Project PALM

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1 Introduction

1.1 Notation

Parantheses generally contain the author (and/or the class where these notes were used), brackets contain the specific page range I'm referring to.

2 Geometry

2.1 Real Basics [hmm]

This is for the poor sods who have never done geometry in their life. At this stage WOOT is a pretty good option, since it does try to provide a lot of motivation and advice.

- (WOOT 2011) Circle Geometry - includes similar triangles, homothety, cyclic quads, ptolemy, power, orthocenter, Simson and tangential quads
- (WOOT 2012) Similar Triangles and Power of a Point - what it says on the tin + cyclic quads and ptolemy

2.2 Fundamentals

Don't get this wrong: these are not the basics. This is about the mastery of the fundamentals: the little things that matter in geometry, other than theorems and configurations.

- (Carlos Shine @ MOP 2010) Angle Chasing - shows you several ways equal angles can appear: from the standard cyclic quads to spirals to isgonal conjugates
- (Evan Chen) Directed Angles - if you want to know why directed angles are important. Another reference is (David Arthur @ Canada 2011 Winter) Geometry Fundamentals [1-5].
- (Carlos Shine @ MOP 2011) Constructions [1] - some strategies for constructing auxiliary points/lines.
- (Matthew Brennan @ Canada 2014 Winter) Synthetic Geometry [2-5] - talks about redefining points to make things easier.
- (Evan Chen @ BMC) Parallelograms - more construction. I like this one a lot.
- (Waldemar Pompe) Quadrilaterals.pdf - a lot of nice problems about cyclic/tangential quadrilaterals, written by an ISL proposer.
- (Richard Rusczyk @ WOOT 2008) Geometry - The Stupid Stuff Works - a walkthrough of USAMO 2008 #2, a little bit of problem solving philosophy

Current gaps - (1) No problem set about areas, midpoints etc. (2) Also, far too little non-configurational (and basic) geometry problem sets - have to rely on Chinese books for now. (3) No article that does a good explanation of the "mental Geogebra" heuristic.

2.3 Power of a Point/Radical Axes

There are many, many articles that deal with this, but I would start with (Yufei Zhao) Power of a Point.

Alternative references include:

- (David Arthur @ Canada 2011 Winter) Geometry Fundamentals [5-8] - essentially a more modern version of Yufei's
- (Michal Rolinek, Josef Tkadlec) Power of a Point - illustrates how to use power to bash

2.4 Collinearity/Concurrency

So everyone has heard of Ceva and Menelaus, but just in case you need a quick refresher, (WOOT 2012) Concurrency and Collinearity is the best one (with a lot of other useful facts/advice for this class of problems in general).

More problems also available from (Victoria Krakovna @ Canada 2010 Summer) Concurrency and Collinearity [2].

Personally, I can't recall the last time I used C/M. Usually it's only when other things (like Projective/IC) can't save you from the grind.

2.5 Projective I - perspectivities, cross-ratio, harmonic(, pole/polars?)

Projective geometry is about interpreting synthetic properties in as projective properties, then manipulating the projective properties (usually via *perspectivities*, or projection through a point). For the newcomer used to standard synthetic proofs, these concepts may be difficult to grasp at first since they are so different. (Alexander Remorov @ Canada 2010 Summer) Projective Geometry [1-2] has most of the facts down, but I love the diagrams in (Yufei Zhao @ Canada 2008 Summer) Circles [3-5].

Both sets also talk about poles and polars but I find that it is rarely useful; more of a conceptual tool than a problem solving one.

Current gaps - No notes talking comprehensively about how to use perspectivities. Also, it seems that same-circle perspectivities are currently undocumented.

2.6 Projective II - Pascal/Brianchon, Desargues

The difference between this and Projective I (above) is mainly the use of more advanced theorems as opposed to purely perspectivities. The most important theorem in this group is **Pascal's Theorem**, and (Carl Joshua Quines) Pascal's Theorem is excellent, providing a lot of motivation and examples.

For a quick glossary of the theorems (and some problems), you can look at both (Victoria Krakovna @ Canada 2010 Summer) Concurrency and Collinearity [3-8] and (Alexander Remorov @ Canada 2010 Summer) Projective Geometry Continued [2]. **Current gaps** - The Desargue point-swapping trick.

2.7 Mean Geometry

This refers to the interpretation of points (and in general geometric objects) as algebraic quantities, but not in a coordinate geometry/complex numbers kind of way. Instead, we focus on simple algebraic relationships between points: for instance, the midpoint M of a segment AB can be thought of as $\frac{1}{2}A+\frac{1}{2}B$

Of course, in this territory, the classic work to read is (Zachary Abel) Mean Geometry.

2.8 Transformations

Since there are quite a few, let's list them down:

- **Homothety** - despite it being quite fundamental, there's no dedicated article to this (yet). But (Alexander Remorov @ Canada 2010 Summer) Projective Geometry Continued [2] does have a small segment on it, but it's treated more as a configuration rather than a technique.
- **Isometries** - this refers to reflection, rotation and translation. (u/c)
- **Inversion** - (u/c)
- **Projective/Affine** - (u/c)

2.9 Configurations

Aside from techniques, it is good to be able to recognize recurring configurations that appear in geometry problems. Familiarity with common configurations can cut down solving time and boost your understanding of geometry, which is excellent news for newcomers, but as you become more experience you should be wary not to be over-reliant on recognition as opposed to actual solving.

The absolute classic here is of course (Yufei Zhao @ Canada 2007 Summer) Lemmas in Euclidean Geometry, which covers:

- Symmedian lemma (intersection of tangents lie on symmedian)
- Diameter of incircle lemma, but please see (Evan Chen) The Incenter-Excenter Lemma as well.
- Spiral similarity (briefly)

- “Chicken feet” lemma
- Mixtilinear incircles, but I recommend the more updated and complete (Evan Chen) Mixtilinear Incircles instead.
- Quadrilateral mixtilinear incircle lemma, a broken-down walkthrough of the (rather difficult) main lemma is available at (Yufei Zhao @ Canada 2008 Summer) Circles [2-3]
- Touch chord-median intersection
- Midline-angle bisector-touch chord concurrency
- Orthocenter reflection lemma
- O and H are isogonal conjugates

Other important configurations include:

- **Miquel/Spiral Similarity/Brokard** - (Yufei Zhao @ Canada 2009 Winter) Cyclic Quadrilaterals is a must-read for this. This configuration really appeared a lot a while back.
- **Tangential quadrilateral** - There's a little bit at (Waldemar Pompe) Quadrilaterals.pdf
- **Anti-Steiner points** - nowhere yet.
- **Symmedian/Lemoine Point** - (u/c) Ricky Liu's one
- **Triangle centers** - (u/c) Abel's one
- **Casey's theorem** - For when you want to bash anything with tangent circles in the diagram.

2.10 Dark Arts

Well, when all else fails... there's always bashing. Two important things to take note (for computational methods in general):

1. Please treat computational methods as an alternative rather than a shortcut. This means you should practice bashing a lot, as you would geometry as a whole, and actually memorize all the formulas.
2. Know the limits of your selected computational method(s). You'll know what I mean if you've tried doing mixtilinear stuff with coordinate geometry. Ideally, you should be able to tell when problems/subproblems are bashable.

That being said, computational methods can often be rewarding in competition scenarios, and lately there's been a **hybriding** trend: basically using synthetic observations to repeatedly reduce the questions to a bashable subproblem. This has proved to be extremely potent, even on Q3s.

A quick overview of all computational methods:

- Coordinate geometry. Not the best, but the most introductory. This is a good model to understand the other methods with. There's no article for this, unfortunately.
- Trigonometry. As a general rule, once you've used cosine rule and/or angle addition formulae you're officially in a bash. No article for this, either.
- Length chasing. Depending on the problem and your approach, you may have trigonometric quantities or just pure length ratios (if you stuck to C/M, Stewart etc). For a quick reference, see DA (u/c)
- Barycentric coordinates. (u/c) Evan
- Complex. (u/c) Yi Sun

3 Algebra

3.1 Functional Equations

Probably the most important category of questions under Algebra. In some senses, functional equations are like geometry problems: there are some established tools available, but the really interesting ones still require to invent someting on the spot.

As a starting point, (Evan Chen) Introduction to Functional Equations just about covers everything for $\mathbb{R}$-functional equations, along with detailed examples. An additional example is available at (Adrian Tang @ Canada 2010 Summer) Functional Equations.

Functions on domains other than $\mathbb{R}$ tend to be either solvable with the same techniques (and hence easier due to domain restrictions), or they may admit pathological solutions (again, arising from the domain-specific properties). The best exposition so far of this area is (Evan Chen) Monsters, which mainly focuses on the pathological solutions to Cauchy's Equation, but also includes a walkthrough on a pathological $\mathbb{Z}$ functional equation.

In fact, sometimes functional equations on $\mathbb{N}$ tend to involve number-theoretic properties, like divisibility and number bases, but there's no single article covering that yet.

Current gaps - Pseudolinearity, and NT-specific methods.

3.2 Inequalities

You should not expect to see an old-style inequality at any competition, but I still think it's fruitful for beginners to have a cursory glance through the "standard toolbox", perhaps at (Samin Riasat) Basics of Olympiad Inequalities, or (David Arthur) Cauchy-Schwarz and Other Classical Inequalities

But aside from the standard tools, it's good in general to have a good concept of:

- **convexity** - knowing when sums of convex functions attain their maximum and/or minimum. Even some facts from majorization theory (e.g. Karamata) can be helpful. See the section about Abel Summation below.
- **double roots** - this gives rise to the tangent line and isolated fudging methods: there's always one step where you attempt to balance the coefficients to make the double root come out. See also: (Thomas Mildorf @ MOP 2011) Inspired Inequalities [1-2].
- **smoothing** - adjusting variables to reduce the difference between the two sides of the inequality. Mixing Variables often refer to this idea as well. See also (Thomas Mildorf @ MOP 2011) Inspired Inequalities [2-3], or (Adrian Tang @ Canada 2010 Winter) Mixing Variables.
- **equality cases** - the most important concept of them all. No dedicated article to this, but it's scattered everywhere.

If inequalities do make a come back, it's likely to be some weird sequencey stuff or something that relies more on generic bounding. The following topics may be worth a look:

- **Abel summation** - Abel (u/s)
- **Sum estimations** - basically, inequalities with a combinatorial flavour (!). See (u/c) for a problem set.

Current gaps - Still lacking an article that talks about smoothing on a more conceptual level. Perhaps also looking forward to better discussions for equality cases.

3.3 Polynomials

(u/c)

3.4 Sequences

This is currently quite an under-explored medium for algebra (especially since the advent of functional equations), so the only reference (= 1 example with a massive problem set) is (Alexander Remorov @ Canada 2012 Winter) Sequences.

Other tangible topics include:

- **telescoping (sums/products)** - the simpler cousin of Abel. (Gabriel Carroll @ MOP 2010) Tricky Sums and Products covers this plus a lot of other strategies for evaluating sums.
- **linear recurrences** - See (WOOT 2010) Linearly Recurrent Sequences for a quick introduction.
- **analysis/growth rate** - No articles yet.

Current gaps - Well, there's no conceptual breakdown of this topic at all. An overview of some bounding concepts (for seq-ineqs) or a walkthrough of some "combinatorics-inspired" strategies (think IVT) may be useful.

4 Other generic algebra

(u/c) completeness of reals.

maybe future article about "fractal" bounds.

5 Number Theory

6 Combinatorics