WOOT 2008-09

The Stupid Stuff Works—Geometry Edition

Richard Rusczyk

In an earlier article, I wrote about how the "stupid stuff" works so well in problem solving. Nowhere is this more true, and a more important lesson to learn, than in geometry.

A lot of students are very intimidated by geometry, and fall into one or both of the following traps when they see a hard problem:

1. Hard problem! Must find that one fancy theorem to solve it! Oh no! I know lots of fancy theorems but I *don't know how to make any of them work*!

2. Hard problem! There are a zillion lines I can draw and three zillion angles to chase! Information overload, shut-down sequence commencing immediately!

This is why I love hard geometry problems, and why I think mastering geometry teaches such important general skills. Both of these traps are traps I'm confronted with every day at AoPS. There are all sorts of things going on at AoPS, and all of them are very much like a hard geometry problem—there's no silver bullet to solve the problem, and there are a zillion different things I could try, but only a few that will actually work. Good, hard geometry problems teach you how to work in such an environment.

Here's what I've learned from fighting with hard geometry problems:

- **Check the fancy stuff at the door.** Very few students really know how to use inversion, isogonal conjugates, etc., and the ones who do *mastered the fundamentals first*. So, don't pull out these tools unless you know what you're doing—there won't be a problem on the USAMO that can be blown away with a blind use of these fancy tools. The same goes for advanced geometry theorems. When this is no longer true, then you know that National Math Olympiads have become a trivia competition, and you should treat them accordingly.

- **Work forwards and backwards.** Start from the given information and see what you can deduce. Don't do so blindly though—keep your eye on the ball, and work towards the desired result. But don't just work forwards. Work backwards, too. Start from the desired result, and work backwards to identify relationships that, if you could show they are true, would lead you to the desired result. I typically put my "forwards work" and "backwards work" on different pieces of paper for very hard problems. You don't want to get these mixed up, or you'll find yourself going in circles.

- **Get unstuck by focusing on unused facts.** You've been going at a complicated problem for an hour and feel hopelessly stuck. Stop for a minute, asses the information you've used so far working forwards and working backwards, choose the two paths in both that seem to come closest to meeting, and investigate both to see if there are any facts in the problem you haven't used yet. The missing link is likely related to the unused information.

- **Don't count on a brilliant insight to save you.** Work counts more than genius. Keep trying things; don't just sit there and hope the light shines down on you. It won't. Stop panicking, stop whining, and do something.

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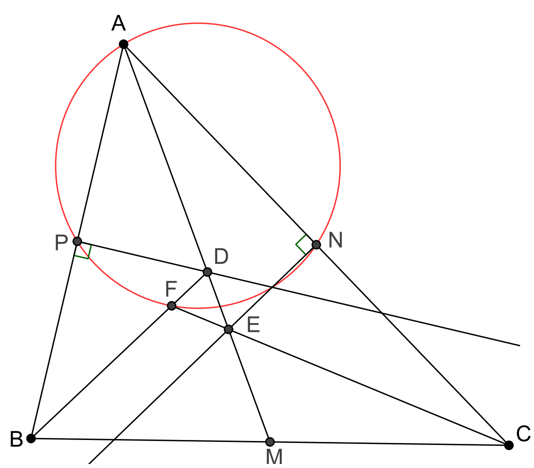
I've used these simple principles to solve lots and lots of problems. Even some geometry problems.

In recent years, Zuming Feng has served up some wonderful geometry problems for the USAMO. Problem #2 on the 2008 USAMO is a great example.

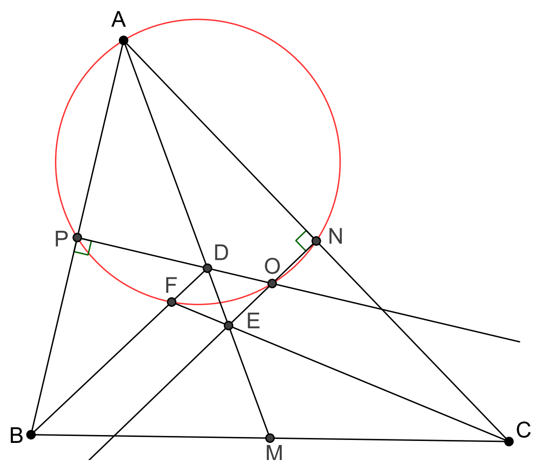
Problem: Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines $\overleftrightarrow{BD}$ and $\overleftrightarrow{CE}$ intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

We start with a diagram. We draw a large, precise diagram. (I recommend bringing extra-large pieces of paper for just this purpose, and drawing diagrams much larger than the one below.) Use a protractor, compass, and straightedge when constructing diagrams. Don't fudge it—a precise diagram will help you find relationships that a sloppy diagram would cause you to overlook.

We start by constructing the basic facts of the problem, and drawing in the alleged circle. Notice that we use a different color for the circle. This is *extremely important*. Circular reasoning is one of the hardest traps to avoid in geometry problems, particularly collinearity, concurrence, and concyclic problems. By drawing the circle in red, we hope to remember that we can never use the red circle as a step in our solution.



Drawing a precise diagram immediately shouts out our first step. It sure looks like $\overline{NE}$ and the extension of $\overline{PD}$ meet on the circle. Since these lines are perpendicular bisectors of the triangle, their intersection is the circumcenter of the triangle, so we label that point O. (You have to know these basic triangle points and lines!)



Next, we prove that O is on our alleged circle. It's not so clear how to link it to F, but tying it to P, A, and N is easy. Since $\angle APO = \angle ANO = 90^\circ$, we have $\angle APO + \angle ANO = 180^\circ$. This means that $APON$ is cyclic. Now, we can use the red circle, but only with points A, P, O, and N. We'll leave the circle red, and try to remember that we cannot use point F with this circle, since we want to prove that F is on the circle.

Finding that O is on the target circle gives us a natural next point for investigation, since it seems like this has to be important. Our goal is to link F and O in some way that allows us to use the fact that O is on the target circle to prove that F is, too.

So, we ask ourselves, "What's special about O?" We'll focus on angle relationships, since showing that a quadrilateral is cyclic usually (but not always!) means angle-chasing. The easiest angle relationships involving the circumcenter of a triangle are the angles formed by connecting the vertices to the circumcenter. We have $\angle AOB = 2\angle C$, $\angle BOC = 2\angle A$, and $\angle COA = 2\angle B$ in $\triangle ABC$. We can see these relationships by considering the circumcircle. Since $\angle BAC$ is inscribed in arc BC of the circumcircle, and $\angle BOC$ is the central angle of this arc, we have $\angle BAC = (\angle BOC)/2$, so $\angle BOC = 2\angle BAC$.

(As a sidenote, you don't need to memorize these angle relationships. They're easy to derive when you need them. Just know that the incenter, orthocenter, and circumcenter of a triangle lead to angles whose measures are easily related to angles of the triangle. These relationships can all be found pretty quickly when you need them, so no need to memorize them. Just know that they exist, so when you're angle-chasing in a problem involving these points, you think to investigate these relationships.)

What does all this have to do with F? We look at how F is related to the vertices of $\triangle ABC$. We look first at $\angle BFC$, since F is already connected to B and C in our diagram. We wish to relate this angle to the angles of $\triangle ABC$. Our first step is through $\triangle FBC$, which gives

$$\angle BFC = 180^\circ - \angle FBC - \angle FCB.$$

Neither of these angles is particularly enlightening, but we do have $\angle FBC = \angle ABC - \angle ABF$ and $\angle FCB =$

$\angle ACB - \angle ACF$. This gives us

$$\angle BFC = 180^\circ - \angle ABC - \angle ACB + \angle ABF + \angle ACF.$$

This is promising, because $180^\circ - \angle ABC - \angle ACB = \angle BAC$, so

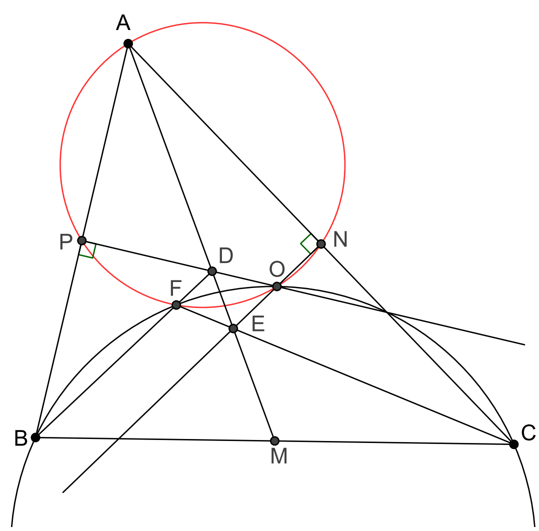
$$\angle BFC = \angle BAC + \angle ABF + \angle ACF.$$

Aha! We know that $\angle BOC = 2\angle BAC$, so if we can show that $\angle ABF + \angle ACF = \angle BAC$, then we'll have $\angle BFC = 2\angle BAC = \angle BOC$, which would mean that $BFOC$ is cyclic. That might be very useful indeed! (Notice here that we're working both backwards and forwards, keeping an eye on useful results we'd like to prove.)

We now want to relate $\angle ABF$ and $\angle ACF$ to $\angle A$. If we don't see immediately how to do it, we might think, "What is special about these angles?" (This is often a question I ask myself when focusing on a particular piece of a geometry problem.) Sides $\overline{AB}$ and $\overline{AC}$ of these angles aren't terribly interesting, but sides $\overline{BF}$ and $\overline{CF}$ are revealing—both are defined in terms of points on the median $\overline{AM}$ and on a perpendicular bisector of a side of the triangle (such as point D on the perpendicular bisector of $\overline{AB}$). Perpendicular bisectors are much more fruitful than medians for chasing angles, so we think about what it means that D is on the perpendicular bisector through P. The answer jumps out immediately: point D is equidistant from A and B, so $\triangle PAD \cong \triangle PBD$, which means $\angle ABF = \angle ABD = \angle BAD$. Similarly, we have $\angle ACF = \angle ACE = \angle CAE$, so we have

$$\angle BFC = \angle BAC + \angle ABF + \angle ACF = \angle BAC + \angle BAD + \angle CAE = 2\angle BAC.$$

Therefore, quadrilateral $BFOC$ is indeed cyclic.



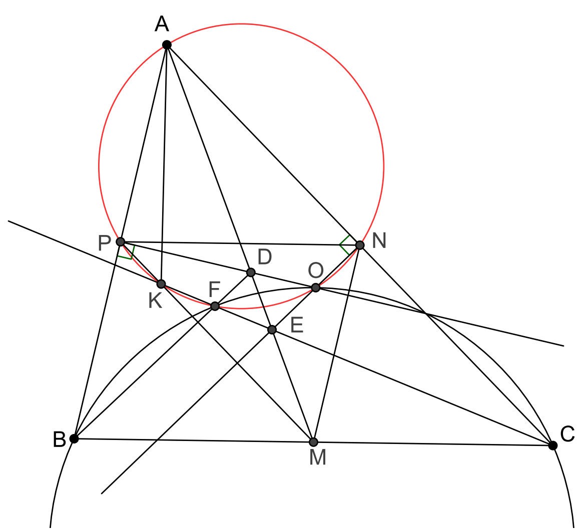
Sweet! We must be just a couple steps away now.

But what are those steps?

When working on the problem, I spent longer than I should have right here. I made a long list of observations (this is a very rich diagram, and there are all sorts of cool relationships). But none of them solved the problem. Finally, I stopped looking blindly, put my pencil down and tried to find a systematic strategy to solve the problem, rather than making blind observations that I thought would be helpful.

After failing to find a way to use the circumcircle of $BFOC$ to show that F is on the target circle, I decided to go hunting for another point to use to try to tie F and O together. So, I went hunting for other potentially important points on the red circle. I didn't see any immediate candidates in the diagram, so I reached for my favorite question when stuck on geometry problems: "What haven't I used yet?" An immediate answer popped up: I hadn't touched the fact that $\overline{AM}$ is a median. So, I started thinking about what was important about the fact that M is the midpoint of $\overline{BC}$, which naturally led to the observation that all three midpoints of the sides of $\triangle ABC$ are in the problem, so maybe the medial triangle (the triangle formed by connecting the midpoints of the sides of the triangle) would be helpful. This wasn't just grasping at straws—I had not used the midpoint M yet, and the medial triangle gives parallel lines, which are enormously helpful for angle-chasing.

When I drew $\overline{MP}$, the problem fell apart. (And it also showed how important it was to have such a precise diagram!)



The line appears to meet the target circle at the exact same point as where $\overleftrightarrow{CF}$ intersects $\overleftrightarrow{MP}$. This can't be a coincidence!

First, we must show that these two lines do indeed meet on the circle. Let point K be the intersection of lines $\overleftrightarrow{CF}$ and $\overleftrightarrow{MP}$. The most obvious angle to focus on at point K is $\angle AKP$. If K is on the target circle, we must have $\angle AKP = \angle ANP$. We have $\angle ANP = \angle ACB$ because $\overline{NP} \parallel \overline{BC}$. So, now we just have to show that $\angle AKP = \angle ACB$. It might seem we just have yet another angle-chasing problem that we could pursue in a million different ways, but we do have something to focus on: we think we have to use the fact that M is the midpoint of $\overline{BC}$. This leads us to note that $\angle AKP = 180^\circ - \angle AKM$, which has us looking at quadrilateral $AKMC$, since we want to relate these angles to $\angle ACB$. Quadrilateral $AKMC$ is a trapezoid, since $\overline{MP} \parallel \overline{AC}$. (We're particularly excited about this observation, because it uses the fact that M is the midpoint of $\overline{BC}$!)

We want to show that $\angle AKM = 180^\circ - \angle ACB$, since this gives us $\angle AKP = \angle ACB$ and puts K on the target circle. But $\angle AKM = 180^\circ - \angle ACB$ if and only if $AKMC$ is an isosceles trapezoid. Is it? We look for what's special about this trapezoid, and we see that perpendicular bisector of base $\overline{AC}$ runs right through the intersection of the diagonals. Aha! Maybe that can only happen in an isosceles trapezoid! The perpendicular bisector gives us $\triangle AEN \cong \triangle CEN$, so $\angle EAN = \angle ECN$ and $AE = EC$. From the equal angles, we have $\angle EKM = \angle ECN = \angle EAN = \angle EMK$, so $EK = EM$, which gives us $AM = AE + EM = CE + EK = CK$, so the diagonals of trapezoid $AKMC$ are equal, which means it is indeed an isosceles trapezoid.

From isosceles trapezoid $AKMC$, we have $\angle AKP = 180^\circ - \angle AKM = \angle ACM = \angle ANP$, and K is indeed on our target circle. So, now we just have to tie K to F.

But how? Well, again, we ask ourselves, what haven't we used? Answer: we found that $BFOC$ is cyclic, but we haven't touched that circle. So, we'll probably use that. We look at $\angle KFO$, since that will allow us to tie K to that circle, and to the target circle. To show that F is on the target circle, we must show that

$$\angle KFO + \angle KAO = 180^\circ.$$

Before we go after $\angle KFO$, we check that we can do something useful with $\angle KAO$. We try to relate it to angles in $\triangle ABC$, since we know we can relate these angles to angles in so many parts of the diagram.

We have $\angle KAN = \angle AKP = \angle ACB$, and from isosceles triangle OAC, we have

$$\angle OAC = (180^\circ - \angle AOC)/2 = 90^\circ - \angle ABC,$$

since $\angle AOC = 2\angle ABC$. Therefore,

$$\angle KAO = \angle KAN - \angle OAN = \angle ACB + \angle ABC - 90^\circ.$$

We might think, uh-oh, what if this comes out to be negative? Well, $\angle ACB + \angle ABC < 90^\circ$ only if $\angle BAC$ is greater than 90°. But the problem tells us that the triangle is acute. This gives us a lot of confidence that we're on the right track, since it allows us to see why the triangle had to be acute in the problem.

So, now we want to express $\angle KFO$ in terms of angles in $\triangle ABC$. We can't use the target circle, but we can use the other lines at F. We have

$$\angle KFO = 360^\circ - \angle KFB - \angle OFB.$$

From cyclic quadrilateral $OFBC$, we have

$$\begin{aligned}\angle OFB &= 180^\circ - \angle OCB \\ &= 180^\circ - (180^\circ - \angle BOC)/2 \\ &= 180^\circ - (180^\circ - 2\angle BAC)/2 \\ &= 90^\circ + \angle BAC.\end{aligned}$$

Finally, we have $\angle KFB = 180^\circ - \angle BFC = 180^\circ - \angle BOC = 180^\circ - 2\angle BAC$, so

$$\begin{aligned}\angle KFO &= 360^\circ - \angle KFB - \angle OFB \\ &= 360^\circ - (180^\circ - 2\angle BAC) - (90^\circ + \angle BAC) \\ &= 90^\circ + \angle BAC\end{aligned}$$

Putting this together with $\angle KAO$, we have

$$\angle KFO + \angle KAO = (90^\circ + \angle BAC) + (\angle ACB + \angle ABC - 90^\circ = \angle BAC + \angle ACB + \angle ABC.$$

Oh yeah, that's 180°, so $KAOF$ is cyclic, and we're done!

On the AoPSWiki, you'll find several solutions to this problem. Many use somewhat fancy tools, or extremely fancy tools. The first solution, like this one, uses nothing but elementary tools. Before reading that solution, I'll give you a few hints, and you can try to find it yourself.

Hint 1: Instead of trying to find an extra point to tie F and O together, try to find another angle. A natural target to try to show that F is on the target circle is to prove that $\angle FPO = \angle FNO$. Try to find another angle in the diagram that equals these angles. Rather than going after this step geometrically, do it as a problem solver. In other words, "cheat" and work backwards. Here again is where a precise diagram is critical. Break out the protractor and start measuring angles. Find one that is equal to these two. Then, try to prove that it is. This is considerably harder, and you might want to keep that protractor handy to help guide you to other equal angles.

Hint 2: The circles aren't the only way to prove angles are equal. Similar triangles are a very powerful tool for pursuing angle equalities. Use that protractor to find some equal angles that might surprise you, then go hunting for similar triangles to tie everything together.

Hint 3: Use our first solution as inspiration—specifically, the answers to the "What have we not used yet?" questions. (M is a midpoint, $BFOC$ is cyclic.)

Happy hunting, and special thanks to AoPSer tjhance, to whom this latter solution is credited in the AoPSWiki.