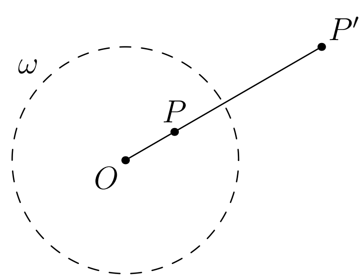
Inversion

1 Introduction

Geometric inversion is a technique for transforming diagrams, like using homothety or reflection. (In fact, inversion can be thought of as reflecting through a circle, as opposed to reflecting through a line.) The power of inversion lies in the fact that an inverted diagram has many of the same properties as the original diagram, but it can be much simpler to work with. In this handout, we describe the properties of inversion, and how it can be applied to solve geometry problems.

2 Definitions

Given a circle ω centered at O with radius r, and a point P other than O, let P' be the point on ray $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$. Then P' is the *inverse* of P with respect to the circle ω. (The inverse of O is the point at infinity, denoted by P_∞.)

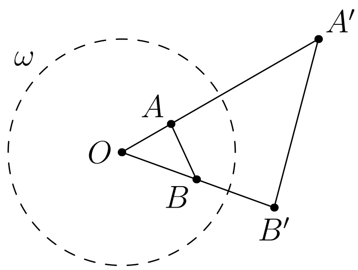


The following properties are evident:

- If P' is the inverse of P, then P is the inverse of P'. Thus, we can refer to pairs of inverse points.
- A point is its own inverse if and only if it lies on the circle.
- A point lies inside the circle if and only if its inverse lies outside the circle.

3 Inverses of Two Points

Let A and B be two points, and let A' and B' be their inverses with respect to circle ω, respectively.



Then

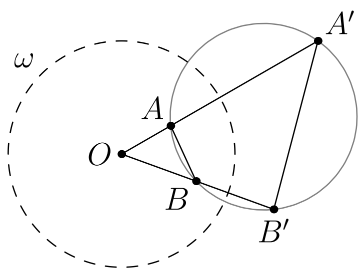
$$\frac{OA'}{OB'} = \frac{r^2/OA}{r^2/OB} = \frac{OB}{OA},$$

and $\angle A'OB' = \angle BOA$, so triangles $A'OB'$ and BOA are similar. Hence, the distance $A'B'$ is given by

$$A'B' = \frac{OA'}{OB} \cdot AB = \frac{r^2/OA}{OB} \cdot AB = \frac{r^2 \cdot AB}{OA \cdot OB}.$$

We will refer to this formula as the **inversion distance formula**.

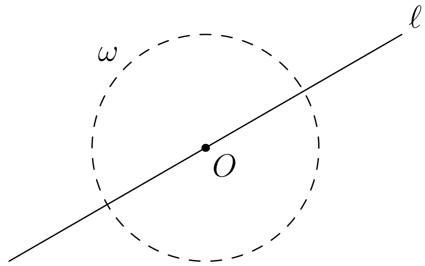
Furthermore, $\angle AA'B' = \angle OA'B' = \angle OBA = 180^\circ - \angle ABB'$, so quadrilateral $ABB'A'$ is cyclic. (This also follows from $OA \cdot OA' = OB \cdot OB' = r^2$.)



4 Inverses of Lines and Circle

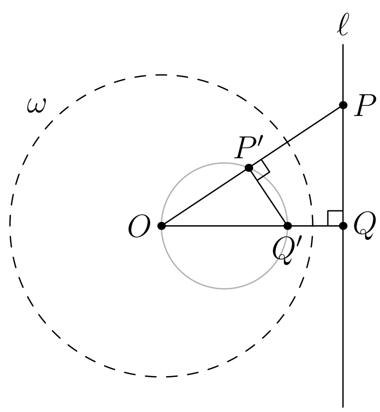
4.1 The Inverse of a Line Passing Through the Center

If ℓ is a line passing through the center O of circle ω, then the inverse of ℓ with respect to ω is itself.



4.2 The Inverse of a Line Not Passing Through the Center

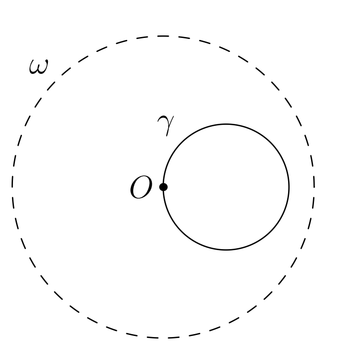
Let ℓ be a line that does not pass through the center O of circle ω. Let P be an arbitrary point on ℓ, and let Q be the projection of O onto ℓ. Let P' and Q' be the inverses of P and Q with respect to ω, respectively. Then quadrilateral $PQQ'P'$ is cyclic.



Since $\angle PQQ' = 90^\circ$, it follows that $\angle PP'Q' = 90^\circ$. Then $\angle OP'Q' = 90^\circ$, which means that P' lies on the circle with diameter OQ'. Conversely, if P' is a point on the circle with diameter OQ' other than O, then let P be the inverse of P'. Then by the same argument, $\angle PQQ' = 90^\circ$, so P lies on ℓ. Therefore, the inverse of ℓ with respect to ω is the circle with diameter OQ'. (In particular, the circle passes through O.)

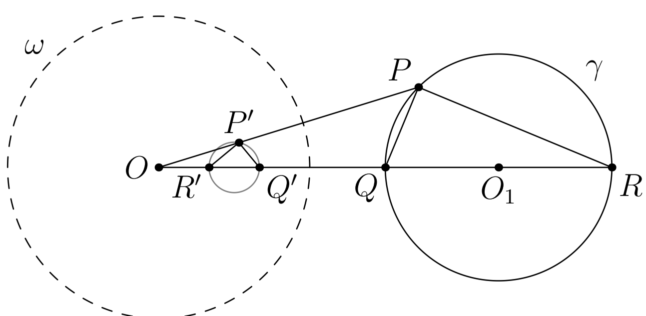
4.3 The Inverse of a Circle Passing Through the Center

Let γ be a circle that passes through the center O of circle ω. Then by the same argument as in the previous subsection, the inverse of γ is a line (not passing through O).



4.4 The Inverse of a Circle Not Passing Through the Center

Let γ be a circle that does not pass through the center O of circle ω. Let O_1 be the center of circle γ, and let OO_1 intersect γ at Q and R. Let P be an arbitrary point on γ, and let P', Q', and R' be the inverses of P, Q, and R with respect to ω, respectively.



Since P lies on γ, $\angle QPR = 90^\circ$. Quadrilateral $PQQ'P'$ is cyclic, so $\angle P'Q'R' = 180^\circ - \angle P'Q'Q = \angle P'PQ$. Quadrilateral $PRR'P'$ is cyclic, so $\angle P'R'Q' = 180^\circ - \angle P'PR$. Then

$$\begin{aligned}
\angle Q'P'R' &= 180^\circ - \angle P'Q'R' - \angle P'R'Q' \\
&= 180^\circ - \angle P'PQ - (180^\circ - \angle P'PR) \\
&= \angle P'PR - \angle P'PQ \\
&= \angle QPR \\
&= 90^\circ,
\end{aligned}$$

which means that P' lies on the circle with diameter $Q'R'$. Conversely, if P' is a point on the circle with diameter $Q'R'$, then let P be the inverse of P'. Then by the same argument, $\angle QPR = 90^\circ$, so P lies on γ. Therefore, the inverse of γ with respect to ω is the circle with diameter $Q'R'$. (In particular, the circle does not pass through O.)

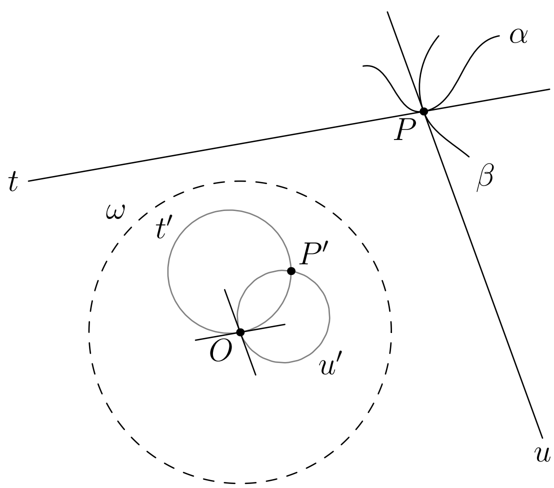
Note: If O_2 is the center of the circle with diameter $Q'R'$, then in general, O_2 is **not** the inverse of O_1 with respect to ω. In other words, when inversion maps circles to circles, it does not necessarily map the centers of those circles to each other.

We can summarize our results as follows:

- The inverse of a line passing through O is itself.
- The inverse of a line not passing through O is a circle (passing through O).
- The inverse of a circle passing through O is a line (not passing through O).
- The inverse of a circle not passing through O is another circle (not passing through O).

5 Inversion and Angles

Let α and β be two curves in the plane that intersect at P, and assume there exist tangents t and u to these curves at P, respectively. Then the angle between curves α and β at P is defined as the angle between tangents t and u.

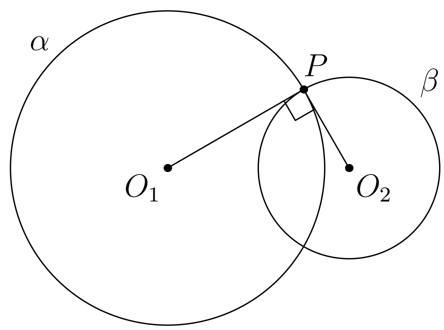


Let P', t', and u' be the inverses of P, t, and u with respect to circle ω, respectively. (We assume that the center O of circle ω does not lie on t or u; these cases can be dealt with easily, using the same argument.) Then t' and u' are circles that intersect at O and P'. Note that the angle between t' and u' at P' is equal to the angle between t' and u' at O.

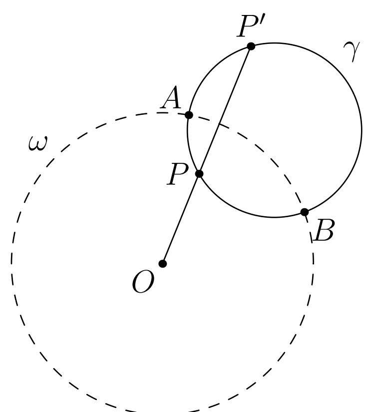
The tangent to t' at O is parallel to t, and the tangent to u' at O is parallel to u. (These observations follow from our proof in Subsection 4.2.) Then the angle between t' and u' at O is equal to the angle between t and u. Therefore, the angle between t' and u' at P' is also equal to the angle between t and u. Hence, inversion preserves angles.

Note: We are not saying that if A', B', and C' are the inverses of A, B, and C, respectively, then $\angle B'A'C' = \angle BAC$. We are saying that inversion preserves angles at intersections of curves.

We say that two curves are *orthogonal* if they intersect at right angles. In particular, two circles α and β with centers O_1 and O_2, respectively, are orthogonal if and only if $\angle O_1PO_2 = 90^\circ$, where P is a point where the two circles intersect. Equivalently, the tangent from O_1 to β is equal to the radius of α (and the tangent from O_2 to α is equal to the radius of β).



Let γ be a circle that is orthogonal to ω. Let γ and ω intersect at A and B, and let γ' be the inverse of circle γ with respect to ω. Then γ' is also a circle that passes through A and B.



But since inversion preserves angles as intersections of curves, and ω is its own inverse, γ' is also orthogonal to ω. Since there is only one circle passing through A and B that is orthogonal to ω, circle γ is its own inverse. (In other words, if P is a point on γ, then the inverse P' of P with respect to ω also lies on γ.) Conversely, by the same argument, any circle (other than ω) that is its own inverse with respect to ω is orthogonal to ω.

We can prove another converse of this result: Let P and P' be two distinct points that are inverses with respect to ω, and let γ be a circle that passes through P and P'. We will show that γ is its own inverse with respect to ω.

Since P and P' are distinct, and inverses with respect to ω, one of them lies inside ω, and the other lies outside ω. Hence, γ must intersect ω at two points, say A and B.

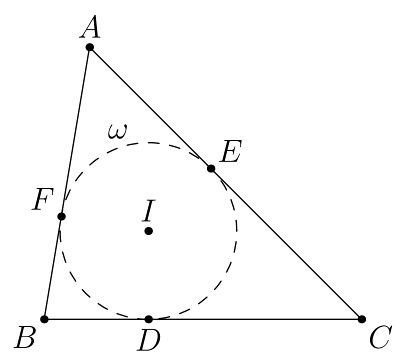
Let γ' be the inverse of γ with respect to ω. Then γ' passes through P, P', A, and B. But γ also passes through these points, and four points uniquely determine a circle, so again, circles γ and γ' coincide, which means that γ is its own inverse.

For a circle γ other than ω, we have established that the following statements are equivalent:

- The circle γ is orthogonal to ω.
- The circle γ is its own inverse with respect to ω.
- The circle γ passes through two distinct points P and P' that are inverses with respect to ω.

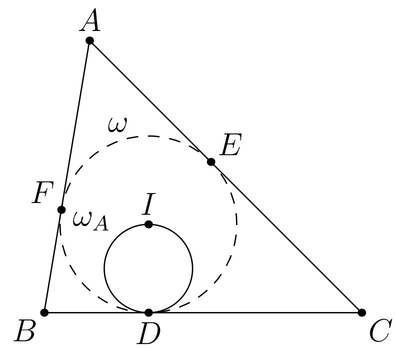
6 Inverting Through the Incircle

Let ω, I, and r denote the incircle, incenter, and inradius of triangle ABC, respectively. Let the incircle ω be tangent to sides BC, CA, and AB at D, E, and F, respectively.



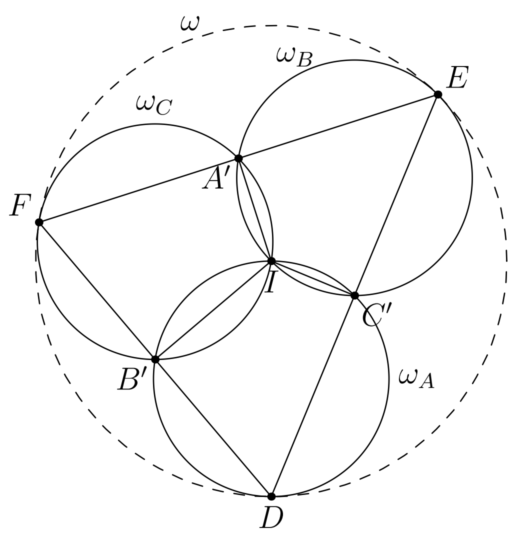
We invert the diagram through the incircle ω.

First, we consider the inverse of line BC. The inverse of line BC is a circle passing through I, say ω_A. Since BC is tangent to ω at D, the circle ω_A is also tangent to ω at D. Therefore, ω_A must be the circle with diameter ID.



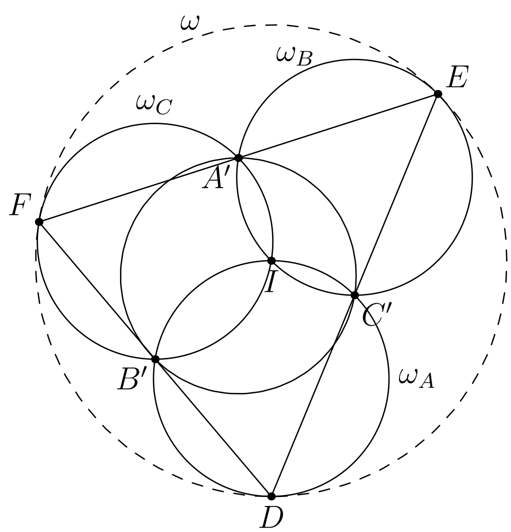
Similarly, the inverse of AC is the circle with diameter IE, say ω_B, and the inverse of AB is the circle with diameter IF, say ω_C.

Now, let A', B', and C' be the inverses of A, B, and C, respectively. Since A is the intersection of AB and AC, A' is the intersection of ω_B and ω_C other than I. Similarly, B' is the intersection of ω_A and ω_C other than I, and C' is the intersection of ω_A and ω_B other than I.



Since A' lies on ω_B, the circle with diameter IE, $\angle IA'E = 90^\circ$. But A' also lies on ω_C, the circle with diameter IF, so $\angle IA'F = 90^\circ$. Hence, E, A', and F are collinear. In fact, A' is the midpoint of EF, since

IA' is perpendicular to chord EF. By the same argument, B' is the midpoint of DF, and C' is the midpoint of DE. We conclude that the circumcircle of triangle $A'B'C'$ (which is the inverse of the circumcircle of triangle ABC) is also the nine-point circle of triangle DEF.



Hence, the circumradius of triangle $A'B'C'$ is $r/2$, half the circumradius of triangle DEF. If O and R are the circumcenter and circumradius of triangle ABC, respectively, then clearly $IO < R$, so by Exercise 1 (at the end of the handout),

$$\frac{r}{2} = \frac{r^2 \cdot R}{R^2 - IO^2},$$

which implies that

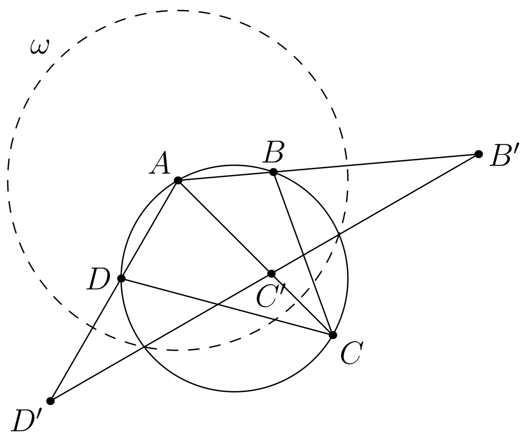
$$IO^2 = R^2 - 2Rr.$$

7 Worked-out Examples

Problem. (Ptolemy's Theorem) Let $ABCD$ be a cyclic quadrilateral. Prove that

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

Solution. Let ω be a circle centered at A with radius r, and let B', C', and D' be the inverses of B, C, and D with respect to ω, respectively.



Since B, C, and D lie on a circle passing through A, the points B', C', and D' lie on a line, in that order, so $B'C' + C'D' = B'D'$. Then from the inversion distance formula,

$$\frac{r^2 \cdot BC}{AB \cdot AC} + \frac{r^2 \cdot CD}{AC \cdot AD} = \frac{r^2 \cdot BD}{AB \cdot AD}.$$

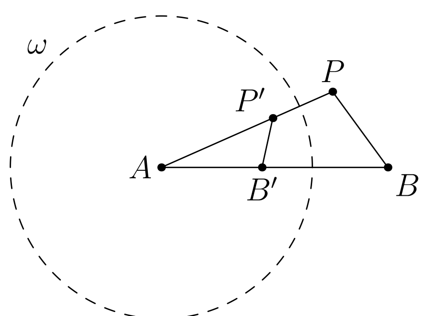
Multiplying both sides by $AB \cdot AC \cdot AD$ and dividing by r^2, we get

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

■

Problem. (Apollonius Circle) Let A and B be two points in the plane, and let $k \neq 1$ be a positive real number. Show that the locus of points P such that $AP/PB = k$ is a circle.

Solution. Let P be a point such that $AP/PB = k$. Let ω be a circle centered at A with radius r, and let B' and P' be the inverses of B and P with respect to ω, respectively.



Then by inversion distance formula,

$$B'P' = \frac{r^2 \cdot BP}{AB \cdot AP} = \frac{r^2}{k \cdot AB},$$

which is a fixed quantity. Hence, the locus of P' is a circle centered at B'. Furthermore,

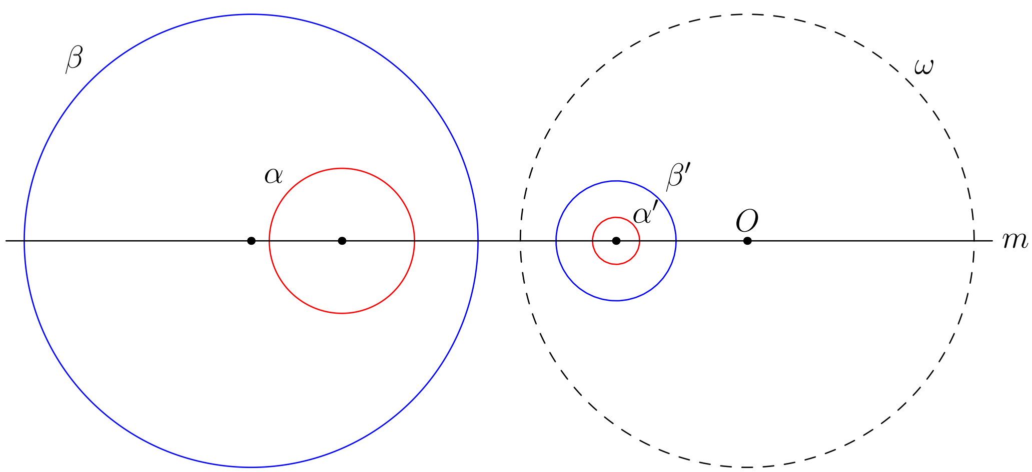
$$AB' = \frac{r^2}{AB},$$

which is not equal to $B'P'$ since $k \neq 1$. Therefore, the locus of P' is a circle that does not pass through A, which means that locus of P is a circle. ■

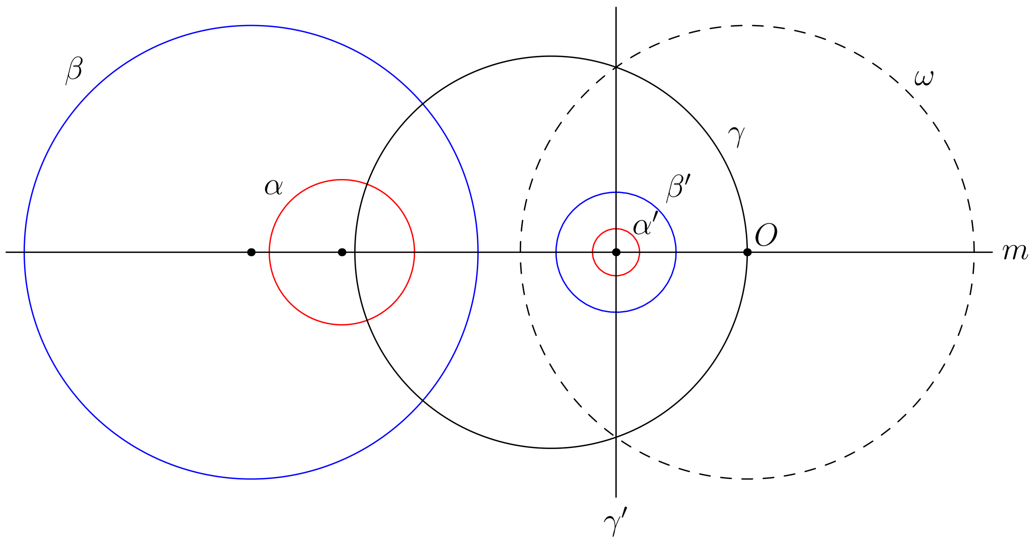
Problem. Let α and β be two circles that do not intersect. Prove that there exists a circle ω such that the inverses α' and β' of α and β with respect to ω, respectively, are concentric.

Solution. If α and β are already concentric, then we can take ω to be any circle that is concentric with α and β, so assume that α and β are not concentric.

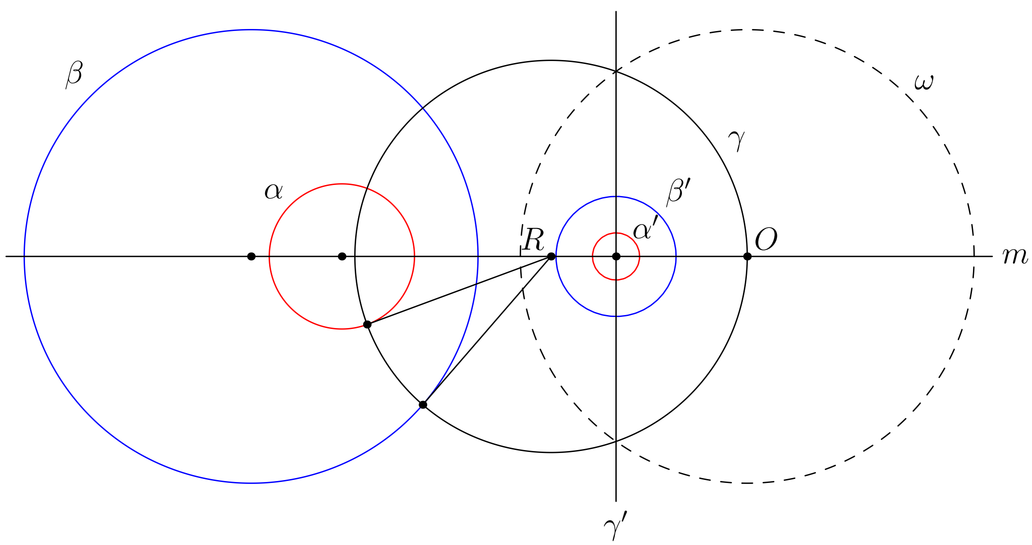
We start by looking at the completed diagram. Let m be the line joining the centers of α and β. By symmetry, it makes sense to look for a circle ω whose center O lies on m.



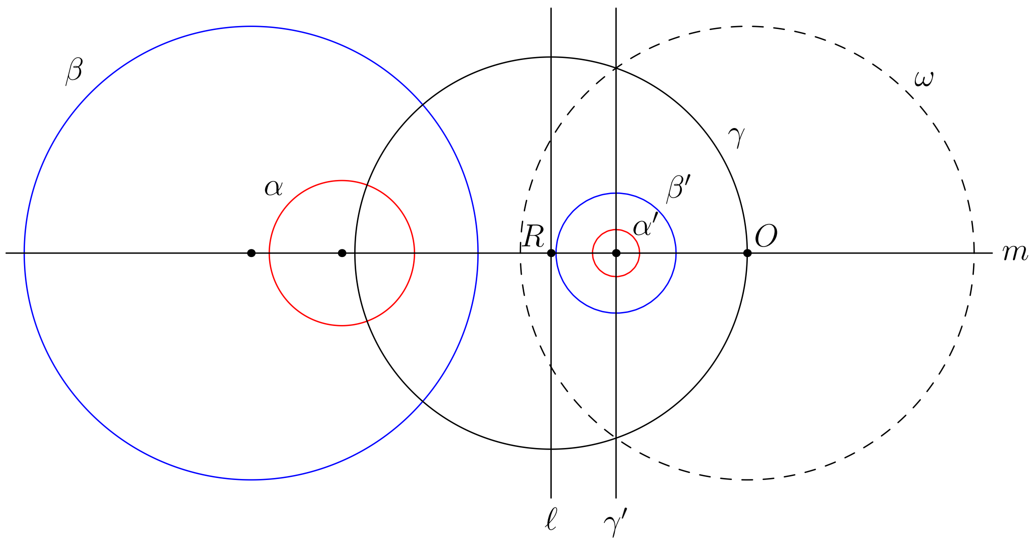
To help us locate O, let γ' be the line passing through the common centers of α' and β' that is perpendicular to m. Then line γ' is orthogonal to both α' and β', so γ is a circle that is orthogonal to both α and β. (Also, γ passes through O.) Thus, we must find a circle γ that is orthogonal to both α and β.



Let R be the center of circle γ. If γ is orthogonal to α, then the tangent from R to α is equal to the radius of γ. Similarly, if γ is orthogonal to β, then the tangent from R to β is also equal to the radius of γ. Therefore, R must lie on the radical axis of α and β.



We can now construct the circle ω as follows: Let m be the line joining the centers of α and β, let ℓ be the radical axis of α and β, and let R be the intersection of ℓ and m.

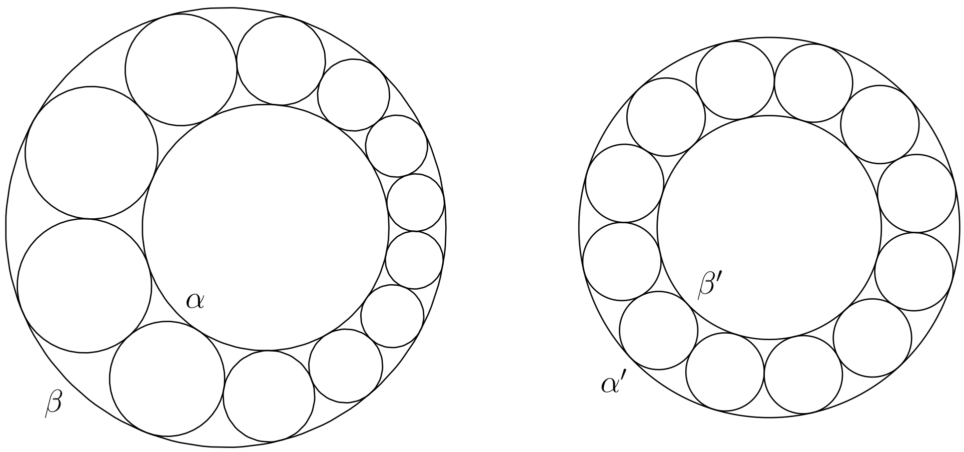


Since R lies on the radical axis ℓ, the tangents from R to α and β are equal. (Since α and β do not intersect, ℓ lies outside both α and β, so R lies outside both α and β.) Let γ be the circle centered at R whose radius is equal to the common tangent. Then γ is orthogonal to both α and β.

Let O be an intersection of γ and m, and let ω be any circle centered at O. Let α' and β' be the inverses of α and β with respect to ω, respectively. Since the centers of both α and β lie on m, the centers of both α' and β' also lie on m.

Since γ is a circle that passes through O and whose center R lies on m, the inverse of γ with respect to ω is a line γ' that is perpendicular to m. And since α and β are orthogonal to γ, α' and β' are orthogonal to γ', which means that the centers of both α' and β' lie on γ'. Therefore, circles α' and β' are concentric. (Both circles are centered at the intersection of m and γ'.) ■

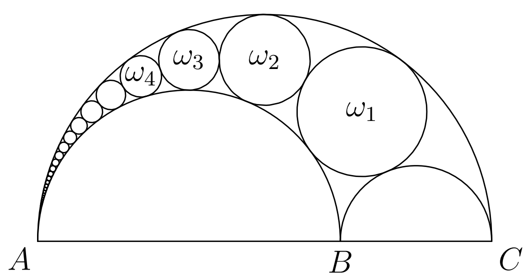
An elegant corollary of this problem is a result known as Steiner's Porism: Let α and β be two circles, so that α lies inside β. Given a circle that is externally tangent to α and internally tangent to β, we construct a chain of circles starting with that circle, so that each circle in the chain is also externally tangent to α and internally tangent to β, and tangent to the previous circle in the chain. If the chain closes for some initial circle, then the chain closes for any initial circle.



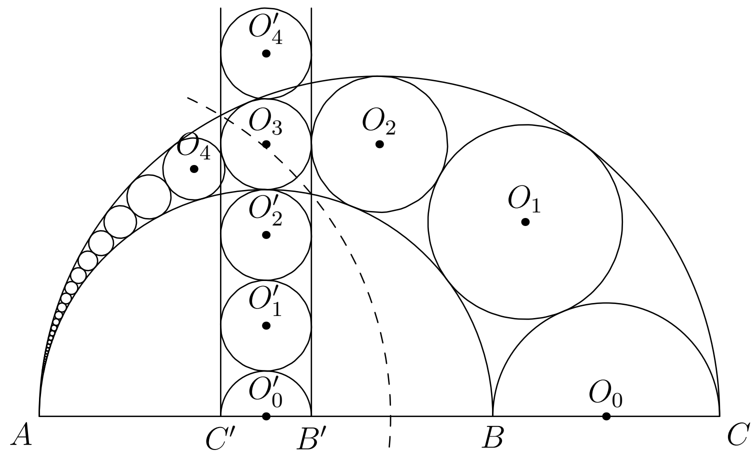
We can invert circles α and β to obtain two concentric circles α' and β', and the result follows.

Problem. Let A, B, and C be three collinear points. We construct the semicircles with diameters AB, AC, and BC. Let ω_0 be the circle with diameter BC, and for all $n \geq 1$, let ω_n be the circle that is externally tangent to circle ω_{n-1} and semicircle BC, and internally tangent to semicircle AC, as shown below.

Let O_n and r_n denote the center and radius of circle ω_n, respectively. Show that the distance from O_n to AC is $2nr_n$ for all n.



Solution. We invert the diagram through a circle centered at A, so that circle ω_n is its own inverse. (For example, we can take the radius of inversion to be the length of the tangent from A to circle ω_n.) The case $n = 3$ is illustrated below.



Let B' and C' be the inverses of B and C, respectively. The inverse of semicircle AB is a ray starting at B' that is perpendicular to AC. Similarly, the inverse of semicircle AC is a ray starting at C' that is perpendicular to AC.

Let ω_i' be the inverse of circle ω_i. Each circle ω_i is tangent to semicircles AB and AC, so each circle ω_i' is tangent to both rays. Hence, all circles ω_i' are congruent. But circle ω_n is its own inverse, so each circle ω_i' has radius r_n. Furthermore, consecutive circles are externally tangent, and the center of circle ω_0' lies on AC, so the distance from O_n to AC is $2nr_n$. ■

8 Tips on Using Inversion

- Inversion works best on problems that involve circles that are either tangent to each other or tangent to lines.
- Ideally, an inverted diagram should be simpler than the original diagram. Since inversion can take circles to lines, and lines are easier to work with than circles, try to choose a point that produces as many lines (and reduces as many circles) as possible. Points that generally achieve this are points of tangency, or points that many circles pass through.
- In most cases, the radius of inversion is not important, because changing the radius of inversion only changes the scale of the inverted diagram. However, in other cases, it is useful to superimpose the inverted diagram on top of the original diagram. (See the problem on page 15 for an example.) If there are elements in the original diagram and inverted diagram that can be aligned nicely, this is a sign that the choice of radius may be important.

9 Exercises

1. Let ω be a circle centered at O with radius r. Let ω_1 be a circle centered at O_1 with radius r_1, and let $d_1 = OO_1$. Prove the following:

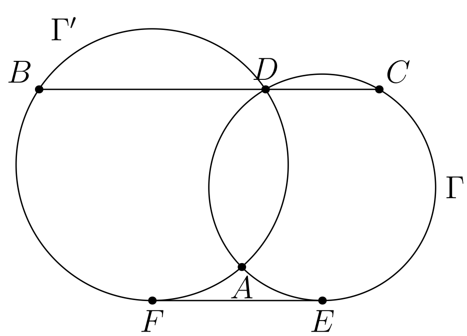
   (a) If $d_1 > r_1$, then the inverse of circle ω_1 with respect to ω is a circle centered at O_2 with radius r_2, where
   $$OO_2 = \frac{r^2 d_1}{d_1^2 - r_1^2},$$
   and
   $$r_2 = \frac{r^2 r_1}{d_1^2 - r_1^2}.$$

   (b) If $d_1 = r_1$, i.e. circle ω_1 passes through O, then the inverse of circle ω_1 with respect to ω is a line whose distance from O is
   $$\frac{r^2}{2r_1}.$$

   (c) If $d_1 < r_1$, then the inverse of circle ω_1 with respect to ω is a circle centered at O_2 with radius r_2, where
   $$OO_2 = \frac{r^2 d_1}{r_1^2 - d_1^2},$$
   and
   $$r_2 = \frac{r^2 r_1}{r_1^2 - d_1^2}.$$

2. Let D be a point in triangle ABC such that $AB = ab$, $AC = ac$, $AD = ad$, $BC = bc$, $BD = bd$, and $CD = cd$. Prove that $\angle ABD + \angle ACD = \pi/3$. (Singapore, 2004)

3. Two circles Γ and Γ' intersect at A and D, as shown below. A line is tangent to Γ and Γ' at E and F, respectively. The line through D parallel to EF intersects Γ and Γ' again at C and B, respectively. Show that the circumcircles of triangles BDE and CDF intersect again on the line AD.

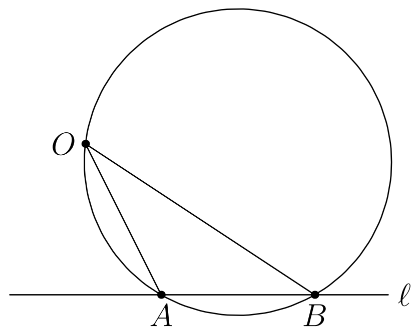


4. Let ω be a circle centered at O. Let A', B', and C' be the inverses of A, B, and C with respect to ω, respectively.

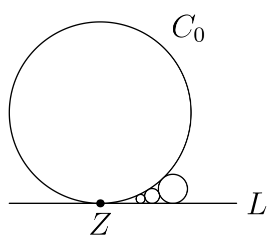
   (a) Let I be the incenter of triangle ABC, and let I' be the inverse of I with respect to ω. Prove that I' is the A-excenter of triangle $AB'C'$.

   (b) Show that O is the orthocenter of triangle ABC if and only if O is the incenter or an excenter of triangle $A'B'C'$.

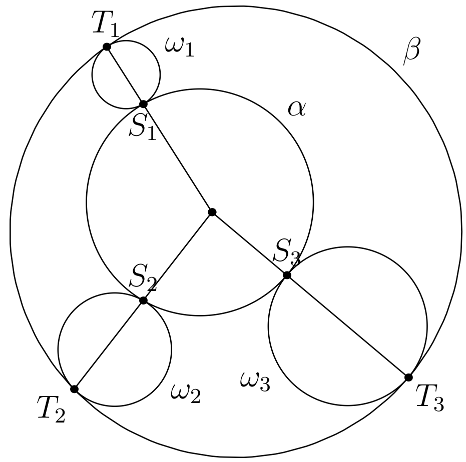
5. Let ℓ be a fixed line, and let O be a point not on ℓ. Let A and B be variable points on ℓ such that $\angle AOB$ is constant. Show that the circumcircle of triangle OAB is tangent to a fixed circle.



6. A circle C_0 of radius 1 km is tangent to a line L at Z. A circle C_1 of radius 1 mm is drawn tangent to both C_0 and L, on the right-hand side of C_0. A family of circles C_i is constructed outwardly to the right side so that each C_i is tangent to C_0, L, and to the previous circle C_{i-1}. Eventually the members become so large that it is impossible to enlarge the family any further. How many circles can be drawn before this happens?



7. Let α and β be two circles, so that α lies inside β. Let ω_1, ω_2, and ω_3 be three circles, such that each circle ω_i is externally tangent to α and internally tangent to β. For $1 \leq i \leq 3$, let ω_i be tangent to α and β at S_i and T_i, respectively. Prove that S_1T_1, S_2T_2, and S_3T_3 are concurrent.



8. Watch the YouTube video "Moebius Transformations Revealed":
`http://www.youtube.com/watch?v=JX3VmDgiFnY`