Induction

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1 Introduction

Everyone is familiar with induction. There are three steps to an induction proof:

- Come up with the inductive hypothesis.
- Proof the base case(s).
- Proof the general case based on reducing to one or more previous cases.

The base case is normally fairly easy, but both of the other two parts can be challenging.

When coming up with the inductive hypothesis, you should probably start with the final result you are trying to prove. However, sometimes this doesn't work. You sometimes will need to prove a *stronger* result. This is counter-intuitive but a stronger result can allow you to complete the inductive step where a weaker one would fail. Here is a classic example:

Example 1. *Prove that*

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

Solution. If you try to prove the claim directly with induction, you will run into problems. The claim actually gets stronger as n increases, so there is nothing you can do! Fortunately, we can prove a slightly stronger claim instead for which this is not true. We prove:

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.$$

When $n = 1$, it is true. Now if it holds for $n - 1$, then we have

$$\begin{aligned}
\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} &\leq \frac{1}{\sqrt{3n+1}} \cdot \frac{2n-1}{2n} \\
&= \sqrt{\frac{4n^2-4n+1}{12n^3+4n^2}} \\
&\leq \sqrt{\frac{4n^2-4n+1}{12n^3+4n^2-13n+4}} \\
&= \sqrt{\frac{4n^2-4n+1}{(4n^2-4n+1)(3n+4)}} \\
&= \frac{1}{\sqrt{3n+1}}.
\end{aligned}$$

□

Generally you want to start with the obvious inductive claim, ask yourself what you *wished* was true, and then see if that can be folded into the inductive claim. You may then need even more to prove it, and you will be out of luck. However, sometimes it just works. Here is another example:

Example 2. *A different real number is written in every square of an $n \times n$ grid. It takes you one second to read the number on any square. Prove that you can find a number which is less than all of its neighbours (up, down, left, and right) in at most $8n$ seconds.*

Solution. We are looking for a local minimum on the grid. We want to sub-divide the board into 4 grids of size $\frac{n}{2} \times \frac{n}{2}$, and recursively solve on one of those grids. However, we run into problems if we find a number on the boundary. It may not be a local minimum in the wider picture.

The key is to strengthen the inductive hypothesis. Suppose we have already found one square in an $n \times m$ grid with value x. We claim we can find a local minimum *with value at most x* in at most $4(n + m)$ seconds. Certainly this is true if either of n or m is at most 2.

For the inductive step, first sub-divide the board into 4 sub-grids each of size at most $\frac{n+1}{2} \times \frac{m+1}{2}$. Read the numbers on the inner boundaries B of all such squares. This takes $2(n + m) - 4$ seconds. Let the smallest number on the boundaries be y, and let $z = \min(x, y)$.

Recursively find a local minimum with value at most z on the sub-grid containing z using at most $4(\frac{n+1}{2} + \frac{m+1}{2}) = 2(n + m) + 4$ seconds. If that local minimum is not on B, it is *also* a local minimum of the complete grid, and we are done. Otherwise, since the local minimum is at most z, it can only be y, and it is still a local minimum of the complete grid (because y is less than the other numbers it is adjacent to in B). Therefore, we have found a local minimum in at most $2(n + m) - 4 + 2(n + m) + 4 = 4(n + m)$ seconds, as required. □

The other way in which an induction problem can be hard is figuring out how to reduce to previous cases. Do not try the most obvious reduction and then give up if it doesn't work! If you know a claim is true in *every* situation, that is a lot of information! Put it all together and see what you can get from that.

Example 3. *Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \le a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that*

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \ge a_n$$

for each positive integer n.[1]

Solution. Certainly the result holds for $n = 1$. Now suppose the result holds for values less than n, and we try to show it for n. We know $a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots + \frac{a_{n-1}}{n-1} \ge a_{n-1}$, so the obvious thing to try is to show $a_{n-1} + \frac{a_n}{n} \ge a_n$. However, it turns out that is not true. Suppose $a_{1,2,3,4} = \{5, 6, 6, 11\}$. You can check this satisfies the conditions of the problem, but $a_3 + \frac{a_4}{4} = 8.75 < 11 = a_4$. Oops. It is tempting to give up on induction at this point, but not all is lost!

[1]APMO 1999 #2

Using the result for each value less than n, we have:

$$\begin{aligned} a_1 &\geq a_1 \\ a_1 + \frac{a_2}{2} &\geq a_2 \\ a_1 + \frac{a_2}{2} + \frac{a_3}{3} &\geq a_3 \\ &\dots \\ a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots + \frac{a_{n-1}}{n-1} &\geq a_{n-1} \end{aligned}$$

If we add these all up, we get

$$\begin{aligned} & (n-1)a_1 + \frac{(n-2)a_2}{2} + \frac{(n-3)a_3}{3} + \ldots + \frac{a_{n-1}}{n-1} \geq a_1 + a_2 + \ldots + a_{n-1} \\ \implies \quad & na_1 + \frac{na_2}{2} + \frac{na_3}{3} + \ldots + \frac{na_{n-1}}{n-1} \geq 2(a_1 + a_2 + \ldots + a_{n-1}) \end{aligned}$$

However, the problem condition ensures $2(a_1 + a_2 + \ldots + a_{n-1}) \geq (n-1)a_n$, so we are now done. □

Example 4. *Let E be a set with n elements. Suppose that $A_1, A_2, \ldots, A_k$ are k distinct nonempty subsets of E such that $A_i \cap A_j = \emptyset, A_i \subset A_j$ or $A_j \subset A_i$ for each $1 \leq i < j \leq k$. What is the maximum value for k?*[2]

Solution. The answer is $2n-1$. For example, if $E = \{1, 2, \ldots, n\}$, this can be achieved by taking $\{1\}, \{2\}, \ldots, \{n\}$ and $\{1, 2\}, \{1, 2, 3\}, \ldots \{1, 2, 3, \ldots, n\}$. The main challenge is proving that you cannot do better.

Towards that end, suppose the result is true for values less than n, and consider $A_1, A_2, \ldots, A_k$ maximizing k. Again, reducing to just $n-1$ for the "obvious" induction is difficult. You would probably consider sets containing a single element and prove there is at most 2 of them. I could not make this work. However... we might as well have E be on the sets. Assume without loss of generality that it is A_1 and A_2 is the second-largest set (breaking ties arbitrarily).

Excluding A_1, no A_i can contain A_2 (since A_2 is maximal). Therefore, all other sets are contained in A_2 or are disjoint from it. The first collection of sets solves the conditions of the problem with E replaced by $|A_2|$, and so there are at most $2|A_2|-1$ such sets (including A_2 itself). The second collection of sets solves the conditions of the problem for E replaced by $E - A_2$, and so there are at most $2(n - |A_2|) - 1$ such sets. Adding them all together gives $k \leq 1 + 2|A_2| - 1 + 2(n - |A_2|) - 1 = 2n - 1$, as required. □

2 Problems

1. Let n be an integer. Consider all points (a, b) in the plane with integer co-ordinates such that $0 \leq a, 0 \leq b$, and $a + b \leq n$. Show that if these points are covered by straight lines then there are at least $n+1$ such lines.

[2] China West 2012 #3

2. For every finite set of positive integers A, define the collection $S(A)$ to be

$$S(A) = \{a + b | a, b \in A, a \neq b\}.$$

For example, if $A = \{1, 2, 3, 4\}$, then $S(A) = \{3, 4, 5, 5, 6, 7\}$. (Note that we are allowed to repeat elements in a collection.)

Prove that there are infinitely many positive integers n for which we can find distinct sets A, B with $|A| = |B| = n$ and $S(A) = S(B)$.

3. A sequence of polygons is derived as follows. The first polygon is a regular hexagon of area 1. Thereafter each polygon is derived from its predecessor by joining to adjacent edge midpoints and cutting off the corner. Show that all the polygons have area greater than $\frac{1}{3}$.

4. (a) Prove that every positive integer n, except a finite number of them, can be represented as a sum of 2015 positive integers: $n = a_1 + a_2 + \ldots + a_{2015}$, where $1 \leq a_1 < a_2 < \ldots < a_{2015}$, and $a_i | a_{i+1}$ for all $1 \leq i \leq 2014$.

   (b) Prove that every positive integer not exceeding $n!$ can be expressed as the sum of at most n distinct positive integers each of which is a divisor of $n!$.

5. Suppose that T is a tree with k edges. Prove that the k-dimensional cube can be partitioned into graphs isomorphic to T. (If you do not understand the question, feel free to ask!)

6. Does there exist a set S of 10 distinct integers such that $(x - y) | x$ for all $x, y \in S$?

7. Given a simple polygon in the plane whose vertices lie on lattice points, show that the area of the polygon is given by $I + B/2 - 1$, where I is the number of lattice points entirely within the polygon and B is the number of lattice points that lie on the boundary of the polygon.

8. Positive integers $x_1, x_2, \ldots, x_n$ $(n \geq 4)$ are arranged in a circle such that each x_i divides the sum of the neighbors; that is

$$\frac{x_{i-1} + x_{i+1}}{x_i} = k_i$$

is an integer for each i, where $x_0 = x_n$, $x_{n+1} = x_1$. Prove that

$$2n \leq k_1 + k_2 + \cdots + k_n < 3n.$$

9. Im playing the colour-country game against Jacob. We take turns; on my turn, I draw in a country. On Jacobs turn, he chooses any colour for the country, but he must make sure that no adjacent countries share the same colour. Is it possible for me to force Jacob to use more than 9000 colors?

10. Prove that every planar graph can have its edges directed such that the in-degree of each vertex is at most 3.

11. Suppose that each positive integer not greater than $n(n^2 - 2n + 3)/2$ is coloured one of two colours (red or blue). Show that there must be an n-term monochromatic sequence $a_1 < a_2 < \ldots < a_n$ satisfying $a_2 - a_1 \leq a_3 - a_2 \leq \ldots \leq a_n - a_{n-1}$.

12. Let $r \geq 2$ be a fixed positive integer, and let F be an infinite family of sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r-1$ that meets each set in F.

13. Each square of a $(2^n - 1) \times (2^n - 1)$ board contains either 1 or -1. Such an arrangement is called successful if each number is the product of its neighbors. Find the number of successful arrangements.

14. I have several coins, arranged into piles at integer positions. I can do the following moves:

    - If there are at least two coins on pile k, I can move one of these coins to pile $k-1$ and the other to pile $k+1$.
    - If there is at least one coin in both of piles $k-1$ and k, I can move them to piles $k-2$ and $k+1$.

    I keep making moves until it is no longer possible. Prove that the process eventually terminates, and the final configuration does not depend on what moves I do.

15. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, and let M be any set of $n1$ positive integers which does not contain $S = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis from 0 to S making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on a point in M.

16. Prove that it is possible to colour every vertex of a planar graph with 5 colours such that any two adjacent vertices are different colours.

3 Hints

1. Is it possible without making one of the lines $a + b = n$? (Folklore)
2. It is only possible if n is a power of 2. Given A, B, can you construct A', B' that are twice as big? (Folklore)
3. Prove that the figure stays convex and you can never get rid of an edge entirely. (USAMO 1997 #4)
4. (a) Try taking $a_1 = 1$ and inducting on the later numbers. (China 2014)
   (b) Choose the first integer so that the difference is a multiple of n.
5. You need to strengthen the inductive hypothesis. Pick a "special" vertex on the tree. Argue that you can solve the problem with one special vertex at every other vertex. (Iran TST 2008)
6. The answer is yes. If $(a_1, a_2, \ldots, a_k)$ works, try adding a number and then playing with transformations like $(Ma_1 + D, Ma_2 + D, \ldots, Ma_k + D)$. (Folklore, but see USAMO 1998 #5)
7. Use induction to reduce the problem to triangles. Now embed the triangles into rectangles with sides parallel to the axes. (Picks Theorem)
8. The lower bound is just AM-GM. For the upper bound, what happens if you remove the largest integer from the list? (Taiwan 2014 TST #1)
9. Yes. Strengthen the hypothesis to ensure that the last country I drew is connected to the outer region of the plane.
10. You might as well assume everything is triangulated. Prove more strongly that the outer points have in-degree at most 2. (Some British thing?)
11. I didn't try it yet. Looks fun though! (MOP 1997)
12. Call a set A "good" if it is a subset of infinitely many elements of F. What is the largest good set? (IMO Shortlist 2002 C5)
13. The only solution is when all numbers are 1. Given a solution of a different type, try making it symmetric. Now reduce to $n - 1$. (Russia 1998 Grade 10 #8)
14. First prove it terminates from any given starting configuration in a bounded number of moves. Now prove that if you make two different moves from the same position, then you *can* get them back to the same position. And now induct.
15. Consider withholding the largest a_i and solving inductively. What can go wrong, and can you repair it? (IMO 2009 #6)
16. From Euler's formula $E \leq 3V - 6$, so some vertex has degree ≤ 5. If it is less than 5, colour the rest, then colour this vertex last. Otherwise, look at its neighbors, and two of them must not be adjacent to each other. Use that.