Problems for Travelling

1. Sasha tries to determine some positive integer $X \leq 100$. He choose any two positive integers M and N that are less than 100 and ask the question "What is the greatest common divisor of the numbers $X+M$ and N?" Prove that Sasha can determine the value of X after 7 questions.

2. Let O be the centre of the circumcircle ω of an acute-angled triangle ABC. The circle ω_1 with center K passes through the points A, O, C and intersects sides $\overline{AB}$ and $\overline{BC}$ at points M and N. Let L be the reflection of K across line MN. Prove that $\overline{BL}$ is perpendicular to $\overline{AC}$.

3. There are several cities in a state and a set of roads, where each road connects two cities and no two roads connect the same pair of cities. It is known that at least 3 roads go out of every city. Prove that there exists a cyclic path (that is, a path where the last road ends where the first road begins) such that the number of roads in the path is not divisible by 3.

4. Let x_1, x_2, …, x_n be real numbers ($n \geq 2$), satisfying the conditions $-1 < x_1 < x_2 < \cdots < x_n < 1$ and

$$x_1^{13} + x_2^{13} + \cdots + x_n^{13} = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$x_1^{13}y_1 + x_2^{13}y_2 + \cdots + x_n^{13}y_n = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

for any real numbers $y_1 < y_2 < \cdots < y_n$.

5. Let $\overline{AA_1}$ and $\overline{CC_1}$ be the altitudes of an acute-angled non-isosceles triangle ABC. The bisector of the acute angles between lines AA_1 and CC_1 intersects sies $\overline{AB}$ and $\overline{BC}$ at P and Q, respectively. Let H be the orthocentre of triangle ABC and let M be the midpoint of $\overline{AC}$; and let the bisector of angle ABC intersect $\overline{HM}$ at R. Prove that quadrilateral $PBQR$ is cyclic.

6. Five stones which appear identical all have different weights; Oleg knows the weight of each stone. Given any stone x, let $m(x)$ denote its weight. Dmitrii tries to determine the order of the weights of the stones. He is allowed to choose any three stones A, B, C, and ask Oleg the question "Is it true that $m(A) < m(B) < m(C)$?" Oleg responds "yes" or "no". Can Dmitrii determine the order of the weights with at most nine questions?

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(x+y) + f(y+z) + f(z+x) \geq 3f(x+2y+3z)$$

for all x, y, z, $\in \mathbb{R}$.

8. Prove that the set of all positive integers can be partitioned into 100 non-empty subsets such that if three positive integers a, b, c, satisfy $a + 99b = c$, then at least two of them belong to the same subset.

9. Let $ABCDE$ be a convex pentagon on the coordinate plane. Each of its vertices are lattice points. The five diagonals of $ABCDE$ form a convex pentagon $A_1B_1C_1D_1E_1$ inside of $ABCDE$. Prove that this smaller pentagon contains a lattice point on its boundary or within its interior.

10. Let a_1, a_2, ..., a_n be a sequence of non-negative real numbers, not all zero. For $1 \leq k \leq n$, let

$$m_k = \max_{1 \leq i \leq k} \frac{a_{k-i+1} + a_{k-i+2} + \cdots + a_k}{i},$$

Prove that for any $\alpha > 0$, the number of integers k which satisfy $m_k > \alpha$ is less than $\dfrac{a_1 + a_2 + \cdots + a_n}{\alpha}$.

11. Let a_1, a_2, a_3, ... be a sequence with $a_1 = 1$ satisfying the recursion

$$a_{n+1} = \begin{cases} a_n - 2 & \text{if } a_n - 2 \notin \{a_1, a_2, \ldots, a_n\} \text{ and } a_n - 2 > 0 \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Prove that for every positive integer $k > 1$, we have $a_n = k^2 = a_{n-1} + 3$ for some n.

12. There are black and white checkers on some squares of a $2n \times 2n$ board, with at most one checker on each square. First we remove every black checker that is in the same column as any white checker. Next, we remove every white checker that is in the same row as any remaining black checker. Prove that for some colour, at most n^2 checkers of this colour remain.

13. One hundred positive integers, with no common divisor greater than one, are arranged in a circle. To any number, we can add the greatest common divisor of its neighbouring numbers. Prove that using this operation, we can transform these numbers into a new set of pairwise co-prime numbers.

14. M is a finite set of real numbers such that given three distinct elements from M, we can choose two of them whose sum also belongs to M. What is the largest number of elements that M can have?

15. A positive integer is called *perfect* if the sum of al its positive divisors, excluding n itself, equals n. For example, 6 is perfect because $6 = 1 + 2 + 3$. Prove that

(a) if a perfect number larger than 6 is divisible by 7, then it is also divisible by 9.
(b) if a perfect number larger than 28 is divisible by 7, then it is also divisible by 49.