WOOT 2012–13

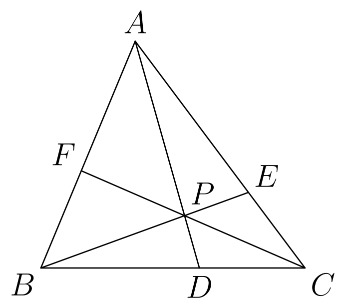
Concurrency & Collinearity

1 Ceva's Theorem and Menelaus's Theorem

Ceva's Theorem. In triangle ABC, let D, E, and F be points on BC, AC, and AB, respectively. Then AD, BE, and CF are concurrent if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Proof. First, assume that AD, BE, and CF concur at a point P.



Triangles AFP and BFP have the same altitude with respect to bases AF and FB, respectively, so

$$\frac{AF}{FB} = \frac{[AFP]}{[BFP]}.$$

Triangles AFC and BFC also have the same altitude with respect to bases AF and FB, respectively, so

$$\frac{AF}{FB} = \frac{[AFC]}{[BFC]}.$$

Therefore,

$$\frac{AF}{FB} = \frac{[AFC] - [AFP]}{[BFC] - [BFP]} = \frac{[PAC]}{[PBC]}.$$

Similarly,

$$\frac{BD}{DC} = \frac{[PAB]}{[PAC]} \quad \text{and} \quad \frac{CE}{EA} = \frac{[PBC]}{[PAB]},$$

so

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{[PAC]}{[PBC]} \cdot \frac{[PAB]}{[PAC]} \cdot \frac{[PBC]}{[PAB]} = 1.$$

Now assume that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

We wish to prove that AD, BE, and CF are concurrent. Let P be the intersection of BE and CF, and let D' be the intersection of AP and BC. Then by our work above,

$$\frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE}{EA} = 1.$$

Hence,

$$\frac{BD}{DC} = \frac{BD'}{D'C},$$

so points D and D' divide segment BC in the same ratio. Therefore, points D and D' coincide, so AD, BE, and CF are concurrent. ∎

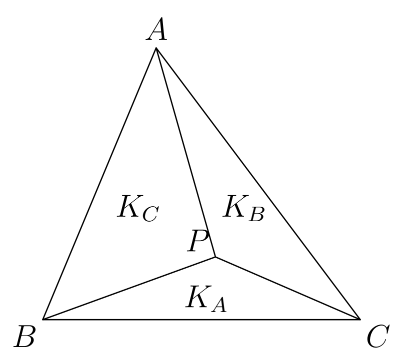
Menelaus's Theorem. In triangle ABC, let P, Q, and R be points on BC, AC, and AB, respectively. Then P, Q, and R are collinear if and only if

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1.$$

(The -1 arises from the use of directed line segments.)

The proof of Menelaus's theorem is left as an exercise.

Given triangle ABC and a point P, let $K_A = [PBC]$, $K_B = [PCA]$, and $K_C = [PAB]$. These areas are understood to be signed, so for example, if P and A are on opposite sides of BC, then K_A is negative. Among other reasons, this preserves the relationship $K_A + K_B + K_C = K = [ABC]$.



We have that

$$\frac{BD}{DC} = \frac{K_C}{K_B} \quad \text{and} \quad \frac{AP}{PD} = \frac{K_B + K_C}{K_A}.$$

Other ratios can be expressed similarly in terms of K_A, K_B, and K_C.

Problem. Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, AC, AB at D, E, F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

if and only if P lies on at least one of the medians of triangle ABC. (USA TST, 2003)

Solution. We have that

$$[PAF] = \frac{K_B}{K_A + K_B}[PAB] = \frac{K_B K_C}{K_A + K_B}.$$

Similarly,

$$[PBD] = \frac{K_A K_C}{K_B + K_C} \quad \text{and} \quad [PCE] = \frac{K_A K_B}{K_A + K_C}.$$

Hence, the given equation becomes

$$\frac{K_B K_C}{K_A + K_B} + \frac{K_A K_C}{K_B + K_C} + \frac{K_A K_B}{K_A + K_C} = \frac{K_A + K_B + K_C}{2}.$$

This equation simplifies as

$$K_A^3 K_B - K_A K_B^3 + K_B^3 K_C - K_B^3 K_C + K_C^3 K_A - K_C K_A^3 = 0,$$

which in turn factors as

$$(K_A + K_B + K_C)(K_A - K_B)(K_B - K_C)(K_C - K_A) = 0.$$

The result follows, because $K_A = K_B$ if and only if P lies on the median from C, etc. ■

Exercises

1. Prove Menelaus's theorem.

2. In triangle ABC, cevians AD, BE, and CF intersect at point P. The areas of triangles PAF, PFB, PBD, and PCE are 40, 30, 35, and 84, respectively. Find the area of triangle ABC. (AIME, 1985)

3. Show that in triangle ABC, altitude AA', median BB', and angle bisector CC' concur if and only if $\sin A = \cos B \tan C$.

4. (Monge's Theorem) Let ω_1, ω_2, and ω_3 be three circles, so that no circle contains another circle. Let P_1 be the intersection of the external common tangents of ω_2 and ω_3, and define points P_2 and P_3 similarly. Show that P_1, P_2, and P_3 are collinear.

5. In triangle ABC, let D, E, and F be points on BC, AC, and AB, respectively. Prove that AD, BE, and CF are concurrent if and only if

$$\frac{\sin \angle BAD}{\sin \angle ABE} \cdot \frac{\sin \angle CBE}{\sin \angle BCF} \cdot \frac{\sin \angle ACF}{\sin \angle CAD} = 1.$$

(This result is known as the angle version of Ceva's theorem.)

6. Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \quad \frac{P_2P}{PQ_2}, \quad \frac{P_3P}{PQ_3},$$

at least one is ≤ 2 and at least one is ≥ 2. (IMO, 1961)

7. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear. (IMO, 1982)

8. In triangle ABC, A', B', and C' are on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Given that $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at the point O, and that

$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92,$$

find the value of

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}.$$

(AIME, 1992)

9. Let ABC be a triangle with circumradius R. Let A', B', C' be points on sides BC, CA, AB, such that AA', BB', CC' are concurrent. Prove that

$$\frac{AB' \cdot BC' \cdot CA'}{[A'B'C']} = 2R.$$

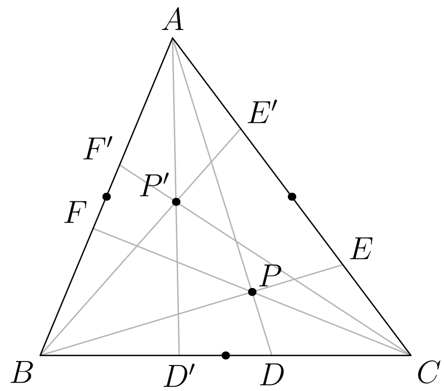
(Ireland, 1992)

2 Isotomic Conjugate and Isogonal Conjugate

Given triangle ABC and a point P, let AP, BP, and CP intersect BC, AC, and AB at D, E, and F, respectively. Then by Ceva's theorem,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Let D', E', and F' be the reflections of D, E, and F in the midpoints of BC, AC, and AB, respectively.

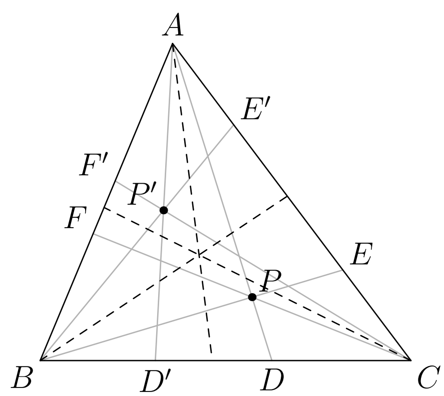


Then

$$\frac{AF'}{F'B} \cdot \frac{BD'}{D'C} \cdot \frac{CE'}{E'A} = \frac{BF}{FA} \cdot \frac{CD}{DB} \cdot \frac{AE}{EC} = 1,$$

so by Ceva's theorem, AD', BE', and CF' are concurrent, say at P'. The point P' is called the *isotomic conjugate* of P (with respect to triangle ABC).

Now, re-define points D', E', and F', so that D' is the intersection of BC and the reflection of AD in the angle bisector of $\angle A$. Define points E' and F' similarly.



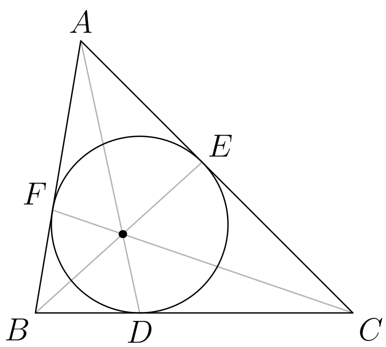
Then by the angle version of Ceva's theorem,

$$\frac{\sin \angle BAD'}{\sin \angle ABE'} \cdot \frac{\sin \angle CBE'}{\sin \angle BCF'} \cdot \frac{\sin \angle ACF'}{\sin \angle CAD'} = \frac{\sin \angle CAD}{\sin \angle CBE} \cdot \frac{\sin \angle ABE}{\sin \angle ACF} \cdot \frac{\sin \angle BCF}{\sin \angle BAD} = 1,$$

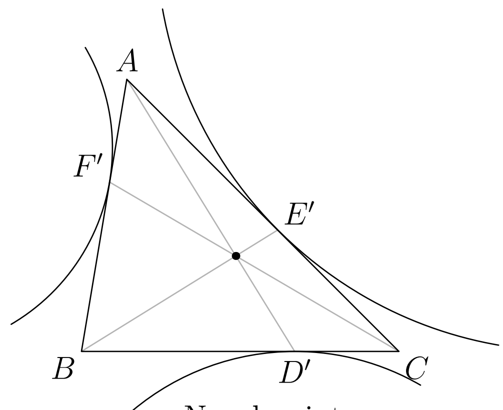
so AD', BE', and CF' are again concurrent at some point P'. In this case, the point P' is called the *isogonal conjugate* of P (with respect to triangle ABC).

Example. The orthocenter and circumcenter of a triangle are isogonal conjugates.

Example. Given triangle ABC, let the incircle touch sides BC, AC, and AB at D, E, and F, respectively. Let the A-excircle touch side BC at D', and define points E' and F' similarly. Then by Ceva's theorem, AD, BE, and CF concur at the *Gergonne point* of triangle ABC, and AD', BE' and CF' concur at the *Nagel point* of triangle ABC. The Gergonne point and Nagel point are isotomic conjugates.



Gergonne point



Nagel point

Exercises

1. Given triangle ABC, which points inside the triangle are their own isotomic conjugate? Which points inside the triangle are their own isogonal conjugate?

2. Given triangle ABC and points P and P', let $K_A = [PBC]$, $K_B = [PCA]$, $K_C = [PAB]$, $K'_A = [P'BC]$, $K'_B = [P'CA]$, and $K'_C = [P'AB]$.

   (a) Show that if P' is the isotomic conjugate of P, then

   $$K'_A : K'_B : K'_C = \frac{1}{K_A} : \frac{1}{K_B} : \frac{1}{K_C}.$$

   (b) Show that if P' is the isogonal conjugate of P, then

   $$K'_A : K'_B : K'_C = \frac{a^2}{K_A} : \frac{b^2}{K_B} : \frac{c^2}{K_C}.$$

3. In triangle ABC, $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on side BC with $CD = 6$. Point E is on side BC such that $\angle BAE = \angle CAD$. Given that $BE = p/q$, where p and q are relatively prime positive integers, find q. (AIME II, 2005)

4. Given triangle ABC, the tangents to the circumcircle of triangle ABC at B and C intersect at T_A. Points T_B and T_C are defined similarly. Show that AT_A, BT_B, and CT_C concur at the isogonal conjugate of the centroid of triangle ABC. (This point of concurrence is known as the *Symmedian*

point of triangle ABC, and the cevians AT_A, BT_B, and CT_C are known as the *symmedians* of triangle ABC.)

5. Let M and N be points inside triangle ABC that are isogonal conjugates. Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{AB \cdot BC} + \frac{CM \cdot CN}{AC \cdot BC} = 1.$$

(IMO Short List, 1998)

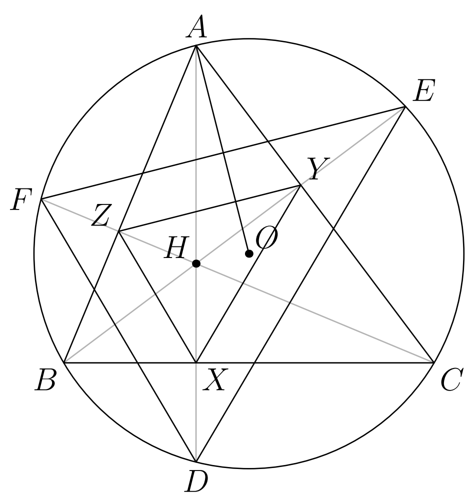
6. What can you say about the isogonal conjugate of a point on the circumcircle of triangle ABC?

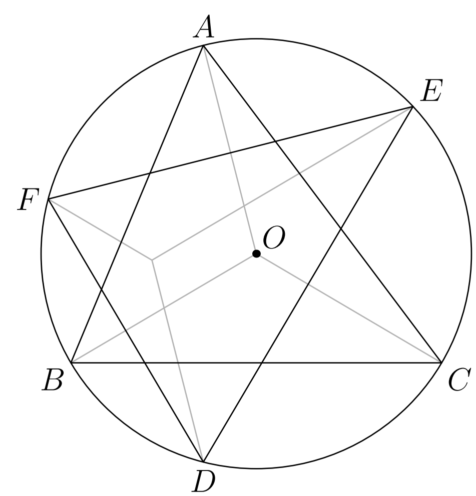
3 Identifying the Point of Concurrency

Often (but not always), lines that are concurrent meet at a significant point, such as the orthocenter of a triangle. (If you think about it, many of the major centers of a triangle are defined as a point of concurrency.) Of course, if you can show that the lines in question are, say, the altitudes of the triangle, then concurrency follows immediately.

Problem. Let O and H be the circumcenter and orthocenter of triangle ABC, respectively, and let AH, BH, CH meet the circumcircle in D, E, F, respectively. Prove that the lines through D, E, F parallel to OA, OB, OC, respectively, are concurrent.

Solution. Let X, Y, and Z be the feet of the altitudes from A, B, and C, respectively. We know that triangle DEF is the image of triangle XYZ under a homothety, centered at H, with scale factor 2. Hence, YZ is parallel to EF. Furthermore, triangle XYZ is the orthic triangle of triangle ABC, so OA is perpendicular to YZ. Therefore, OA is perpendicular to EF.

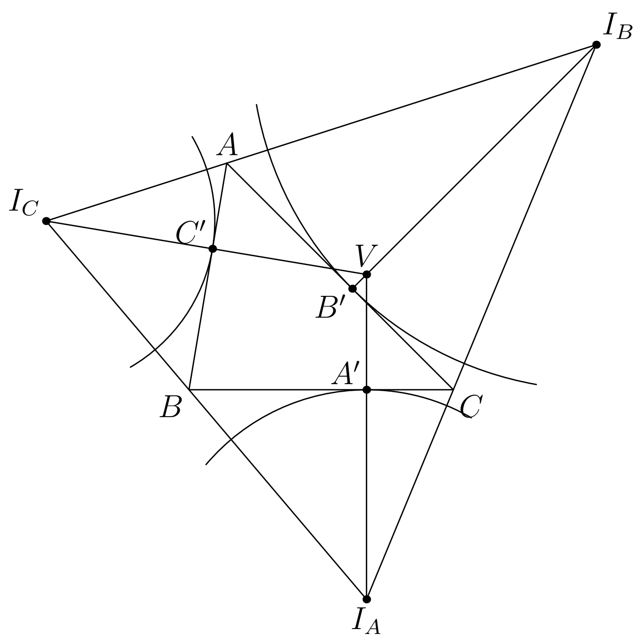




Then the line through D parallel to OA is the altitude from D to side EF in triangle DEF. Similarly, the other two lines are also altitudes of triangle DEF. Hence, all three lines concur at the orthocenter of triangle DEF. ■

Problem. In triangle ABC, let A' be the point where the excircle opposite vertex A touches side BC, and let l_a be the line passing through A' perpendicular to BC. Define points B', C', and lines l_b, l_c similarly. Prove that l_a, l_b, and l_c are concurrent.

Solution. Let I_A, I_B, and I_C denote the A-, B-, and C-excenter of triangle ABC, respectively. Let V be the circumcenter of triangle $I_AI_BI_C$.



We know that A, B, and C are the feet of the altitudes of triangle $I_AI_BI_C$. Thus, triangle ABC is the orthic triangle of triangle $I_AI_BI_C$. Hence, VI_A is perpendicular to BC, i.e. V lies on l_a. Similarly, V lies on l_b and l_c. Hence, all three lines concur at V. ■

The point V (the circumcenter of the excentral triangle) is known as the *Bevan point* of triangle ABC.

Exercises

1. Let ABC be an acute-angled triangle. Three lines L_A, L_B, L_C are constructed through the vertices A, B, and C, respectively, according to the following prescription: let H be the foot of the altitude drawn from the vertex A to the side BC; let S_A be the circle with diameter AH; let S_A meet the sides AB and AC at M and N, respectively, where M and N are distinct from A; then L_A is the line through

A perpendicular to MN. The lines L_B and L_C are constructed similarly. Prove that L_A, L_B, and L_C are concurrent. (IMO Short List, 1988)

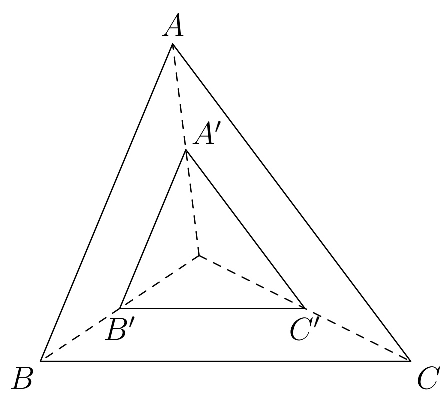
2. For triangle ABC, let A' denote the reflection of A in side BC, and define points B' and C' similarly. Let O_A, O_B, and O_C be the circumcenters of triangles $A'BC$, $AB'C$, ABC', respectively. Show that AO_A, BO_B, and CO_C are concurrent.

3. In triangle ABC, let X, Y, and Z be the midpoints of the altitudes from vertices A, B, and C, respectively. Let the excircle opposite vertex A touch side BC at A', and define points B' and C' similarly. Show that $A'X$, $B'Y$, and $C'Z$ are concurrent.

4. Consider triangle ABC with circumcenter O and orthocenter H. Let A_1 be the projection of A onto BC and let D be the intersection of AO with BC. Denote by A_2 the midpoint of AD. Similarly, we define B_1, B_2 and C_1, C_2. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent. (Mathematical Reflections)

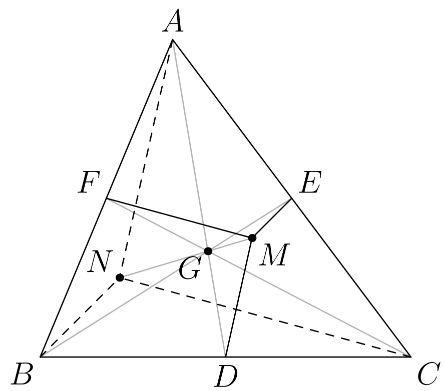
4 Homothety and Homothetic Triangles

If several lines are concurrent, then their images under the same homothety are also concurrent. Two triangles ABC and $A'B'C'$ are homothetic (that is, one is the image of the other under a homothety) if and only if their corresponding sides are parallel. In such a case, AA', BB', and CC' concur at the center of homothety.



Problem. Let ABC be a triangle, and let M be an arbitrary point. Let D, E, and F be the midpoints of sides BC, AC, and AB, respectively. Drawn through the points A, B, and C are straight lines parallel to the straight lines MD, ME, and MF, respectively. Prove that these lines are concurrent.

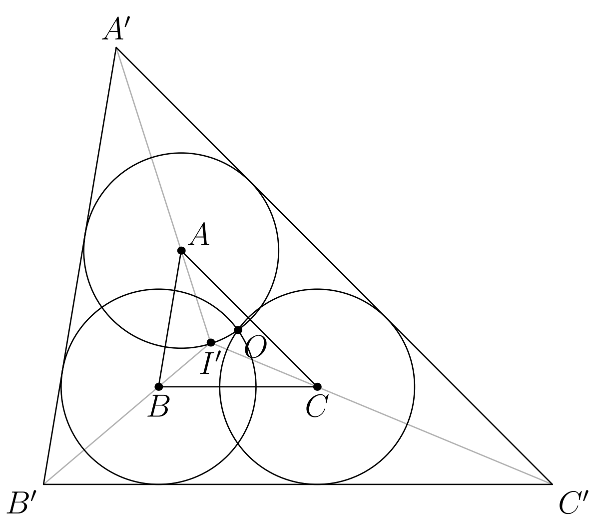
Solution. Let h denote the homothety, centered at G, with scale factor -2. Then $h(D) = A$, $h(E) = B$, and $h(F) = C$. Let $N = h(M)$.



The image of line MD under the homothety h is NA, which is parallel to MD. Similarly, NB is parallel to ME, and NC is parallel to MF. Hence, NA, NB, and NC are the three lines in the problem, and they clearly concur at N. ■

Problem. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and circumcenter of the triangle and the point O are collinear. (IMO, 1981)

Solution. Let A, B, and C be the centers of the circles, and let the triangle be $A'B'C'$, as shown below.



Since A and B are equidistant from side $A'B'$, AB is parallel to $A'B'$. Similarly, AC is parallel to $A'C'$, and BC is parallel to $B'C'$. Hence, triangles ABC and $A'B'C'$ are homothetic, so AA', BB', and CC' are concurrent.

Furthermore, A is equidistant from sides $A'B'$ and $A'C'$, so A lies on the angle bisector of $\angle A'$. Similarly, B lies on the angle bisector of $\angle B'$, and C lies on the angle bisector of $\angle C'$, so AA', BB', and CC' concur at the incenter I' of triangle $A'B'C'$. Let h denote the homothety that takes triangle ABC to triangle $A'B'C'$, so h is centered at I'.

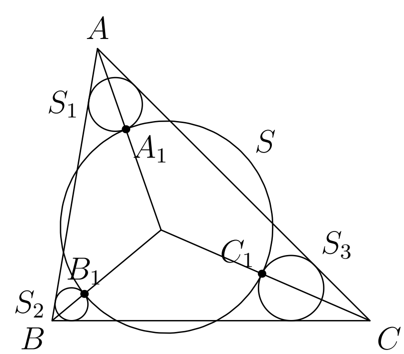
Since the three circles are congruent, and have O as a common point, $OA = OB = OC$. Thus, O is the circumcenter of triangle ABC. Let O' be the circumcenter of triangle $A'B'C'$. Then h takes O to O', so I', O, and O' are collinear. ■

Exercises

1. Let H be the orthocenter of triangle ABC, and let D, E, and F be the circumcenters of triangles BHC, CHA, and AHB, respectively. Prove that AD, BE, and CF are concurrent.

2. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1$, 2, 3, M_i is the midpoint of side a_i, and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent. (IMO, 1982)

3. A triangle ABC and circle S is given in the plane. Circle S_1 is tangent to sides AB and AC, and to S at A_1. Circles S_2 and S_3 and points B_1 and C_1 are defined similarly. Prove that the lines AA_1, BB_1, and CC_1 are concurrent. (Russia, 1994)



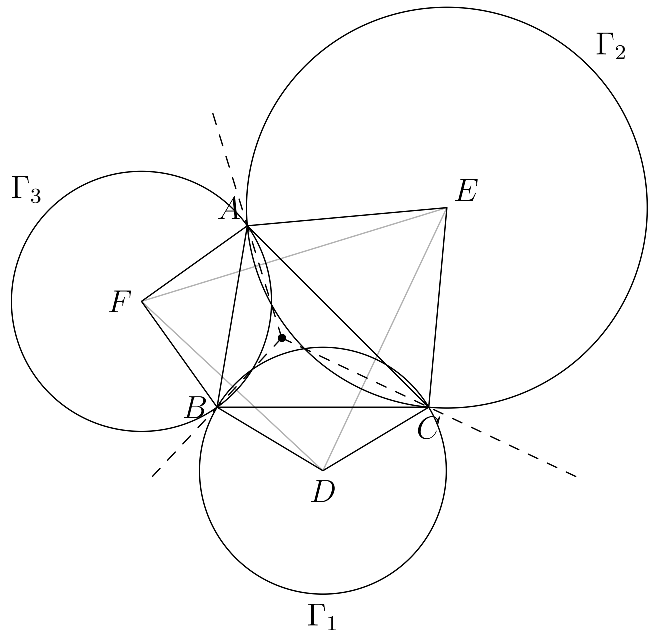
5 Radical Axis and Radical Center

Given two circles, the *radical axis* of the two circles is the set of points whose powers with respect to both circles are equal. If the centers of the two circles are distinct, then the radical axis is a line. Thus, the radical axis can be used to show that a set of points is collinear.

Given three circles, the three radical axes of the three circles, taken in pairs, which concur at the *radical center* of the three circles.

Problem. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent. (USAMO, 1997)

Solution. Let Γ_1 be the circle centered at D with radius $DB = DC$, let Γ_2 be the circle centered at E with radius $EA = EC$, and let Γ_3 be the circle centered at F with radius $FA = FB$.

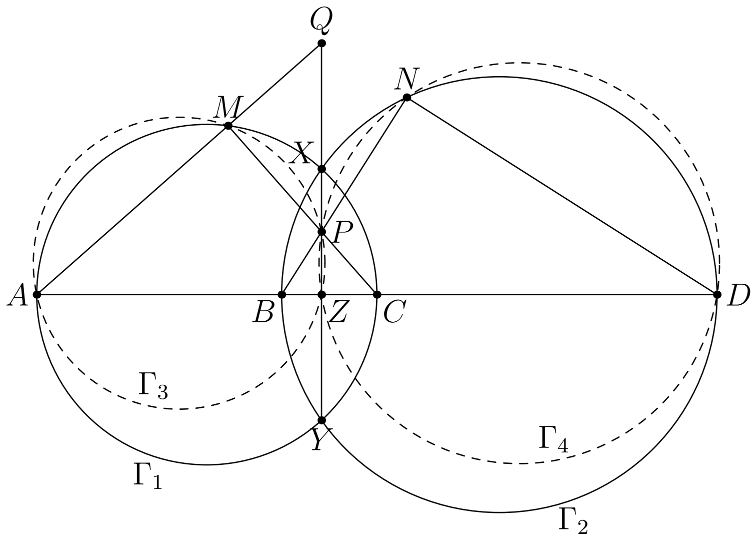


Then the radical axis of Γ_2 and Γ_3 passes through A and is perpendicular to EF, the radical axis of Γ_1 and Γ_3 passes through B and is perpendicular to DF, and the radical axis of Γ_1 and Γ_2 passes through C and is perpendicular to DE.

Hence, the three lines in the problem are the radical axes of the three circles taken in pairs, and they concur at the radical center of the three circles. ■

Problem. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY are concurrent. (IMO, 1995)

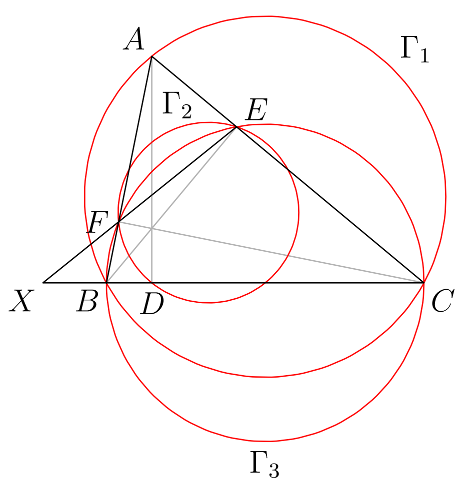
Solution. Let Γ_1, Γ_2, Γ_3, and Γ_4 denote the circles with diameters AC, BD, AP, and DP, respectively. Let AM and XY intersect at Q.



The radical axis of Γ_1 and Γ_3 is AM, and the radical axis of Γ_3 and Γ_4 is PZ (which is also XY), so Q is the radical center of Γ_1, Γ_3, and Γ_4. But PZ is also the radical axis of Γ_1 and Γ_2. Hence, Q is the radical center of all four circles. Since DN is the radical axis of Γ_2 and Γ_4, Q lies on DN. Thus, AM, DN, and XY concur at Q. ■

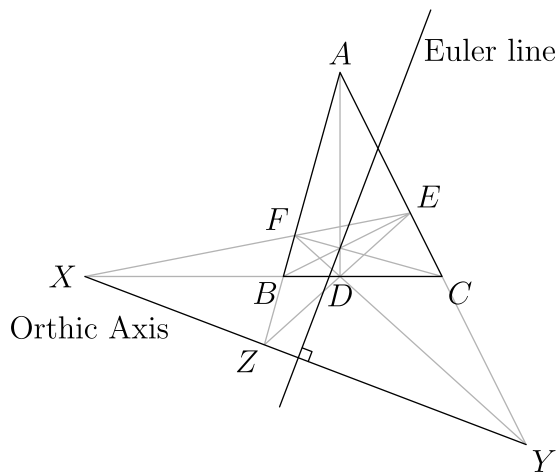
Problem. Let AD, BE, and CF be the altitudes of triangle ABC. Let X be the intersection of BC and EF, Y the intersection of AC and DF, and Z the intersection of AB and DE. Prove that X, Y, and Z are collinear.

Solution. Let Γ_1 be the circumcircle of triangle ABC, Γ_2 the nine-point circle of triangle ABC, and Γ_3 the circle with diameter BC.



Then the radical axis of Γ_1 and Γ_3 is BC, and the radical axis of Γ_2 and Γ_3 is EF, so X is the radical center of the three circles. In particular, X lies on the radical axis of Γ_1 and Γ_2. Similarly, so do Y and Z. Thus, X, Y, and Z are collinear. ■

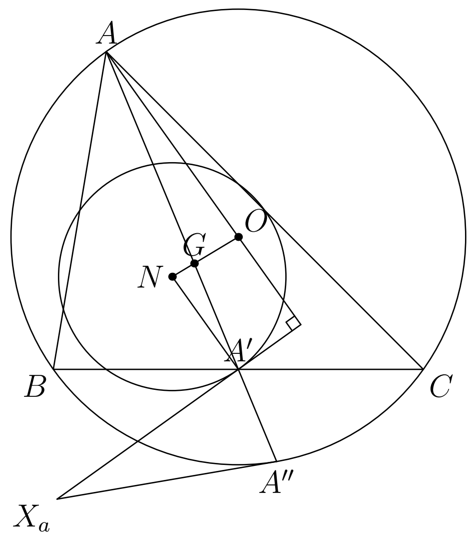
The radical axis of the circumcircle and nine-point circle of triangle ABC is known as the *orthic axis* of triangle ABC. Since the centers of the circumcircle and the nine-point circle both lie on the Euler line of triangle ABC, the orthic axis is perpendicular to the Euler line.



Problem. Let ABC be an acute triangle, with circumenter O. Extend median AA' to the circumcircle at A''. The tangent to the circumcircle at A'' meets the line through A' perpendicular to AO at X_a. Define points X_b and X_c similarly. Prove that X_a, X_b, and X_c are collinear. (Iran, 2004)

Solution. Let O, G, and N denote the circumcenter, centroid, and nine-point center of triangle ABC.

Let l_a denote the line passing through A' perpendicular to AO. Consider the homothety, centered at G, with scale factor -2. This homothety takes A' to A and N to O, so it takes $A'N$ to AO. Hence, $A'N$ is parallel to AO. Then l_a is also perpendicular to $A'N$, which means l_a is the tangent to the nine-point circle at A'.



Furthermore, since l_a is perpendicular to AO, l_a is parallel to the tangent to the circumcircle at A. Then $\angle X_aA'A'' = \angle X_aA''A'$, so triangle $X_aA'A''$ is isosceles with $X_aA' = X_aA''$. Then the tangents from X to the circumcircle and the nine-point circle are equal in length, which means X_a lies on the radical axis of the circumcircle and the nine-point circle, i.e. the orthic axis. Similarly, X_b and X_c also lie on the orthic axis. Thus, X_a, X_b, and X_c are collinear. ■

Exercises

1. Two circles in the plane are exterior to each other. Prove that the midpoints of the two common external tangents and two common internal tangents are collinear.

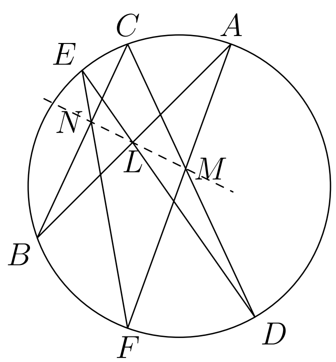
2. Let AD, BE, and CF be the altitudes of triangles ABC, and let Γ_1 and Γ_2 denote the circumcircles of triangles ABC and DEF, respectively. The tangent to Γ_1 at A and the tangent to Γ_2 at D intersect at P. Points Q and R are defined similarly. Show that P, Q, and R are collinear.

3. ABC is a triangle. The tangents from A touch the circle with diameter BC at P and Q. Show that the orthocenter of triangle ABC lies on PQ. (China, 1996)

4. Let l_1, l_2, l_3, and l_4 be four lines in the plane. Let H_1 denote the orthocenter of the triangle determined by the lines l_2, l_3, and l_4, and define points H_2, H_3, and H_4 similarly. Show that H_1, H_2, H_3, and H_4 are collinear.

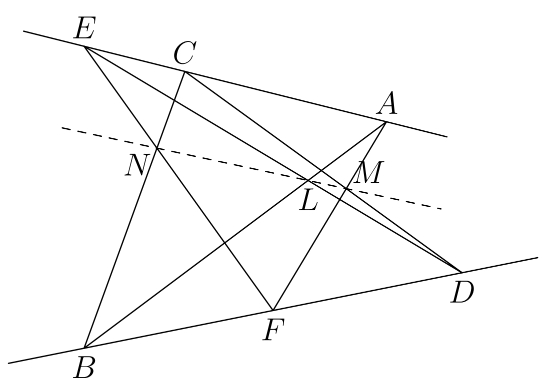
6 Pascal, Pappus, Brianchon, and Desargues

Pascal's Theorem. Let A, B, C, D, E, and F be six points lying on a circle. Let L be the intersection of AB and DE, let M be the intersection of CD and FA, and let N be the intersection of BC and EF. Then L, M, and N are collinear.

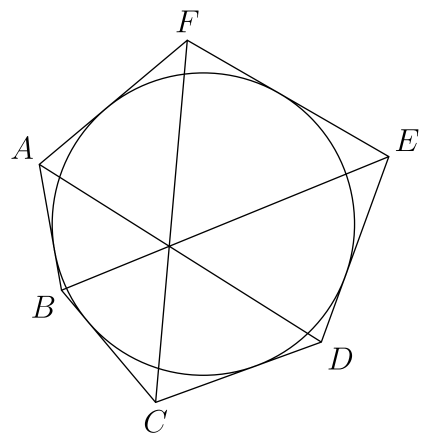


In a more general version of Pascal's theorem, the circle can be replaced by any conic section (i.e. ellipse, parabola, or hyperbola). In particular, if the conic section is a set of two lines (which can be seen as a degenerate hyperbola), then we obtain Pappus's Theorem.

Pappus's Theorem. Points A, C, and E lie on one line, and points B, D, and F lie on another line. Let L be the intersection of AB and DE, let M be the intersection of CD and FA, and let N be the intersection of BC and EF. Then L, M, and N are collinear.

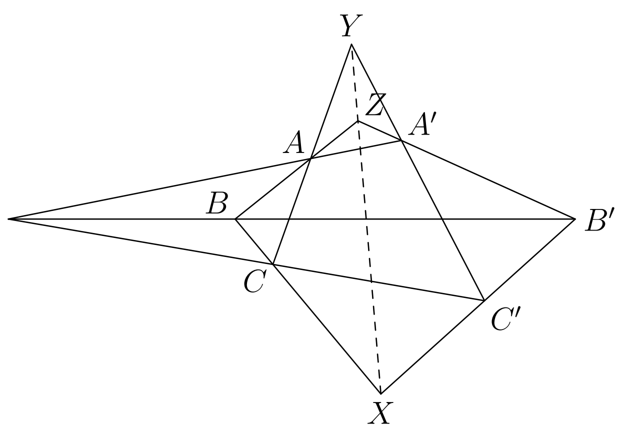


Brianchon's Theorem. Let $ABCDEF$ be a hexagon, such that each side is tangent to a circle. Then AD, BE, and CF are concurrent.



As with Pascal's theorem, the circle in Brianchon's theorem can be replaced by any conic section.

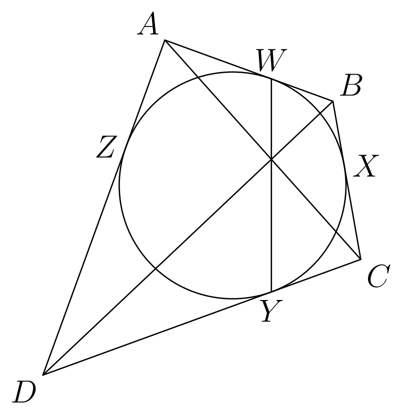
Desargues's Theorem. Let ABC and $A'B'C'$ be triangles. Let X be the intersection of BC and $B'C'$, let Y be the intersection of AC and $A'C'$, and let Z be the intersection of AB and $A'B'$. Then X, Y, and Z are collinear if and only if AA', BB', and CC' are concurrent. In other words, two triangles are perspective from a line if and only if they are perspective from a point.



The theorems listed in this section belong to a branch of geometry known as **projective geometry**, but they can be used effectively to solve problems in Euclidean geometry.

Problem. Quadrilateral $ABCD$ circumscribes a circle, which touches sides AB, BC, CD, and DA at W, X, Y, and Z, respectively. Show that AC, BD, WY, and XZ are concurrent.

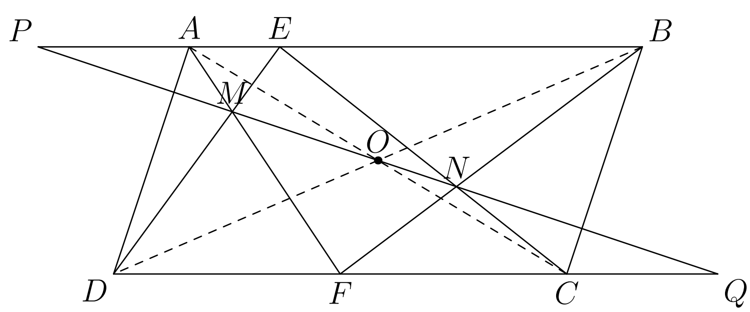
Solution. Each side of (degenerate) hexagon $AWBCYD$ is tangent to the circle, so by Brianchon's theorem, AC, BD, and WY are concurrent.



Similarly, each side of (degenerate) hexagon $ABXCDZ$ is tangent to the circle, so by Brianchon's theorem, AC, BD, and XZ are concurrent. Hence, AC, BD, WY, and XZ are all concurrent. ■

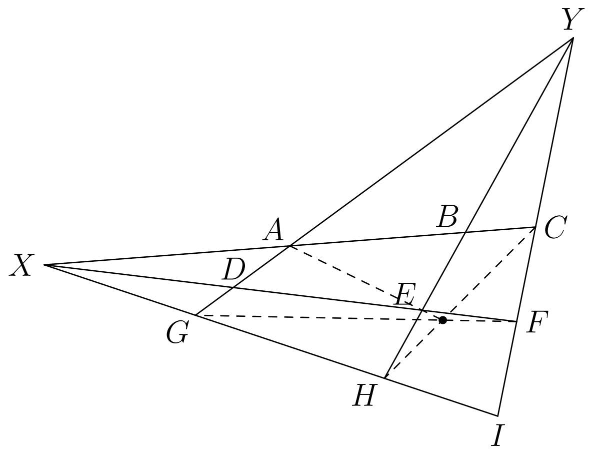
Problem. Let $ABCD$ be a parallelogram. Let E be a point on AB, and let F be a point on CD. Let AF and DE intersect at M, and let BF and CE intersect at N. Let MN intersect AB and CD at P and Q, respectively. Prove that $AP = CQ$.

Solution. Let O be the center of parallelogram $ABCD$. The points A, B, and E are collinear, and the point D, F, and C are also collinear. Then by Pappus's theorem, $M = AF \cap DE$, $N = BF \cap CE$, and $O = AC \cap BD$ are collinear. In other words, MN passes through O.



Consider the rotation of 180° about O. This rotation takes A to C and B to D, so it takes P to Q. Therefore, $AP = CQ$. ■

Problem. Let X and Y be two points in the plane. Three lines from X and three lines from Y intersect at A, B, C, D, E, F, G, H, and I, as shown below.

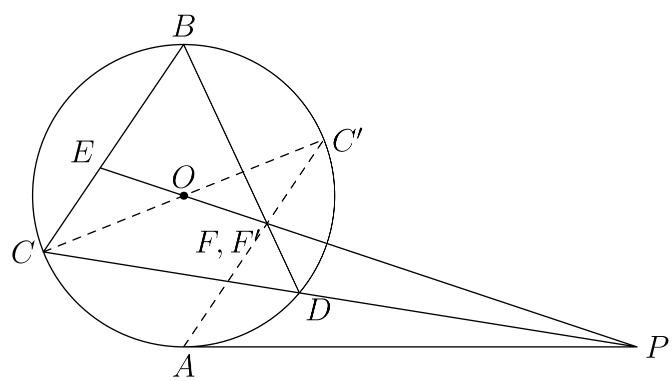


Show that AE, CH, and FG are concurrent.

Solution. The points X, G, and H are collinear, and the points Y, C, and F are collinear. Then by Pappus's theorem, the points $A = CX \cap GY$, $E = FX \cap HY$, and $CH \cap FG$ are collinear. In other words, AE, CH, and FG are concurrent. ■

Problem. Segment AB is a diameter of a circle with center O. Let AP be a tangent, and let D be a point on the circle. Line PD intersects the circle again at C. Lines BC and BD intersect PO at E and F, respectively. Prove that $OE = OF$.

Solution. Let C' be the point diametrically opposite C, and let $F' = BD \cap AC'$.



Then by Pascal's theorem, $P = AA \cap CD$, $F' = BD \cap AC'$, and $O = AB \cap CC'$ are collinear. (Note that AA denotes the tangent to the circle at A.) In other words, F' is the intersection of BD and PO. Hence, points F and F' coincide.

Consider the rotation of 180° about O. This rotation takes line BC to line AC', and line EF to itself, so it takes E to F. Hence, $OE = OF$. ■

Exercises

1. Let $ABCD$ be a quadrilateral that has both a circumcircle and an incircle. Let O be the circumcenter, let I be the incenter, and let P be the intersection of the diagonals AC and BD. Prove that O, I, and P are collinear. (IMO Short List, 1989)

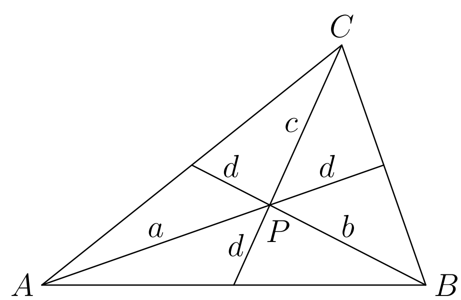
2. A circle meets the three sides BC, CA, AB of triangle ABC at points D_1, D_2; E_1, E_2; and F_1, F_2, in turn. The line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, and line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the three lines AL, BM, and CN are concurrent. (China, 2005)

3. Let H and O denote the orthocenter and circumcenter of triangle ABC, and let ω denote the circumcircle of triangle ABC. Let AO intersect ω at A_1, let A_1H intersect ω at A', and let AH intersect ω at A''. Define points B_1, C_1, B', C', B'', and C'', similarly.

   Prove that $A'A''$, $B'B''$, and $C'C''$ are concurrent at a point on the Euler line of triangle ABC. (Iran, 2005)

7 Miscellaneous Problems

1. Let P be an interior point of triangle ABC and extend the lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$. (AIME, 1988)



2. Given triangle ABC, a straight line intersects the sides BC, AC, and AB at D, E, and F, respectively. Prove that the midpoints of the line segments AD, BE, and CF are collinear.

3. Let $ABCDEF$ be a convex, cyclic hexagon. Show that diagonals AD, BE, and CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

4. A circle intersects side AB of triangle ABC at C_1, C_2, side AC at B_1, B_2, and side BC at A_1, A_2. Prove that AA_1, BB_1, CC_1 are concurrent if and only if AA_2, BB_2, CC_2 are concurrent.

5. Let ω be the circumcircle of triangle ABC. The tangent to ω at A meets BC at P_A, and the points P_B and P_C are defined similarly. Prove that P_A, P_B, and P_C are collinear.

6. Let D, E, and F be points in the same plane as triangle ABC. Let D_B, D_C be the projection of D onto sides AC and AB, respectively, and define E_A, E_C, F_A, and F_B similarly. Prove that AD, BE, and CF are concurrent if and only if
$$\frac{DD_B}{DD_C} \cdot \frac{EE_C}{EE_A} \cdot \frac{FF_A}{FF_B} = 1.$$

7. Given a non-isosceles, non-right triangle ABC, let O denote the center of the circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that triangle OAA_1 is similar to OA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent. (USAMO, 1995)

8. Let Γ be the circumcircle of triangle ABC, and let P, Q, R be the midpoints of sides BC, AC, AB, respectively. Let l_1 be the perpendicular dropped from P to the tangent to Γ at A, and define l_2 and l_3 similarly. Show that l_1, l_2, and l_3 are concurrent.

9. Let I be the incenter of triangle ABC, and let A', B', and C' be the points of tangency of the incircle with sides BC, AC, and AB, respectively. Let D, E, and F be points on IA', IB', and IC' such that $ID = IE = IF$. Show that AD, BE and CF are concurrent.

10. (a) Given a triangle ABC and a point P, let AP, BP, and CP intersect BC, AC, and AC at D, E, and F, respectively. Let $X = EF \cap BC$, $Y = DF \cap AC$, and $Z = EF \cap AB$. Prove that X, Y, and Z are collinear. (Line XYZ is called the *trilinear polar* of P with respect to triangle ABC.)

    (b) Given a triangle ABC and a line l, does there always exist a point P such that l is the trilinear polar of P? (In such a case, P is called the *trilinear pole* of l with respect to triangle ABC.)