# Geometric Transformations

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All of the problems below involve or can be solved using one or more of the four classical geometric transformations: translation, reflection, rotation, and dilation.

1. Let $ABCD$ be a quadrilateral, and let $P$ be the midpoint of $AB$. Prove that the area of triangle $PCD$ is at least half the area of $ABCD$. When does equality hold?

2. Let $ABC$ be a triangle with medians $AD$ and $BE$. Prove that if $\angle DAC = \angle CBE = 30°$, then $ABC$ is equilateral.

3. Let $ABCD$ be a quadrilateral, and let $K, L, M$, and $N$ be the midpoints of $AB, BC, CD$, and $DA$, respectively. Given $a = AB$, $b = BC$, $c = CD$, and $d = DA$, what are the maximum possible values of $KM$ and $LN$?

4. Let $ABC$ be a triangle with incenter $I$. The incircle of $ABC$ touches side $BC$ at $D$. Show that the line connecting the midpoints of $AD$ and $BC$ passes through $I$.

5. Let $ABC$ be a triangle, and let $P, Q$, and $R$ lie on sides $BC$, $CA$, and $AB$, respectively. Show that the circumcenters of $AQR$, $BRP$, and $CPQ$ form a triangle similar to $ABC$.

6. Suppose that a parabola is tangent to the three sides of a triangle. Show that its focus lies on the circumcircle of the triangle.

7. (Miquel) Four lines intersect to form four triangles. Show that the circumcircles of these triangles pass through a common point. Furthermore, show that the centers of these circumcircles lie on a circle containing this point.

8. (Monge) Given a tetrahedron, show that the six planes passing through the midpoint of an edge perpendicular to the opposite edge have a common point.

9. Let $ABC$ and $PQR$ be triangles. Construct points $P'$, $Q'$, and $R'$ on the sides of $ABC$ such that $P'Q'R'$ is congruent to $PQR$.

10. Given a point $A$ and two circles $\omega_1$ and $\omega_2$, construct a line through $A$ such that the chords cut out on $\omega_1$ and $\omega_2$ have equal lengths.

11. Let $X$ and $Y$ lie in the interior of acute angle $AOB$. Construct $Z$ on ray $OA$ such that lines $ZX$ and $ZY$ form two sides of an isosceles triangle with base along $OB$.

12. Construct a quadrilateral given its angles and a) two opposite sides; b) its diagonals.

13. Given a triangle $ABC$, construct a line dividing both its perimeter and its area in half.

14. Let $ABC$ be a triangle. Let $D$, $E$, and $F$ be the centers of the spiral similarities that take $CA$ to $AB$, $AB$ to $BC$, and $BC$ to $CA$, respectively. Show that the circumcircle of $DEF$ passes through the circumcenter of $ABC$.

15. (IMO 1999) Two circles $\omega_1$ and $\omega_2$ are contained inside the circle $\omega$ and are tangent to $\omega$ at distinct points $M$ and $N$, respectively, such that $\omega_1$ passes through the center of $\omega_2$. The line passing through the two points of intersection of $\omega_1$ and $\omega_2$ meets $\omega$ at $A$ and $B$. The lines $MA$ and $MB$ meet $\omega_1$ at $C$ and $D$, respectively. Prove that $CD$ is tangent to $\omega_2$.

16. (IMO 2000) Let $AH_1$, $BH_2$, and $CH_3$ be the altitudes of acute triangle $ABC$. The incircle of $ABC$ touches the sides $BC$, $CA$, and $AB$ at $T_1$, $T_2$, and $T_3$, respectively. Let the lines $\ell_1$, $\ell_2$, and $\ell_3$ be the reflections of $H_2H_3$, $H_3H_1$, and $H_1H_2$ in $T_2T_3$, $T_3T_1$, and $T_1T_2$, respectively. Prove that $\ell_1$, $\ell_2$, and $\ell_3$ determine a triangle with vertices lie on the incircle of $ABC$.