My Favorite Triangle Center

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"The symmedian point is one of the crown jewels of modern geometry."
–Ross Honsberger

A *symmedian* of a triangle is the reflection of a median over the corresponding angle bisector. Basic properties/constructions of the symmedian:

1. Given a triangle ABC, we say that the segment $B'C'$ is an *antiparallel* to BC if B' lies on AC, C' lies on AB, and $\angle ABC = \angle AB'C'$. Prove that the symmedian from A bisects all antiparallels to BC.

2. Let S lie on side BC of triangle ABC such that AS is a symmedian. Calculate $\frac{BS}{CS}$ and $\frac{\sin \angle BAS}{\sin \angle SAC}$ in terms of the sides of ABC.

3. Let ω be the circumcircle of ABC, and let the tangents to ω at B and C intersect at D. Prove that AD is a symmedian of ABC. (Thus the Gergonne point of a triangle is the symmedian point of its intouch triangle.)

4. Let ω be the circumcircle of ABC, and let the tangent to ω at A intersect line BC at D. Let the other tangent from D intersect ω at E. Prove that AE is a symmedian of ABC.

5. Let two circles ω_1 and ω_2 intersect at A and D. The tangent to ω_1 at A intersects ω_2 again at B, and the tangent to ω_2 at A intersects ω_1 again at C. Prove that AD is a symmedian of ABC. Moreover, suppose AD intersects the circumcircle of ABC again at E. Prove that D is the midpoint of AE.

6. Let $ABCD$ be a cyclic quadrilateral such that the products of the lengths of its opposite sides are equal. Prove that AC is a symmedian of ABD.

The three symmedians of ABC intersect at the *symmedian point* of ABC (also called the *Lemoine point* or *Grebe point*). For the remainder, we will let K denote the Lemoine point of ABC.

7. Compute the barycentric/trilinear coordinates of K.

8. Show that K is the circumcenter of the reflections of the centroid of ABC over its sides.

9. (a) Let D, E, and F be the projections of K to the sides of ABC. Show that the sides of DEF are perpendicular to the medians of ABC.
   (b) Show that K is the centroid of DEF.

10. (a) The antiparallels through K to the sides of ABC intersect the sides at six points. Prove that these six points lie on a circle. Where is its center?
    (b) The parallels through K to the sides of ABC intersect the sides at six points. Prove that these six points lie on a circle. Show that its center is the midpoint of KO, where O is the circumcenter of ABC.
    (c) For any point P_1 on AB, let the parallel to BC through P_1 intersect AC at P_2. Let the antiparallel to AB through P_2 intersect BC at P_3. Let the parallel to AC through P_3 intersect AB at P_4. Let the antiparallel to BC through P_4 intersect AC at P_5. Let the parallel to AB through P_5 intersect BC at P_6. Let the antiparallel to AC through P_6 intersect AB at P_7. Prove that $P_1 = P_7$ and that the points P_i all lie on a circle whose center lies on the line KO.

11. Show that K lies on the lines connecting the midpoints of a side to the midpoints of the corresponding altitude.

Other assorted problems (in no particular order):

12. Where is the symmedian point of a right triangle?

13. Construct squares outwardly on the sides of ABC. The sides of these squares parallel to the sides of ABC can be extended to form a larger triangle $A'B'C'$. Prove that AA', BB', and CC' intersect at the common symmedian point of ABC and $A'B'C'$.

14. Show that K is the point inside ABC that minimizes the sum of the squares of the distances to the sides.

15. Let ABC be a triangle, and let the feet of the perpendiculars from B and C to the angle bisector at A be P and Q, respectively. The line through P parallel to AB and the line through Q parallel to AC intersect at R. Prove that AR is a symmedian of ABC.

16. Let circles ω_1 and ω_2 be tangent to line BC at B and C, respectively. Suppose ω_1 and ω_2 intersect at P, and let PC intersect ω_1 again at A. Let M be the midpoint of AB, and let CM intersect ω_2 at again at N. Prove that $\angle BNC = \angle CNP$.

17. Let ABC be an isosceles triangle with vertex A inscribed in circle ω. Let the tangent to ω at C intersect line AB at D, and let the other tangent to ω through D intersect BC at E. Compute $\frac{EB}{BC}$.

18. Let ABC be a triangle, and let D, E, and F be the feet of the altitudes. Show that the six feet of the perpendiculars from D, E, and F to the other sides of ABC lie on a circle.

19. Let the symmedians of ABC intersect the circumcircle of ABC again at D, E, and F. Prove that AD, BE and CF are also symmedians of DEF.

20. Let ABC be a triangle with incenter I, and let the incircle of ABC touch BC, CA, and AB at D, E, and F, respectively. Let AD intersect BE at X. The lines through X parallel to the sides of DEF intersect the sides of ABC at six points. Prove that these six points lie on a circle with center I.

21. Let ABC be a triangle with centroid G. Let D, E, F, and O be the circumcenters of BCG, CAG, ABG, and ABC, respectively. Prove that O is the centroid of DEF.

22. Let P lie inside ABC such that $\angle PAB = \angle PBC = \angle PCA$. Prove that AP, the symmedian from B, and the median from C concur.

23. Let ABC be a triangle with angle bisectors AD, BE, and CF and incenter I. Let the line through A and the midpoint of EF intersect the line through B and the midpoint of CF at J. Prove that I, J, and K are collinear. Show that the symmedian point of the triangle formed by the excenters of ABC also lies on this line.

24. Fix a triangle ABC, and let $0° < \theta < 180°$. Construct similar isosceles triangles C_1AB and A_1BC outwardly with vertex angles $\angle AC_1B = \angle BA_1C = \theta$. Let AA_1 intersect CC_1 at X. Analogously construct isosceles triangles C_2AB and A_2BC inwardly with vertex angle θ, and let AA_2 intersect CC_2 at Y. Show that as θ varies, XY always passes through a fixed point depending only on ABC.

25. (a) Let X, Y, and Z be the centers of the spiral similarities sending CA to AB, AB to BC, and BC to CA, and let O be the circumcenter of ABC. Show that X, Y, Z, and O lie on a circle.
    (b) Let P and Q lie inside ABC such that $\angle PAB = \angle PBC = \angle PCA$ and $\angle QBA = \angle QCB = \angle QAC$. Prove that P and Q also lie on this circle and that $OP = OQ$.

26. (IMO Shortlist 2000) The tangents to B and A to the circumcircle of acute triangle ABC meet the tangent at C at T and U, respectively. Let AT meet BC at P, and let Q be the midpoint of AP. Likewise let BU meet CA at R, and let S be the midpoint of BR. Prove that $\angle ABQ = \angle BAS$. Also determine, in terms of ratios of side lengths, the triangles for which this angle is maximized.