Complex Numbers in Geometry

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1 Warmup

Problem 1. In cyclic quadrilateral $ABCD$, let H_a, H_b, H_c, and H_d be the orthocenters of triangles BCD, CDA, DAB, and ABC, respectively. Prove that $H_aH_bH_cH_d$ is cyclic.

2 How to Use Complex Numbers

In this handout, we will identify the real plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Instead of viewing points in vector form, we instead consider the corresponding complex number.

Note. Throughout this handout, we use a lowercase letter to denote the complex number that represents the point labeled by the corresponding uppercase letter.

This allows us to compute geometric transformations easily. Sometimes, this can lead to solutions that are quite elegant:

Problem 2 (TST 2006/6)**.** Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

When synthetic approaches have failed, we can use computation with complex numbers:

Problem 3 (MOP 2010)**.** In cyclic quadrilateral $ABCD$, diagonals AC and BD meet at E, and lines AD and BC meet at F. Let G and H be the midpoints of sides AB and CD, respectively. Prove that EF is tangent to the circumcircle of triangle EGH.

There are a number of general principles for this:

- To solve a problem using complex numbers, the general approach is reduce the desired statement to a calculation. This can be done in a number of ways:
    1. Checking a condition (for parallel lines, cyclic quadrilaterals, collinearity) or checking a claim about angles or distances given in the problem. In this case, you should just calculate and then check the condition.
    2. Reducing the claim to the fact that some two points are the same. This is usually used to show concurrency or collinearity. The calculation can sometimes be reduced a bit in this case; if the problem involves a triangle ABC and asks to show that some lines ℓ_A, ℓ_B, ℓ_C defined symmetrically in terms of A, B, C are concurrent, then it is enough to show that the coordinates of $\ell_A \cap \ell_B$ are symmetric in a, b, c.

- Before checking any conditions, it is important to set up a good set of variable points to compute with. Some pointers:

  1. If there is a circle that figures prominently in the problem, you should set this to be the unit circle so that conjugates of points are also explicitly computable. In particular, if a triangle or quadrilateral is circumscribed rather than inscribed, take the points of tangency as your variables.
  2. Any synthetic observations you make (most commonly, finding a spiral similarity) might allow you to reduce the number of unknown variables.

- Sometimes, solutions will involve a fair amount of algebraic manipulation. It might seem obvious, but you try to keep the algebra as clean as possible. I recommend the following: (1) Whenever possible, factor. (2) Keep your terms homogeneous so you can spot errors easily. (3) Before you begin, try to estimate the number of terms you will end up with, and prepare accordingly. Everyone has a personal system for dealing with large expressions; you should find what works for you.

3 When to Use Complex Numbers

There are a number of signs that suggest a problem can be approached using complex numbers:

- There is one primary circle in the diagram. If this is the case, you can set this to be the unit circle, and calculate all other points in terms of some values on the circle. In particular, the common formulas will not contain conjugates, allowing for simpler manipulations.

- A small number of points is sufficient to define the diagram. In particular, if the coordinates of each point in the diagram may be computed easily, then solving the problem with complex numbers should be simple. This most often occurs with a cyclic complete quadrilateral.

Perhaps more important is when **NOT** to use complex numbers. Do not use complex numbers if:

- You have just started working on a problem. Complex numbers should only be a method of last resort, and you should almost always try synthetic approaches first. Regardless of what type of approach you are trying, it is almost always essential that you make some synthetic observations before plunging into calculation.

- The problem involves a large number of circles. The condition for cyclic points is rarely very useful as a given, so the extra circle information will be quite difficult to use.

- There are "too many" steps in the calculation. Each additional layer of construction will add to the number of terms that you need to deal with, and at some point, it will become impossible handle. After using this method a couple times, you will gradually see what your limit is. Before using complex numbers, make sure the number of terms will not exceed this!

4 Useful Facts

Below are some useful facts. You should be able to derive these easily on your own, and you might want to prove some of the more complicated ones on an actual olympiad solution.

Fact 1. A complex number z is real iff $z = \bar{z}$ and is pure imaginary iff $z = -\bar{z}$. If $|z| = 1$, we have $\bar{z} = 1/z$.

Fact 2. For *any* points A, B, C, D, we have:

- $AB \parallel CD \iff \dfrac{a-b}{c-d} \in \mathbb{R}$
- ABC collinear $\iff \dfrac{a-b}{b-c} \in \mathbb{R}$
- $AB \perp CD \iff \dfrac{a-b}{c-d} \in i\,\mathbb{R}$
- $ABCD$ is cyclic $\iff \dfrac{b-a}{c-a} \bigg/ \dfrac{b-d}{c-d} \in \mathbb{R}$

Fact 3. Angles $\angle ABC$ and $\angle XYZ$ are equal iff $\dfrac{a-b}{c-b} \bigg/ \dfrac{x-y}{z-y} \in \mathbb{R}$.

Fact 4. For points A, B, C, D *on the unit circle*, we have:

- the equation of chord AB is $z = a + b - ab\bar{z}$
- the intersection of chords AB and CD is $\dfrac{ab(c+d) - cd(a+b)}{ab - cd}$
- the equation of the tangent at A is $z = 2a - a^2\bar{z}$
- the intersection of the tangents at A and B is $\frac{2ab}{a+b}$

Fact 5. For a triangle ABC inscribed in the unit circle, we have:

- the centroid is given by $g = \frac{a+b+c}{3}$
- the orthocenter is given by $h = a + b + c$

Fact 6. For a chord AB on the unit circle, the projection of any point Z onto AB is $\frac{1}{2}(a+b+z-ab\bar{z})$

5 Problems

The problems below should lend themselves well to complex number approaches. Be careful to make sure you have the right coordinate system before starting your computation!

5.1 Imaginary Problems

Problem 4. Solve your favorite geometry problem using complex numbers.

Problem 5. Find the area of a triangle ABC in terms of the complex numbers a, b, and c.

Problem 6 (TST 2008/7)**.** Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB, respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

Problem 7 (MOP 2006/ϵ/2)**.** Point H is the orthocenter of triangle ABC. Points D, E, and F lie on the circumcircle of triangle ABC such that $AD \parallel BE \parallel CF$. Points S, T, and U are the respective reflections of D, E, and F across the lines BC, CA, and AB. Prove that S, T, U, and H are cyclic.

Problem 8 (WOP 2004/3/4)**.** Convex quadrilateral $ABCD$ is inscribed in circle ω. Let M and N be the midpoints of diagonals AC and BD, respectively. Lines AB and CD meet at F, and lines AD and BC meet at F. Prove that

$$\frac{2MN}{EF} = \left|\frac{AC}{BD} - \frac{BD}{AC}\right|.$$

Problem 9 (WOOT 2006/4/5)**.** Let O be the circumcenter of triangle ABC. A line through O intersects sides AB and AC at M and N, respectively. Let S and R be the midpoints of BN and CM, respectively. Prove that $\angle ROS = \angle BAC$.

Problem 10 (MOP 2006/4/1)**.** Convex quadrilateral $ABCD$ is inscribed in circle ω centered at O. Point O does not lie on the sides of $ABCD$. Let O_1, O_2, O_3, O_4 denote the circumcenters of triangles OAB, OBC, OCD, and ODA, respectively. Diagonals AC and BD meet at P. Prove that O_1O_3, O_2O_4, and OP are concurrent.

Problem 11 (USAMO 2006/6)**.** Let $ABCD$ be a quadrilateral and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

Problem 12 (China 1996)**.** Let H be the orthocenter of the triangle ABC. The tangents from A to the circle with diameter BC intersect the circle at the points P and Q. Prove that the points P, Q, and H are collinear.

Problem 13 (Putnam 2008)**.** Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n - 1$.

5.2 Real Problems

Problem 14 (TST 2000/2)**.** Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

Problem 15 (WOP 2004/1/2)**.** Let $ABCD$ be a convex quadrilateral with AB not parallel to CD, and let X be a point insider $ABCD$ such that $\angle ADX = \angle BCX < 90^\circ$ and $\angle DAX = \angle CBX < 90^\circ$. If the perpendicular bisectors of segments AB and CD intersect at Y, prove that $\angle AYB = 2\angle ADX$.

Problem 16 (MOP 2006/5/3)**.** Let $ABCD$ be a quadrilateral circumscribed about a circle with center O. Let line AO intersect the perpendicular from C to BD at E, line CO intersect the perpendicular from A to BD at F, and let AC and BD intersect at G, prove that E, F, and G are collinear.

Problem 17 (MOP 2006/β/3)**.** Triangle ABC is inscribed in circle ω. Point P lies inside the triangle. Rays AP, BP, and CP meet ω again at A_1, B_1, and C_1, respectively. Let A_2, B_2, and C_2 be the reflections of A_1, B_1, and C_1 across the midpoints of sides BC, CA, and AB, respectively. Prove that the circumcircle of triangle $A_2B_2C_2$ passes through the orthocenter of triangle ABC.

Problem 18 (IMO 2002/2). BC is a diameter of a circle with center O. A is a point on the circle with $\angle AOC > 60°$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of minor arc AB. The line through O parallel to AD meets AC again at J. Show that J is the incenter of triangle CEF.

Problem 19 (IMO 2004/5). In the convex quadrilateral $ABCD$ the diagonal BD is not the bisector of any of the angles ABC and CDA. Let P be the point in the interior of $ABCD$ such that

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA.$$

Prove that the quadrilateral $ABCD$ is cyclic if and only if $AP = CP$.

Problem 20 (Iran 2005). Let ABC be an isosceles triangle such that $AB = AC$. Let P be on the extension of the side BC and X and Y on AB and AC such that

$$PX \parallel AC \text{ and } PY \parallel AB.$$

Let T be the midpoint of the arc BC. Prove that $PT \perp XY$.

Problem 21 (ISL 2004). Let $A_1, A_2, \ldots, A_n$ be a regular n-gon. Assume that the points $B_1, B_2, \ldots, B_{n-1}$ are determined in the following way:

- for $i = 1$ or $i = n-1$, B_i is the midpoint of the segment A_iA_{i+1};
- for $i \neq 1$, $i \neq n-1$, and S the intersection of A_iA_{i+1} and A_nA_i, B_i is the intersection of the bisectors fo the angle A_iS_{i+1} with A_iA_{i+1}.

Prove that $\angle A_1B_1A_n + \angle A_1B_2A_n + \cdots + \angle A_1B_{n-1}A_n = 180°$.

Problem 22 (ISL 1998). Let ABC be a triangle such that $\angle ACB = 2\angle ABC$. Let D be the point of the segment BC such that $CD = 2BD$. The segment AD is extended over the point D to the point E for which $AD = DE$. Prove that $\angle ECD + 180° = 2\angle EBC$.

Problem 23 (Iran 2005). Let n be a prime number and H_1 a convex n-gon. Label the vertices of H_1 with $0, \ldots, n-1$ clockwise around H_1. The polygons $H_2, \ldots, H_n$ are defined recursively as follows: vertex i of polygon H_{k+1} is obtained by reflecting vertex i of H_k through vertex $i+k$ of H_k, where we consider vertex labels modulo n. Prove that H_1 and H_n are similar.

5.3 Complex Problems

Problem 24 (USAMO 2004/6). A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

Prove that $ABCD$ is an isosceles trapezoid.

Problem 25 (IMO 2000/6). Let AH_1, BH_2, and CH_3 be the altitudes of an acute triangle ABC. The incircle ω of triangle ABC touches the sides BC, CA, and AB at T_1, T_2, and T_3, respectively. Consider the symmetric images of lines H_1H_2, H_2H_3, and H_3H_1 with respect to lines T_1T_2, T_2T_3, and T_3T_1. Prove that these images form a triangle whose vertices lie on circle ω.

Problem 26 (ISL 2004)**.** Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, and N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

Problem 27 (IMO 2008/6)**.** Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangles ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD.
Prove that the common external tangents to k_1 and k_2 intersect on k.

Problem 28 (Vietnam 2003)**.** The circles k_1 and k_2 touch each other at the point M. The radius of the circle k_1 is bigger than the radius of the circle k_2. Let A be an arbitrary point of k_2 which doesn't belong to the line connecting the centers of the circles. Let B and C be the points of k_1 such that AB and AC are its tangents. The lines BM and CM intersect k_2 again at E and F, respectively. The point D is the intersection of the tangent at A with the line EF. Prove that the locus of points D (as A moves along the circle) is a line.