describing excursions with regard to the vague topology.

This implies a convergence of all excursions of sufficient length on finite intervals. For a proof of Theorem 1.7 to hold, it remains to be shown that we do not need to worry about "mass wandering off to infinity", meaning the rescaled starting points of excursions of sufficient length to diverge to infinity as $n \to \infty$. Another problem that might arise is the mass of all small excursions prohibiting a convergence in $l^2_\searrow$. We deal with both problems in Section 5.2 and at last prove our main theorem in Section 5.3.

5.1 Matching components

We start with a deterministic Lemma. Given a continuous function f with properties similar to a Brownian motion and a sequence of functions f_n converging to f uniformly on bounded intervals, we can define a set of excursions on each f_n such that the point processes of starts and lengths of excursions converge vaguely to the point process of starts and lengths of excursions on f.

Lemma 5.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function. Let E be the set of non-empty intervals $e = [l, r] \subset \mathbb{R}_{\geq 0}$ such that*

$$f(r) = f(l) = \min_{s \leq l} f(s), \tag{5.1}$$

$$f(l) < f(s) \quad \forall l < s < r. \tag{5.2}$$

Define $\Xi := \{(l, r-l) \mid (l, r) \in E\}$.

Suppose that for intervals $(l_1, r_1), (l_2, r_2) \in E$ with $l_1 < l_2$ we have

$$f(l_1) > f(l_2) \tag{5.3}$$

and the complement of $\cup_{e \in E}(l, r)$ *has Lebesgue measure zero,*

$$\mu\left((\cup_{e \in E}(l, r))^c\right) = 0. \tag{5.4}$$

Let $f_n \to f$ *as* $n \to \infty$ *uniformly on bounded intervals. Let* $\mathcal{T}_n := (t_{n,i}, i \geq 1)$ *be a sequence of sets of points satisfying the following conditions:*

$$0 = t_{n,1} < t_{n,2} < \dots \text{ and } \lim_{i \to \infty} t_{n,i} = \infty, \tag{5.5}$$

$$f_n(t_{n,i}) = \min_{u \leq t_{n,i}} f_n(u), \tag{5.6}$$

$$\max_{t_{n,i} \leq s_0} \left(f_n(t_{n,i}) - f_n(t_{n,i+1})\right) \to 0 \text{ as } n \to \infty, \text{ for all } s_0 < \infty. \tag{5.7}$$

Define $\Xi^{(n)} := \{(t_{n,i}, t_{n,i+1} - t_{n,i}) \mid i \geq 1\}$. *Then* $\Xi^{(n)} \to_v \Xi$ *as* $n \to \infty$.

Note. The convergence $\Xi^{(n)} \to_v \Xi$ is to be interpreted as the vague convergence of counting measures which, by Lemma 2.19, is equivalent to the convergence

$$\Xi^{(n)}(C) \to \Xi(C)$$

for all relatively compact subsets $C \subseteq [0, \infty) \times (0, \infty)$, with $\Xi(\partial C) = 0$. In this case, a relatively compact subset is a pair of intervals $[T_1, T_2] \times [d_1, d_2]$, with $T_1, T_2 \geq 0$ and $d_1, d_2 > 0$. The condition on the measure of the boundary means that the limit process must not have any excursions starting exactly at T_1 or T_2, or any excursions of length exactly d_1 or d_2. An exception is the case of $T_1 = 0$. Since the domain of f starts at 0 and the condition on the boundary is needed to prevent the case of points in $\Xi^{(n)}$ converging to some point on the boundary "from outside", we do not need to consider 0 as part of the boundary of an interval $[0, T_2] \times [d_1, d_2]$.

Proof: To prove the Lemma, we fix some bounded subset of $[0, \infty) \times (0, \infty)$, $C := [T_1, T_2] \times [d_1, d_2]$, and show that, for sufficiently large n, we have $\Xi^{(n)}(C) = \Xi(C)$. That is, there are exactly as many excursions of f_n starting in (T_1, T_2), with length in (d_1, d_2), as similar excursions of the limit function f.

We will first show that every excursion of f is eventually matched by some excursion of f_n, then show that there can not be any more excursion of f_n of sufficient length.

Part 1 ($\Xi(C) \leq \Xi^{(n)}(C)$): Let us first establish some facts about excursions of the limit function f. In the interval $[T_1, T_2]$, there can only be a finite number of excursion starting with length of at least d_1, at most $(T_2 - T_1)/d_1$. We call excursions of length greater than d_1 large, all other excursions small. Let

$$E^* := \{(l_i, r_i) \mid i = 1, \dots, k\}$$

be the set of these excursions. Consider a $(l, r) \in E^*$. Since $l - r > d_1$, we can find an $\varepsilon > 0$ such that the length of the interval $[l + \varepsilon, r - \varepsilon]$ is still greater than d_1.

We show that we can find an $\varepsilon > 0$ sufficiently small, so that for every $x \in [0, l-\varepsilon] \cup [l+\varepsilon, r-\varepsilon]$,

$$f(x) > f(l) + \delta \tag{5.8}$$

holds for some $\delta > 0$:

Assume this does not hold to the left of the ε-neighbourhood of l. Then for all $\delta > 0$ exists $x_\delta \in [0, l-\varepsilon]$ such that $f(x_\delta) \leq f(l)+\delta$. Take $\delta_n = \frac{1}{n}$. Then x_{δ_n} is a sequence in the compact interval $[0, l-\varepsilon]$ and the Bolzano–Weierstrass theorem states that there exists a convergent subsequence with limit $x \in [0, l-\varepsilon]$. Then $f(x) \leq f(l) + \delta$ for all $\delta > 0$. This implies $f(x) \leq f(l)$ and by condition (5.1), $f(l) = \min_{u \leq l} f(u)$, so $f(x) = f(l)$. That leaves two possibilities: First, f is constant on the interval $[x, l]$. But this would be an interval of non-zero length without any excursions on it, a contradiction to (5.4). Second, there is an interval $[x', l']$, with $x \leq x' < l' \leq l$, such that $f(y) > f(x)$ for all $y \in (x', l')$. That makes (x', l') another excursion in E, but $f(x') = f(l)$, which is a contradiction to condition (5.3).

Using the same logic, assuming (5.8) does not hold in $[l+\varepsilon, r-\varepsilon]$ leads to a point $x \in [l+\varepsilon, r-\varepsilon]$ such that $f(x) \leq f(l)$, which is a contradiction to condition (5.2).

Now consider the behaviour of f on $[r, r+\varepsilon]$. As previously stated, condition (5.4) prevents f from being constant. If $f(x) \geq f(r)$ for all $x \in [r, r+\varepsilon]$, there would be another excursion that contradicts (5.3). So there must exist an $r^* \in (r, r+\varepsilon]$ with $f(r^*) < f(r)$.

We take a look at f_n. Fix an $x^* \in [l+\varepsilon, r-\varepsilon]$. We will now show that there exist points $l_n(x^*) \in [l-\varepsilon, l+\varepsilon]$ and $r_n(x^*) \in [r-\varepsilon, r+\varepsilon]$ for which condition (5.6) holds, making $(l_n(x^*), r_n(x^*) - l_n(x^*))$ a possible element of $\Xi^{(n)}$.

We define

$$l_n(x^*) := \min\{\arg\min_{u \leq x^*} f_n(u)\} \tag{5.9}$$

and

$$\delta^* := \min\{\delta, f(r) - f(r^*)\}, \tag{5.10}$$

where δ is the constant used in the discussion of $[0, l-\varepsilon]$ and $[l+\varepsilon, r-\varepsilon]$ above.

By the convergence $f_n \to f$, which is uniform on the compact interval $[T_1, T_2]$, we can find an $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \delta^*/3 \tag{5.11}$$

for all $x \in [T_1, T_2]$ and $n \geq N$. So for every point $x \in [0, l-\varepsilon] \cup [l+\varepsilon, r-\varepsilon]$,

$$\begin{aligned} f_n(x) &> f(x) - \delta^*/3 && \text{by (5.11)} \\ &> f(l) + \delta^* - \delta^*/3 && \text{by (5.10)} \\ &> f(l) + \delta^*/3 && \\ &> f_n(l) && \text{by (5.11).} \end{aligned} \tag{5.12}$$

So f_n takes its minimum over $[0, l-\varepsilon] \cup [l+\varepsilon, r-\varepsilon]$ at l or somewhere in $[l-\varepsilon, l+\varepsilon]$. Therefore $l_n(x^*) \in [l-\varepsilon, l+\varepsilon]$ and $f_n(l_n(x^*)) = \min_{u \leq l_n(x^*)} f_n(u)$.

For the right side of the interval, define

$$r_n(x^*) := \inf\{x > x^* \mid f_n(x) = f_n(l_n(x^*))\}. \tag{5.13}$$

By (5.12), the function f_n can not reach its past minimum $f_n(l_n(x^*))$ before $r - \varepsilon$. As previously discussed, there exists a $r^* \in (r, r+\varepsilon]$ such that $f(r) > f(r^*)$ and by (5.10) we have $f(r) - f(r^*) \geq \delta^*$. We calculate

$$\begin{aligned} f_n(l_n(x^*)) &> f(l) - \delta^*/3 && \text{by (5.11)} \\ &= f(r) - \delta^*/3 && \text{by (5.1)} \\ &\geq f(r^*) + \delta^* - \delta^*/3 && \text{by (5.10)} \\ &> f(r^*) + \delta^*/3 \\ &> f_n(r^*) && \text{by (5.11),} \end{aligned}$$

which implies that $r_n(x^*)$ must be smaller than r^*, since f_n is a continuous function and must cross $f_n(l_n(x^*))$ before becoming smaller at r^*. Therefore $r_n(x^*) \in [r-\varepsilon, r+\varepsilon]$. Since $f_n(l_n(x)) = \min_{u \leq l_n(x^*)} f_n(u)$ and $r_n(x^*)$ is the first time this previous minimum is reached, $f_n(r_n(x^*)) = \min_{u \leq l_n(x^*)} f_n(u)$.

We have shown that the only points satisfying condition (5.6) must lay near the beginning and end of excursions of f. This is not sufficient as proof of the Lemma, since the sequence $\mathcal{T}_n$ might not contain any points between two large excursions, or even not contain any points, thus skipping one or more eligible excursions. We will now show that (5.5) and (5.7) imply that any set satisfying these two conditions must contain at least one element in between two large excursions of f. First of all, (5.5) ensures there must exist points in $\mathcal{T}_n$ and no last element of $\mathcal{T}_n$ can exist.

Consider two consecutive large excursions of f, (l_1, r_1) and (l_2, r_2) and the space between r_1 and l_2. Suppose there is no element of $\mathcal{T}_n$ in $[r_1 - \varepsilon, l_2 + \varepsilon]$ for all $\varepsilon > 0$. The latest element of $\mathcal{T}_n$ was located around l_1, the next will be around r_2. We know $f(r_1) = f(l_1) > f(l_2)$, so there is $\delta > 0$ such that for all large n, $|f_n(l^*) - f_n(r^*)| > \delta$ for all l^* and r^* in the sufficiently small ε-neighbourhoods around l_1 and r_2. Any previous element of $\mathcal{T}_n$ will only yield a larger, any element after r_2 only a smaller f_n-value. This is a contradiction to (5.7). So there must be at least one point of $\mathcal{T}_n$ in between these two large excursions, no excursion can be skipped.

For any excursion (l, r) of f, there exists $l_n(x^*), r_n(x^*) \in \mathcal{T}_n$ in the respective ε-neighbourhoods, so the excursion of f is matched by an excursion of f_n of similar length. Since $l - r + 2\varepsilon > d_1$, we have found $t_{n,i} = l_n(x^*)$, $t_{n,i+1} = r_n(x^*)$, such that $(t_{n,i}, t_{n,i+1} - t_{n,i}) \in \Xi^{(n)} \cap C$ for all n greater than some $N_i = N \in \mathbb{N}$. Now let $N^* := \max\{N_1, \ldots, N_k\}$, where k is the finite number of elements of E^*, and every excursion of f in C is matched by an excursion of f_n in C for $n \geq N^*$. Therefore eventually $\Xi(C) \leq \Xi^{(n)}(C)$.

Part 2 ($\Xi^{(n)}(C) \leq \Xi(C)$): Now that every excursion of f is matched, we need to show that there can not exist any additional large excursion of f_n. Considering the fact that excursions can not overlap, the only possibility for an additional large excursion is the space between two large excursions. For a pair of large excursions, (l_1, r_1) and (l_2, r_2), there is only enough space in between them if $l_2 - r_1 > d_1$. Assume the interval (r_2, l_1) is of length greater than d_1. By condition (5.4) the space must be filled with smaller excursions of f. There is an at most countable number of these (around each rational number q in the interval we can construct only one excursion, we elaborate on this argument in the proof of Lemma 5.2), therefore let $E^* := \{(l_i, r_i) \mid i \in \mathbb{N}\}$ be the set of such excursions starting in (r_2, l_1) with length less than or equal to d_1. We know

$$\sum_{i=1}^{\infty} r_i - l_i = l_2 - r_1 > d_1,$$

so we can choose a finite set of excursions $\{(l_{i_j}, r_{i_j}) \mid j = 1, \ldots, K\}$ such that, if we exclude these from the interval (r_2, l_1), the space remaining is less than d_1:

$$l_2 - r_1 - \sum_{j=1}^{K} r_{i_j} - l_{i_j} < d_1.$$

Let $d^* < \min\{r_{i_j} - l_{i_j} \mid j = 1, \ldots, K\}$ and apply the logic of Part 5.1 to the compact set $[r, l] \times [d^*, d_1]$. For sufficiently large n, every one of these K excursions of f will be matched with an excursion of f_n, so that there will be no space left for a large excursion of f_n in $[r, l]$. Applying this logic to every one of the finitely many gaps between large excursions of f, we see that there can not exist any more large excursions of f_n than those already matching f. Therefore $\Xi^{(n)}(C) \leq \Xi(C)$ for sufficiently large n, which completes the proof. □

The following lemma will now link excursions of $\bar{Z}_n^\theta$ and W^θ in the language of Lemma 5.1. We define a random point process containing the starts and lengths of excursions of B^θ together with a sequence of random point processes describing the starts and sizes of components discovered by $\bar{Z}_n^\theta$. Let $\mathcal{C}_{n,i}$ be the size of the i-th component and define $\gamma_n(i) \in \{1, \ldots, n\}$ as the index for which $v_{\gamma_n(i)}$ is the last vertex of the $i-1$-th component encountered by the breadth-first walk Z_n, with $\gamma_n(1) = 0$.

For this lemma, we need to extend the definitions of γ_n and $\mathcal{C}_{n,i}$ beyond the n-th vertex and the last component. Let i_n^* be the index of the last component and define $\gamma_n(i_n^* + 1) := n$ and consequently $\gamma_n(j) := \gamma_n(j-1) + 1$ for all $j > i_n^* + 1$. For component sizes, let $\mathcal{C}_{n,i} := 1$ for all $i > i^*$. Vertex n is the last vertex of the last component and with every step beyond n, $\gamma_n(j)$ increases by one with a component of size one.

Lemma 5.2. *Let Ξ be the point process with points corresponding to excursions of B^θ,*

$$\Xi := \{(l(\gamma), |\gamma|) \mid \gamma \text{ excursion of } B^\theta\}. \tag{5.14}$$

Let $\Xi^{(n)}$ be the rescaled point process with points corresponding to excursions of the breadth-first walk

$$\Xi^{(n)} := \{(n^{-2/3}\gamma_n(i), n^{-2/3}\mathcal{C}_{n,i}) \mid i \geq 1\}. \tag{5.15}$$

Then $\Xi^{(n)} \to_d \Xi$ as $n \to \infty$ with regard to the vague topology.

Proof: First, let us give some remarks on the structure of the objects in use here and their convergence. Lemma 5.1 states that, under certain conditions, two deterministic point processes converge in the vague topology. This Lemma now introduces $\Xi^{(n)}$ and Ξ, which are random variables mapping into the space of point processes, again equipped with the vague topology.

Both $\Xi^{(n)}$ and Ξ are random in the sense that they depend on their underlying processes, $\bar{Z}_n^\theta$ and W^θ respectively, so to clarify we define the sets

$$\begin{aligned}\Xi_{W^\theta} &:= \{(l(\gamma), |\gamma|) \mid \gamma \text{ excursion of } B^\theta\},\\ \Xi_{Z_n^\theta}^{(n)} &:= \{(n^{-2/3}\gamma_n(i), n^{-2/3}\mathcal{C}_{n,i}) \mid i \geq 1\},\\ \bar{\Xi}_{\bar{Z}_n^\theta}^{(n)} &:= \{(\bar{\gamma}_n(i), \bar{\mathcal{C}}_{n,i}) \mid i \geq 1\},\end{aligned} \tag{5.16}$$

which are dependent on the specific realisation of the random processes W^θ and Z_n^θ, where $\bar{\gamma}_n$ and $\bar{\mathcal{C}}_{n,i}$ describe starting points and excursion lengths of $\bar{Z}_n^\theta$. Since $\bar{Z}_n^\theta(s) = n^{-1/3}Z_n^\theta(n^{2/3}s)$ we have $\Xi_{Z_n^\theta}^{(n)} = \bar{\Xi}_{\bar{Z}_n^\theta}^{(n)}$ and we can use $\Xi^{(n)}$ and $\bar{\Xi}^{(n)}$ interchangeably when appropriate.

Now (5.14) and (5.15) can be redefined as the random variables

$$\begin{aligned}\Xi &: \omega \mapsto \Xi_{W^\theta(\omega)},\\ \Xi^{(n)} &: \omega \mapsto \Xi_{Z_n^\theta(\omega)}^{(n)}.\end{aligned} \tag{5.17}$$

We apply the Skorohod representation theorem once more to construct random variables $\bar{\mathcal{Z}}_n^\theta$ and $\mathcal{W}^\theta$ on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, which converge almost surely. Almost sure convergence implies

$$\sup_{s \leq s_0} |\bar{\mathcal{Z}}_n^\theta(s) - \mathcal{W}^\theta(s)| \to 0 \tag{5.18}$$

for all $s_0 < \infty$ almost surely, so $\bar{\mathcal{Z}}_n^\theta(\omega') \to \mathcal{W}^\theta(\omega')$ uniformly on bounded intervals for almost all $\omega' \in \Omega'$. Analogously we have

$$\mathcal{B}^\theta(s) := \mathcal{W}^\theta(s) - \min_{u \leq s} \mathcal{W}^\theta(s). \tag{5.19}$$

On the same probability space, define $\Xi'^{(n)}$ and Ξ' as in (5.16) and (5.17). Since $\bar{Z}_n^\theta \sim \bar{\mathcal{Z}}_n^\theta$ and $W^\theta \sim \mathcal{W}^\theta$ we have $\Xi^{(n)} \sim \Xi'^{(n)}$ and $\Xi \sim \Xi'$.

By the definition of $\mathcal{B}^\theta$, Ξ' is the Ξ in Lemma 5.1 with $f = \mathcal{W}^\theta$. As f_n we take $\bar{Z}_n^\theta$ and define $t_{n,i} := n^{-2/3}\gamma(n,i)$, that is, the elements of $\mathcal{T}_n$ to be the end-points of components, rescaled to match $\bar{\mathcal{Z}}_n^\theta$. This way $\Xi'^{(n)}$ coincides with $\Xi^{(n)}$ in Lemma 5.1.

We still need to show that conditions (5.1) to (5.4) hold for $\mathcal{W}^\theta$ and conditions (5.5) to (5.7) hold for the breadth-first walk. We start with the former. We first define the set $\mathcal{E}$ for the Brownian motion. Consider the set of positive rational numbers. For every $q \in \mathbb{Q}^+$, define

$$l(q) := \sup\{\arg\min_{s<q} \mathcal{W}^\theta(s)\} \tag{5.20}$$

and

$$r(q) := \inf\{s > q \mid \mathcal{W}^\theta(s) = \mathcal{W}^\theta(l(q))\}. \tag{5.21}$$

Now every rational number belongs to one excursion $(l(q), r(q))$, while one excursion contains multiple rational numbers. We define the set of excursions

$$\mathcal{E} := \bigcup_{q \in \mathbb{Q}_+} \{(l(q), r(q))\} \tag{5.22}$$

and note that it is countable.

The following properties will be proven on a standard Brownian motion W, we later use Girsanov's theorem to apply them to $\mathcal{W}^\theta$. Consider two excursions $(l_1, r_2), (l_2, r_2)$ with $l_1 < l_2$. We want to show that, almost surely, for any two excursions $(l_1, r_2), (l_2, r_2)$ with $l_1 < l_2$

$$W(l_2) = \min_{s \le l_2} W(s) < W(l_1) \tag{5.23}$$

holds. If not, there exists an excursion (l, r) such that another excursion begins immediately at r or that W is constant for some time after r and then starts a new excursion. Fix $q \in \mathbb{Q}_+$. By (5.21) $r(q)$ is a stopping time and by the strong Markov property of Brownian motion, see [19, Theorem 2.16, p.43], $W(s) - W(r(q))$ behaves like a standard Brownian motion for $s \ge r(q)$. By [19, Theorem 2.8, p.38], W has positive and negative values on any interval $[0, \delta]$ for $\delta > 0$ almost surely. Therefore for all $\delta > 0$ there exists $x \in (r(q), r(q) + \delta)$ with $W(x) < W(r(q))$. Thus, (5.23) holds almost surely for each $(l, r) = (l(q), r(q))$ and since $\mathbb{Q}$ is countable, condition (5.3) holds with probability 1.

The complement of all excursion is the set of intervals on which the Brownian motion is monotonously decreasing. By [19, Theorem 1.22, p.18], W is not monotonous on any interval $[a, b]$ with $0 \le a < b < \infty$ almost surely. Therefore almost surely there is no interval in between any two excursions on which W is monotonously decreasing and condition (5.4) holds almost surely.

A standard Brownian motion W satisfies the conditions of Lemma 5.1 almost surely, therefore $\mathcal{W}^\theta$ does so almost surely under $\tilde{\mathbb{P}}$, the equivalent probability measure defined in (2.9), and by Girsanov's theorem (Theorem 2.22) likewise under $\mathbb{P}$.

We now show that conditions (5.5) to (5.7) hold for the random walk Z_n and $t_{n,i} = n^{-2/3}\gamma_n(i)$. The breadth-first walk Z_n^θ is a discrete process, therefore for all $n \in \mathbb{N}$ and $i \ge 2$:

$$t_{n,i} - t_{n,i-1} = n^{-2/3}\gamma_n(i) - n^{-2/3}\gamma_n(i-1) \ge n^{-2/3} > 0.$$

By definition of $\gamma_n(i)$, $\lim_{i\to\infty} \gamma_n(i) = \infty$ for all n which establishes condition (5.5).

The breadth-first walk attains a new minimum at the end of every component, which ensures (5.6). The difference between the levels of Z_n^θ at the end of two consecutive components is always 1, so

$$\max_{i: t_{n,i} \le s_0} (\bar{\mathcal{Z}}_n^\theta(t_{n,i}) - \bar{\mathcal{Z}}_n^\theta(t_{n,i+1})) = n^{-1/3} \to 0$$

for all $s_0 > 0$ as $n \to \infty$.

For almost all realizations $\bar{\mathcal{Z}}_n^\theta(\omega')$ and $\mathcal{W}^\theta(\omega')$ the processes and sets defined meet all conditions of Lemma 5.1 and we can establish the convergence

$$\Xi'^{(n)}_{\bar{\mathcal{Z}}_n^\theta(\omega')} \to_v \Xi'_{\mathcal{W}^\theta(\omega')}.$$

This convergence holds almost surely, so

$$\Xi'^{(n)} \to_{a.s.} \Xi'$$

with regard to the vague topology. Since almost sure convergence implies convergence in distribution we have

$$\Xi'^{(n)} \to_d \Xi'$$

and therefore

$$\Xi^{(n)} \to_d \Xi$$

with regard to the vague topology, which completes the proof. □

5.2 Analysis of atypical excursions

From the previous section we assert that for any $C, \delta > 0$ all excursions before time $Cn^{2/3}$ and larger than $\delta n^{2/3}$ eventually converge. This does not suffice as proof of the convergence with regard to the product topology of Folk Theorem 1.6 or with regard to the $l^2_\searrow$ topology of Theorem 1.7. In this section we analyse the behaviour of Z_n^θ and its excursions after time $Cn^{2/3}$ or of size less than $\delta n^{2/3}$.

Lemma 5.3. *Let $\mathbb{P}(\mathcal{E}_{\delta,C,n})$ be the probability that the breadth-first walk Z_n^θ makes an excursion γ_n of length $|\gamma_n| > \delta n^{2/3}$ starting after step $Cn^{2/3}$. Then for all $\varepsilon > 0$ and $\delta > 0$ exists $C > 0$ such that $\sup_{n\in\mathbb{N}} \mathbb{P}(\mathcal{E}_{\delta,C,n}) < \varepsilon$.*

Note. As mentioned before, in [3, Lemma 9, p.826] a similar statement is proven by citing bounds on the numbers of tree components, unicyclic components and complex components given in [20]. We present a proof independent of existing random graph results.

Proof: By the law of total expectation,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_{\delta,C,n}) &\le \mathbb{E}\left[\text{Number of excursions } \gamma_n : \ |\gamma_n| \ge \delta n^{2/3}, l(\gamma_n) \ge Cn^{2/3}\ \right] \\
&= \mathbb{E}\left[\sum_{\gamma_n:\, l(\gamma_n)\ge Cn^{2/3}} \mathbb{1}_{\{|\gamma_n|\ge\delta n^{2/3}\}}\right] \\
&\le \mathbb{E}\left[\sum_{\gamma_n:\, l(\gamma_n)\ge Cn^{2/3}} \frac{|\gamma_n|^2}{\delta^2 n^{4/3}}\right] \\
&= \frac{1}{\delta^2 n^{4/3}}\mathbb{E}\left[\sum_{\gamma_n:\, l(\gamma_n)\ge Cn^{2/3}} |\gamma_n|^2\right].
\end{aligned} \tag{5.24}$$

Let T be the time the last excursion starting before $Cn^{2/3}$ ends. The behaviour of the breadth-first walk after T will be the same as the behaviour of a new walk on $\mathscr{G}(n-T, p_n)$. We write $\mathcal{C} \in \mathscr{G}$ to denote a component $\mathcal{C}$ contained in the random graph $\mathscr{G}$, and $|\mathcal{C}|$ for its size. Since the notions of excursions of the breadth-first walk and components in the underlying graph are interchangeable, we can rewrite (5.24) as

$$\begin{aligned}\mathbb{P}(\mathcal{E}_{\delta,C,n}) &\leq \frac{1}{\delta^2 n^{4/3}}\mathbb{E}\left[\sum_{\mathcal{C}\in\mathscr{G}(n-T,p_n)} |\mathcal{C}|^2\right] \\ &\leq \frac{1}{\delta^2 n^{4/3}}\mathbb{E}\left[\sum_{\mathcal{C}\in\mathscr{G}(n-Cn^{2/3},p_n)} |\mathcal{C}|^2\right].\end{aligned} \tag{5.25}$$

For ease of notation we consider the graph $\mathscr{G}(k, p_n)$ and calculate

$$\begin{aligned}\mathbb{E}\left[\sum_{\mathcal{C}\in\mathscr{G}(k,p_n)} |\mathcal{C}|^2\right] &= \mathbb{E}\left[\sum_{\mathcal{C}\in\mathscr{G}(k,p_n)} |\mathcal{C}|\sum_{v\in\mathcal{C}} 1\right] \\ &= \mathbb{E}\left[\sum_{\mathcal{C}\in\mathscr{G}(k,p_n)} |\mathcal{C}| \sum_{v\in\mathscr{G}(k,p_n)} \mathbb{1}_{\{v\in\mathcal{C}\}}\right] \\ &= \sum_{v\in\mathscr{G}(k,p_n)} \mathbb{E}\left[\sum_{\mathcal{C}\in\mathscr{G}(k,p_n)} |\mathcal{C}|\mathbb{1}_{\{v\in\mathcal{C}\}}\right] \\ &= \sum_{v\in\mathscr{G}(k,p_n)} \mathbb{E}\left[|\mathcal{C}(v)|\right] \\ &= k\mathbb{E}\left[|\mathcal{C}(1)|\right],\end{aligned} \tag{5.26}$$

where $\mathcal{C}(v)$ denotes the component containing the vertex v and the last inequality stems from the interchangeability of the vertex labels. We will bound the expectation of the size of this component from above by a suitable branching process $(Y_i,\ i \geq 0)$. Starting at time 0 with one vertex, we have $Y_0 = 1$. The number of children of this vertex is a $\mathrm{Bin}(k-1, p_n)$ distributed random variable, Y_1. In the next step, each child-vertex will itself have children, each Binomially distributed on the remaining set of vertices with probability p_n. We compute

$$\begin{aligned}Y_{2,1} &\sim \mathrm{Bin}(k-1-Y_1, p_n), \\ Y_{2,2} &\sim \mathrm{Bin}(k-1-Y_1-Y_{2,1}, p_n), \\ &\cdots \\ Y_{2,Y_1} &\sim \mathrm{Bin}(k-1-Y_1-Y_{2,1}-\cdots-Y_{2,Y_1-1}, p_n) \\ Y_2 &= \sum_{i=1}^{Y_1} Y_{2,i}.\end{aligned}$$

The size of the component will then be the total amount of explored vertices, which is the sum of all Y_j, $j \geq 0$. To provide an upper bound we consider the branching process where each amount of children is $\text{Bin}(k, p_n)$ distributed. Define the process as follows,

$$\begin{aligned} Z_0 &:= 1, \\ Z_j &:= \sum_{i=1}^{Z_{j-1}} Z_{j,i}, \end{aligned}$$

where $Z_{j,i} \sim \text{Bin}(k, p)$ for all $i, j \geq 1$. Then the process $(Y_i,\ i \geq 0)$ is stochastically dominated by $(Z_i,\ i \geq 0)$ and

$$|\mathcal{C}(1)| \leq_{\text{st.}} Z_0 + Z_1 + Z_2 + \dots$$

which gives

$$\mathbb{E}\left[|\mathcal{C}(1)|\right] \leq \sum_{j=0}^{\infty} \mathbb{E}\left[Z_j\right]. \tag{5.27}$$

For $j \geq 0$ we calculate the expectation of Z_j by

$$\begin{aligned} \mathbb{E}\left[Z_j\right] &= \mathbb{E}\left[Z_{j-1}\right] kp_n \\ &= \dots \\ &= \mathbb{E}\left[Z_0\right] (kp_n)^j \\ &= (kp_n)^j. \end{aligned}$$

Substituting in (5.27) gives

$$\begin{aligned} \mathbb{E}\left[|\mathcal{C}(1)|\right] &\leq \sum_{j=0}^{\infty} (kp_n)^j \\ &\leq \frac{1}{(1 - kp_n)_+}, \end{aligned} \tag{5.28}$$

where $(1 - kp_n)_+$ denotes the positive part of $1 - kp_n$. We continue the calculation in (5.25) using (5.26) and (5.28), which yields

$$\begin{aligned} \mathbb{P}(\mathcal{E}_{\delta,C,n}) &\leq \frac{n - Cn^{2/3}}{\delta^2 n^{4/3}} \frac{1}{1 - (n - Cn^{2/3})p_n} \\ &= \frac{n - Cn^{2/3}}{\delta^2 n^{4/3}} \frac{1}{1 - (n - Cn^{2/3})(n^{-1} + \theta n^{-4/3})} \\ &= \frac{n - Cn^{2/3}}{\delta^2 n^{4/3}} \frac{n^{1/3}}{C - \theta + C\theta n^{-1/3}} \\ &\leq \frac{n}{\delta^2 n^{4/3}} \frac{n^{1/3}}{C - \theta + C\theta n^{-1/3}} \\ &= \delta^{-2} \frac{1}{C - \theta + C\theta n^{-1/3}}. \end{aligned}$$

For fixed C, δ and θ this expression converges asymptotically to $\delta^{-2}\frac{1}{C-\theta}$ as $n \to \infty$. Therefore for all $\varepsilon > 0$ we can choose $C > 0$ such that

$$\mathbb{P}(\mathcal{E}_{\delta,C,n}) \le \delta^{-2}\frac{1}{C-\theta+C\theta n^{-1/3}} < \varepsilon$$

for all n. This completes the proof. $\square$

While this lemma provides enough information to prove the convergence with regard to the product topology of Folk Theorem 1.6, Theorem 1.7 requires additional information. From [3, Lemma 25, p.843] we cite the following lemma on excursion lengths of Brownian motion. Aldous' paper provides a complete proof using stochastic calculus methods.

Lemma 5.4. *Let Γ^θ be the set of excursions of B^θ and let $|\gamma|$ be the length of an excursion γ. Then*

$$\mathbb{E}\left[\sum_{\gamma\in\Gamma^\theta}|\gamma|^2\right] < \infty.$$

Since $\sum_{\gamma\in\Gamma^\theta}|\gamma|^2$ is a convergent series almost surely, for all $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\mathbb{E}\left[\sum_{|\gamma|<\delta}|\gamma|^2\right] < \varepsilon. \tag{5.29}$$

Then by Markov's inequality for all $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ there exists $\delta > 0$ such that

$$\mathbb{P}\left(\sum_{|\gamma|<\delta}|\gamma|^2 > \varepsilon_1\right) < \varepsilon_0. \tag{5.30}$$

An immediate corollary is the following equivalent of Lemma 5.3 for the Brownian motion.

Lemma 5.5. *Let $\mathbb{P}(\mathcal{E}_{\delta,C})$ be the probability that B^θ makes an excursion γ of length $|\gamma| > \delta$ with $l(\gamma) > C$. Then for all $\varepsilon > 0$ and $\delta > 0$ there exist $C > 0$ such that $\mathbb{P}(\mathcal{E}_{\delta,C}) < \varepsilon$.*

We state and prove a result similar to (5.29) on excursion lengths of the rescaled breadth-first walk.

Lemma 5.6. *Denote by $\bar\gamma_n$ the excursions of $\bar Z_n^\theta$. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sum_{|\bar\gamma_n|<\delta}|\bar\gamma_n|^2\right] < \varepsilon.$$

Proof: For $C > 0$ we have

$$\mathbb{E}\left[\sum_{|\bar\gamma_n|<\delta}|\bar\gamma_n|^2\right] \le \mathbb{E}\left[\sum_{l(\bar\gamma_n)>C}|\bar\gamma_n|^2\right] + \mathbb{E}\left[\sum_{\substack{l(\bar\gamma_n)\le C\\ |\bar\gamma_n|<\delta}}|\bar\gamma_n|^2\right].$$

Consider the second term:

$$\mathbb{E}\left[\sum_{\substack{l(\bar{\gamma}_n)\leq C\\|\bar{\gamma}_n|<\delta}}|\bar{\gamma}_n|^2\right]=\mathbb{E}\left[\sum_{k=0}^{\infty}\sum_{\substack{l(\bar{\gamma}_n)\leq C\\|\bar{\gamma}_n|\in I_{\delta,k}}}|\bar{\gamma}_n|^2\right]$$

where $I_{\delta,k}=[\frac{\delta}{2^{k+1}},\frac{\delta}{2^k}]$. For $k\in\mathbb{N}$ we have at most $C2^{k+1}\delta^{-1}$ excursions in $[0,C]$ with length in $I_{\delta,k}$. Therefore

$$\sum_{\substack{l(\bar{\gamma}_n)\leq C\\|\bar{\gamma}_n|\in I_{\delta,k}}}|\bar{\gamma}_n|^2\leq\left(\frac{\delta}{2^k}\right)^2\frac{C2^{k+1}}{\delta}$$

and since this is now an entirely deterministic term we have

$$\begin{aligned}\mathbb{E}\left[\sum_{\substack{l(\bar{\gamma}_n)\leq C\\|\bar{\gamma}_n|<\delta}}|\bar{\gamma}_n|^2\right]&\leq\sum_{k=0}^{\infty}\left(\frac{\delta}{2^k}\right)^2\frac{C2^{k+1}}{\delta}\\&=\sum_{k=0}^{\infty}\frac{2C\delta}{2^k}\\&=4C\delta.\end{aligned}$$

From the proof of Lemma 5.3 we know for all $\varepsilon>0$ we can choose $C>0$ such that

$$\begin{aligned}\mathbb{E}\left[\sum_{l(\bar{\gamma}_n)>C}|\bar{\gamma}_n|^2\right]&=n^{-4/3}\mathbb{E}\left[\sum_{l(\gamma_n)>Cn^{2/3}}|\gamma_n|^2\right]\\&\leq\varepsilon/2.\end{aligned}$$

For this C choose $\delta>0$ such that $4C\delta<\varepsilon/2$. Then

$$\mathbb{E}\left[\sum_{|\bar{\gamma}_n|<\delta}|\bar{\gamma}_n|^2\right]<\varepsilon.$$

Since δ and C are independent of n we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sum_{|\bar{\gamma}_n|<\delta}|\bar{\gamma}_n|^2\right]<\varepsilon.$$

$\square$

5.3 Finishing the proofs

After these technical considerations we are now ready to prove our main theorem.

Proof of Theorem 1.7: We first prove the convergence of component sizes and surplus edges to excursions and mark counts of B^θ in the product topology and then the convergence of component sizes to Brownian excursions in the $l^2_\searrow$ topology.

Part 1: We again define $\bar{\mathcal{Z}}_n^\theta \sim \bar{Z}_n^\theta$ and $\mathcal{W}^\theta \sim W^\theta$ on the same probability space, where $\bar{\mathcal{Z}}_n^\theta \to_{a.s.} \mathcal{W}^\theta$. Define

$$Y_n := ((|\bar{\gamma}_{n,1}|, \bar{\sigma}_1), (|\bar{\gamma}_{n,2}|, \bar{\sigma}_2), \dots) \tag{5.31}$$

as the excursion lengths of $\bar{\mathcal{Z}}_n^\theta$ and corresponding counts of $\bar{\mathcal{N}}_n^\theta$, as well as

$$Y := ((|\gamma_1|, \mu(\gamma_1)), (|\gamma_2|, \mu(\gamma_2)), \dots) \tag{5.32}$$

for the lengths of excursions and corresponding mark counts of $\mathcal{B}^\theta$, each ordered by decreasing excursion length. Denote by

$$\begin{aligned} Y_n^{(k)} &:= ((|\bar{\gamma}_{n,1}|, \bar{\sigma}_1), \dots, (|\bar{\gamma}_{n,k}|, \bar{\sigma}_k)) \\ Y^{(k)} &:= ((|\gamma_1|, \mu(\gamma_1)), \dots, (|\gamma_k|, \mu(\gamma_k))) \end{aligned} \tag{5.33}$$

the vector of the first k components of the corresponding vector. We want to show that for all $k \in \mathbb{N}$ we have

$$Y_n^{(k)} \to_d Y^{(k)}.$$

Fix $k \in \mathbb{N}$. Since $Y^{(k)}$ is a vector of finite length, for all $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\mathbb{P}(|\gamma_k| < \delta) < \varepsilon. \tag{5.34}$$

From Lemma 5.3 we know that for all $\varepsilon > 0$ and $\delta > 0$ there exists $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}\left(\exists \bar{\gamma}_n : \ l(\bar{\gamma}_n) > C, |\bar{\gamma}_n| > \delta\right) < \varepsilon \tag{5.35}$$

and Lemma 5.5 gives

$$\mathbb{P}(\exists \gamma : \ l(\gamma) > C, |\gamma| > \delta) < \varepsilon. \tag{5.36}$$

Since $\bar{\mathcal{Z}}_n^\theta \to_{a.s.} \mathcal{W}^\theta$ for all $\beta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\mathbb{P}\left(\sup_{s \in [0,C]} |\bar{\mathcal{Z}}_n^\theta(s) - \mathcal{W}^\theta(s)| > \beta\right) < \varepsilon. \tag{5.37}$$

We show that $Y_n^{(k)} \to_p Y^{(k)}$, i.e. for all $\alpha > 0$ we have

$$\mathbb{P}(d(Y_n^{(k)}, Y^{(k)}) > \alpha) \to 0$$

as $n \to \infty$, where d is a distance on $\mathbb{R}^{2,k}$ compatible with the description of convergence in the product topology in Folk Theorem 1.6. Denote by $\mathcal{E}_{Y^{(k)}}, \mathcal{E}_{\bar{\gamma}_n}, \mathcal{E}_\gamma, \mathcal{E}_\beta$ the events in (5.34), (5.35), (5.36) and (5.37) respectively. Then

$$\begin{aligned}\mathbb{P}(d(Y_n^{(k)}, Y^{(k)}) > \alpha) \leq \mathbb{P}(d(Y_n^{(k)}, Y^{(k)}) > \alpha \mid \neg\mathcal{E}_{Y^{(k)}}, \neg\mathcal{E}_{\bar{\gamma}_n}, \neg\mathcal{E}_\gamma, \neg\mathcal{E}_\beta) \\ + \mathbb{P}(\mathcal{E}_{Y^{(k)}}) + \mathbb{P}(\mathcal{E}_{\bar{\gamma}_n}) + \mathbb{P}(\mathcal{E}_\gamma) + \mathbb{P}(\mathcal{E}_\beta).\end{aligned}$$

The conditions in the first probability assert that there are no large excursions of $\bar{\mathcal{Z}}_n^\theta$ or $\mathcal{W}^\theta$ after C, that the k-th excursion of $\mathcal{W}^\theta$ is large and $\bar{\mathcal{Z}}_n^\theta$ and $\mathcal{W}^\theta$ are close. Since excursions of $\bar{\mathcal{Z}}_n^\theta$ and $\mathcal{W}^\theta$ eventually match up, all k relevant excursions must happen before C. With shrinking distance β between $\bar{\mathcal{Z}}_n^\theta$ and $\mathcal{W}^\theta$ the distance between excursion lengths must become smaller as well and for sufficiently large n the number of surplus edges in $Y_n^{(k)}$ coincides exactly with number of marks in $Y^{(k)}$. Therefore for all $\alpha > 0$ we can find $\beta = \beta(\alpha)$ such that

$$\mathbb{P}(d(Y_n^{(k)}, Y^{(k)}) > \alpha \mid \neg\mathcal{E}_{Y^{(k)}}, \neg\mathcal{E}_{\bar{\gamma}_n}, \neg\mathcal{E}_\gamma, \neg\mathcal{E}_\beta) = 0.$$

Then for all $\varepsilon > 0$ we choose $\delta > 0$ such that $\mathbb{P}(\mathcal{E}_{Y^{(k)}}) < \varepsilon/4$, for these δ and ε we find $C > 0$ such that $\mathbb{P}(\mathcal{E}_{\bar{\gamma}_n}) < \varepsilon/4$ and $\mathbb{P}(\mathcal{E}_\gamma) < \varepsilon/4$ and finally $N \in \mathbb{N}$ such that $\mathbb{P}(\mathcal{E}_\beta) < \varepsilon$ for all $n \geq N$. Then

$$\mathbb{P}(d(Y_n^{(k)}, Y^{(k)}) > \alpha) \leq \varepsilon$$

which proves $Y_n^{(k)} \to_p Y^{(k)}$. Now convergence in probability implies convergence in distribution which completes the proof of the first and second statement of Theorem 1.7.

Part 2: The only thing left to show is that the convergence of component sizes holds with respect to the $l^2_\searrow$ topology. Define

$$\begin{aligned} Y_n' &:= (|\bar{\gamma}_{n,1}|, |\bar{\gamma}_{n,2}|, \dots), \\ Y' &:= (|\gamma_1|, |\gamma_2|, \dots). \end{aligned} \tag{5.38}$$

For $\delta > 0$ and $\varepsilon_1 > 0$, denote by $\mathcal{E}_{\sum\gamma}$ the event that $\sum_{|\gamma|<\delta} |\gamma|^2 > \varepsilon_1$. By Lemma 5.4 for all $\varepsilon > 0$ there exist $\delta > 0$ and $\varepsilon_1 > 0$ such that

$$\mathbb{P}\left(\mathcal{E}_{\sum\gamma}\right) = \mathbb{P}\left(\sum_{|\gamma|<\delta} |\gamma|^2 > \varepsilon_1\right) < \varepsilon.$$

Similarly, denote by $\mathcal{E}_{\sum\bar{\gamma}_n}$ the event that $\sum_{|\bar{\gamma}_n|<\delta} |\bar{\gamma}_n|^2 > \varepsilon_2$. By Lemma 5.6 for all $\varepsilon > 0$ there exist $\delta > 0$ and $\varepsilon_2 > 0$ such that

$$\mathbb{P}\left(\mathcal{E}_{\sum\bar{\gamma}_n}\right) = \mathbb{P}\left(\sum_{|\bar{\gamma}_n|<\delta} |\bar{\gamma}_n|^2 > \varepsilon_2\right) < \varepsilon.$$

Fix $\varepsilon > 0$. Then for all $\alpha > 0$ we have

$$\begin{aligned}\mathbb{P}\left(d_2(Y_n', Y') > \alpha\right) \leq\ & \mathbb{P}\left(d_2(Y_n', Y') > \alpha \mid \neg\mathcal{E}_{\sum\bar{\gamma}_n}, \neg\mathcal{E}_{\sum\gamma}, \neg\mathcal{E}_{\gamma}, \neg\mathcal{E}_{\bar{\gamma}_n}, \neg\mathcal{E}_{\beta}\right) \\ & + \mathbb{P}(\mathcal{E}_{\sum\bar{\gamma}_n}) + \mathbb{P}(\mathcal{E}_{\sum\gamma}) + \mathbb{P}(\mathcal{E}_{\gamma}) + \mathbb{P}(\mathcal{E}_{\bar{\gamma}_n}) + \mathbb{P}(\mathcal{E}_{\beta}).\end{aligned} \tag{5.39}$$

Choose $\delta > 0$ and $C > 0$ such that $\mathbb{P}(\mathcal{E}_{\gamma}) < \varepsilon/5$ and $\mathbb{P}(\mathcal{E}_{\bar{\gamma}_n}) < \varepsilon/5$. For this δ, choose $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 < \alpha$ such that $\mathbb{P}(\mathcal{E}_{\sum\bar{\gamma}_n}) < \varepsilon/5$ and $\mathbb{P}(\mathcal{E}_{\sum\gamma}) < \varepsilon/5$. Now consider the conditioned probability in (5.39). The distance between all entries of Y_n' and Y' not starting before C or with size smaller than δ will be at most $\varepsilon_1 + \varepsilon_2$. As $\beta \to 0$ the distance of all other excursions, say d_β, vanishes as well. Therefore choose $\beta > 0$ such that $\mathbb{P}(\mathcal{E}_{\beta}) < \varepsilon/5$ and $\varepsilon_1 + \varepsilon_2 + d_\beta < \alpha$. Then

$$\mathbb{P}\left(d_2(Y_n', Y') > \alpha \mid \neg\mathcal{E}_{\sum\bar{\gamma}_n}, \neg\mathcal{E}_{\sum\gamma}, \neg\mathcal{E}_{\gamma}, \neg\mathcal{E}_{\bar{\gamma}_n}, \mathcal{E}_{\beta}\right) = 0$$

and $\mathbb{P}(d_2(Y_n', Y') > \alpha) < \varepsilon$. Therefore $Y_n' \to_p Y'$ which implies $Y_n' \to_d Y'$ with regard to the $l^2_{\searrow}$ topology. $\square$

6 The nonuniform random graph

In this thesis we discussed a uniform random graph model in which all vertices were treated equally, meaning the probability of drawing an edge (v_i, v_j) was p_n regardless of the choice of vertices v_i and v_j. As it turns out, Theorem 1.7 extends to a nonuniform random graph, in which every vertex has a weight or size and an edge is more likely to be drawn between two large vertices. All concepts introduced here can be found in Chapters 1.4 and 3 of [3].

6.1 The nonuniform model

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a positive real vector and $q > 0$. We define a random graph $\mathscr{W}(\boldsymbol{x}, q)$ on vertices $\{1, \ldots, n\}$ similarly to the uniform Erdős-Rényi model. A pair (i, j) of vertices is an edge with probability

$$p_{i,j} = 1 - \exp(-q x_i x_j),$$

independently for distinct pairs. Interpreting x_i as the size or weight of vertex i, we say a component $\mathcal{C}$ of $\mathscr{W}(\boldsymbol{x}, q)$ has size

$$C = \sum_{i \in \mathcal{C}} x_i.$$

We now state the equivalent of Theorem 1.7 for the nonuniform model. Recall that we had to rescale the breadth-first walk on the graphs of the uniform model to fit the Brownian motion B^θ. In this case, we build the scaling into the restrictions on the random graph, such that the component sizes themselves may converge in distribution to excursions of the Brownian motion with drift. For a vector $\boldsymbol{x}$, define

$$\sigma_r := \sum_i x_i^r \quad \text{for } r \geq 1,$$
$$x_* := \max_i x_i.$$

Theorem 6.1. *For each n, let $\boldsymbol{x}^{(n)}$ be a finite positive vector and let $q^{(n)} > 0$. Suppose*

$$\frac{\sigma_3^{(n)}}{(\sigma_2^{(n)})^3} \to 1, \tag{6.1}$$

$$q^{(n)} - \frac{1}{\sigma_2^{(n)}} \to \theta, \tag{6.2}$$

$$\frac{x_*^{(n)}}{\sigma_2^{(n)}} \to 0, \tag{6.3}$$

as $n \to \infty$ for some $\theta \in (-\infty, \infty)$. Let $(C^{(n)}(j); j \geq 1)$ be the ordered component sizes of $\mathscr{W}(\boldsymbol{x}^{(n)}, q^{(n)})$. Then

$$(C^{(n)}(j); j \geq 1) \to_d (C^{\theta}(j); j \geq 1)$$

with respect to the $l^2_{\searrow}$ topology, where $(C^{\theta}(j); j \geq 1)$ are the ordered excursion lengths of B^{θ}.

Note that this theorem does not imply the convergence of the number of vertices in each component, but its total size which is dependent on the number of vertices and their individual sizes.

We begin a brief discussion of the conditions on the theorem above. Consider the model $\mathscr{W}(a\boldsymbol{x}, a^{-2}q)$. For a pair (i, j) the edge probability

$$p_{i,j} = 1 - \exp(-a^{-2}qax_i ax_j) = 1 - \exp(-qx_i x_j)$$

is the same as in $\mathscr{W}(\boldsymbol{x}, q)$, therefore their component sizes will have the same distribution, scaled by the factor a. Assume $\mathscr{W}(\boldsymbol{x}, q)$ satisfies (6.2) and (6.3), while $\sigma_3^{(n)}(\sigma_2^{(n)})^{-3} \to c$ for some $c \neq 0$. Then for $a = c^{1/3}$ the random graph $\mathscr{W}(a\boldsymbol{x}, a^{-2}q)$ satisfies

$$\begin{aligned} \frac{\sigma_3^{(n)}}{(\sigma_2^{(n)})^3} &\to 1, \\ q^{(n)} - \frac{1}{\sigma_2^{(n)}} &\to a^{-2}\theta, \\ \frac{x_*^{(n)}}{\sigma_2^{(n)}} &\to 0, \end{aligned}$$

and its component sizes, and therefore the rescaled component sizes of $\mathscr{W}(\boldsymbol{x}, q)$, converge in distribution to excursion lengths of $B^{a^{-2}\theta}$. As long as $\sigma_3^{(n)}(\sigma_2^{(n)})^{-3}$ does not diverge or converge to zero, we can assume (6.1) by rescaling. Condition (6.2) then links the edge probability to the parameter θ. Assuming (6.1) holds we can rewrite (6.3) as

$$\frac{x_*^3}{\sigma_3^{(n)}} \to 0$$

and see that individual terms contribute to $\sigma_3^{(n)}$ only negligibly for large n.

How do the individual edge probabilities behave as $n \to \infty$? Clearly $\sigma_3^{(n)} \leq x_*^{(n)}\sigma_2^{(n)}$, so we have

$$\frac{\sigma_3^{(n)}}{(\sigma_2^{(n)})^3} \leq \frac{x_*^{(n)}\sigma_2^{(n)}}{(\sigma_2^{(n)})^3} = \frac{x_*^{(n)}}{\sigma_2^{(n)}}\frac{1}{\sigma_2^{(n)}}.$$

Using conditions (6.1) and (6.3) gives $(\sigma_2^{(n)})^{-1} \to \infty$ and $\sigma_2^{(n)} \to 0$. Then, by (6.2),

$$q = O\left(\frac{1}{\sigma_2^{(n)}}\right)$$

and since $x_i \leq \sqrt{\sigma_2^{(n)}}$ for all i we conclude

$$\begin{aligned}\max_{i,j} q x_i x_j &\leq q x_*^{(n)} \sqrt{\sigma_2^{(n)}} \\ &= O\left(\frac{x_*^{(n)}}{\sigma_2^{(n)}} \sqrt{\sigma_2^{(n)}}\right) \\ &\to 0\end{aligned}$$

by (6.3). Therefore

$$p_{i,j} = 1 - \exp(q x_i x_j) \to 0,$$

the individual edge probabilities converge to zero as $n \to \infty$ and are asymptotically close to $q x_i x_j$ since $1 - \exp(-x) \approx x$ for very small x.

The classical model $\mathscr{G}(n, n^{-1} + \theta n^{-4/3})$ can be derived by setting

$$\begin{aligned}x_i^{(n)} &= n^{-2/3}, \\ \sigma_2^{(n)} &= \sum_{i=1}^{n} (n^{-2/3})^2 = n^{-1/3}, \\ \sigma_3^{(n)} &= n^{-1}, \\ q^{(n)} &= n^{1/3} + \theta,\end{aligned}$$

since then $p_{i,j} = 1 - \exp(-(n^{1/3} + \theta) n^{-4/3}) = 1 - \exp(-p_n) \approx p_n$ for large n.

6.2 The breadth-first walk

Theorem 6.1 is proven again by defining the breadth-first walk on this new random graph and proving its convergence to the Brownian motion with drift. In Chapter 3 we defined this walk for a deterministic unweighted graph and examined its properties when applied to a random construction of a graph. For this model it is a better fit to describe a construction of $\mathscr{W}(\boldsymbol{x}, q)$ with a simultaneous corresponding breadth-first walk.

For each ordered pair (i, j) of vertices, $i \neq j$, define a random variable $\xi_{i,j} \sim \operatorname{Exp}(q x_j)$. The first vertex v_1 is chosen via size-biased sampling, meaning the probability of choosing any vertex is proportional to its size (see Section 3.3 of [3] for details). The children of v_1 are now all vertices v for which $\xi_{v_1,v} \leq x_{v_1}$. Order these vertices as $v_2, v_3, \ldots$ such that ξ_{v_1,v_i} is increasing in i.

We begin defining the breadth-first walk $Z(s)$. Start with $Z(0) = 0$ and for $0 \leq s \leq x_{v_1}$ let

$$Z(s) = -s + \sum_{v} x_v \mathbb{1}_{\{\xi_{v_1,v} \leq s\}}. \tag{6.4}$$

Now

$$Z(x_{v_1}) = -x_{v_1} + \sum_{v \text{ child of } v_1} x_v. \tag{6.5}$$

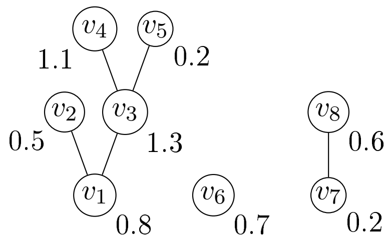
While the jumps of the breadth-first walk were always of size 1 for the uniform model, now they are equal to the size of the vertex connecting to the current node.

We define children of further nodes inductively: If v_i is a node in the same component as v_1, consider the set of vertices which are children to one of the vertices $v_1, \dots, v_{i-1}$ but are not part of the set $\{v_1, \dots, v_{i-1}\}$ themselves. These vertices are labelled $v_i, \dots, v_{l(i)}$ for some $l(i) \geq i$. Now define the children of v_i as the vertices $v \notin \{v_1, \dots, v_{l(i)}\}$ for which $\xi_{v_i,v} \leq x_{v_i}$, label them as $v_{l(i)+1}, v_{l(i)+2}, \dots$ and again order them such that ξ_{v_i,v_j} is increasing in j.

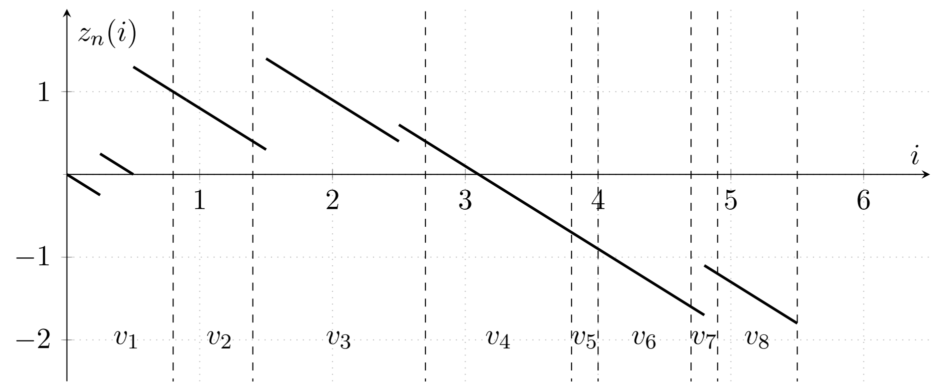
Write $\tau_{i-1} = \sum_{j=1}^{i-1} x_{v_j}$ and set

$$Z(\tau_{i-1} + s) = Z(\tau_{i-1}) - s + \sum_{v \text{ child of } v_i} x_v \mathbb{1}_{\{\xi_{v_i,v} \leq s\}}$$

for $0 \leq s \leq x_{v_i}$. After having exhausted every vertex in the first component, choose the root of the next component by size-biased sampling and continue.



(a) Components of $\mathscr{W}(\boldsymbol{x}, q)$ with corresponding vertex sizes.



(b) The associated continuous-time breadth-first walk

Figure 6.1: Continuous breadth-first walk components of the nonuniform random graph. Dashed lines indicate the end of vertices.

This construction yields a random forest on the vertices $\{1, \dots, n\}$ with an ordering $\{v_1, \dots, v_n\}$ as well as an associated breadth-first walk in continuous time $(Z(s); 0 \leq s \leq \sum_v x_v)$.

For surplus edges, we add each edge (v_i, v_j) such that $i < j \leq l(i)$ and $\xi_{v_i,v_j} \leq x_{v_i}$. It is easily seen how the resulting random graph is $\mathscr{W}(\boldsymbol{x}, q)$: For a pair of vertices (i, j), one must be traversed first by the breadth-first walk, say i, and the resulting probability of the edge (i,j) being present in the final graph is equal to

$$\mathbb{P}(\xi_{i,j} \leq x_i) = 1 - \exp(qx_ix_j)$$

since $\xi_{i,j} \sim \mathrm{Exp}(qx_j)$. Once j is choosing its children it can not form a new connection to i any more.

Figure 6.1 shows this construction for some sample components. We may think of the vertices $v_1, v_2, \ldots$ occupying consecutive intervals on the time axis, each the length of the interval for each vertex equal to its size. During this time interval we search for children of the current vertex and each "birth" results in a jump of height equal to the size of the new vertex.

There now exists a direct equivalent to Theorem 3.1 for this nonuniform graph model and its breadth-first walk.

Theorem 6.2. *Let $Z_n(s), 0 \leq s \leq \sum_v x_v^{(n)}$ be the breadth-first walk on $\mathscr{W}(\boldsymbol{x}^{(n)}, q^{(n)})$. Define the rescaled breadth-first walk*

$$\bar{Z}_n(s) := \sqrt{\frac{\sigma_2^{(n)}}{\sigma_3^{(n)}}} Z_n(s).$$

Then $\bar{Z}_n \to_d W^\theta$.

For a complete proof of both theorems we refer to [3, p.828ff.].

7 Outlook

This chapter will provide an overview over the other central issue of [3], the multiplicative coalescent introduced in Chapters 1.5 and 4. So far, we always considered θ as a constant which lead to a change of p_n only in n. We will now introduce a process that describes the random graph $\mathscr{W}(\boldsymbol{x}, q)$ for fixed $\boldsymbol{x}$ and n but variable q and therefore $p_{i,j}$.

We change the notation slightly. Fix $\boldsymbol{x} \in l^2_\searrow$. For a pair (i,j), with $i < j$, define an exponentially distributed random variable $\xi_{i,j} \sim \mathrm{Exp}(1)$, independent for distinct pairs. For $\theta \in [0,\infty)$, define the random graph model $\mathscr{W}(\boldsymbol{x}, \theta)$ as the graph in which there exists an edge (i,j) iff $\xi_{i,j} \leq \theta x_i x_j$. This way, we constructed a random graph process $(\mathscr{W}(\boldsymbol{x}, \theta); 0 \leq \theta < \infty)$ where the number of edges of $\mathscr{W}(\boldsymbol{x}, \theta)$ is increasing in θ. Let $X_i(\boldsymbol{x}, \theta)$ be the size of the i-th largest component of $\mathscr{W}(\boldsymbol{x}, \theta)$ and let $\boldsymbol{X}(\boldsymbol{x}, \theta) = (X_i(\boldsymbol{x}, \theta); i \geq 1)$.

For a vector of vertex sizes $\boldsymbol{x}$, $(\boldsymbol{X}(\boldsymbol{x}, \theta); 0 \leq \theta < \infty)$ is a continuous time Markov chain on a finite state space which is entirely dependent on $\boldsymbol{x}$. A state at some time θ_0 is a vector of component sizes, which consist of the sizes of the individual vertices:

$$\boldsymbol{X}(\boldsymbol{x}, \theta_0) = (X_1(\boldsymbol{x}, \theta_0), X_2(\boldsymbol{x}, \theta_0), \dots) = (|\mathcal{C}_1|, |\mathcal{C}_2|, \dots)$$

where $|\mathcal{C}_i| = \sum_{v \in \mathcal{C}_i} x_v$.

The dynamics of this Markov chain can be described as follows. With increasing θ we expect new edges to form and therefore separate components to merge into larger components. Take two components of sizes $|\mathcal{C}_1| = x$ and $|\mathcal{C}_2| = y$. There are xy different possible edges between these two that may form to directly connect them and create a large component of size $x + y$. We call a Markov process with finite state space in $l^2_\searrow$ and dynamics as described above a *multiplicative coalescent*. For an initial state $\boldsymbol{x} = (x, 0, 0, \dots)$ we have a "constant" multiplicative coalescent with

$$\boldsymbol{X}(\theta) = (x, 0, 0, \dots) \text{ for all } -\infty < \theta < \infty.$$

We say a vector $\boldsymbol{x} \in l^2_\searrow$ is of finite length if $x_i = 0$ for all $i \geq N$ for some $N \in \mathbb{N}$. For a size vector $\boldsymbol{x}$ of finite length, the dynamics of the multiplicative coalescent starting in $\boldsymbol{X}(0) = \boldsymbol{x}$ can be expressed in martingale form as follows. For any vector $\boldsymbol{x}$, let $\boldsymbol{x}^{(i+j)}$ be the configuration obtained by merging the i-th and j-th components of $\boldsymbol{x}$, such that there is a new cluster of size $x_i + x_j$ inserted next to some cluster x_u:

$$\boldsymbol{x}^{(i+j)} = (x_1, \dots, x_{u-1}, x_i + x_j, x_u, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots).$$

Define the filtration $\mathcal{F}_\theta = \sigma\{\boldsymbol{X}(u); u \leq \theta\}$. Then for all test functions $g : l^2_\searrow \to \mathbb{R}$ we have

$$\mathbb{E}\left[\Delta g(\boldsymbol{X}(\theta)) \mid \mathcal{F}_\theta\right] = \sum_i \sum_{j > i} X_i(\theta) X_j(\theta) \left(g(\boldsymbol{X}^{(i+j)}(\theta)) - g(\boldsymbol{X}(\theta))\right) d\theta,$$

where we use the infinitesimal notation

$$\mathbb{E}\left[\Delta Y(t) \mid \mathcal{F}_t\right] = A(t)dt$$
$$\iff M(t) = Y(t) - \int_0^t A(s)ds \text{ is a local martingale.}$$

A key result of [3] is the existence of at least one multiplicative coalescent process, the steps of which are distributed as the limit process found in Theorem 1.7. We call a multiplicative coalescent *eternal* if it is defined for all $-\infty < \theta < \infty$.

Theorem 7.1. *There exists a multiplicative coalescent process* $(\boldsymbol{X}^*(\theta); -\infty < \theta < \infty)$ *on* $l^2_\searrow$*, called the* standard eternal multiplicative coalescent*, such that for each* θ *we have* $\boldsymbol{X}^*(\theta) =_d \boldsymbol{C}^\theta$*, where* $\boldsymbol{C}^\theta$ *is the joint distribution of excursion lengths of* B^θ *as used in Theorem 1.7.*

A proof of this theorem is featured in Chapter 4 of [3]. A further analysis of multiplicative coalescent processes can be found in a companion paper by David Aldous and Vlada Limic, [21].

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