Some Latin geometry problems

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1. (Brazil 2009) Let ABC be a triangle and O its circumcenter. Lines AB and AC meet the circumcircle of OBC again in $B_1 \neq B$ and $C_1 \neq C$, respectively, lines BA and BC meet the circumcircle of OAC again in $A_2 \neq A$ and $C_2 \neq C$, respectively, and lines CA and CB meet the circumcircle of OAB in $A_3 \neq A$ and $B_3 \neq B$, respectively. Prove that lines A_2A_3, B_1B_3 and C_1C_2 have a common point.

2. (Brazil 2010) Let $ABCD$ be a convex quadrilateral with $\angle B \neq 90^\circ$, and M and N be the midpoints of the sides CD and AD, respectively. The lines perpendicular to AB passing through M and to BC passing through N intersect at point P. Prove that P lies on the diagonal BD if and only if the diagonals AC and BD are perpendicular.

3. (Brazil 2006) Let P be a convex 2006-gon. The 1003 diagonals connecting opposite vertices and the 1003 lines connecting the midpoints of opposite sides are concurrent, that is, all 2006 lines have a common point. Prove that the opposite sides of P are parallel and congruent.

4. (Brazil 2003) Let $ABCD$ be a rhombus. Let E, F, G, H be points on sides AB, BC, CD, DA respectively so that EF and GH are tangent to the incircle of $ABCD$. Show that EH and FG are parallel.

5. (Brazil 2004) Let $ABCD$ be a convex quadrilateral. Prove that the incircles of the triangles ABC, BCD, CDA and DAB have a point in common if, and only if, $ABCD$ is a rhombus.

6. (Peru Cono Sur TST 2008) Let $ABCDEF$ be a convex hexagon such that $\angle FAB = \angle CDE = 90^\circ$ and the quadrilateral $BCEF$ has an inscribed circle. Prove that $AD \leq BC + FE$.

7. (Iberoamerican 2003) In a square $ABCD$, let P and Q be points on the sides BC and CD respectively, different from its endpoints, such that $BP = CQ$. Consider points X and Y such that $X \neq Y$, in the segments AP and AQ respectively. Show that, for every X and Y chosen, there exists a triangle whose sides have lengths BX, XY and DY.

8. (Iberoamerican 1998) The incircle of the triangle ABC touches sides BC, CA and AB at D, E and F, respectively. Line AD intersect the incircle at Q. Show that the line EQ meets the segment AF at its midpoint if and only if $AC = BC$.

9. (Iberoamerican 2008) Let ABC be a triangle and X, Y and Z be points on sides BC, CA and AB, respectively. Let A', B' and C' be the circumcenters of triangles AZY, BXZ, and CYX, respectively. Prove that $4[A'B'C'] \geq [ABC]$ with equality if and only if AA', BB' and CC' meet at a single point.

10. (Olympic Revenge 2007) The triangles BCD and ACE are externally constructed to sides BC and CA of a triangle ABC such that $AE = BD$ and $\angle BDC + \angle AEC = 180^\circ$. Let F be a point on segment AB such that $\frac{AF}{FB} = \frac{CD}{CE}$. Prove that $\frac{DE}{CD+CE} = \frac{EF}{BC} = \frac{FD}{AC}$.

11. (Peru Cono Sur TST 2006) Let AA_1 and BB_1 be altitudes of the acute triangle ABC (A_1 on BC, B_1 on AC) and A_0 and B_0 be the midpoints of sides BC and AC, respectively. Segments A_0B_0 and A_1B_1 meet at C'. Prove that CC' is perpendicular to the line connecting the orthocenter and the circumcenter of triangle ABC.

12. (Peru TST 2006) Let ABC be an acute triangle, ω be its circumcircle, O be its circumcenter, Γ be the circumcircle of triangle AOC and OQ be a diameter of Γ. Let M and N be points on lines AQ and AC, respectively, such that $AMBN$ is a parallelogram.

    Prove that the intersection of lines MN and BQ lies on Γ.

13. (Peru Cono Sur TST 2008) Given a triangle ABC, let P and Q be points on sides AB and AC, respectively, such that PQ is parallel to BC. Let M be the midpoint of BC and X be the orthogonal projection of Q onto PM. Prove that $\angle AXQ = \angle QXC$.

14. (Iberoamerican Shortlist 2010, Brazil TST 2011) Two circles C_1 and C_2, with centers O_1 and O_2 respectively, intersect at points A and B. Let X and Y be points on C_1 distinct from A and B. Lines XA and YA meet C_2 again at Z and W, respectively.

    Let M be the midpoint of O_1O_2, S be the midpoint of XA and T be the midpoint of WA. Prove that $MS = MT$ if and only if X, Y, Z and W lie on the same circle.

15. (Iberoamerican Shortlist 2010) Let ABC be an acute triangle and AA_1, BB_1 and CC_1 be its altitudes (A_1, B_1, C_1 lying on the sides). Let A_2 be a point on segment AA_1 such that $\angle BA_2C = 90°$; define B_2 and C_2 in the same fashion. Lines B_2C and BC_2 meet at A_3; define B_3 and C_3 in an analogous way. Prove that A_2A_3, B_2B_3 and C_2C_3 have a point in common.

16. (Peru Cono Sur TST 2010) Let ABC be an acute triangle. Let M and N be points on sides AC and AB, respectively. Line segments BM and CN meet at P and Q is a point inside quadrilateral $ANPM$ such that $\angle BQC = 90°$ and $\angle BQP = \angle BMQ$. Given that $ANPM$ is a cyclic quadrilateral, prove that $\angle QNC = \angle PQC$.

17. (Iberoamerican 2009) Let C_1 and C_2 be two congruent circles centered at O_1 and O_2, which intersect at A and B. Take a point P on the arc AB of C_2 which is contained in C_1. AP meets C_1 again at C, CB meets C_2 again at D and the bisector of $\angle CAD$ intersects C_1 and C_2 again at E and L, respectively. Let F be the symmetric point of D with respect to the midpoint of PE. Prove that there exists a point X satisfying $\angle XFL = \angle XDC = 30°$ and $CX = O_1O_2$.

18. (Peru Cono Sur TST 2006) Let ABC be a triangle with $AC \neq BC$. The excircles touch side BC at A_1 and side AC at B_1. Let I be the incenter of ABC. Line CI meets the circumcircle of ABC at P and the line perpendicular to CP passing through P meets line AB at Q. Prove that lines IQ and A_1B_1 are parallel.

19. (Iberoamerican 2006) Let ω be the incircle of circumscribed quadrilateral $ABCD$. Line BD intersects ω at X and Y. Let M be the midpoint of XY. Sides DA and DC touch ω at P and Q, respectively. Prove that $\angle AMP = \angle CMQ$.

20. (Olympic Revenge 2011) Let $ABCD$ be a quadrilateral inscribed in a circle Γ. Lines r and s touch Γ at B and C, respectively, and meet line AD at M and N, respectively. Lines BN and CM meet at E, lines AE and BC meet at F and L is the midpoint of BC. Prove that the circumcircle of triangle DLF is tangent to Γ.

21. (Olympic Revenge 2010) Let ABC be a fixed triangle. Let A', B' and C' be variable points such that, at every instant, A' moves countinuously and with fixed velocity v towards the point A'' such that $A''B'C'$ is similar (in this order) to ABC, B' moves countinuously and with fixed velocity v towards the point B'' such that $A'B''C'$ is similar (in this order) to ABC and C' moves countinuously and with fixed velocity v towards the point C'' such that $A'B'C''$ is similar (in this order) to ABC. Does there exist an instant such that, at this exact moment, A', B' and C' are the vertices of a triangle similar to ABC?

22. (Olympic Revenge 2009) Let ABC be a triangle and I its incenter. Let ω_A, ω_B and ω_C be the incircles of BIC, CIA and AIB, respectively. Circle ω_A touches BC at T. Prove that the common internal tangent line to ω_B and ω_C, different from AI, contains T.

23. (Iberoamerican 2007) Two teams, A and B, fight for a territory limited by a circumference.

    A has n blue flags and B has n white flags ($n \geq 2$ fixed). They play alternatively and A begins the game. Each team, in its turn, places one of his flags in a point of the circumference that has not been used in a previous play. Each flag, once placed, cannot be moved.

    Once all $2n$ flags have been placed, territory is divided between the two teams. A point of the territory belongs to A if the closest flag to it is blue, and it belongs to B if the closest flag to it is white. If the closest blue flag to a point is at the same distance than the closest white flag to that point, the point is neutral (not from A nor from B). A team wins the game is their points cover a greater area that that covered by the points of the other team. If both cover equal areas we consider the game a draw.

    Prove that, for every n, team B has a winning strategy.

24. (Iberoamerican 2007) Let $\mathcal{F}$ be a family of convex hexagons H satisfying the following properties:

    (i) H has parallel opposite sides.

    (ii) Any three vertices of H can be covered with a strip of width 1.

    Determine the least $\ell \in \mathbb{R}$ such that every hexagon belonging to $\mathcal{F}$ can be covered with a strip of width ℓ.

    Note: A strip of width ℓ is the region bounded by two parallel lines separated by a distance ℓ. The lines belong to the strip, too.

25. (Olympic Revenge 2007, IOI) Let $A_1A_2B_1B_2$ be a convex quadrilateral. At adjacent vertices A_1 and A_2 there are two Argentinian cities. At adjacent vertices B_1 and B_2 there are two Brazilian cities. There are a Argentinian cities and b Brazilian cities inside the quadrilateral, no three of which collinear. Determine if it's possible, independently from the cities position, to build straight roads, each of which connects two Argentinian cities ou two Brazilian cities, such that:

    - Two roads does not intersect in a point which is not a city;
    - It's possible to reach any Argentinian city from any Argentinian city using the roads; and
    - It's possible to reach any Brazilian city from any Brazilian city using the roads.

    If it's always possible, construct an algorithm that builds a possible set of roads.

26. (Olympic Revenge 2011) Let E be an infinite set of congruent ellipses in the plane and r be a fixed line. It is known that every line parallel to r intersects at least one ellipse from E. Prove that there exist infinite triples of distinct ellipses from E such that there is a line that intersects all three of the ellipses in the triple.