Enumerative Combinatorics - Creative Counting

Combinatorics problems often ask you to find the number of elements in a possibly large, but finite set. For example, how many functions $f\colon \{1,2,3\} \to \{1,2,3,4\}$ are strictly increasing?

In all of these our main strategy is to divide and conquer. That can be done by writing the set as a union or a Cartesian product of other sets and in that case you can use the principle of inclusion-exclusion to get a sum of products. In these notes we will look at another powerful way of breaking up a problem: by recursion.

Recurrence Relations

Let (a_i) for $i \in \mathbb{N}$ be a sequence indexed by the natural numbers. A *recurrence relation* for this sequence is given by a function f such that

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-p}, n)$$

for some constant p (independent of n). It is easy to see that for any set of the *initial values* $a_1, a_2, \ldots, a_p$ this relation determines any other value a_n. This is also true for any other collection of values $a_{i_1}, \ldots, a_{i_p}$, but it is much harder to find the other values in that case, and some of the methods described in these notes may help.

Here is a set of counting problems that can be modeled using a recurrence relation.

(1) Let S be a set containing n elements. A *partition of S into k subsets* is a collection $A_1, A_2, \ldots, A_k$ of non-empty disjoint subsets of S such that $S = \bigcup_{i=1}^{k} A_i$. The number of ways in which you can do this is called $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, the Stirling numbers of the second kind. Find a recurrence relation for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$. (Note that this is a function in two variables - there will be more about this type of recurrence relations in the section on generating functions.)

(2) Let B_n be the total number of partitions of a set of n elements. (These numbers are called the Bell numbers, after Eric Temple Bell (1883-1960).) Find a recurrence relation for B_n.

(3) A mail carrier has one piece of mail for each house in a street with n houses and wants to deliver the mail in such a way that each mailbox receives one piece of mail, but no mail gets delivered to the correct address. Let D_n be the number of ways in which he can do this. (The D_n are called the *derangement numbers*.) Find a recurrence relation for the D_n.

(4) The *Tower of Hanoi*. There are three diamond needles and n gold disks of different sizes are placed on the first needle in order so that the largest disk is on the bottom and each disk is only above disks of a larger size. T_n is the number of moves required to move the disks to the third needle (with the use of the second one) such that in no stage of the process a disk is placed on a needle with a disk of smaller size below it. Find a recurrence relation for T_n.

(5) A sequence in which every term is one of m symbols is called an m-ary sequence. Find a recurrence relation for the number a_n of binary sequences of length n that have no consecutive zeros.

(6) Find a recurrence relation for the number b_n of ternary sequences of length n containing a 2 such that there are no zeros after it in the sequence.
(7) Find a recurrence relation for the number c_n of ternary sequences of length n with no consecutive zeros.
(8) Cars are parked in line as they come off 3 different assembly lines, ready for later road testing. Today's production produces 3 models, each of a single colour. Each red car takes 2 spaces, each blue car takes 2 spaces, and each green car takes only 1 space. Find a recurrence relation for the number q_n of ways of filling the first n parking spaces.
(9) Find a recurrence relation for the number e_n of comparisons that must be made between pairs of numbers in order to determine the maximum and the minimum of a set of 2^n distinct real numbers.
(10) Let $a_{r,n}$ be the number of ways to distribute r balls into n distinguishable boxes with $2, 3$, or 4 balls in each box if there are at least $4n$ red balls, $4n$ white balls, and $4n$ blue balls available.

Homogeneous Linear Recurrence Relations. A recurrence relation is homogeneous and linear if it is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{p-1} a_{n-p+1} + c_p a_{n-p} \tag{1}$$

for $n \geq p$, and constants c_i. Suppose that $c_p \neq 0$. To solve a relation of this kind, we first look for solutions of the form $a_k = \lambda^k$. Substituting this into our equation (1) we get a polynomial equation

$$\lambda^p = c_1 \lambda^{p-1} + c_2 \lambda^{p-2} + \cdots + c_{p-1} \lambda + c_p,$$

which we call the *characteristic equation* of (1).

It is easy to see that any linear combination of solutions to (1) is again a solution. So if $\lambda_1, \lambda_2, \ldots, \lambda_p$ are roots of the characteristic equation, any linear combination $a_k = r_1 \lambda_1^k + r_2 \lambda_2^k + \cdots + r_p \lambda_p^k$ is a solution of the equation (1). If all the roots $\lambda_1, \ldots, \lambda_r$ are distinct, all solutions of (1) are of this form. (However, even though we are working with sequences of integer numbers, don't be surprised if you obtain roots that are not integers, and may even be complex numbers.)

If there are repeated roots, say root λ_i has multiplicity m_i (this means that m_i is the highest power of $(\lambda - \lambda_i)$ that divides the characteristic equation), then $k\lambda_i^k$, $k^2 \lambda_i^k, \ldots, k^{m_i - 1} \lambda_i^k$ are also solutions of (1).

Example 1. Consider $a_n = 3a_{n-1} - 4a_{n-3}$ for $n \geq 3$ with $a_0 = 0$, $a_1 = 2$, and $a_2 = -1$. Find a closed formula for a_n.

Nonhomogeneous Linear Recurrence Relations. Sometimes you may need to consider recurrence relations of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{p-1} a_{n-p+1} + c_p a_{n-p} + g(n).$$

Note that if we have two different solutions of this equation, their difference will be a solution of the corresponding homogeneous system:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{p-1} a_{n-p+1} + c_p a_{n-p}.$$

So if we have the general solution of the homogeneous system, and a special solution of the nonhomogeneous system, their sum will give us the general solution for the nonhomogeneous system. So the question is how to find a special solution for the nonhomogeneous equation.

If $g(n)$ is a polynomial of degree m, you can find a special solution of degree m, i.e. $a_n = d_m n^m + \cdots d_1 n + d_0$. However, if the equation is $a_n = a_{n-1} + g(n)$, you need to take a polynomial of degree $m+1$ for your special solution.

If $g(n) = x \cdot y^n$, a special solution of the form $a_n = k \cdot y^n$ usually works. It doesn't work if y is a root of the characteristic equation of the corresponding homogeneous system. If y is a root of multiplicity r, a special solution of the form $kn^r y^n$ exists.

If $g(n)$ is a sum of a polynomial and an exponential function in n, you can use both techniques separately, and add the two special functions. For more complicated $g(n)$ and non-linear recurrence relations, we may not always be able to find a closed formula, but the technique of generating functions can be very helpful.

Exercises 2.

(1) Give a closed formula for each of the following recurrence relations with their initial conditions.
   (a) $a_n = a_{n-1} + 6a_{n-2} + 3n$ when $a_0 = a_1 = 0$.
   (b) $a_n = a_{n-1} + 6a_{n-2} + 2^n$ when $a_0 = a_1 = 0$.
   (c) $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ when $a_0 = a_1 = 1$.

(2) Show that the total number of triangles in an equilateral triangle of side n tiled by equilateral triangles of side 1 is given by the formula

$$\frac{4n^3 + 10n^2 + 4n - 1 + (-1)^n}{16}.$$

Generating Functions

A different way to reason about a sequence (a_n) for $n \geq 0$, is by considering its *generating function*

$$G_{(a_n)}(x) = \sum_{n=0}^{\infty} a_n x^n.$$

This type of function is a *formal power series*. You may be worried that $G(x)$ may not be defined for all values of x, but for our purposes that generally won't matter. We view formal power series only as algebraic gadgets, they are just clotheslines to hang our sequences on. We will perform various algebraic operations on the formal power series and will generally just be interested in their effect on the coefficients (the clothes on the clothesline). So we introduce the notation $[x^n]F(x)$ for the coefficient of x^n in the formal power series $F(x)$. In particular, $a_n = [x^n]G_{(a_n)}(x)$.

To get an idea about how generating functions may help us, let us consider a simple example of a linear recurrence relation, $a_{n+1} = 2a_n + 1$ for $n \geq 0$ with $a_0 = 0$. Take a moment to see whether you can quickly guess the solution for the closed form formula of this sequence.

Now let's see what the generating function can tell us. We do this by multiplying the recurrence relation by x^n and summing over all values of n for which it is valid, and then we try to relate both sides to the generating function $G(x)$. So we obtain

$$\sum_{n\geq 0} a_{n+1}x^n = \sum_{n\geq 0} 2a_n x^n + \sum_{n\geq 0} x^n.$$

The left-hand side is $\sum_{n\geq 0} a_{n+1}x^n = \frac{G(x)-a_0}{x} = \frac{G(x)}{x}$, since $a_0 = 0$ in this problem. The right hand side is $\sum_{n\geq 0} 2a_n x^n + \sum_{n\geq 0} x^n = 2G(x) + \sum_{n\geq 0} x^n$. So we have that $\frac{G(x)}{x} = 2G(x) + \sum_{n\geq 0} x^n$, and therefore: $\frac{(1-2x)G(x)}{x} = \sum_{n\geq 0} x^n$ and $G(x) = \frac{x}{1-2x} \sum_{n\geq 0} x^n$.

At this point it is convenient to know that

$$\sum_{n\geq 0} x^n = \frac{1}{1-x}. \tag{2}$$

This gives us that $G(x) = \frac{x}{(1-2x)(1-x)}$. The geometric series equation (2) implies that $\sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x}$, and it is easier to add two formal power series than to multiply them, so we use partial fractions to turn our product into a sum of fractions: $\frac{x}{(1-2x)(1-x)} = x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right)$. So

$$\begin{aligned} G(x) &= x\left(2\sum_{n\geq 0} 2^n x^n - \sum_{n\geq 0} x^n\right) \\ &= (2x + 2^2x^2 + 2^3x^3 + \cdots) - (x + x^2 + x^3 + \cdots) \\ &= (2-1)x + (2^2-1)x^2 + (2^3-1)x^3 + \cdots \end{aligned}$$

and it is clear that $a_n = 2^n - 1$ for each $n \geq 0$.

Some Useful Power Series. As we saw in our first problem already, it is convenient to be able to recognize the power series expansions of certain functions. First,

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n$$

which implies that

$$\frac{1}{1-ax} = \sum_{n\geq 0} a^n x^n$$

for any non-zero constant a. Using derivatives (or Taylor series), we can derive that

$$\frac{1}{(1-x)^{k+1}} = \sum_n \binom{n+k}{n} x^n$$

Also, using Taylor series we can show for any positive real number α that

$$(1+x)^\alpha = \sum_k \binom{\alpha}{k} x^k$$

where $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$. Four more Taylor series results:

$$\ln\frac{1}{1-x} = \sum_{n\geq 1} \frac{x^n}{n}$$

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{(2n)!}$$

Exercises 3. (1) Show that $\frac{1}{\sqrt{1-4x}} = \sum_k \binom{2k}{k} x^k$.

(2) Show that $\frac{1}{2x}(1 - \sqrt{1-4x}) = \sum_n \frac{1}{n+1}\binom{2n}{n} x^n$.

The Algebra of Formal Power Series. As we have seen already, a linear combination of two formal power series can be calculated as

$$u\sum_n a_n x^n + v\sum_n b_n x^n = \sum_n (ua_n + vb_n)x^n.$$

Power series can be multiplied using the Cauchy convolution product rule:

$$\sum_n a_n x^n \cdot \sum_n b_n x^n = \sum_n \left(\sum_k a_k b_{n-k}\right) x^n.$$

For example, $(1-x)(1+x+x^2+x^3+\cdots) = 1$. This says that $1-x$ and $1+x+x^2+x^3+\cdots$ are reciprocal power series. In general, we have the following result:

Proposition 4. *A formal power series $f = \sum_{n\geq 0} a_n x^n$ has a reciprocal if and only if $a_0 \neq 0$ and the reciprocal is*

$$\sum_{n\geq 0} b_n x^n$$

with $b_0 = 1/a_0$, and

$$b_n = -\frac{1}{a_0}\sum_{k\geq 1} a_k b_{n-k} \text{ for } n \geq 1.$$

Application to Counting Problems. Note that we have seen that $\frac{1}{(1-x)^k} = \sum_n \binom{n+k-1}{n} x^n$. By the Cauchy formula for the multiplication of formal power series this implies that the number of different ways of writing $n = a_1 + a_2 + \ldots + a_k$ where the a_i are non-negative integers is equal to $\binom{n+k-1}{n}$ (which we can of course also see directly). In other words, $\frac{1}{(1-x)^k}$ is the generating function for the number of different ways of writing n as a sum of k non-negative integers.

Following this line of reasoning, we obtain that the generating function for the number of ways to write n as a sum of 5 odd integers is $(x+x^3+x^5+\cdots)^5 = \left(\frac{x}{1-x^2}\right)^5$.

Exercises 5.
1. (1) Determine the generating function for the number of ways to distribute n identical candies (for a large number n) to four children so that the first two children receive each an odd number of candies, the third child receives at least three candies and the fourth child receives at most two candies.
2. (2) (Challenge) Show that the number of ways to make $10m$ cents change using only pennies, nickels, and dimes is $(m+1)^2$.
3. (3) Use generating functions to find b_n, the number of ways that $n \geq 0$ identical candies can be distributed among 4 children and 1 adult in such a way that each child receives an odd number of candies and the adult receives 0, 1, or 2 candies.
4. (4) In a certain game it is possible to score 1, 2, or 4 points at each turn. Find the generating function for the number of ways to score n points in a game in which there are at least two turns where 4 points are scored.

The Calculus of Formal Power Series. When we use formal power series to solve recurrence relations, a useful operation is the following collection of *shifts*:

If $f(x)$ is the nerating function for $(a_n)_0^\infty$, then the generating function for $(a_{n+1})_0^\infty$ is $\frac{f(x)-a_0}{x}$; the generating function for $(a_{n+2})_0^\infty$ is $\frac{f(x)-a_0-a_1x}{x^2}$; and in general,

Rule 6 (Shift)**.** *If $f(x)$ is the generating function for $(a_n)_0^\infty$, then for any integer $h > 0$, the generating functioni for $(a_{n+h})_0^\infty$ is* $\frac{f(x)-a_0-a_1x-a_2x^2-\cdots-a_{h-1}x^{h-1}}{x^h}$.

Another question you may want to ask is: if $f(x)$ is the generating function for $(a_n)_0^\infty$, then what is the generating function for $(na_n)_0^\infty$? First note that if $f = \sum_n a_n x^n$, then $f' = \sum_n n a_n x^{n-1}$. We will write D for the operation of taking a derivative (so $Df = f'$). Then we see that $xDf(x)$ is the power series for $(na_n)_0^\infty$ and in general $(xD)^k f$ is the power series for $(n^k a_n)_0^\infty$. This generalizes to the following rule:

Rule 7 ($P(n)$-Rule)**.** *If $f(x)$ is the generating function for $(a_n)_0^\infty$ and P is a polynomial, then $P(xD)f$ is the generating function for $(P(n)a_n)_0^\infty$.*

From the Cauchy product rule for formal power series we can derive:

Rule 8 (Partial Sums)**.** *If $f(x)$ is the formal power series for $(a_n)_0^\infty$, then $\frac{f(x)}{1-x}$ is the formal power series for the sequence of partial sums* $\left(\sum_{j=0}^n a_j\right)_0^\infty$.

Two Independent Variables. Consider again the recursion for the Stirling numbers of the second kind, $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$$

where $\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1$. If we want to use generating functions to study this sequence, we have three options:

$$A_n(y) = \sum_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} y^k$$

$$B_k(x) = \sum_n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^n$$

$$C(x,y) = \sum_{n,k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^n y^k.$$

Because of the occurrence of the factor k in the recurrence, the B_k-approach is the easiest. This translates the recurrence relation into

$$B_k(x) = xB_{k-1}(x) + kxB_k(x) \quad (k \geq 1; B_0(x) = 1).$$

This leads to

$$B_k(x) = \frac{x}{1-kx} B_{k-1}(x) \quad (k \geq 1; B_0(x) = 1).$$

Unwinding this gives us

$$B_k(x) = \sum_n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^n = \frac{x^k}{(1-x)(1-2x)(1-3x)\cdots(1-kx)} \quad (k \geq 0).$$

So our last task is to find the partial fractions expansion

$$\frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-kx)} = \sum_{j=1}^{k} \frac{\alpha_j}{1-jx}.$$

Show that

$$\begin{aligned}\alpha_r &= \frac{1}{(1-1/r)(1-2/r)\cdots(1-(r-1)/r)(1-(r+1)/r)\cdots(1-k/r)} \\ &= (-1)^{k-r}\frac{r^{k-1}}{(r-1)!(k-r)!}\end{aligned}$$

for $1 \le r \le k$. So we get that

$$\begin{aligned}\left\{ \begin{matrix} n \\ k \end{matrix} \right\} &= [x^n]\left\{\frac{x^k}{(1-x)(1-2x)(1-3x)\cdots(1-kx)}\right\} \\ &= [x^{n-k}]\left\{\frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-kx)}\right\} \\ &= [x^{n-k}]\sum_{r=1}^{k}\frac{\alpha_r}{1-rx} \\ &= \sum_{r=1}^{k}\alpha_r[x^{n-k}]\frac{1}{1-rx} \\ &= \sum_{r=1}^{k}\alpha_r r^{n-k} \\ &= \sum_{r=1}^{k}(-1)^{k-r}\frac{r^{k-1}}{(r-1)!(k-r)!}r^{n-k} \\ &= \sum_{r=1}^{k}(-1)^{k-r}\frac{r^n}{r!(k-r)!}\end{aligned}$$

Recall how the Bell numbers are related to the Stirling numbers of the second kind (as sums). So we can get a closed formula for them from the closed formula of the Stirling numbers. However, we could also use generating functions to find a new recurrence relation for the Bell numbers. This would take the exponential generating functions introduced in the next subsection.

The Exponential Generating Functions. In this section we introduce a new clothesline with new algebraic properties. The *exponential generating function* for a sequence $(a_n)_0^\infty$ is the formal power series

$$\sum_{n\ge 0}\frac{a_n}{n!}x^n.$$

Lets see how the operations for generating functions translate in this situation. For example, what is the generating function for the shifted sequence $(a_{n+1})_0^\infty$? It turns out that this is f', i.e., Df. So our first new rule is

Rule 9 (Exponential Shift)**.** *If f is the exponential generating function for $(a_n)_0^\infty$, and h a non-negative integer, then $D^h f$ is the exponential generating function for $(a_{n+h})_0^\infty$.*

Multiplying by a polynomial in n works the same way as before:

Rule 10 (Exponential $P(n)$-Rule)**.** *If f is the exponential generating function for $(a_n)_0^\infty$, and P a given polynomial, then $P(xD)f$ is the generating function for $(P(n)a_n)_0^\infty$.*

Products work differently than before. Recall that for ordinary generating functions, the product of two functions had coefficients given by Cauchy convolutions of the terms of the orginal sequences. The new rule is

Rule 11 (Exponential Product Rule)**.** *Let f be the exponential generating function of $(a_n)_0^\infty$ and g the exponential generating function of $(b_n)_0^\infty$. Then fg is the exponential generating function of*

$$\left(\sum_r \binom{n}{r} a_r b_{n-r}\right)_0^\infty .$$

Ordinary versus Exponential Generating Functions. The main difference between the two types of generating functions discussed so far is the sequence represented by the product of two generating functions.

For example, ordinary generating functions can be used to find the number of ways we can make change when we have coins of certain values, but exponential generating functions can be used when we want to make words out of certain letters (and the order is important).

We will now consider two examples of situations where the recursion formula indicates which type of generating function will be most appropriate.

Example 12. We want to count the number $f(n)$ of ways of arranging n pairs of left and right parentheses into a legal string. To find a recurrence relation, we attach to each legal string a positive integer k which is the smallest positive integer k such that the first $2k$ characters of the string also form a legal string of parentheses. For example, the string ()(()) has $k = 1$, but (())() has $k = 2$ and ((())()) has $k = 3$. A legal string of length $2n$ is called *primitive* if $k = n$.

Note that the number of legal strings of length $2n$ with integer k is equal to $f(k-1)f(n-k)$, since the matching right parenthesis for the first left parenthesis is at position $2k$ and in between those two positions we may place any legal string of length $2k-2$; furthermore, this primitive string must be followed by any legal string of lenth $2n-2k$. So we find that

$$f(n) \sum_k f(k-1)f(n-k) \quad (n \neq 0; f(0) = 1)$$

where we assume that f of a negative argument is zero.

Note that this sum suggests that we may want to consider the ordinary power series for $f(n)$, since the formula on the right looks like the convolution product. To be precise, if $F(x) = \sum_n f(n)x^n$, then the sum above is the coefficient of x^n in the product of the series F and the series $\sum_k f(k-1)x^k = xF(x)$. So we derive that $F(x) - 1 = xF(x)^2$. To solve this, we need to solve a quadratic. A priori, we obtain two solutions:

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

However, for $x = 0$, we know that our generating function takes the value 1. This is only the case for $F(x) = \frac{1-\sqrt{1-4x}}{2x}$. The coefficients of this sequence are $[x^n]F(x) = \frac{1}{n+1}\binom{2n}{n}$, the famous Catalan numbers.

Example 13. For our second example, we will look more closely at the number of derangements of n letters (permutations without any fixed points). Note that the total number of permutations of n letters is $n!$. Also the number of permutations with k fixed points is $\sum \binom{n}{k} D_{n-k}$. This leads us to the equation

$$n! = \sum_k \binom{n}{k} D_{n-k} \quad (n \geq 0).$$

Taking the exponential generating function on both sides gives

$$\frac{1}{1-x} = e^x D(x),$$

where $D(x)$ is the exponential generating function for the derangement numbers. So we see that $D(x) = e^{-x}/(1-x)$, and the coefficient for x^n gives

$$\frac{D_n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}.$$

The Snake Oil Method

First a couple of reminders of conventions: we will assume that $\binom{n}{k} = 0$ whenever $k < 0$ or when $0 \leq n < k$, and we will assume that all sums for which the limits are not explicitly stated are summed from $-\infty$ to ∞. So we have for example,

$$\sum_k \binom{n}{k} = 2^n,$$

and we can write

$$\sum_k \binom{n}{r+k} x^k = x^{-r} \sum_k \binom{n}{r+k} x^{r+k} = x^{-r} \sum_s \binom{n}{s} x^s = x^{-r}(1+x)^n,$$

without needing to worry about the boundaries of the sums. Finally, the basic power series we will use most are

$$\begin{aligned}
\sum_{r \geq 0} \binom{r}{k} x^r &= \frac{x^k}{(1-x)^{k+1}} \quad (k \geq 0) \\
\sum_r \binom{n}{r} x^r &= (1+x)^n \\
\sum_n \frac{1}{n+1} \binom{2n}{n} x^n &= \frac{1}{2x}(1 - \sqrt{1-4x}).
\end{aligned}$$

The snake oil method works for sums with an additional free variable, such as

$$\sum_{k \geq 0} \binom{k}{n-k} \quad (n = 0, 1, 2, \cdots)$$

The variable n is free in this expression, so we could write $f(n) = \sum_{k \geq 0} \binom{k}{n-k}$ and we would like to obtain a simpler description for this function. A lot of traditional methods would work internally: manipulate the expression inside the summation.

Instead, snake oil works externally: it takes the generating function $F(x)$ for the $f(n)$. So,

$$F(x) = \sum_n x^n \sum_{k \geq 0} \binom{k}{n-k}.$$

Note that we can bring x^n inside the second sum and then we can interchange the two sum symbols, so

$$F(x) = \sum_{k \geq 0} \sum_n \binom{k}{n-k} x^n.$$

To view the new internal sum as an instance of the binomial sum, we need to adjust the power of x,

$$F(x) = \sum_{k \geq 0} x^k \sum_n \binom{k}{n-k} x^{n-k}.$$

Taking r as the new dummy variable for the internal sum gives

$$F(x) = \sum_{k \geq 0} x^k \sum_r \binom{k}{r} x^r = \sum_{k \geq 0} x^k (1+x)^k = \sum_{k \geq 0} (x+x^2)^k = \frac{1}{1-x-x^2}.$$

We recognize this as the generating function for the Fibonacci numbers, so $f(n) = F_{n+1}$.

Here is an overview for the steps of the snake oil method to sweep all combinatorial sums under the rug:

(1) Identify the free variable, say n, that the sum depends on, and call the sum $f(n)$.
(2) Let $F(x)$ be the ordinary generating function for the sequence $(f(n))_0^\infty$.
(3) Note that $F(x)$ is now a double sum. Move the exponent of x inside and interchange the order of the summation.
(4) Manipulate the exponent of x so that you can evaluate the inside sum in simple closed form using one of the standard power series given above.
(5) Try to identify the coefficients of the resulting generating function.

Here are some examples to practice on. Evaluate the following combinatorial sums:

(1) $\displaystyle\sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$ with $n, m \geq 0$. (Hint: choose n as the free variable.)
(2) $\displaystyle\sum_{k \leq \frac{n}{2}} (-1)^k \binom{n-k}{k} y^{n-2k}$ for $n \geq 0$.
(3) $\displaystyle\sum_k \binom{n+k}{2k} 2^{n-k}$ for $n \geq 0$.

Example 14. Snake oil can also be used to show that two combinatorial sums are equal without evaluating either one of them. Try it to prove the following identity:

$$\sum_k \binom{m}{k} \binom{n+k}{m} = \sum_k \binom{m}{k} \binom{n}{k} 2^k$$

Inversion Formulas. Snake oil can also be used to find inversion formulas. But you need to choose the type of generating function you want to use carefully. Suppose that two sequences $(a_r)_0^\infty$ and $(b_s)_0^\infty$ are related by the formula

$$a_r = \sum_s \binom{r}{s} b_s$$

An inversion of this formula would give b_s as function of a_r. Because of the multiplication rule for exponential generating functions, and the fact that the exponential generating function for $(1)_0^\infty$ is e^x, we see that if we compare the exponential generating function $A(x)$ for (a_r) with the exponential generating function $B(x)$ for (b_s), we get $A(x) = e^x B(x)$. So we see that $B(x) = e^{-x} A(x)$, and this translates into

$$b_n = \sum_m \binom{n}{m} (-1)^{n-m} a_m \quad (n \geq 0).$$

Exercises

(1) Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be sets such that $A_1 = \emptyset$, $B_1 = \{0\}$, and

$$A_{n+1} = \{x+1 | \, x \in B_n\} \quad \text{and} \quad B_{n+1} = A_n \cup B_n - A_n \cap B_n,$$

for all positive integers n. Determine all positive integers n such that $B_n = \{0\}$.

(2) [China 1996] Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

(3) [China 2000, Yuming Huang] The sequence $(a_n)_{n \geq 1}$ satisfies the conditions $a_1 = 0$, $a_2 = 1$,

$$a_n = \frac{1}{2} n a_{n-1} + \frac{1}{2} n(n-1) a_{n-2} + (-1)^n \left(1 - \frac{n}{2}\right),$$

$n \geq 3$. Determine the explicit form of

$$f_n = a_n + 2\binom{n}{1} a_{n-1} + 3\binom{n}{2} a_{n-2} + \cdots + (n-1)\binom{n}{n-2} a_2 + n\binom{n}{n-1} a_1.$$

(4) [Highschool Mathematics, 1994/1, Qihong Xie] Find the number of subsets of $\{1, 2, \ldots, 2000\}$, the sum of whose elements is divisible by 5.

(5) [AIME 2001] A mail carrier delivers mail to nineteen houses on the east side of Elm Street. The carrier notes that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

(6) A function f is defined for all $n \geq 1$ by the relations (a) $f(1) = 1$; (b) $f(2n) = f(n)$; and (c) $f(2n+1) = f(n) + f(n+1)$. Find its generating function.

(7) Find the number $f(n, k)$ of k-subsets of $[n] = \{1, 2, 3, \ldots, n\}$.

(8) Let $f(n, m, k)$ be the number of strings of n 0's and 1's that contain exactly m 1's, no k of which are consecutive.
   (a) Find a recurrence formula for f.
   (b) Find, in simple closed form, the generating functions

$$F_k(x, y) = \sum_{n,m \geq 0} f(n, m, k) x^n y^m \quad (k = 1, 2, \ldots)$$

(c) Find an explicit formula for $f(n, m, k)$ from the generating function.

(9) Let $D(n)$ be the number of derangements of n letters.

(a) Find in simple explicit form the exponential generating function of $(D(n))_0^\infty$.

(b) Prove by any method that

$$D(n+1) = (n+1)D(n) + (-1)^{n+1} \quad (n \geq 0; D(0) = 1)$$

(c) Prove by any method that

$$D(n+1) = n(D(n) + D(n-1)) \quad (n \geq 1; D(0) = 1; D(1) = 0)$$

(d) Show that the number of permutations of n letters that have exactly 1 fixed point differs from the number with no fixed point by ± 1.

(e) Let $D_k(n)$ be the number of permutations of n letters that have exactly k fixed points. Show that

$$\sum_{k,n \geq 0} D_k(n) \frac{x^n y^k}{n!} = \frac{e^{-x(1-y)}}{1-x}.$$

(f) Show that

$$\sum_r \binom{n}{\lfloor \frac{r}{2} \rfloor} x^r = (1+x)(1+x^2)^n.$$

Then use Snake Oil to evaluate

$$\sum_k \binom{n}{k} \binom{n-k}{\lfloor \frac{r}{n-k} \rfloor} y^k$$

explicitly, when $y = \pm 2$ (due to D.E. Knuth).

(g) Find the generating function of these sums, whatever the value of y.

(10) (a) Find an explicit formula, not involving sums, for the polynomial

$$\sum_{k \geq 0} \binom{k}{n-k} t^k.$$

(b) Evaluate

$$\sum_k \binom{2n+1}{2p+2k+1} \binom{p+k}{k}.$$

(c) Show that

$$\sum_m \binom{r}{m} \binom{s}{t-m} = \binom{r+s}{t}.$$

Then evaluate

$$\sum_k \binom{n}{k}^2.$$

(d) Show that (Graham and Riordan)

$$\sum \binom{2n+1}{2k} \binom{m+k}{2n} = \binom{2m+1}{2n}.$$

(e) Show that for all $n \geq 0$,

$$\sum_k \binom{n}{k} \binom{k}{j} x^k = \binom{n}{j} x^j (1+x)^{n-j}.$$

(f) Show that for all $n \geq 0$,

$$x \sum_k \binom{n+k}{2k} \left(\frac{x^2-1}{4}\right)^{n-k} = \left(\frac{x-1}{2}\right)^{2n+1} + \left(\frac{x+1}{2}\right)^{2n+1}$$

(g) Show that for all $n \geq 1$,

$$\sum_{k \geq 1} \binom{n+k-1}{2k-1} \frac{(x-1)^{2k} x^{n-k}}{k} = \frac{(x^n-1)^2}{n}.$$

(11) Prove that

$$\sum_k (-1)^{n-k} \binom{2n}{k}^2 = \binom{2n}{n}$$

(12) To evaluate a sum that is of the form

$$S(n) = \sum_k f(k) g(n-k),$$

the natural method is to recognize $S(n)$ as $[x^n]\{F(x)G(x)\}$, where F and G are the ordinary generating functions of (f_n) and (g_n). Use this method to evaluate

$$S(n) = \sum_k \frac{1}{k+1} \binom{2k}{k} \frac{1}{n-k+1} \binom{2n-2k}{n-k}.$$

(13) [Vietnam 1998, Category A] Let $(a_n)_0^\infty$ be a sequence of positive integers defined recursively by

$$a_0 = 20, a_1 = 100, a_{n+2} = 4a_{n+1} + 5a_n + 20.$$

Determine the smallest positive integer h for which $a_{n+h} - a_n$ is divisible by 1998 for every non-negative integer n.

(14) [Vietnam 1998, Category B] Let a, b be integers. Define a sequence $(a_n)_0^\infty$ of integers defined by

$$a_0 = a, a_1 = b, a_2 = 2b - a + 2, a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$$

for $n \geq 0$.

(a) Find the general term of the sequence.

(b) Determine all integers a and b for which a_n is a perfect square for all $n \geq 1998$.

Further Reading

These notes are largely based on the book Generatingfunctionology by Herbert S. Wilf, which may be downloaded from http://www.math.upenn.edu/ wilf/DownldGF.html Other useful books on combinatorics that discuss generating functions are:

- John M. Harris, Jeffry L. Hirst, Michael J. Mossinghoff, *Combinatorics and Graph Theory*, Springer Verlag, New York 2000
- Daniel A. Marcus, *Combinatorics A Problem Oriented Approach*, The MAA, 1998
- George E. Martin, *Counting: The Art of Enumerative Combinatorics*, Springer Verlag, New York 2001