Tricky Sums (and Maybe Products)

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1 Warm-ups

1. [Russia, 1998] Two identical decks have 36 cards each. One deck is shuffled and put on top of the second. For each card of the top deck, we count the number of cards between it and the corresponding card of the bottom deck. What is the sum of these numbers?

2. For x a real number and n a positive integer, express in closed form the value of

$$x + 2x^2 + 3x^3 + \cdots + nx^n.$$

3. For any integer $n > 1$, prove that

$$2(\sqrt{n+1} - 1) < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1.$$

2 Overview

This lecture is a workshop in algebraic manipulation. It's useful to be skilled in evaluating and manipulating big sums (and maybe products) — both as an end in itself, and as a step in the process of solving other kinds of problems in which such sums appear.

You should know the notation $\sum$ for sums and $\prod$ for products. An expression such as $\sum_{i=1}^{n} f(i)$ means the sum of $f(i)$ as i ranges (over integers) from 1 to n; but the index of summation does not always have to be an integer — the problems include examples of other kinds of summation.

Here are some common methods for handling sums:

- Use induction
- Pair up (or otherwise group together) terms
- Telescope
- Expand using partial fractions

- Break each term up into pieces, introducing a new variable if need be
- Switch the order of summation (if multiple variables)
- Be on the lookout for sums that factor
- To extract only the even terms (or only the odd terms) of a sequence, use alternating signs and non-alternating signs, and average
- Find a combinatorial interpretation
- Find a polynomial for which the sum is a convenient coefficient (a.k.a generating function)

3 Problems

1. Let $a_1, \dots, a_n$ be nonnegative integers, all less than or equal to m. For each $j = 1, \dots, m$, let b_j be the number of values of i for which $a_i \geq j$. Prove that
$$a_1 + \cdots + a_n = b_1 + \cdots + b_m.$$
2. [Germany, 1997] Let $u(k)$ be the largest odd divisor of k. Prove that
$$\frac{1}{2^n} \cdot \sum_{k=1}^{2^n} \frac{u(k)}{k} \geq \frac{2}{3}.$$
3. [Putnam, 2001] Let n be an even positive integer. The numbers 1 through n^2 are written in the squares of an $n \times n$ grid, with $1, 2, \dots, n$ in order along the first row, $n+1, n+2, \dots, 2n$ along the second row, and so forth. The squares are arbitrarily colored red and blue so that, in each row and column, half the squares are red. Prove that the sum of numbers in the red squares equals the sum of numbers in the blue squares.
4. Compute $\sum_{i=1}^{99} 1/(2^i + 2^{50})$.
5. [Korea, 1997] Express $\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor$ explicitly, in terms of n and $\lfloor \sqrt{n} \rfloor$.
6. [Russia, 1999] Let $\{x\}$ denote the fractional part of x. Prove that for every natural number n,
$$\sum_{k=1}^{n^2} \{\sqrt{k}\} \leq (n^2 - 1)/2.$$
7. [AIME, 1983] For any finite set S of positive integers, its *alternating sum* is the value obtained by writing its elements in decreasing order and alternately adding and subtracting. For example, if $S = \{1, 3, 4, 6, 8\}$ then the alternating sum is $8 - 6 + 4 - 3 + 1 = 4$. Find the sum of the alternating sums of all nonempty subsets of $\{1, 2, \dots, 10\}$.

8. [Canada, 1997] Prove that

$$\frac{1}{1999} < \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{1997}{1998} < \frac{1}{44}.$$

9. Let $\tau(n)$ denote the number of divisors of n, and $\sigma(n)$ the sum of the divisors of n.

   (a) Prove that

$$\tau(1) + \tau(2) + \cdots + \tau(n) = \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor.$$

   (b) Prove that

$$\sigma(1) + \sigma(2) + \cdots + \sigma(n) = \left\lfloor \frac{n}{1} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + \cdots + n\left\lfloor \frac{n}{n} \right\rfloor.$$

   (c) Prove that

$$\tau(\gcd(1,n)) + \tau(\gcd(2,n)) + \cdots + \tau(\gcd(n,n)) = \sigma(n).$$

10. [APMO, 1998] Let F be the set of all n-tuples $(A_1, \ldots, A_n)$ of subsets of $\{1, 2, \ldots, 1998\}$. Find

$$\sum_{(A_1,\ldots,A_n)\in F} |A_1 \cup \cdots \cup A_n|,$$

where $|A|$ denotes the number of elements of the set A.

11. [APMO, 1997] Let

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \cdots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}},$$

where the denominators contain partial sums of reciprocals of triangular numbers. Prove that $S > 1001$.

12. [IMO, 1988] Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

13. Let $0 \leq m \leq n$ be integers with m even. Prove:

$$\left| \binom{n}{0}\binom{n}{m} - \binom{n}{1}\binom{n}{m-1} + \binom{n}{2}\binom{n}{m-2} - \cdots + \binom{n}{m}\binom{n}{0} \right| = \binom{n}{m/2}.$$

14. [Putnam, 2001] You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of landing heads. If the n coins are tossed, what is the probability that the number of heads is odd?

15. [Putnam, 1997] Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all integers $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

16. [Putnam, 1998] Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

17. [USAMO, 2010] Let $q = (3p-5)/2$, where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $1/p - 2S_q = m/n$ for integers m and n, then $m - n$ is divisible by p.

18. [USAMO, 1991] For any nonempty set S of numbers, let $\sigma(S)$ denote the sum of its elements and $\pi(S)$ the product of its elements. Prove that

$$\sum_S \frac{\sigma(S)}{\pi(S)} = (n^2 + 2n) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)(n+1),$$

where the sum on the left-hand side is over all nonempty subsets S of $\{1, 2, \ldots, n\}$.

19. (a) [Principle of Inclusion-Exclusion] Let $A_1, \ldots, A_n$ be finite sets. Prove that

$$|A_1 \cup \cdots \cup A_n| = \sum_I (-1)^{|I|-1} |\cap_{i \in I} A_i|,$$

where the sum is over all nonempty subsets I of $\{1, \ldots, n\}$.

(b) [Möbius Inversion Formula] Let $f(n)$ be a real-valued function defined on the positive integers, and define $g(n) = \sum_{d \mid n} f(d)$. Prove that

$$f(n) = \sum_{d \mid n} \mu(n/d) g(d),$$

where the function μ is defined as follows: $\mu(1) = 1$; $\mu(n) = (-1)^k$ if n is a product of k distinct primes; and $\mu(n) = 0$ otherwise.

20. [TST, 2001] Evaluate

$$\sum_{k=0}^{n} (-1)^k (n-k)!(n+k)!$$

in closed form.