Some useful constructions

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Abstract

Suppose you have a geometry problem and some really weird relation between sides and/or angles. What do you do then? Trig-bash? Complex-bash? Bash? You **can** do that, of course, but you can save a lot of computations just by making a simple construction.

1 Some tricks

- **Midpoints and angle chasing.** Midpoints can be really annoying in a geometry problem, specially if mixed with a lot of angles. How do you relate midpoints with angles? There are at least two possibilities:
  - **Midlines.** You should know that if M and N are the midpoints of segments AB and AC then BC is parallel to MN. So you might want to construct another midpoint!
  - **Paralellograms.** In paralellograms, the diagonals meet at their respective midpoints. You might want to take advantage of that.
  - Another nice ratio is $2:1$: it is the centroid ratio. So if a point X divides AB in a $1:2$ ratio, you can try to construct a triangle with centroid X and median AB.
- **Divide and conquer.** You can divide some segments and/or angles in parts, especially when you have relations like "this angle is three times that other" or $AB = AC + AD$.
- **More on segments.** If a broken line has the same length as a segment or another broken line, a good idea is to straighten all broken lines. This usually generates a lot of isosceles triangles, and you know good things happen when isosceles triangles are around.
- **Circles with radius 0.** Sometimes it's handy to consider a point a circle with radius 0. Especially if you apply power of a point.
- **Segment ratios and angles.** Suppose you have some segment ratios, but still have to deal with angles. One good bridge between these two apparently disparate worlds is the Apolonius circle: let A and B be two distinct fixed points; the locus of the points X such that $\frac{AX}{BX}$ is fixed and different from 1 is a circle with AB as an axis of symmetry.
- **Point redefinition.** You want a point to have a property, but can't get a proof of it! So try to construct a point with the same property and prove that it is actually the same point.
- **Projecting a point onto the sides of a polygon.** I know it sounds a bit random, but projecting a point onto the sides of a polygon generates a lot of cyclic quads, and some nice angles may appear.

2 Problems

1. Prove the *Erdős-Mordell inequality*: Let P be a point inside triangle ABC and d_a, d_b, d_c be the distances from P to sides BC, CA, AB respectively. Then $PA + PB + PC \geq 2(d_a + d_b + d_c)$.

2. (IMO 1996) Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

3. (IMO 1995) Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \frac{\pi}{3}$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = \frac{2\pi}{3}$. Prove that $AG + GB + GH + DH + HE \geq CF$.

4. (Brazil 2006) Let ABC be a triangle. The internal bisector of $\angle B$ meets AC in P. Let I be the incenter of ABC. Prove that if $AP + AB = CB$, then API is an isosceles triangle.

5. (IMO 1997) It is known that $\angle BAC$ is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T.

   Show that $AU = TB + TC$.

6. (IMO 2001) Let ABC be a triangle with $\angle BAC = 60°$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

7. (Brazil 2001) In a convex quadrilateral, the *altitude* relative to a side is defined to be the line perpendicular to this side through the midpoint of the opposite side. Prove that the four altitudes have a common point if and only if the quadrilateral is cyclic, that is, if and only if, there exists a circle which contains its four vertices.

8. (IMO 2007) Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = FG = EC$. Prove that ℓ is the bisector of angle $\angle DAB$.

9. (IMOSL 2007) Denote by M the midpoint of side BC in an isosceles triangle ABC with $AC = AB$. Take a point X on the smaller arc MA of circumcircle of triangle ABM. Denote by T the point inside of angle $\angle BMA$ such that $\angle TMX = 90°$ and $TX = BX$.

   Prove that $\angle MTB - \angle CTM$ does not depend on choice of X.

10. Triangle ABC is such that $AB = AC$. Let D be a point on side BC such that $BD = 2DC$. Point P lies on segment AD and satisfies $\angle ABP = \angle PAC$. Prove that $\angle BAC = 2\angle DPC$.

11. (Bosnia and Herzegovina 2011) Let ABC be a triangle such that $AB + AC = 2BC$. Prove that the midpoints M of AB and N of AC, the incenter I of ABC and A lie on the same circle.

12. (Iran 2011) Let ABC be a triangle. Line r intersects the extension of AB at D (B between A and D) and the extension of AC in E (C between A and E). Suppose that reflection of line ℓ with respect to the perpendicular bisector of side BC intersects the mentioned extensions in D' and E' respectively. Prove that if $BD + CE = DE$, then $BD' + CE' = D'E'$.

13. Let A be a point in the interior of triangle BCD such that $AB \cdot CD = AD \cdot BC$. Point P is the symmetrical of point A with respect to BD. Prove that $\angle PCB = \angle ACD$.

14. (USAMO 2005) Let ABC be an acute-angled triangle, and let P and Q be two points on its side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and the points C_1 and Q lie on opposite sides of the line AB. Construct a point B_1 in such a way that the convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and the points B_1 and Q lie on opposite sides of the line AC.

    Prove that the points B_1, C_1, P, and Q lie on a circle.

15. (USAMO 2009) Trapezoid ABC, with $AB \parallel CD$, is inscribed in circle ω and point G lies inside triangle BCD. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to AB intersect BD and BC at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if BG bisects $\angle CBD$.

16. (Iran TST 2011) In acute triangle ABC, $\angle B > \angle C$. Let M be the midpoint of BC, D and E be the feet of the altitudes from C and B respectively. K and L are midpoints of ME and MD respectively. If KL intersects the line through A parallel to BC at T, prove that $TA = TM$.

17. (USAMO 1999) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

18. (USA TST, 2000) $ABCD$ is a cyclic quadrilateral. The projections of the intersection of its diagonals to sides AB and CD are E, F respectively. Show that the line EF is perpendicular to the line containing the midpoints of the sides of BC and DA.

19. (USA TST, 2001) In triangle ABC, $\angle B = 2\angle C$. Let P and Q be points on the perpendicular bisector of segment BC such that rays AP and AQ trisect $\angle BAC$. Prove that $PQ < AB$ if and only if $\angle ABC$ is obtuse.

20. (IMO Shortlist, 2009) Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms.

    Prove that $GR = GS$.

21. (IMO Shortlist, 2009) Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

22. (IMO Shortlist, 2008) Let $ABCD$ be a convex quadrilateral. Prove that there exists a point P inside the quadrilateral such that

    $$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

    if and only if the diagonals AC and BD are perpendicular.