Art of Problem Solving

WOOT 2012–13

Similar Triangles/ Power of a Point

In many challenging problems, one of the biggest difficulties can be finding something to try. In most geometry problems, however, our greatest hurdle is quite the opposite: there are zillions of things we could try. The hard part is figuring out which one might be fruitful.

In this article we outline five general problem solving tools that are useful in geometry problems. Of course, this is by no means an exhaustive list! You're probably familiar with many of these tools, as well as with some of the results we offer as examples or exercises. As you read through these, and as you look at geometry problems in the future, you shouldn't just focus on learning exactly what the tools are and how to use them. It's very important, *perhaps even more important*, to think about when and why you use the tools. So, when you read a solution to a problem you couldn't solve on your own, don't just soak up the steps. Think about what clues exist in the problem that would suggest to you to try each step.

Five general tools we use to tackle a great many geometry problems are:

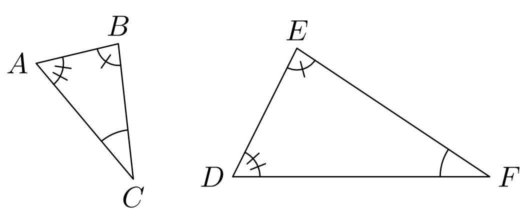
1. Similar Triangles
2. Homothety
3. Power of a Point
4. The Radical Axis
5. Cyclic Quadrilaterals

In this article, we'll offer a brief description of important facts, a list of tips for when to use each one, and some examples and exercises. Some of this information will consist of links to articles that have been written on Art of Problem Solving and other sites, such as the excellent cut-the-knot site by Alexander Bogomolny.

1 Similar Triangles

In a great many challenging geometry problems, a key step is finding a pair of similar triangles. (Congruent triangles are a subset of similar triangles; we'll focus on similar triangles here.)

Below, we have $\triangle ABC \sim \triangle DEF$.



Because the triangles are similar, we have

$$\angle A = \angle D, \qquad \angle B = \angle E, \qquad \angle C = \angle F,$$

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

Of course, we don't have to prove all of these facts to deduce that $\triangle ABC$ and $\triangle DEF$ are similar. The most common ways to deduce that triangles ABC and DEF are similar are:

- **AA Similarity**. If $\angle A = \angle D$ and $\angle B = \angle E$, then $\triangle ABC \sim \triangle DEF$. This is by far the most common way to determine that two triangles are similar. Also, this is part of the reason that it's important to mark equal angles in your diagram as you find them. Then, the similar triangles will stand out more clearly. Once you find similar triangles, you should write down the fact that they are similar so you won't forget – you'll often learn facts in geometry problems before you need them. So, keep a list of what you discover that you can refer to when you're stuck.
- **SAS Similarity**. If $\angle A = \angle D$ and $AB/DE = AC/DF$, then $\triangle ABC \sim \triangle DEF$. This is obviously of limited use, but when you have a problem in which you are given information regarding ratios, you might think of trying to use this at some point in the problem. SAS Similarity can also come into play in problems in which two triangles share an angle. (See Example 1 below.)
- **SSS Similarity**. If $AB/DE = AC/DF = BC/EF$, then $\triangle ABC \sim \triangle DEF$. This one's of pretty limited usefulness. Obviously, you'd have to have lots of information about lengths for this to be useful.

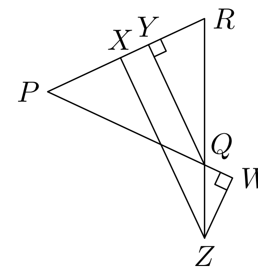
Clues that it's time to look for similar triangles include:

- **Parallel lines**. Parallel lines mean equal angles. Equal angles mean similar triangles. Parallel lines are so useful in finding similar triangles that sometimes we add a parallel line to a diagram just to make similar triangles. (Try it on Example 3.)
- **Perpendicular lines**. If you have several right angles in a problem, then you almost certainly have a bunch of right triangles. In each pair of these triangles, you already have one pair of equal angles, so you only have to find one more pair of equal angles to find similar triangles. (This isn't quite the tip-off that parallel lines are, since perpendicular lines are a huge tip-off for other tools as well, such as the Pythagorean Theorem and cyclic quadrilaterals.)
- **Angles inscribed in the same arc**. If two angles are inscribed in the same arc, these angles are equal. Look for triangles including these angles, and try to find other pairs of equal angles.
- **You have information about ratios of segment lengths**. If you are given an equality involving ratios of sides, this is a sure sign to look for similar triangles.
- **You need a length and nothing else has worked**. You've tried the Pythagorean Theorem. You've tried areas. You've even (gasp) tried coordinates. But you still can't find a needed length in a problem. Try building similar triangles by drawing in lines parallel to lines in the problem, or by dropping perpendiculars. There's always something to try!

Here are some exercises:

1. Let X be on $\overline{AB}$ and Y be on $\overline{AC}$ such that $AX/AB = AY/AC$. Show that $\overline{XY} \parallel \overline{BC}$.

2. In the diagram at right, $PQ = PR$, $\overline{ZX} \parallel \overline{QY}$, $\overline{QY} \perp \overline{PR}$, and $\overline{PQ}$ is extended to W such that $\overline{WZ} \perp \overline{PW}$.



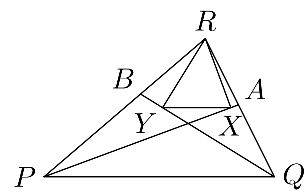
   (a) Show that $\triangle QWZ \sim \triangle RXZ$. Hints: 1

   (b) Show that $YQ = ZX - ZW$. Hints: 5, 2

3. **The Angle Bisector Theorem**. If D is on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle BAC$, then $AB/BD = AC/CD$. Take a look here for hints.

4. Prove that SAS Similarity works without using the law of sines.

5. $\overline{PA}$ and $\overline{BQ}$ bisect angles $\angle RPQ$ and $\angle RQP$, respectively. Given that $\overline{RX} \perp \overline{PA}$ and $\overline{RY} \perp \overline{BQ}$, prove that $\overline{XY} \parallel \overline{PQ}$. Hints: 9, 4



2 Homothety

Homothety is a special type of similarity. You may know it as dilation, or as 'homothecy'. Visit this site for an article outlining the basics of homothety.

Tips to use homothety in a problem include:

- **Tangent circles**. Any two circles are homothetic, but when you have two tangent circles, you also have useful information about one center of homothety (the point of tangency).
- **Common tangents to two circles**. If the common tangents to two circles are part of a problem, then you should consider homothety because these two tangents will intersect at a center of homothety that maps one circle to the other.
- **The Medial Triangle**. The triangle connecting the midpoints of the sides of a triangle is homothetic to the original triangle. (Prove it!) Any problem involving the medial triangle or the centroid of a triangle is a candidate for homothety.
- **Intersecting lines pass through endpoints of parallel segments**. If $\overline{AB} \parallel \overline{XY}$, and lines $\overleftrightarrow{AX}$ and $\overleftrightarrow{BY}$ meet at P, then P is the center of a homothety that maps $\overline{AB}$ to $\overline{XY}$.
- **Collinearity problems**. If we must prove that points A, P, and X are collinear, one way to do so is to show that A and X are corresponding points on two figures that are homothetic under a homothety with center P. Usually you can tell pretty quickly if you have two figures that might be candidates for such a homothety. So, on a collinearity problem, you can briefly consider this approach and usually discard it quickly if it obviously won't work.

- **Concurrency problems**. Lines $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$, and $\overleftrightarrow{CZ}$ are concurrent if they connect corresponding points on two homothetic figures (the center of homothety is the point at which they meet). As with collinearity, it's usually quickly obvious whether or not this approach has any hope of working.

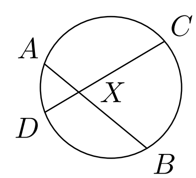
Homothety in general is an example of why it's very important on tough geometry problems to draw large, accurate diagrams. If you draw a precise diagram, it will usually be pretty obvious when two lines are probably parallel, and parallel lines often lead to homothetic triangles. However, if you just make a rough sketch, you might miss a homothety that blows a problem away.

If you don't quite grasp what the big deal is about homothety, don't worry about it just yet. Master the other tools first. Homothety is largely just a fancy way of saying 'similar triangles.' In general, much of what we can do with homothety can be done (with a bunch more steps) with similar triangles and other Euclidean geometry tools.

3 Power of a Point

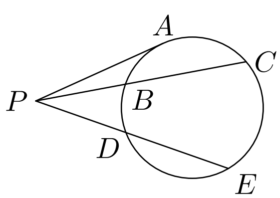
We mentioned 'inscribed angles' in our list of clues to look for similar triangles. Power of a Point is the fancy name given to a powerful example of such similar triangles.

Suppose a line through a point P intersects a circle in two points, U and V. The **Power of a Point Theorem** states that for all such lines, the product $(PU)(PV)$ is constant. We call this product the power of point P.



For example, in the figure at right, applying Power of a Point to X with respect to the circle shown gives

$$(XA)(XB) = (XC)(XD).$$



In the figure at left, the power of point P with respect to the circle gives us

$$PA^2 = (PB)(PC) = (PD)(PE).$$

The tip-offs to use power of a point are not surprising:

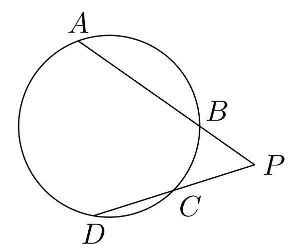
- **Circles and lengths**. You have circles, and you have (or want) information about lengths. You should be considering ways to use power of a point.
- **Circles and ratios**. You have circles, and you have information about ratios. You should be thinking about power of a point.

Notice, however, that Power of a Point will probably not be so useful if you're dealing with an inscribed polygon and have information about side lengths. Power of a Point is useful with secants, tangents, and intersecting chords. It's not so helpful with nonintersecting chords (unless they are parts of secants that meet at a useful point) or with chords that share an endpoint.

6. Chords $\overline{AB}$ and $\overline{CD}$ meet at X. Prove that $(XA)(XB) = (XC)(XD)$. (No citing the Power of a Point Theorem! You're being asked here to prove it!) Hints: 3

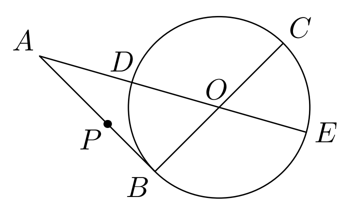
7. Point P is outside a circle and A is on the circle such that $\overline{PA}$ is tangent to the circle. A line through P meets the circle at B and C. Prove that $PA^2 = (PB)(PC)$. Use this to show that if a second line through P meets the circle at D and E, then $PA^2 = (PD)(PE)$. (Again, you can't just cite Power of a Point!) Hints: 8

8. In the diagram at right, we have $BP = 8$, $AB = 10$, $CD = 7$, and $\angle APC = 60°$. Find the area of the circle. *Source: AHSME*



9. Point O is the center of the circle, $\overline{AB} \perp \overline{BC}$, $AP = AD$, and $\overline{AB}$ has length twice the radius of the circle. Prove that $AP^2 = (PB)(AB)$.

10. (This one can be particularly useful.) Show that if P is outside circle $\odot O$, and that the radius of $\odot O$ is r, then the power of point P is $OP^2 - r^2$. What if P is inside the circle?



4 The Radical Axis

The set of all points that have the same power with respect to two different circles is called the **radical axis** of the two circles. That these points together form a straight line is not an obvious fact! You'll be challenged to prove it on the message board.

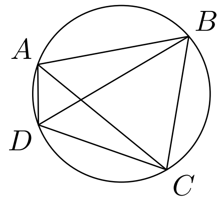
The fact that these points form a straight line can be extremely useful, sometimes offering very quick proofs to otherwise very tricky problems. By far the most common tip-off to think about the radical axis of a pair of circles is if you have intersecting circles in a problem. Exercise 11 suggests why.

11. Describe the radical axis of two intersecting circles.

12. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ meet at A and B. Circles $\mathcal{C}_2$ and $\mathcal{C}_3$ meet at C and D. Circles $\mathcal{C}_1$ and $\mathcal{C}_3$ meet at E and F. Show that $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$ are concurrent.

5 Cyclic Quadrilaterals

A quadrilateral that can be inscribed in a circle is a **cyclic quadrilateral**. Discovering cyclic quadrilaterals is a key step in a great many tough geometry problems. Below, $ABCD$ is a cyclic quadrilateral.



Here are a variety of ways to prove that a quadrilateral is cyclic. The first two, involving angles of a quadrilateral, are the most commonly used.

- **Equal Inscribed Angles**. If $\angle ACB = \angle ADB$ in convex quadrilateral $ABCD$, then $ABCD$ is cyclic. This is yet another reason marking equal angles is so important; if you find two equal angles that intersect as described above, then you have a cyclic quadrilateral.
- **Supplementary Opposite Angles**. If $\angle BAD + \angle BCD = 180^\circ$ in convex quadrilateral $ABCD$, then $ABCD$ is cyclic. This one is particularly useful when you have a lot of right angles in a problem. If two of these angles intersect, odds are you have a cyclic quadrilateral.
- **Converse of Power of a Point**. If P is the intersection of AB and CD, and $PA \cdot PB = PC \cdot PD$, then $ABCD$ is a cyclic quadrilateral. This is simply Power of a Point in reverse. Obviously, we'll only reach for this one if we have information about lengths (or ratios of lengths). Often we use this in conjunction with similar triangles, which give us the necessary information about ratios of lengths.

Cyclic quadrilaterals are extremely useful in problems that require angle-chasing (finding equal angles). This is because once we prove a quadrilateral is cyclic, then we have a bunch of pairs of equal angles from the angles that inscribe common arcs. For example, once we know that $ABCD$ is cyclic, we immediately know that $\angle ACB = \angle ADB$, $\angle ABD = \angle ACD$, and so on.

Here are some clues to look for cyclic quadrilaterals:

- **You are angle chasing**. A very common way to show that two angles are equal is to first find a cyclic quadrilateral involving the two angles, then use the resulting circle to show the two angles are inscribed in the same arc. This is obviously only useful if the angles intersect appropriately (such as $\angle ACB$ and $\angle ADB$ above).
- **You're trying to show that two appropriately intersecting angles are supplementary**. If your target angles intersect as $\angle BAD$ and $\angle BDC$ do in our sample cyclic quadrilateral, and you must show that these angles are supplementary, then showing that you have a cyclic quadrilateral, like $ABCD$ above, will do the job.

- **Right angles**. Any two intersecting right angles will give you a cyclic quadrilateral. If you have a bunch of right angles in a problem, you have a bunch of cyclic quadrilaterals.
- **You're doing a medium or tough national olympiad problem**. Finding a cyclic quadrilateral is a step in a large portion of medium-to-tough national olympiad problems. Often, you'll have to find several cyclic quadrilaterals.

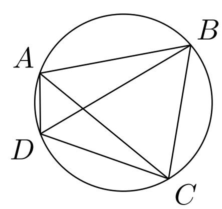
These are only a few of the reasons you might look for cyclic quadrilaterals, but they capture most of the main reasons you might seek them. Above all, circles are very helpful with angle chasing, due to the relationships between angles and the arcs they intercept. So, discovering a circle that involves multiple points in your diagram can be extremely useful.

Here are some basic exercises that require finding a cyclic quadrilateral as a key step.

13. Altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ of acute triangle $\triangle ABC$ intersect at H. Show that $\angle BFD = \angle ACB$. Hints: 7
14. Points A, C, D, and B are in that order on a circle with center O such that $\overline{AB}$ is a diameter of the circle and $\angle DOB = 50^\circ$. Point P is on $\overline{AB}$ such that $\angle PCO = \angle PDO = 10^\circ$. Find the measure in degrees of arc CD. Hints: 6
15. Prove that each of the methods above does indeed show that a quadrilateral is cyclic.

Ptolemy's Theorem

One occasionally useful theorem regarding cyclic quadrilaterals is Ptolemy's Theorem.



Ptolemy's Theorem states that $ABCD$ is a cyclic quadrilateral if and only if

$$(AC)(BD) = (AB)(CD) + (AD)(BC).$$

Note that Ptolemy's Theorem is an 'if and only if' theorem, so if the lengths of the sides and diagonals of $ABCD$ satisfy the equation above, then $ABCD$ is cyclic. Of course, it's pretty rare that you have all that information about a quadrilateral and haven't yet solved the problem. It's much more common to apply Ptolemy's Theorem to problems in which you already know a quadrilateral is cyclic, and need to find some information about the sides.

We usually think to try using Ptolemy when we have an inscribed polygon and we need (or have) information about the side lengths and/or diagonal lengths of the polygon.

16. Prove Ptolemy's Theorem. This is pretty tough to do if you've never seen the proof before, so I'll give you the first step. Ptolemy's Theorem involves a bunch of side lengths and a circle, so we might think of Power of a Point. Unfortunately, most of the side lengths are chords of the circle and are not well situated to use Power of a Point. However, circles give us equal angles, which can help find similar triangles. So, here's the first step: take a point M on diagonal $\overline{BD}$ such that $\angle ACB = \angle MCD$. Now, find some similar triangles. Then mark equal angles and find more similar triangles.

17. Let ABC be an equilateral triangle and let P be a point on its circumference such that PC is greater than both PA and PB. Prove that $PA + PB = PC$.

(Solutions to both of these problems can be found on this website.)

Note: Ptolemy's Theorem is the equality condition of Ptolemy's Inequality, which states that for any four points A, B, C, D, we have

$$(AC)(BD) \leq (AB)(CD) + (AD)(BC).$$

Equality holds only when quadrilateral $ABCD$ is cyclic.

6 Some General Strategies

You'll see me preach about many of these in class. These are general guidelines for approaching geometry problems. As I noted earlier, one of the largest challenges in olympiad geometry problems is that there are usually zillions of different things you can try on any given problem. The trouble is managing all the information you have, and all the information you can find.

- **Draw large precise diagrams**. A great many solutions hinge on discovering one or two surprising facts. By drawing a large, precise diagram, you might identify three lines that look concurrent, or lines that look parallel, or segments that look equal, or angles that look right, etc. Moreover, in trying to figure out how to construct a diagram with straightedge and compass, you might discover some facts you can use to solve the problem.
- **Draw more than one diagram**. Just because your first diagram makes two lines look parallel doesn't mean a second will. Before you spend 20 minutes trying to show that two lines are parallel, draw a second diagram and see if whatever your first diagram suggests is true still looks true in the second diagram.
- **Don't just stare**. Mark equal angles and equal segments as you find them. Assign variables to lengths of segments or to measures of angles, and use your geometry toolchest to find other lengths or measures in terms of those of those variables. Then, write this information on your diagram. Having all this information on your diagram will help you see relationships you wouldn't see if you just hold this information in your head.

- **Use the clues**. Use the general information you have in the problem to determine what general tools are likely to be useful. Don't just forget about all other tools, but focus on those that are most likely to be fruitful first.

- **Work backwards**. In many geometry problems, particularly proofs, you can work backwards from what you want until you hit something you know how to find (or prove). Don't just work forwards from the given information.

- **Keep all your observations organized**. On most hard problems, you'll work both forwards and backwards. So, you'll have a list of 'What I Know' and a list of 'What I Need'. When something pops up on both lists, then you're done. However, if you mix up your lists, you're almost certain to mess up (usually, you'll end up with circular reasoning somewhere). So, keep these lists separate – on different papers for more complicated problems.

- **Use all your information**. If you're stuck on a problem, look back at the given information. If there's anything there you haven't used to make more observations, that's where you should look. Also, look at your 'What I Know' list; anything there that looks fruitful, but is yet unused, is something you should investigate further. 'What haven't I used yet?' is the first question that comes into my mind whenever I am stuck.

- **Don't fall for the fancy stuff**. You know inversion. You know complex numbers. You know vectors. You know projective geometry. Big deal. Most olympiad problems are specifically chosen so that the fancy-shmancy tools are no more helpful than the basic tools of Euclidean geometry. You shouldn't spend more than a few minutes on any of your fancy stuff – if the fancy stuff helps, it will help almost immediately. But if you spend 30 minutes playing with inversion, then you're probably barking up the wrong tree.

- **Don't spend too much time on algebra**. You really love analytic geometry and pounding away with equations. I won't say you should never use analytic geometry, but you should use it very judiciously. Set up the problem so that your expressions are as simple as possible. If it isn't obvious within 5-10 minutes that analytic geometry will provide a solution (even if it will take you 10 minutes to wade through the algebra), then you should probably stop and try other approaches. Don't just blindly push equations around and hope something good happens. Only slog through algebra on a geometry problem when you have a clear plan to get the answer.

Here are some hints to the problems:

1. What angles equal $\angle ZQW$? What angles equal $\angle R$?
2. Solve for RZ and QZ in terms of RQ and the lengths in the equation we want to prove.
3. Draw $\overline{AD}$ and $\overline{CB}$.
4. Prove that $\triangle RYX \sim \triangle RCD$. (You may need to find some congruent triangles first!)
5. Note that $RQ = RZ - QZ$, which looks a lot like what we want.

6. What kind of quadrilateral is $PCDO$?
7. Show that $BDHF$ is a cyclic quadrilateral.
8. Draw $\overline{AB}$ and $\overline{AC}$. Find a pair of similar triangles.
9. Extend rays RY and RX to meet $\overline{PQ}$ at C and D, respectively.