Geometric Inequalities (Black Group)

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June 28, 2010

Introduction and Useful Techniques

Because geometric inequalities combine two subjects, they come in several varieties, some more geometric, some more algebraic. Nonetheless, there is a rather short list of general methods and results that often apply.

1. **Calculation and conversion of the problem into a "normal" inequality.** The key in using this method is to express the quantities in the problem in terms of variables that will make the inequality nice. Common choices are **lengths of sides, trigonometric functions of angles, and areas**. Often, the inequality will be intractable with one choice of variables but quite easy with another. In easier problems, the choice of variables is obvious and expressing the inequality in terms of those variables is immediate. In harder problems, you may need to do some geometry or a fair amount of computation to re-express the inequality. The **Laws of Sines and Cosines, Angle Bisector theorem, Ceva, trig Ceva, Menelaus,** and **Stewart's theorem** may come in handy.

   The distinguishing characteristic of this category of geometric inequalities is that the *inequality* is ultimately achieved through algebra; all of the previous steps involved *equalities*. Once the inequality is set up, all of the techniques for solving regular inequalities apply. Some appear much more frequently than others, however. Whereas standard inequalities (especially the harder ones) often require algebra tricks, the list of frequently-used techniques in geometric inequalities is much more mundane. In fact, **smoothing** and **multiplying out** are probably the most common. When multiplying out an inequality involving the sides of a triangle, the constraints of the triangle inequality can be removed by using the **triangle substitution** $a = y + z, b = z + x, c = x + y$, where $x = s - a, y = s - b, z = s - c$.

   In most cases, however, the inequality will arise from geometry rather than algebra, and the next few methods address this case.

2. **Triangle inequality.** Although this inequality may seem trivial, its use is surprisingly prevalent—even in some of the hardest problems! Of the problems below, roughly one-fourth involve the triangle inequality. The biggest clue is summed lengths. In geometry problems, the trick is usually to put such pairs of lengths along the same line; in geometric inequalities, moving the lengths to segments that share an endpoint suffices.

3. **Ptolemy's inequality.** Here, the hints are sums of products or ratios of lengths, the latter of which can be obtained by taking Ptolemy's inequality and dividing out by some length. Another hint is an equality condition involving concyclic points. As a side note, Ptolemy's

inequality itself can be seen to be equivalent to the triangle inequality upon inversion. (Prove this as an exercise if you aren't familiar with this fact.)

Like the triangle inequality, Ptolemy's inequality comes in handy surprisingly often. A good example that combines these two methods is the proof that the **Fermat point** F for which $\angle AFB = \angle BFC = \angle CFA = 120°$ minimizes $AF + BF + CF$. In fact, the proof of this result is more important to remember than the result itself.

4. **Projection.** Once more, we have an obvious geometric fact that can be remarkably useful: the distance from a point to a line is minimized along the altitude. Equivalently, the projection of a segment on a line is at most as long as the original segment. The best example of this technique is the proof of the **Erdős-Mordell inequality**, which states that for an acute triangle ABC and point P in its interior,

$$AP + BP + CP \geq 2(PD + PE + PF),$$

where D, E, F are the feet of the perpendiculars from P to BC, CA, AB, respectively.

5. **The distance from a point on a circle to a chord is maximized at one of the midpoints of the arcs.** Not much more to say about this one; it's useful more in proving small lemmas on the way to solving a problem.

6. **Vectors.** These are sometimes good for problems in which the configuration of the diagram is unclear, and occasionally for proving acuteness or obtuseness.

7. **Equalities that are actually inequalities.** The tip-off here is a problem in which the diagram seems to have too many degrees of freedom to imply the result. Usually, the case is that the equation given is the equality case of an inequality.

In general, looking at **equality cases** is extremely important, as this will often suggest a technique to use, or on the flip side, which techniques can't be used.

Aside from this, one last point worth emphasizing is that doing a little geometric work will frequently save much algebraic work, so drawing good diagrams and "thinking geometrically," so to speak, is important, just as in normal geometry problems.

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The problems below are taken from the USAMO, IMO, and IMO Shortlists from 1996-2003. As a piece of trivia, geometric inequalities (or disguised geometric inequalities) appeared on the USAMO in 96, 99, 00, 01, 02 and on the IMO in 96, 98, 01, 02, 03.

Easier Problems

1. [S 01] Let M be a point in the interior of triangle ABC. Let A' lie on BC with MA' perpendicular to BC. Define B' and C' analogously. Define

$$p(M) = \frac{MA' \cdot MB' \cdot MC'}{MA \cdot MB \cdot MC}.$$

Determine, with proof, the location of M such that $p(M)$ is maximal. Let $\mu(ABC)$ denote this maximal value. For which triangles ABC is the value of $\mu(ABC)$ maximal?

2. [S 96] Let ABC be an equilateral triangle and let P be a point in its interior. Let lines AP, BP, CP meet sides BC, CA, AB at points A_1, B_1, C_1, respectively. Prove that

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A.$$

3. [S 97] Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$, $EF = FA$. Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}$$

and determine when equality holds.

4. [USAMO 00/2] Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP} + \frac{1}{BQ} + \frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA, respectively. Prove that all triangles in S are isosceles and similar to one another.

5. [USAMO 02/2] Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2 + \left(2\cot\frac{B}{2}\right)^2 + \left(3\cot\frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisor, and determine those integers.

6. [USAMO 96/3] Let ABC be a triangle. Prove that there is a line ℓ (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in ℓ has area more than $2/3$ the area of triangle ABC.

7. [USAMO 01/4] Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.

8. [IMO 01/1] Let ABC be an acute triangle with circumcenter O. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30°$. Prove that $\angle CAB + \angle COP < 90°$.

9. [USAMO 99/2] Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

Not-as-easy Problems

1. [S 01] Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that $AP \cdot AG + BP \cdot BG + CP \cdot CG$ is minimized, and express the minimum value in terms of the side lengths of ABC.

2. [S 96] Let ABC be an acute-angled triangle with circumcenter O and circumradius R. Let AO meet circle BOC again at A', and define B' and C' analogously. Prove that

$$OA' \cdot OB' \cdot OC' \geq 8R^3$$

and determine when equality holds.

3. [S 02] Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Rays BF and CF meet sides AC and AB at D and E, respectively. Prove that

$$AB + AC \geq 4DE.$$

4. [S 96] Let $ABCD$ be a convex quadrilateral, and let R_A, R_B, R_C, R_D denote the circumradii of triangles DAB, ABC, BCD, CDA, respectively. Prove that

$$R_A + R_C > R_B + R_D \quad \text{if and only if} \quad \angle A + \angle C > \angle B + \angle D.$$

5. [IMO 98/5] Let I be the incenter of triangle ABC. Let the incircle of ABC touch sides BC, CA, AB at K, L, M, respectively. The line through B parallel to MK meets lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

6. [S 96] On the plane are given a point O and a polygon F (not necessarily convex). Let P denote the perimeter of F, D the sum of the distances from O to the vertices of F, and H the sum of the distances from O to the lines containing the sides of F. Prove that

$$D^2 - H^2 \geq \frac{P^2}{4}.$$

7. [S 99] Let M be a point in the interior of triangle ABC. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

Problems that Might Make You Uneasy

1. [S 99] For a triangle $T = ABC$ we take the point X on side AB such that $AX/XB = 4/5$, the point Y on segment CX such that $CY = 2YX$, and, if possible, the point Z on ray CA such that $\angle CXZ = 180° - \angle ABC$. We denote by S the set of all triangles T for which $\angle XYZ = 45°$. Prove that all the triangles in S are similar to each other and find the measure of their smallest angle.

2. [S 01] Let O be an interior point of acute triangle ABC. Let A_1 lie on BC with OA_1 perpendicular to BC. Define B_1 on CA and C_1 on AB similarly. Prove that O is the circumcenter of ABC if and only if the perimeter of $A_1B_1C_1$ is not less than any one of the perimeters of AB_1C_1, BC_1A_1, and CA_1B_1.

3. [IMO 03/3] A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

4. [IMO 96/5] Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that
$$R_A + R_C + R_E \geq \frac{P}{2}.$$

5. [IMO 02/6] Let n be an integer with $n \geq 3$. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be unit circles in the plane, with centers $O_1, O_2, \ldots, O_n$, respectively. If no line meets more than two of the circles, prove that
$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

6. [S 03] Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, and AB are drawn on the outside of triangle ABC. The circle tangent to all three semicircles has radius t. Prove that
$$\frac{s}{2} < t \leq \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

Easier Problems

1. Trig Ceva; Jensen.
2. Ceva and Law of Cosines, or projection.
3. Ptolemy.
4. Tangent trig identity; Cauchy or smoothing or quadratic formula.
5. Similar to previous.
6. Angle bisector theorem.
7. Vectors or Ptolemy.
8. Trig (various solutions).
9. Similar triangles; triangle inequality.

Not-as-easy Problems

1. Vectors.
2. Similar triangles; areas.
3. Fermat point equilateral triangle construction; arc midpoint property; triangle inequality.
4. One direction of the inequality is enough. Assume $\angle A + \angle C > \angle B + \angle D$. Then $\angle ABD > \angle ACD$. Also, $\angle CDB > \angle CAB$. Assuming all of these angles are acute, we have $R_C > R_B$ and $R_A > R_D$. If one of them is not, then do the same with the other possible pair of triangles and observe that the angles here must be acute.
5. Observe that $\triangle RBM \sim \triangle KBS$. Thus, $BM = BK$ is the geometric mean of BR and BS. On the other hand, $BI > BM, BK$ and is perpendicular to RS.
6. Draw rays from O to the vertices of F, decomposing the polygon into triangles (which may overlap). Rearrange the inequality we want as $2D \geq \sqrt{(2H)^2 + P^2}$ and write out D, H, and P as sums of lengths. Aligning corresponding lengths properly, this becomes exactly the (algebraic) triangle inequality—the "segments" being added don't correspond to any geometry from the original picture, though.
7. (Smoothing.) Lengthen the shortest of MA, MB, MC by moving M along that ray until the shortest two of MA, MB, MC are equal. Next, notice that as we move from a midpoint of a side to the circumcenter (along the perpendicular bisector), the derivative of our expression increases. Hence, it suffices to check that the inequality holds at the midpoints of the sides and at the circumcenter (if the triangle is acute). For the former, draw in a midline and use the triangle inequality twice. For the latter, unsmooth to show that $4R < 2R(\sin A + \sin B + \sin C)$.

Problems that Might Make You Uneasy

1.

2. The forward direction is obvious. For the reverse, let DEF be the medial triangle of ABC. Without loss of generality, assume $A_1 \in \overline{BD}$ and $B_1 \in \overline{AE}$. (A little thought will show that this can indeed be assumed.) Also assume that $C_1 \in \overline{AF}$. Then O lies in (or on) one of the triangles APF or APE, where P is the circumcenter of ABC. We claim that in either case, $P_{CA_1B_1} \geq P_{A_1B_1C_1}$, with equality holding only if $O = P$. It is easy to see that $A_1C \geq DC = FE \geq C_1B_1$, so it suffices to show that $B_1C \geq C_1A_1$.

   If O lies in APF, then $d(C_1, BC) \leq d(B_1, BC)$. Also, $\angle C_1A_1B \geq \angle C$, so $B_1C \geq C_1A_1$ as wanted. If O lies in APE, then $\angle B_1OC \geq \angle EPC = \angle B$. Thus, $B_1C = OC \sin \angle B_1OC \geq OC \sin \angle B \geq OB \sin \angle B = C_1A_1$. Consideration of the equality cases shows that equality is only achieved when $O = P$.

   Alternate solution (the official one, I believe): extend triangle CB_1A_1 to a parallelogram; if it contains triangle $A_1B_1C_1$, then we are done. Show by "walking around" the angles $\angle AB_1C_1$, etc., that one of these containments must hold (obtain a contradiction $AO > BO > CO > AO$).

3. Draw in the main diagonals of the hexagon. If the triangle formed in the center has an angle greater than 60 degrees, connect that vertex to the midpoints of its corresponding pair of opposite sides of the hexagon. Use the arc midpoint property and the triangle inequality to obtain a contradiction.

4. Emulate the proof of Erdős-Mordell (use projection).

5. It is enough to consider the case in which $O_1O_2 \ldots O_n$ is a convex polygon; centers inside the convex hull can then be added in iteratively. In the convex case, use $\tan\theta < \theta$ to convert ratios into angles, and by drawing in both internal and external tangents to pairs of circles, compare the half-angles formed by internal tangents to angles $\angle O_iO_jO_k$. In this way, obtain the inequality $\sum_{i \neq j} 4/O_iO_j - \sum_i 2/O_iO_{i+1} < (n-2)\pi$. Combine this with $\sum_i 4/O_iO_{i+1} < 2\pi$, proved similarly by considering exterior angles of the polygon, to obtain the desired result.

6. Solution due to Ricky Liu:

   Let a, b, and c be the side lengths of ABC, and let $A'B'C'$ be the medial triangle of ABC. Draw circles centered at A', B', and C' with radii $s/2 - a/2$, $s/2 - b/2$, and $s/2 - c/2$; these are externally tangent. Then the inner Soddy circle of $A'B'C'$ is the circle externally tangent to all three. Denote this circle by ω, its center by O, and its radius by x. Then the circle centered at O with radius $x + s/2$ is in fact the given circle, since the distance from O to A' is $x + s/2 - a/2$, and the semicircle centered at A' has radius $a/2$. Then $t - s/2 = x$, and clearly $x > 0$.

   Let Γ be the incircle of $A'B'C'$ with center I and radius $r' = r/2$. We wish to show that $x \leq (2 - \sqrt{3})r'$. Suppose $O = I$. Then the circles centered at A', B', and C' are each orthogonal to Γ and tangent to ω. They must therefore be congruent, and a straightforward calculation shows that in this case $x = (2 - \sqrt{3})r'$.

   Suppose O does not equal I. Choose point D such that inverting about D makes ω and Γ concentric with ω inside Γ. To see that this point exists, draw the line connecting O and I, and coordinatize it such that ω and Γ intersect the line at coordinates i, j, k, and l, with $i < j < k < l$ and $j - i > l - k$. Then consider the function $f(z) = (1/(z-l) - 1/(z-k))/(1/(z-j) - 1/(z-i))$. Then as z approaches l from above, $f(z)$ approaches infinity, and as z approaches infinity, $f(z)$ approaches $(l-k)/(j-i) < 1$, so we can choose D at $z > l$ such that $f(z) = 1$. Then $1/(z-i)+1/(z-l) = 1/(z-j)+1/(z-k)$, so the images of ω and Γ under inversion by D are concentric. Moreover, since $0 < 1/(z-i) < 1/(z-j) < 1/(z-k) < 1/(z-l)$, $(1/(z-i))(1/(z-l)) < (1/(z-j))(1/(z-k))$. This implies that

$$(1/(z-k)-1/(z-j))/(1/(z-l)-1/(z-i)) = (k-j)/(l-i)\cdot(z-i)(z-l)/(z-j)(z-k) > (k-j)/(l-i) = x/r'.$$

In other words, the ratio between the radius of ω and Γ increases after the inversion. Since the circles centered at A', B', and C' invert to circles orthogonal to the image of Γ and tangent to the image of ω, we have that the ratio after inversion is just $2-\sqrt{3}$ as above, which proves the result. Equality holds if and only if $O = I$, that is, if ABC is equilateral.