Hall's marriage theorem

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We write $N(W)$ to denote the neighborhood of W: the set of all vertices adjacent to some vertex of W. A matching in a graph is a set of edges without common vertices. A matching covers a set of vertices if it contains all the vertices in the set.

Theorem (Hall's marriage theorem)
Let G be a bipartite graph with bipartite sets X and Y. Then there exists a matching that covers X if and only if for each subset W of X, $|W| \leq |N(W)|$.

1 Tips

- Hall's marriage theorem can come up in expected and unexpected ways. The key is to look for two sets which we want to match with one another: finding these sets is often half of the problem.
- If a Hall's problem involves a table, we might be interested in choosing the bipartite sets as the rows and columns themselves. Then we might draw an edge if the entry in the table is a certain value.
- Since the marriage condition is necessary and sufficient, then the inverse of Hall's is true as well. If a matching does not exist, then there must be some subset of vertices that have a larger neighborhood.
- In some cases the sets we want to match is obvious, the trouble is proving that the marriage condition holds. Sometimes it's obvious. For the harder ones, we usually try contradiction: assume that a subset that doesn't satisfy the condition holds, then this has to lead to a contradiction.
- For the harder problems in this set, using Hall's theorem will probably not be the main idea in the proof, but rather only used as an intermediate step in the solution. This is a combinatorics handout, so try lots of things.

2 Problems

While some of the problems can be solved without Hall's, the hints assume use of Hall's theorem. Straightforward problems usually involve a clear application with a clear proof of the marriage condition, involved problems require a bit more ingenuity, while challenging problems are difficult enough to be some of the hardest olympiad problems. Problems are very roughly sorted by difficulty.

2.1 Straightforward

1. (Combinatorial Hall's) Let $S_1, S_2, \dots, S_m$ be sets. A transversal is an ordered set $(s_1, s_2, \dots, s_m)$ such that each of the s_i's are different and $s_i \in S_i$. Prove that there exists a transversal if and only if the union of any k sets has at least k elements.

2. Prove that if all the vertices of a bipartite graph have the same degree, then it has a perfect matching.

3. We have a regular deck of 52 playing cards, with exactly 4 cards of each of the 13 ranks. The cards have been randomly dealt into 13 piles, each with 4 cards in it. Prove that there is a way to take a card from each pile so that after we take a card from every pile, we have exactly a card of every rank.

   Prove that, in fact, we can go further: after taking a card of every rank, there are 3 cards left in each pile. We can then take a card of every rank once more, leaving 2 cards in each pile. Finally, we do it once more, and the remaining card in each pile must be of every rank.

4. (Kazakhstan 2003) We are given two square sheets of paper with area 2003. Suppose we divide each of these papers into 2003 polygons, each of area 1. (The divisions for the two piece of papers may be distinct.) Then we place the two sheets of paper directly on top of each other. Show that we can place 2003 pins on the pieces of paper so that all 4006 polygons have been pierced.

5. (Putnam 2012/B3 [1]) Suppose $2m$ teams play in a round-robin tournament. Over a period of $2m-1$ days, every team plays every other team exactly once. There are no ties. Show that for each day we can select a winning team, without selecting the same team twice.

6. A class of 100 is participating in an oral exam. The committee consists of 25 members. Each student is interviewed by one member of the committee. It is known that each student likes at least 10 committee members. Prove that we can arrange the exam schedule such that each student is interviewed by one of the committee members that he likes, and each committee member interviews at most 10 students.

2.2 Involved

1. (AMSP C3 2014) Let $n \in \{1, 2, \dots, 8\}$. Consider an 8×8 chessboard with the property that on each column and each row there are exactly n pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column.

2. Let G be a connected bipartite graph with bipartite sets X and Y. Suppose that for any two adjacent vertices $x \in X$ and $y \in Y$, we have $\deg(x) \geq \deg(y)$. Prove that G has a matching that covers X.

3. (Making Latin squares) We are given an $n \times k$ board with $k < n$ such that in every square there is a number from 1 to n. We know that in every column and every row no number repeats itself. Show that we can extend this to an $n \times n$

board with a number from 1 to n in each square, such that in every row and every column no number repeats itself.

4. (Baltic Way 2013/6 [2]) Santa Claus has at least n gifts for n children. For $i \in \{1, 2, ..., n\}$, the ith child considers $x_i > 0$ of these items to be desirable. Assume that
$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \le 1.$$
Prove that Santa Claus can give each child a gift that this child likes.

5. (IMC 2011/2/2b [3]) An alien race has three genders: male, female and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual – if x likes y, then y likes x.

   The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

   Show that if $k \ge 3n/4$, then it is always possible to create n disjoint married triples, thus marrying all of the expedition members.

6. (AMSP C3 2014 [4]) An $n \times n$ table is filled with 0 and 1 so that if we choose any n cells, no two of them on the same row or column, then at least one contains 1. Prove that we can find i rows and j columns so that $i + j \ge n + 1$ and their intersection contains only 1s.

2.3 Challenging

1. (Vietnam TST 2001/3 [5]) A club has 42 members. Suppose that for any 31 members in this club, there exists a boy and a girl among these 31 members who know each other. Show that we can form 12 disjoint pairs of people, each pair having one boy and one girl, such that the people in each pair know each other.

2. (Canada 2006/3 [6]) In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

3. (ISL 2010/C2 [7]) On some planet, there are 2^N countries, where $N \ge 4$. Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

   We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

4. (WOOT 2011) A table has m rows and n columns with $m, n > 1$. The following permutations of its mn elements are permitted: any permutation leaving each

element in the same row (a "horizontal move"), and any permutation leaving each element in the same column (a "vertical move"). Find the smallest integer k such that any permutation of the mn elements can be realized by at most k permitted moves.

5. (ISL 2012/C5 [8]) The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, ..., 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is $0, 1$ or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

   Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

6. (ISL 2006/C6 [9]) A holey triangle is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \leq k \leq n$.

3 Hints

3.1 Straightforward

1. Let the elements of X be the sets and the elements of Y be the elements of the union of all the sets.
2. First we show that X and Y have the same number of vertices. Hall's can then be used directly.
3. The elements of X are the thirteen ranks and the elements of Y are the thirteen piles.
4. The sets are the polygons in each sheet of paper: we only need to prove the marriage condition.
5. The sets are the $2m$ teams and the $2m-1$ days. Prove the marriage condition by contradiction.
6. Instead of considering the 25 commitee members, consider that each commitee member has 10 "time slots", to account for the fact each member interviews at most 10 students.

3.2 Involved

1. Let X be the rows and Y be the columns, and draw an edge between $x \in X$ and $y \in Y$ if a piece is on the intersection of x and y.
2. How can we use $\deg(x) \geq \deg(y)$ to prove the marriage condition?
3. This is an induction problem: we create an $n \times (k+1)$ board from an $n \times k$ board. How can finding a matching between rows and columns help us do this?
4. The obvious choice of sets works; prove the marriage condition by contradiction.
5. First marry males to females, then marry the male-female pairs to emales. Prove the marriage condition by contradiction.
6. Again, our sets are rows and columns, and an edge is drawn if the entry is 1. Prove the contrapositive using Hall's.

3.3 Challenging

1. The natural choice of boys and girls for sets, with an edge drawn between people who know each other, works. Prove the marriage condiiton by contradiction.
2. Match rows to columns to show $m \leq n$, then use the same argument (which should be symmetric!) to show $m \geq n$, these imply $m = n$.
3. The answer is $M = 2^{N-2} + 1$. To prove the upper bound, construct two graphs with the rows and columns as bipartite sets: one graph has an edge if the square is yellow, the other has an edge if the square is blue.

4. We only need three moves. The strategy is to get all the pieces in their proper rows in two moves so we can use the last move to place it in the permutation we want.

5. Without loss of generality, we only need to prove that the A-tokens can be moved to distinct A-squares that have distance at most $d+2$ from their original places. We need to match tokens to squares.

6. We're matching downward unit triangles to adjacent upward unit triangles. Sufficiency is equivalent to Hall's theorem, why? Prove necessity by contradiction: take the largest set W such that $|W| < |N(W)|$, and use the condition to construct a smaller such W.

References

[1] kskedlaya.org/putnam-archive/2012s.pdf

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[3] imc-math.org.uk/imc2011/imc2011-day2-solutions.pdf

[4] aops.com/community/c6h1154372

[5] aops.com/community/c6h42400

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[8] aops.com/community/c6h546172

[9] aops.com/community/c6h155700

If anyone knows the sources to any problem without one, see an error, have a correction, or have a question, please do not hestitate to contact me at cjquines0@gmail.com.