

# Concurrency and Collinearity

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## 1 Elementary Tools

Here are some tips for concurrency and collinearity questions:

1. You can often restate a concurrency question as a collinearity question, and vice versa. For example, proving that  $AB$ ,  $CD$  and  $EF$  are concurrent is equivalent to proving that  $E$ ,  $F$  and  $AB \cap CD$  are collinear.
2. A common way of proving concurrency is to consider the pairwise intersections of the lines, and then show that they are the same. A common way of proving collinearity is to show that the three points form an angle of  $180^\circ$ .
3. A special case of concurrency is parallel lines meeting at the point at infinity. Make sure to be mindful of this case in your solutions of contest problems.

We will be discussing several powerful tools in this lecture: Pappus', Pascal's and Desargues' Theorems. However, you should remember that in questions on concurrency and collinearity, your best friends are good old Ceva and Menelaus. They are universal, and in fact the proofs of Pascal and Desargues consist of repeated applications of Menelaus.

**Theorem 1. (Ceva's Theorem.)** In  $\triangle ABC$ , let  $D$ ,  $E$ ,  $F$  be points on lines  $BC$ ,  $AC$ ,  $AB$  respectively. Then  $AD$ ,  $BE$ ,  $CF$  are concurrent iff

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1$$

**Theorem 2. (Ceva's Theorem, trig version.)** In  $\triangle ABC$ , let  $D$ ,  $E$ ,  $F$  be points on lines  $BC$ ,  $AC$ ,  $AB$  respectively. Then  $AD$ ,  $BE$ ,  $CF$  are concurrent iff

$$\frac{\sin(\angle BAD)}{\sin(\angle CAD)} \cdot \frac{\sin(\angle ACF)}{\sin(\angle BCF)} \cdot \frac{\sin(\angle CBE)}{\sin(\angle ABE)} = 1$$

**Theorem 3. (Menelaus Theorem.)** In  $\triangle ABC$ , let  $D$ ,  $E$ ,  $F$  be points on lines  $BC$ ,  $AC$ ,  $AB$  respectively. Then  $AD$ ,  $BE$ ,  $CF$  are concurrent iff

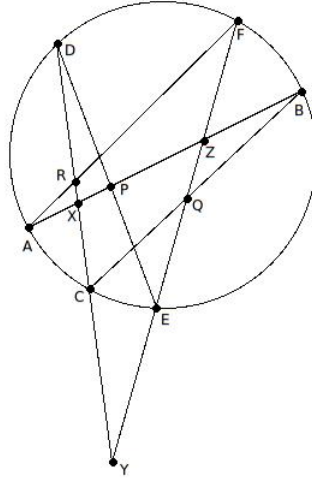
$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1$$

**Problems**

1. (Korea 1997) In an acute triangle  $ABC$  with  $AB \neq AC$ , let  $V$  be the intersection of the angle bisector of  $A$  with  $BC$ , and let  $D$  be the foot of the perpendicular from  $A$  to  $BC$ . If  $E$  and  $F$  are the intersections of the circumcircle of  $\triangle AVD$  with  $AC$  and  $AB$ , respectively, show that the lines  $AD$ ,  $BE$ ,  $CF$  are concurrent.
2. (Iran 1998) Let  $ABC$  be a triangle and  $D$  be the point on the extension of side  $BC$  past  $C$  such that  $CD = AC$ . The circumcircle of  $\triangle ACD$  intersects the circle with diameter  $BC$  again at  $P$ . Let  $BP$  meet  $AC$  at  $E$  and  $CP$  meet  $AB$  at  $F$ . Prove that the points  $D$ ,  $E$ ,  $F$  are collinear.
3. (Turkey 1996) In a parallelogram  $ABCD$  with  $\angle A < 90^\circ$ , the circle with diameter  $AC$  meets the lines  $CB$  and  $CD$  again at  $E$  and  $F$ , respectively, and the tangent to this circle at  $A$  meets  $BD$  at  $P$ . Show that  $P$ ,  $F$ ,  $E$  are collinear.
4. (IMO SL 1995) Let  $ABC$  be a triangle. A circle passing through  $B$  and  $C$  intersects the sides  $AB$  and  $AC$  again at  $C'$  and  $B'$ , respectively. Prove that  $BB'$ ,  $CC'$ , and  $HH'$  are concurrent, where  $H$  and  $H'$  are the orthocenters of triangles  $ABC$  and  $AB'C'$  respectively.
5. (IMO SL 2000) Let  $O$  be the circumcenter and  $H$  the orthocenter of an acute triangle  $ABC$ . Show that there exist points  $D$ ,  $E$ , and  $F$  on sides  $BC$ ,  $CA$ , and  $AB$  respectively such that  $OD + DH = OE + EH = OF + FH$  and the lines  $AD$ ,  $BE$ , and  $CF$  are concurrent.

## 2 Power Tools

**Theorem 4. (*Pascal's theorem*)** Let  $A, B, C, D, E, F$  be points on a circle, in some order. Then if  $P = AB \cap DE$ ,  $Q = BC \cap EF$ ,  $R = CD \cap FA$ , then  $P, Q, R$  are collinear. In other words, if  $ABCDEF$  is a cyclic (not necessarily convex) hexagon, then the intersections of the pairs of opposite sides are collinear.



*Proof.* Let  $X = AB \cap CD$ ,  $Y = CD \cap EF$ ,  $Z = EF \cap AB$ . We apply Menelaus three times, to lines  $BC, DE, FA$  cutting the sides (possibly extended) of  $\triangle XYZ$ :

$$\frac{XB}{BZ} \frac{ZQ}{QY} \frac{YC}{CX} = -1$$

$$\frac{YD}{DX} \frac{XP}{PZ} \frac{ZE}{EY} = -1$$

$$\frac{ZF}{FY} \frac{YR}{RX} \frac{XA}{AZ} = -1$$

We multiply the three equations, and observe that by Power of a Point

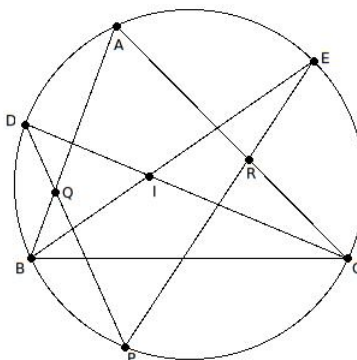
$$XA \cdot XB = CX \cdot DX, YC \cdot YD = EY \cdot FY, ZE \cdot ZF = AZ \cdot BZ,$$

so after cancellation the product becomes

$$\frac{YR}{RX} \frac{XP}{PZ} \frac{ZQ}{QY} = -1$$

which implies by Menelaus in  $\triangle XYZ$  that  $P, Q, R$  are collinear.  $\square$

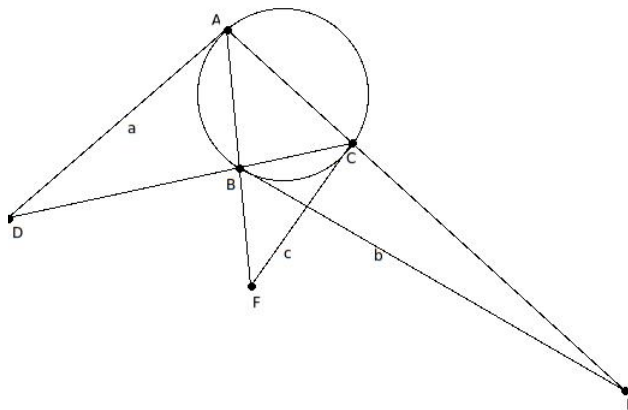
**Example 1.** On the circumcircle of triangle  $ABC$ , let  $D$  be the midpoint of the arc  $AB$  not containing  $C$ , and  $E$  be the midpoint of the arc  $AC$  not containing  $B$ . Let  $P$  be any point on the arc  $BC$  not containing  $A$ ,  $Q = DP \cap AB$  and  $R = EP \cap AC$ . Prove that  $Q$ ,  $R$  and the incenter of triangle  $ABC$  are collinear.



*Solution.* Since  $CD$  and  $BE$  are angle bisectors in  $\triangle ABC$ , they intersect at the incenter  $I$ . Now we want to apply Pascal to points  $A, B, C, D, E, P$  in such an order that  $Q, R, I$  are the intersections of pairs of opposite sides of the resulting hexagon. This is accomplished using the hexagon  $ABEPDC$ .  $\square$

In many problems, you don't have a configuration with six points around a circle. But it is still possible to apply the degenerate case of Pascal's Theorem, where some of the adjacent vertices of the "hexagon" coincide. In the limiting case of vertex  $A$  approaching vertex  $B$ , the line  $AB$  becomes the tangent line at  $B$ . We illustrate this with the following simple example.

**Example 2.** (Macedonian MO 2001) Let  $ABC$  be a scalene triangle. Let  $a, b, c$  be tangent lines to its circumcircle at  $A, B, C$ , respectively. Prove that points  $D = AB \cap c$ ,  $E = AC \cap b$ , and  $F = BC \cap a$  exist, and that they are collinear.



*Solution.* Apply Pascal to the degenerate hexagon  $AABBCC$ . Then the sides (in order) are lines  $a, AB, b, BC, c, CA$ . The desired conclusion follows.  $\square$

**Theorem 5. (Pappus' Theorem)** *Points  $A, C, E$  lie on line  $l_1$ , and points  $B, D, F$  lie on line  $l_2$ . Then  $AB \cap DE, BC \cap EF$ , and  $CD \cap FA$  are collinear.*

*Proof.* Exercise (Use Menelaus lots of times).  $\square$

**Theorem 6. (Desargues' Theorem)** *Given triangles  $ABC$  and  $A'B'C'$ , let  $P = BC \cap B'C'$ ,  $Q = CA \cap C'A'$ ,  $R = AB \cap A'B'$ . Then  $AA', BB', CC'$  concur iff  $P, Q, R$  are collinear. In other words, the lines joining the corresponding vertices are concurrent iff the intersections of pairs of corresponding sides are collinear.*

*Proof.* ( $\Rightarrow$ ) Suppose  $AA', BB', CC'$  concur at a point  $O$ . Apply Menelaus to lines  $A'B', B'C', C'A'$  cutting triangles  $ABO, BCO, CAO$  respectively:

$$\frac{AA'}{A'O} \frac{OB'}{B'B} \frac{BR}{RA} = -1$$

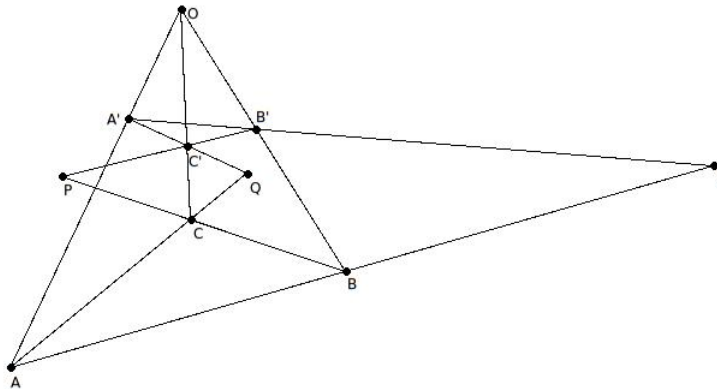
$$\frac{BB'}{B'O} \frac{OC'}{C'C} \frac{CP}{PB} = -1$$

$$\frac{CC'}{C'O} \frac{OA'}{A'A} \frac{AQ}{QC} = -1$$

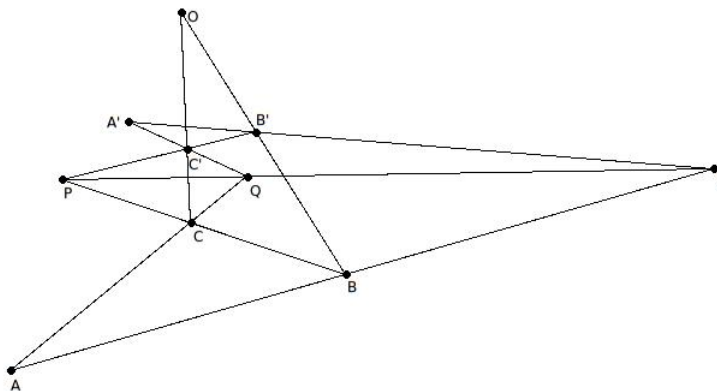
Multiplying the three equations together, we obtain

$$\frac{AQ}{QC} \frac{CP}{PB} \frac{BR}{RA} = -1$$

and so by Menelaus in triangle  $ABC$ , we have that  $P, Q, R$  are collinear.



( $\Leftarrow$ ) Suppose  $P, Q, R$  are collinear. Consider  $\triangle QCC'$  and  $\triangle RBB'$ , and let  $B$  correspond to  $C$ ,  $B'$  correspond to  $C'$ ,  $R$  correspond to  $Q$ . Since the lines through corresponding vertices concur, we apply ( $\Rightarrow$ ) and obtain that  $O = BB' \cap CC'$ ,  $A' = QC' \cap RB'$  and  $A = QC \cap RB$  are collinear. Therefore,  $AA', BB', CC'$  concur at  $O$ .



□

**Example 3.** In a quadrilateral  $ABCD$ ,  $AB \cap CD = P$ ,  $AD \cap BC = Q$ ,  $AC \cap BD = R$ ,  $QR \cap AB = K$ ,  $PR \cap BC = L$ ,  $AC \cap PQ = M$ . Prove that  $K, L, M$  are collinear.

*Solution.* Since  $AQ, BR, CP$  intersect at  $D$ , we apply ( $\Rightarrow$ ) of Desargues' Theorem to triangles  $ABC$  and  $QRP$ . Then  $K = AB \cap QR$ ,  $L = BC \cap RP$ ,  $M = AC \cap QP$  are collinear. □

The great thing about these theorems is that there are no configuration issues whatsoever. Pascal in particular is very versatile - given six points around a circle, the theorem can be applied to them in many ways (depending on the ordering), and some of them are bound to give you useful information.

The following problems use Pascal, Pappus, and Desargues, sometimes repeatedly or in combination. The problems are roughly in order of difficulty: 1-5 are easy, 6-10 are hard.

## Problems

1. Points  $A_1$  and  $A_2$  that lie inside a circle centered at  $O$  are symmetric through point  $O$ . Points  $P_1, P_2, Q_1, Q_2$  lie on the circle such that rays  $A_1P_1$  and  $A_2P_2$  are parallel and in the same direction, and rays  $A_1Q_1$  and  $A_2Q_2$  are also parallel and in the same direction. Prove that lines  $P_1Q_2, P_2Q_1$  and  $A_1A_2$  are concurrent.
2. (Australian MO 2001) Let  $A, B, C, A', B', C'$  be points on a circle, such that  $AA' \perp BC$ ,  $BB' \perp CA$ ,  $CC' \perp AB$ . Let  $D$  be an arbitrary point on the circle, and let  $A'' = DA' \cap BC$ ,  $B'' = DB' \cap CA$  and  $C'' = DC' \cap AB$ . Prove that  $A'', B'', C''$  and the orthocenter of  $\triangle ABC$  are collinear.

3. The extensions of sides  $AB$  and  $CD$  of quadrilateral  $ABCD$  meet at point  $P$ , and the extensions of sides  $BC$  and  $AD$  meet at point  $Q$ . Through point  $P$  a line is drawn that intersects sides  $BC$  and  $AD$  at points  $E$  and  $F$ . Prove that the intersection points of the diagonals of quadrilaterals  $ABCD$ ,  $ABEF$  and  $CDFE$  lie on a line that passes through point  $Q$ .
4. (IMO SL 1991) Let  $P$  be a point inside  $\triangle ABC$ . Let  $E$  and  $F$  be the feet of the perpendiculars from the point  $P$  to the sides  $AC$  and  $AB$  respectively. Let the feet of the perpendiculars from point  $A$  to the lines  $BP$  and  $CP$  be  $M$  and  $N$  respectively. Prove that the lines  $ME$ ,  $NF$ ,  $BC$  are concurrent.
5. Quadrilateral  $ABCD$  is circumscribed about a circle. The circle touches the sides  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  at points  $E$ ,  $F$ ,  $G$ ,  $H$  respectively. Prove that  $AC$ ,  $BD$ ,  $EG$  and  $FH$  are concurrent.
6. (Bulgaria 1997) Let  $ABCD$  be a convex quadrilateral such that  $\angle DAB = \angle ABC = \angle BCD$ . Let  $H$  and  $O$  denote the orthocenter and circumcenter of the triangle  $ABC$ . Prove that  $D$ ,  $O$ ,  $H$  are collinear.
7. In triangle  $ABC$ , let the circumcenter and incenter be  $O$  and  $I$ , and let  $P$  be a point on line  $OI$ . Given that  $A'$  is the midpoint of the arc  $BC$  containing  $A$ , let  $A''$  be the intersection of  $A'P$  and the circumcircle of  $ABC$ . Similarly construct  $B''$  and  $C''$ . Prove that  $AA''$ ,  $BB''$ ,  $CC''$  are concurrent.
8. In triangle  $ABC$ , heights  $AA_1$  and  $BB_1$  and angle bisectors  $AA_2$  and  $BB_2$  are drawn. The inscribed circle is tangent to sides  $BC$  and  $AC$  at points  $A_3$  and  $B_3$ , respectively. Prove that lines  $A_1B_1$ ,  $A_2B_2$  and  $A_3B_3$  are concurrent.
9. (China 2005) A circle meets the three sides  $BC$ ,  $CA$ ,  $AB$  of a triangle  $ABC$  at points  $D_1$ ,  $D_2$ ;  $E_1$ ,  $E_2$ ;  $F_1$ ,  $F_2$  respectively. Furthermore, line segments  $D_1E_1$  and  $D_2F_2$  intersect at point  $L$ , line segments  $E_1F_1$  and  $E_2D_2$  intersect at point  $M$ , line segments  $F_1D_1$  and  $F_2E_2$  intersect at point  $N$ . Prove that the lines  $AL$ ,  $BM$ ,  $CN$  are concurrent.
10. (IMO SL 1997) Let  $A_1A_2A_3$  be a non-isosceles triangle with incenter  $I$ . Let  $\omega_i$ ,  $i = 1, 2, 3$ , be the smaller circle through  $I$  tangent to  $A_iA_{i+1}$  and  $A_iA_{i+2}$  (the addition of indices being mod 3). Let  $B_i$ ,  $i = 1, 2, 3$ , be the second point of intersection of  $\omega_{i+1}$  and  $\omega_{i+2}$ . Prove that the circumcentres of the triangles  $A_1B_1I$ ,  $A_2B_2I$ ,  $A_3B_3I$  are collinear.

## Hints

1. Consider reflections of  $P_2$  and  $Q_2$  across  $O$ .
2. Easy Pascal.
3. This problem uses two of the theorems.
4. There is in fact a circle in this diagram :).
5. Lots of tangent lines here - what theorem does that suggest?
6. Consider the centroid of  $\triangle ABC$  instead of the orthocenter.
7. It might look like we have enough points on this circle, but we need more! Consider reflections of  $A'$ ,  $B'$ ,  $C'$  in  $O$ . It also helps to consider only  $A$ 's and  $B$ 's.
8. Consider the circumcircle of  $ABA_1B_1$ . Two more points in the diagram are on that circle.
9. Lots of extra points and many applications of Pascal are needed, but still not sufficient. The 3-way symmetry suggests another theorem to use.
10. Once again, we have 3-way symmetry. The centers of  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  are important here.

## 3 References

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