Enhance your angle-chasing skills

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Abstract

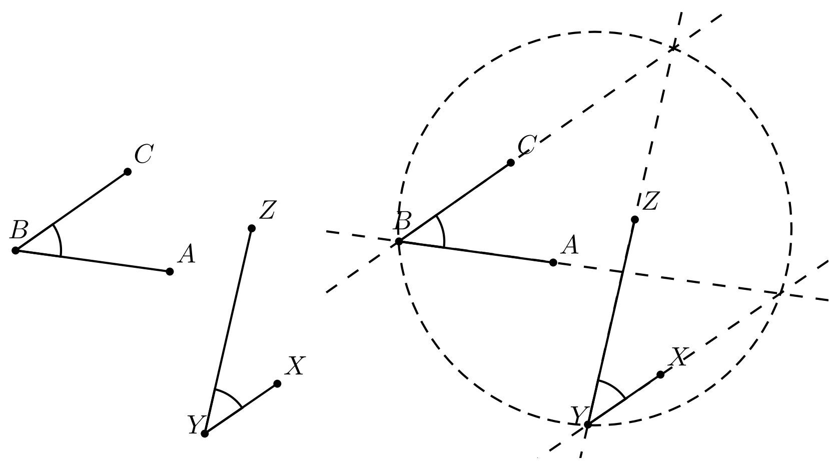
Tired of not spotting cyclic quads and having to resort to bashing methods? Here are some tips.

1 Some useful facts

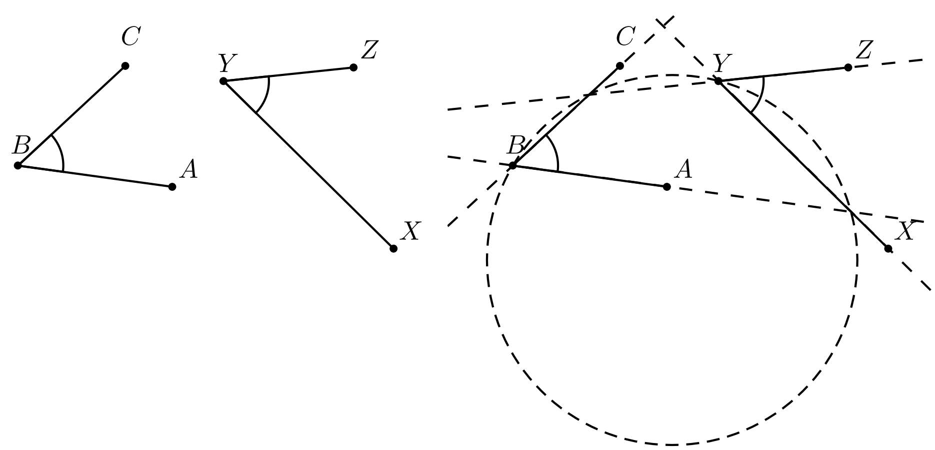
1.1 Equal angles usually induce cyclic quads

If you spot two equal angles in very different parts of the diagram, there is usually a cyclic quad involving them. Here's how to find it:

Fact 1 (The "squinting" method)**.** *If $\angle ABC$ and $\angle XYZ$ are facing the "same way", and A, B, C and X, Y, Z are labelled in the same direction (say, counterclockwise), then B, Y, $AB \cap XY$ and $BC \cap YZ$ are, in most cases, concyclic.*



It also works when the vertex of one of the angles is inside the other angle:



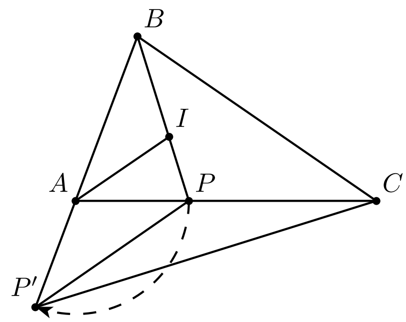
This is far from formal, I know, but it is quite useful, especially as the diagram gets more complex.

1.2 Open the gates!

If you have an equality involving lengths of polygonal lines, you might want to "straight them up", that is, to construct some auxiliar points. It's easier to grasp this idea by looking at an example.

Example 1 (Brazilian Math Olympiad, 2006)**.** Let ABC be a triangle. The internal bisector of $\angle B$ meets AC in P. Let I be the incenter of ABC. Prove that if $AP + AB = CB$, then API is an isosceles triangle.

Solution. Let P' be a point in the extension of ray BA such that $P'A = PA$ (hence we "open the gates"). So $BP' = BA + AP' = AB + AP = BC$, so $P'BC$ is isosceles.



After this, symmetry and some angle chasing do the trick: since BP bisecs $\angle P'BC$, it is also the perpendicular bisector of $P'C$. This means that $PP' = PC$, so $\angle C = \angle BCP = \angle BCP' - \angle PCP' = \angle BP'C - \angle PP'C = \angle BP'P = \angle AP'P$. But $AP = AP'$, so triangle APP' is also isosceles and $\angle AP'P = \frac{1}{2}(\angle AP'P + \angle APP') = \frac{1}{2}\angle BAP = \angle A/2$, so $\angle C = \angle A/2$. Adding to this the fact that $\angle CBP = \angle ABI$, we have that triangles ABI and CBP are similar, thus $\angle AIB = \angle BPC$. Then we are done, since those two angles are supplementary to $\angle AIP$ and $\angle API$, respectively.

1.3 Power of a point

You probably know that if two lines passing through a point P meet a circle at A, B and C, D then $PA \cdot PB = PC \cdot PD$. But you might not know that the converse is true!

Fact 2. *If two lines AB and CD meet at P and $PA \cdot PB = PC \cdot PD$ (signed segments) then A, B, C and D are concyclic.*

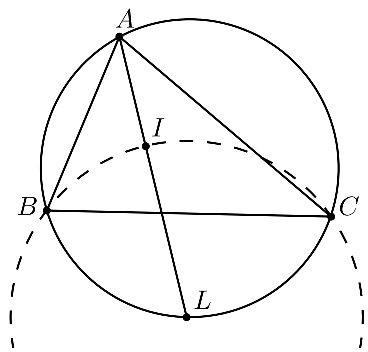
1.4 The ubiquitous incenter property

There is another characterization of the incenter of a triangle that can be useful in at least two situations:

- you need to prove that some point is the incenter of a triangle;
- you need some isosceles triangles or collinear points.

Fact 3 (The ubiquitous incenter property)**.** *Let I be the incenter of a triangle ABC and L be the midpoint of the arc BC that doesn't contain A. Then*

$$LI = LB = LC$$



The proof is just an angle-chase, but you **should write it during a test**, as a lemma: $\angle IBL = \angle IBC + \angle CBL = \frac{1}{2}\angle B + \angle CAL = \frac{1}{2}\angle B + \frac{1}{2}\angle A$ and, $\angle BIL = \angle ABI + \angle IAB = \frac{1}{2}\angle B + \frac{1}{2}\angle A$. So $\angle IBL = \angle BIL$, and thus BIL is isosceles, with $LB = LI$. Analogously, $LC = LI$.

There is a number of ways of interpreting this fact:

1. L is the circumcenter of the triangle BIC;
2. You can find the incenter of a triangle by marking L and drawing a circle with center in L and radius $LB = LC$. The incenter is the intersection of said circle and angle bisector AL. This is particularly useful when you need to prove that I is the incenter: you don't have to prove that it's the intersection of two bisectors.

1.5 Spiral similarities

Spiral similarities are a magnet for similar triangles, and thus, cyclic quads. Here's why:

Fact 4 (Automatic similarities)**.** *Let σ be a spiral similarity with center O, angle α and ratio k. For the sake of simplicity, denote $A' = \sigma(A)$ (that is, σ takes A to A'). Then*

1. *All triangles OAA' are similar.*
2. *For every pair of points A, B, OAB and $OA'B'$ are similar.*

Let's prove this fact: 1 follows directly from the fact that $\frac{OA'}{OA} = k$ and $\angle AOA' = \alpha$ are constant, so all such triangles are similar by the SAS theorem; 2 follows almost in the same way.

Nonetheless, if you find a spiral similarity, bear in mind that you can find a lot of similar triangles without even having to think!

1.6 Isogonal conjugates

Suppose you have two pair of equal angles and you need to prove that a third pair of angles are equal. Then you might have a job for isogonal conjugates!

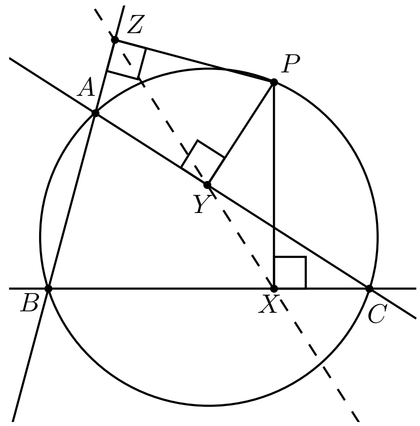
Fact 5 (Buy two, get the third free)**.** *Let ABC be a triangle. If P and Q are points in the plane such that $\angle PAB = \angle CAQ$ and $\angle PBC = \angle ABQ$ (directed angles!) then $\angle PCA = \angle BCQ$.*

You can prove it via trig Ceva.

1.7 Simson's line

If you have a lot of perpendicular lines, you find not only several cyclic quads, but maybe some additional perpendicular lines.

Fact 6 (Simson's line)**.** *Let ABC be a triangle anf P a point in the plane. Points X, Y and Z are the orthogonal projections of P on lines BC, CA and AB, respectively. Then X, Y and Z are collinear if and only if P lies on the circumcircle of ABC.*



You can prove it by another angle chasing. But you can interpret this fact in many ways:

- If P is a point on the circumcircle of triangle BC and you drop perpendiculars to all three sides, they are collinear. This is actually more interesting if the projections have some sort of order: if P is a point on the circumcircle of triangle BC and you drop perpendiculars to two sides, the line connecting its projections meet the other side at the projection of P to this side.
- Considering the quadrilaterals $PYAZ$, $PZBX$ and $PYXC$, if two of them are cyclic them the third one also is.
- You might also use the circles with diameters PA, PB and PC. Think about it!

One final remark about Simson's line: it can be easily generalized to all projections (that is, non-orthogonal projections).

1.8 Apollonius circle

A good way of transforming ratios into angles is using Apollonius circles (or inversion!).

Fact 7 (Transforming ratios into angles)**.** *Let A and B be points. The locus of points P such that $\frac{PA}{PB}$ is a constant different from 1 is a circle symmetric with respect to AB.*

You can prove it by using analytic geometry.

1.9 Some final hints

- This list of facts is really far from complete; there are many other cool facts that you can use!
- Draw big, good diagrams. Use ruler and compass. If mahy points in the diagram are close, don't hesitate and draw another diagram. A good rule of thumb for drawing another diagram is: if you feel restrained in drawing additional elements, it's time to draw another diagram.
- If you realize some lines or circles are useless, don't draw them; you should only draw what's relevant at the moment.
- This is important in all problems, but it is crucial in geometry: **focus on what you want to prove**. Before you find lots of cyclic quads or angles, establish yourself a concrete goal. For example, "I need to find $\angle XYZ$" or "I need to prove that those three lines are concurrent".
- If you are dealing with a triangle problem, you might want to elect a "reference triangle" to work with. Switching the reference triangle might be useful as well.
- Exploit both algebraic and geometry symmetry as much as you can. It usually saves a lot of time.

2 Problems

For now, try not to bash them, i.e., don't use trigonometry, complex numbers or analytic geometry! You can bash them later if you want, though. Contest problems are in chronological order.

1. (Brazil 1996) ABC is acute-angled. D is a variable point on the side BC. O_1 is the circumcenter of ABD, O_2 is the circumcenter of ACD, and O is the circumcenter of AO_1O_2. Find the locus of O.

2. (Iberoamerican 1997) In a triangle ABC, it is drawn a circumference with center in the incenter I and that meet twice each of the sides of the triangle: the segment BC on D and P (where D is nearer to B than P); the segment CA on E and Q (where E is nearer to C than Q); and the segment AB on F and R (where F is nearer to A than R).

   Let S be the point of intersection of the diagonals of the quadrilateral $EQFR$. Let T be the point of intersection of the diagonals of the quadrilateral $FRDP$. Let U be the point of intersection of the diagonals of the quadrilateral $DPEQ$.

   Show that the circuncircles of the triangle FRT, DPU and EQS have a unique point in common.

3. (Brazil 1999) Given a triangle ABC, explain how to construct with ruler and compass a triangle $A'B'C'$ of minimum area such that $C' \in AC$, $A' \in AB$, $B' \in BC$ and $\angle B'A'C' = \angle BAC$, $\angle A'C'B' = \angle ACB$.

4. (IMO 2000) Two circles Γ_1 and Γ_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on Γ_1 and D on Γ_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

5. (IMO 2001) Let ABC be a triangle with $\angle BAC = 60°$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB+BP = AQ+QB$, what are the angles of the triangle?

6. (Brazil 2001) E and F are points of side AB of triangle ABC such that $AE = EF = FB$. D is a point of BC such that BC is perpendicular to ED. AD is perpendicular to CF, $\angle BDF = x$ and $\angle CFA = 3x$. Determine DB/DC.

7. (Brazil 2001) In a convex quadrilateral, the *altitude* relative to a side is defined to be the line perpendicular to this side through the midpoint of the opposite side. Prove that the four altitudes have a common point if and only if the quadrilateral is cyclic, that is, if and only if, there exists a circle which contains its four vertices.

8. (IMO 2002) The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

9. (Iberoamerican 2003) Let C and D be two points on the semicircumference with diameter AB such that B and C are in distinct semiplanes with respect to the line AD. Denote by M, N and P the midpoints of AC, BD and CD respectively. Let O_A and O_B the circumcenters of the triangles ACP and BDP. Show that the lines O_AO_B and MN are parallel.

10. (IMO 2004) In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

    $$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

    Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

11. (IMO 2005) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

    Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

12. (IMO 2005) Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie on the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.

    Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

13. (IMO 2006) Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

    $$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

    Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

14. (IMOSL 2006) Consider a convex pentagon ABCDE such that

    $$\angle BAC = \angle CAD = \angle DAE \quad , \quad \angle ABC = \angle ACD = \angle ADE$$

    Let P be the point of intersection of the lines BD and CE. Prove that the line AP passes through the midpoint of the side CD.

15. (Iberoamerican 2006) Let τ be a circumference inscribed in a circumscribed quadrilateral $ABCD$. Let X and Y be the points where BD intersects τ, and M be the midpoint of XY. Let P and Q be the points of tangency of DA and DC with τ. Prove $\angle AMP = \angle CMQ$.

16. (Olympic Revenge 2007) [1] Triangles BCD and ACE are erected externally to a triangle ABC such that $AE = BD$ and $\angle BDC + \angle AEC = 180°$. Let F be a point on the segment AB such that $\frac{AF}{FB} = \frac{CD}{CE}$. Prove that $\frac{DE}{CD+CE} = \frac{EF}{BC} = \frac{FD}{AC}$.

17. (IMO 2007) Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle $\angle DAB$.

18. (IMOSL 2007) Denote by M the midpoint of side BC in an isosceles triangle ABC with $AC = AB$. Take a point X on the smaller arc MA of circumcircle of triangle ABM. Denote by T the point inside of angle $\angle BMA$ such that $\angle TMX = 90°$ and $TX = BX$.

    Prove that $\angle MTB - \angle CTM$ does not depend on choice of X.

19. (IMOSL 2007) The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

20. (Olympic Revenge 2008) Let ABC be a triangle and M the midpoint of BC. Let D and E be points such that ABC, DBA and EAC are similar (in this order). The symmedians of D with respect to DBA and E with respect to EAC meet at K. Suppose that $\frac{AM}{BC} = \frac{\sqrt{3}}{2}$[2]. Prove that KAB and KCA are similar (in this order).

[1] *Olympic Revenge* is a competition for Brazilian teachers. The students pose the problems for the teachers during a training camp, held every year in January!

[2] This is not necessary; the problem holds without this fact.

21. (Baltic Way 2008) Let $ABCD$ be a parallelogram. The circle with diameter AC intersects the line BD at points P and Q. The perpendicular to the line AC passing through the point C intersects the lines AB and AD at points X and Y, respectively. Prove that the points P, Q, X and Y lie on the same circle.

22. (IMOSL 2008) Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

23. (IMOSL 2008) There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

if and only if the diagonals AC and BD are perpendicular.

24. (Cono Sur Olympiad 2008) Let ABC be an isosceles triangle with $AC = BC$. Let Γ be a semicircle with center on segment AB that touches AC and BC. A line is tangent to Γ and cuts AC and BC at D and E, respectively.

    Suppose that the perpendicular lines from D to AC and from E to BC meet at a point P inside triangle ABC. Let Q be the orthogonal projection of P in line AB. Prove that $\frac{PQ}{CP} = \frac{1}{2} \cdot \frac{AB}{AC}$.

25. (Cono Sur Olympiad 2009) Let A, B and C be three points such that B is the midpoint of AC and P a point such that $\angle PBC = 60^\circ$. Let Q be a point such that PCQ is an equilateral triangle and B and Q are in different semiplanes with respect to PC, and R be a point such that APR is an equilateral triangle and B and R are in the same semiplane with respect to AP. Let X be the intersection point of lines BQ and PC and Y be the point of intersection of lines BR and AP. Prove that the lines XY and AC are parallel.

26. (Iberoamerican 2009) Let C_1 and C_2 be two congruent circles centered at O_1 and O_2, which intersect at A and B. Take a point P on the arc AB of C_2 which is contained in C_1. AP meets C_1 at C, CB meets C_2 at D and the bisector of $\angle CAD$ intersects C_1 and C_2 at E and L, respectively. Let F be the symmetric point of D with respect to the midpoint of PE. Prove that there exists a point X satisfying $\angle XFL = \angle XDC = 30^\circ$ and $CX = O_1O_2$.

27. (Cono Sur, June 2010 – *just out of the oven!*) Let ABC be a triangle. Its incircle touches the sides BC, CA and AB at D, E and F, respectively. The lines DE and DF meet the circumcircle of AEF at $E_a \neq E$ and $F_a \neq F$. Let r_A be the line E_aF_a. Define lines r_B and r_C analogously. Prove that r_A, r_B and r_C determine a triangle whose vertices lie on the sides of the triangle ABC.

28. (Poncelet's Porism, particular case) Triangles ABC and DEF share the same incircle. Prove that if A, B, C, D, E lie on the same circle then F also does.

29. Let A be a point in the interior of triangle BCD such that $AB \cdot CD = AD \cdot BC$. Point P is the symmetrical of point A with respect to BD. Prove that $\angle PCB = \angle ACD$.