1 Introduction

In many challenging problems, one of the biggest difficulties can be finding something to try. In most geometry problems, however, our greatest hurdle is quite the opposite: there are zillions of things we could try. The hard part is figuring out which one might be fruitful.

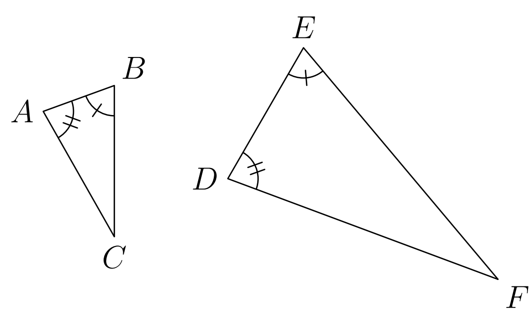
In this article, we outline a number of general problem solving tools that are useful in geometry problems. Of course, this is by no means an exhaustive list! You're probably familiar with many of these tools, as well as with some of the results we offer as examples or exercises. As you read through these, and as you look at geometry problems in the future, you shouldn't just focus on learning exactly what the tools are and how to use them. It's very important, *perhaps even more important*, to think about when and why you use the tools. So, when you read a solution to a problem you couldn't solve on your own, don't just soak up the steps. Think about what clues exist in the problem that would suggest to you to try each step.

In this article, we'll offer a brief description of important facts, a list of tips for when to use each one, and some examples and exercises.

2 Similar Triangles

This article is titled "Geometry of the Circle." But it's just plain hard to do geometry without focusing on triangles. In a great many challenging geometry problems, a key step is finding a pair of similar triangles. (Congruent triangles are a subset of similar triangles; we'll focus on similar triangles here.)

Below, we have similar triangles ABC and DEF.



Because the triangles are similar, we have

$$\angle A = \angle D, \quad \angle B = \angle E, \quad \angle C = \angle F,$$

and

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

Of course, we don't have to prove all of these facts to deduce that triangles ABC and DEF are similar. The most common ways to deduce that triangles ABC and DEF are similar are:

- **AA Similarity**. If $\angle A = \angle D$ and $\angle B = \angle E$, then triangles ABC and DEF are similar. This is by far the most common way to determine that two triangles are similar. Also, this is part of the reason that it's important to mark equal angles in your diagram as you find them. Then, the similar triangles will stand out more clearly. Once you find similar triangles, you should write down the fact that they are similar so you won't forget — you'll often learn facts in geometry problems before you need them. So, keep a list of what you discover that you can refer to when you're stuck.
- **SAS Similarity**. If $\angle A = \angle D$ and $AB/DE = AC/DF$, then triangles ABC and DEF are similar. This is obviously of limited use, but when you have a problem in which you are given information regarding ratios, you might think of trying to use this at some point in the problem. SAS Similarity can also come into play in problems in which two triangles share an angle.
- **SSS Similarity**. If $AB/DE = AC/DF = BC/EF$, then triangles ABC and DEF are similar. This one's of pretty limited usefulness. Obviously, you'd have to have lots of information about lengths for this to be useful.

Clues that it's time to look for similar triangles include:

- **Parallel lines**. Parallel lines mean equal angles. Equal angles mean similar triangles. Parallel lines are so useful in finding similar triangles that sometimes we add a parallel line to a diagram just to make similar triangles.
- **Perpendicular lines**. If you have several right angles in a problem, then you almost certainly have a bunch of right triangles. In each pair of these triangles, you already have one pair of equal angles, so you only have to find one more pair of equal angles to find similar triangles. (This isn't quite the tip-off that parallel lines are, since perpendicular lines are a huge tip-off for other tools as well, such as the Pythagorean Theorem and cyclic quadrilaterals.)
- **Angles inscribed in the same arc**. If two angles are inscribed in the same arc, these angles are equal. Look for triangles including these angles, and try to find other pairs of equal angles.
- **You have information about ratios of segment lengths**. If you are given an equality involving ratios of sides, this is a sure sign to look for similar triangles.
- **You need a length and nothing else has worked**. You've tried the Pythagorean Theorem. You've tried areas. You've even (gasp) tried coordinates. But you still can't find a needed length in a problem. Try building similar triangles by drawing in lines parallel to lines in the problem, or by dropping perpendiculars. There's always something to try!

3 Homothety

Homothety is a special type of similarity. You may know it as dilation, or as 'homothecy.' See the following link for an article outlining the basics of homothety:

`http://www.artofproblemsolving.com/Forum/blog.php?u=1163&b=87`

Tips to use homothety in a problem include:

- **Tangent circles**. Any two circles are homothetic, but when you have two tangent circles, you also have useful information about one center of homothety (the point of tangency).
- **Common tangents to two circles**. If the common tangents to two circles are part of a problem, then you should consider homothety because these two tangents will intersect at a center of homothety that maps one circle to the other.
- **The Medial Triangle**. The triangle connecting the midpoints of the sides of a triangle is homothetic to the original triangle. (Prove it!) Any problem involving the medial triangle or the centroid of a triangle is a candidate for homothety.
- **Intersecting lines pass through endpoints of parallel segments**. If $\overline{AB} \parallel \overline{XY}$, and lines $\overleftrightarrow{AX}$ and $\overleftrightarrow{BY}$ meet at P, then P is the center of a homothety that maps $\overline{AB}$ to $\overline{XY}$.
- **Collinearity problems**. If we must prove that points A, P, and X are collinear, one way to do so is to show that A and X are corresponding points on two figures that are homothetic under a homothety with center P. Usually you can tell pretty quickly if you have two figures that might be candidates for such a homothety. So, on a collinearity problem, you can briefly consider this approach and usually discard it quickly if it obviously won't work.
- **Concurrency problems**. Lines $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$, and $\overleftrightarrow{CZ}$ are concurrent if they connect corresponding points on two homothetic figures (the center of homothety is the point at which they meet). As with collinearity, it's usually quickly obvious whether or not this approach has any hope of working.

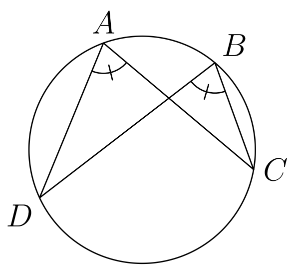
Homothety in general is an example of why it's very important on tough geometry problems to draw large, accurate diagrams. If you draw a precise diagram, it will usually be pretty obvious when two lines are probably parallel, and parallel lines often lead to homothetic triangles. However, if you just make a rough sketch, you might miss a homothety that blows a problem away.

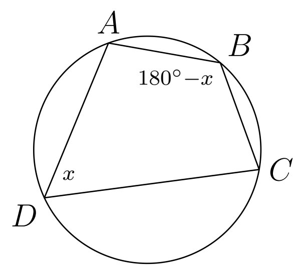
If you don't quite grasp what the big deal is about homothety, don't worry about it just yet. Master the other tools first. Homothety is largely just a fancy way of saying 'similar triangles.' In general, much of what we can do with homothety can be done (with a bunch more steps) with similar triangles and other Euclidean geometry tools.

4 Cyclic Quadrilaterals

A quadrilateral that can be inscribed in a circle is a *cyclic quadrilateral*. Discovering cyclic quadrilaterals is a key step in a great many tough geometry problems. Here are a variety of ways to prove that a quadrilateral is cyclic.

- **Equal Inscribed Angles**. If $\angle CAD = \angle CBD$ in convex quadrilateral $ABCD$, then $ABCD$ is cyclic. This is yet another reason marking equal angles is so important; if you find two equal angles that intersect as described above, then you have a cyclic quadrilateral.





- **Supplementary Opposite Angles**. If $\angle ABC + \angle CDA = 180^\circ$ in convex quadrilateral $ABCD$, then $ABCD$ is cyclic. This one is particularly useful when you have a lot of right angles in a problem. If two of these angles intersect, odds are you have a cyclic quadrilateral.

Cyclic quadrilaterals are extremely useful in problems that require angle-chasing (finding equal angles). This is because once we prove a quadrilateral is cyclic, then we have a bunch of pairs of equal angles from the angles that inscribe common arcs. For example, once we know that $ABCD$ is cyclic, we immediately know that $\angle ACB = \angle ADB$, $\angle ABD = \angle ACD$, and so on.

Here are some clues to look for cyclic quadrilaterals:

- **You are angle chasing**. A very common way to show that two angles are equal is to first find a cyclic quadrilateral involving the two angles, then use the resulting circle to show the two angles are inscribed in the same arc. This is obviously only useful if the angles intersect appropriately (such as $\angle ACB$ and $\angle ADB$ above).
- **You're trying to show that two appropriately intersecting angles are supplementary**. If your target angles intersect as $\angle ABC$ and $\angle CDA$ do in our sample cyclic quadrilateral, and you must show that these angles are supplementary, then showing that you have a cyclic quadrilateral, like $ABCD$ above, will do the job.
- **Right angles**. Any two intersecting right angles will give you a cyclic quadrilateral. If you have a bunch of right angles in a problem, you have a bunch of cyclic quadrilaterals.

- **You're doing a medium or tough national olympiad problem**. Finding a cyclic quadrilateral is a step in a large portion of medium-to-tough national olympiad problems. Often, you'll have to find several cyclic quadrilaterals.

These are only a few of the reasons you might look for cyclic quadrilaterals, but they capture most of the main reasons you might seek them. Above all, circles are very helpful with angle chasing, due to the relationships between angles and the arcs they intercept. So, discovering a circle that involves multiple points in your diagram can be extremely useful.

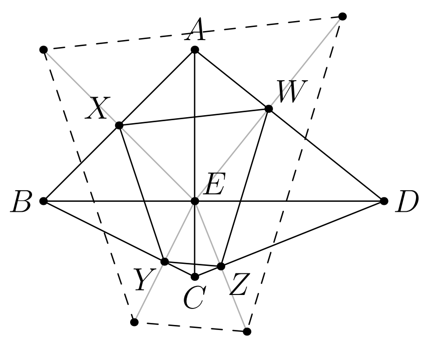
Problem. Let $ABCD$ be a convex quadrilateral such that the diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic. (USAMO, 1993)

Solution. Let X, Y, Z, and W be the projections of E onto AB, BC, CD, and DA, respectively. Let h denote the homothety centered at E with ratio $1/2$. Then quadrilateral $XYZW$ is the image of the quadrilateral in the problem under h, so it suffices to show that quadrilateral $XYZW$ is cyclic. To prove that quadrilateral $XYZW$ is cyclic, it suffices to prove that $\angle WXY + \angle YZW = 180°$.

Since $\angle EWA = \angle EXA = 90°$, quadrilateral $EWAX$ is cyclic. Similarly, quadrilaterals $EXBY$, $EYCZ$, and $EZDW$ are cyclic.

We can split $\angle WXY$ into angles $\angle EXW$ and $\angle EXY$. Since qudarilateral $EWAX$ is cyclic, $\angle EXW = \angle EAW$. Since quadrilateral $EXBY$ is cyclic, $\angle EXY = \angle EBY$. Similarly, we can split $\angle YZW$ into angles $\angle EZY$ and $\angle EZW$. Since quadrilateral $EYCZ$ is cyclic, $\angle EZY = \angle ECY$. Since quadrilateral $EZDW$ is cyclic, $\angle EDW = \angle EDA$. Putting everything together, we find

$$\begin{aligned}\angle WXY + \angle YZW &= \angle EXW + \angle EXY + \angle EZY + \angle EZW\\ &= \angle EAW + \angle EBY + \angle ECY + \angle EDW\\ &= \angle EAD + \angle EBC + \angle ECB + \angle EDA\\ &= 90° + 90°\\ &= 180°.\end{aligned}$$



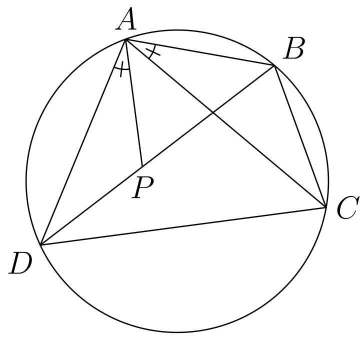
Hence, quadrilateral $XYZW$ is cyclic. ■

4.1 Ptolemy's Theorem

If quadrilateral $ABCD$ is cyclic, then

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

Proof. Take point P on BD such that $\angle DAP = \angle CAB$.



Since $\angle ADP = \angle ADB = \angle ACB$, triangles DAP and CAB are similar, so

$$\frac{DP}{BC} = \frac{AD}{AC}.$$

Also, $\angle APB = \angle 180° - \angle APD = 180° - \angle ABC = \angle ADC$, and $\angle ABP = \angle ABD = \angle ACD$, so triangles APB and ADC are similar. Hence,

$$\frac{BP}{CD} = \frac{AB}{AC}.$$

From these equations, $DP = BC \cdot AD/AC$ and $BP = AB \cdot CD/AC$. Adding these equations, we get

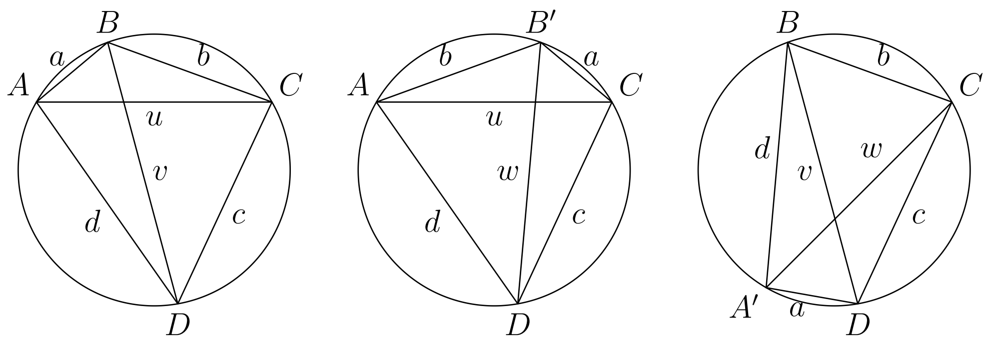
$$\frac{BC \cdot AD + AB \cdot CD}{AC} = BP + DP = BD,$$

so $AB \cdot CD + BC \cdot DA = AC \cdot BD$. ■

Problem. Let $ABCD$ be a cyclic quadrilateral, and let $a = AB$, $b = BC$, $c = CD$, and $d = DA$. Find the length of diagonals AC and BD in terms of a, b, c, and d.

Solution. Let $u = AC$ and $v = BD$. Then by Ptolemy's theorem, $uv = ac + bd$. Of course, we need further equations to determine u and v. To generate further equations, we think about other ways of arranging the sides lengths a, b, c, and d.

Since each side length corresponds to a fixed arc length on the circumcircle of cyclic quadrilateral $ABCD$, we can arrange the side lengths a, b, c, and d in any order, to produce another cyclic quadrilateral with the same circumcircle. For example, if we arrange the side lengths in the order b, a, c, and d, then we produce cyclic quadrilateral $AB'CD$, as shown in the second diagram.



Note that in quadrilateral $AB'CD$, we still have diagonal $AC = u$. Let $w = B'D$. Then by Ptolemy's theorem on quadrilateral $AB'CD$, $uw = ad + bc$.

We can also arrange the side lengths in the order d, b, c, and a, to produce cyclic quadrilateral $A'BCD$, as shown in the third diagram. Note that we still have diagonal $BD = v$. Since the circles in the second and third diagrams have the same radius, the chords of length b and d subtend the same angle. Hence, triangles CBA' and $B'AD$ are congruent, so $A'C = B'D = w$. Then by Ptolemy's theorem on quadrilateral $A'BCD$, $vw = ab + cd$.

Multiplying all three equations produced by Ptolemy's theorem, we get

$$u^2v^2w^2 = (ab + cd)(ac + bd)(ad + bc),$$

so

$$uvw = \sqrt{(ab + cd)(ac + bd)(ad + bc)}.$$

Hence,

$$AC = u = \frac{uvw}{vw} = \sqrt{\frac{(ac + bd)(ad + bc)}{ab + cd}}$$

and

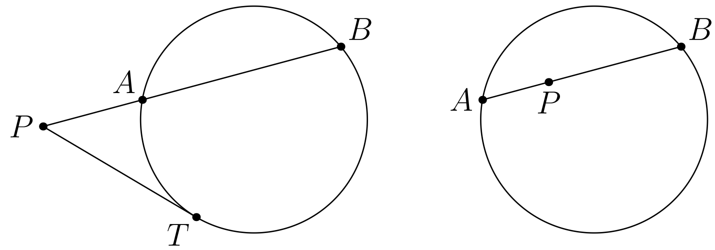
$$BD = v = \frac{uvw}{uw} = \sqrt{\frac{(ac + bd)(ab + cd)}{ad + bc}}.$$

■

5 Power of a Point

We mentioned 'inscribed angles' in our list of clues to look for similar triangles. Power of a point is the fancy name given to a powerful example of such similar triangles.

Given a circle centered at O with radius r, the *power* of point P with respect to the circle is defined as $PO^2 - r^2$. Note that the power is positive, zero, or negative, according to whether P is outside, on, or inside the circle, respectively.



If a line through P intersects the circle at A and B, then $PA \cdot PB$ is equal to the power of P. Note that we are using directed line segments when computing this product. For example, in the first diagram, $PA \cdot PB$ is considered positive because PA and PB "point" in the same direction, but in the second diagram, $PA \cdot PB$ is considered negative because PA and PB point in opposite directions.

If P lies outside the circle and PT is a tangent to the circle, then the power of point P is equal to PT^2.

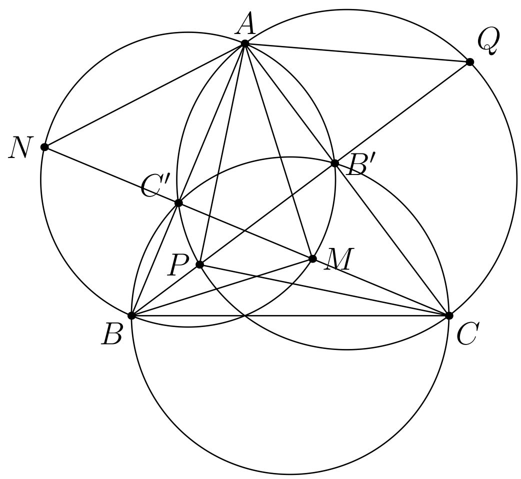
The tip-offs to use power of a point are not surprising:

- **Circles and lengths**. You have circles, and you have (or want) information about lengths. You should be considering ways to use power of a point.
- **Circles and ratios**. You have circles, and you have information about ratios. You should be thinking about power of a point.

Notice, however, that power of a point will probably not be so useful if you're dealing with an inscribed polygon and have information about side lengths. Power of a point is useful with secants, tangents, and intersecting chords. It's not so helpful with chords that do not intersect (unless they are parts of secants that meet at a useful point) or with chords that share an endpoint.

Problem. An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle. (USAMO, 1990)

Solution. We claim that all four points M, N, P, and Q are equidistant from A. By symmetry, AB is the perpendicular bisector of MN, so $AM = AN$. Similarly, AC is the perpendicular bisector of PQ, so $AP = AQ$.



Since M lies on the circle with diameter AB, $\angle AMB = 90^\circ$. Then right triangles $AC'M$ and AMB are similar, so

$$\frac{AC'}{AM} = \frac{AM}{AB},$$

which means $AM^2 = AB \cdot AC'$. Similarly, right triangles $AB'P$ and APC are similar, so

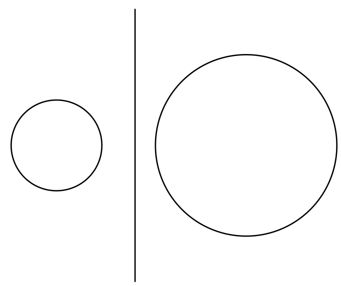
$$\frac{AB'}{AP} = \frac{AP}{AC},$$

which means $AP^2 = AB' \cdot AC$.

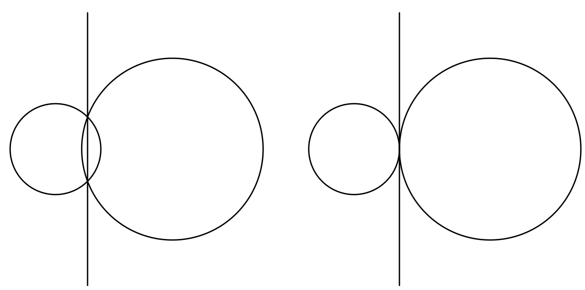
Finally, since $\angle BB'C = \angle BC'C = 90^\circ$, points B' and C' lie on the circle with diameter BC. Hence, quadrilateral $BCB'C'$ is cyclic, so by power of a point, $AB \cdot AC' = AB' \cdot AC$. Therefore, $AM = AP$, which means M, N, P, and Q are all equidistant from A. ■

5.1 Radical Axis

Given two non-concentric circles, the locus of points P such that P has the same power with respect to both circles is called the *radical axis* of the two circles. The radical axis is always a line which is perpendicular to the line joining the center of the two circles.

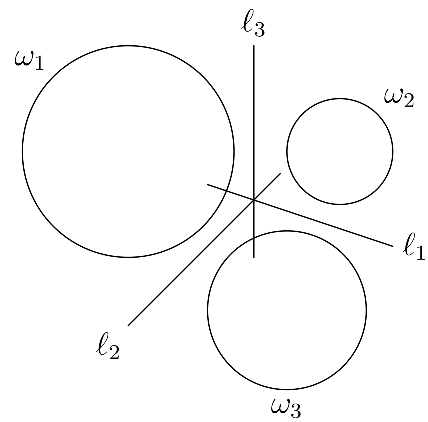


If the two circles intersect at two points, then the radical axis is the line passing through the two points of intersection, and if the two circles are tangent (internally or externally), then the radical axis is the common tangent.



5.2 Radical Center

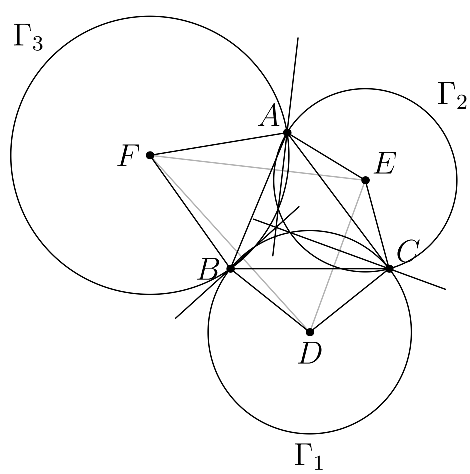
Let ω_1, ω_2, and ω_3 be three circles, where no two circles have the same center. Let ℓ_1, ℓ_2, and ℓ_3 be the radical axes of ω_2 and ω_3, ω_1 and ω_3, and ω_1 and ω_2, respectively, and let R be the intersection of ℓ_1 and ℓ_2.



Since R lies on ℓ_1, R has the same power with respect to ω_2 and ω_3. Since R lies on ℓ_2, R has the same power with respect to ω_1 and ω_3. Therefore, R has the same power with respect to ω_1 and ω_2, which means that R lies on ℓ_3. Hence, ℓ_1, ℓ_2, and ℓ_3 are all concurrent at R. This point of concurrence is called the *radical center* of the three circles.

Problem. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent. (USAMO, 1997)

Solution. Since $DB = DC$, both B and C lie on a circle centered at D. Thus, let Γ_1 be the circle centered at D with radius $DB = DC$. Also, let Γ_2 be the circle centered at E with radius $EA = EC$, and let Γ_3 be the circle centered at F with radius $FA = FB$.



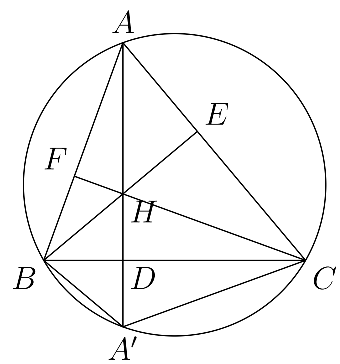
Since E and F are the centers of Γ_2 and Γ_3, respectively, the line through A that is perpendicular to EF is the radical axis of Γ_2 and Γ_3. Similarly, the line through B that is perpendicular to DF is the radical axis of Γ_1 and Γ_3, and the line through C that is perpendicular to DE is the radical axis of Γ_1 and Γ_2. Hence, the three lines in the problem concur at the radical center of the three circles.

The equal distances in the diagram led us to the three circles, and when you have a concurrency problem involving three circles, you should think of the radical center of the circles. ■

6 Circumcircle of a Triangle

6.1 The Orthocenter

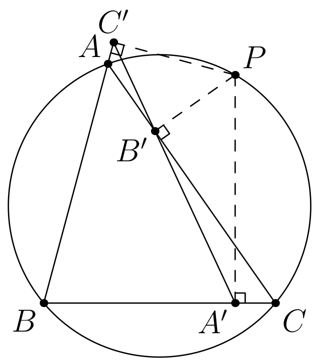
Let AD, BE, and CF be the altitudes of triangle ABC. We know that these altitudes concur at the orthocenter H of triangle ABC. Extend altitude AD to the circumcircle of triangle ABC at A'.



Then $\angle A'BC = \angle A'AC = \angle DAC = 90^\circ - \angle ACB = \angle EBC = \angle HBC$, and $\angle A'CB = \angle A'AB = \angle DAB = 90^\circ - \angle ABC = \angle FCB = \angle HCB$. Therefore, triangles $A'BC$ and BHC are congruent. It follows that the reflection of H in side BC lies on the circumcircle of triangle ABC. By symmetry, the reflections of H in sides AB and AC also lie on the circumcircle of triangle ABC. (Our argument assumes that triangle ABC is acute, but the result holds for right and obtuse triangles as well.)

6.2 The Simson Line

Let P be a point on the circumcircle of triangle ABC, and let A', B', and C' be the projections of P onto sides BC, AC, and AB, respectively. We claim that points A', B', and C' are collinear. We will prove this by showing that $\angle C'B'A = \angle A'B'C$.



Since $\angle AB'P = \angle AC'P = 90^\circ$, quadrilateral $PB'AC'$ is cyclic. Therefore, $\angle C'B'A = \angle C'PA$. But

$$\begin{aligned} \angle C'PA &= 90^\circ - \angle PAC' \\ &= 90^\circ - (180^\circ - \angle PAB) \\ &= 90^\circ - \angle PCB \\ &= \angle A'PC, \end{aligned}$$

so $\angle C'B'A = \angle A'PC$.

Since $\angle CA'P = \angle CB'P = 90^\circ$, quadrilateral $PB'A'C$ is also cyclic. Therefore, $\angle A'PC = \angle A'B'C$, so $\angle C'B'A = \angle A'B'C$, as desired.

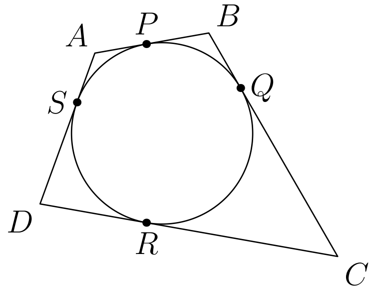
We conclude that points A', B', and C' are collinear. The line $A'B'C'$ is called the *Simson line* of point P with respect to triangle ABC.

7 Circumscribed Quadrilaterals

A convex quadrilateral is *circumscribed* if each side is tangent to a circle inside the quadrilateral, called the *incircle* of the quadrilateral. There is a simple characterization of circumscribed quadrilaterals.

Pitot Theorem. Convex quadrilateral $ABCD$ is circumscribed if and only if $AB + CD = BC + AD$.

Proof. First, we prove that if convex quadrilateral $ABCD$ is circumscribed, then $AB + CD = BC + AD$. Let the incircle of quadrilateral $ABCD$ be tangent to AB, BC, CD, and DA at P, Q, R, and S, respectively.

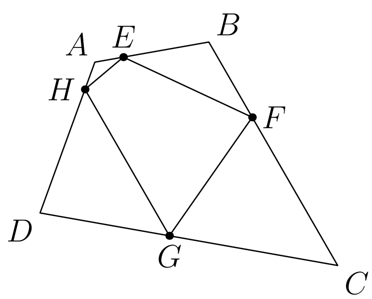


Then

$$\begin{aligned}AB + CD &= (AP + BP) + (CR + DR)\\&= AS + BQ + CQ + DS\\&= (BQ + CQ) + (DS + AS)\\&= BC + AD.\end{aligned}$$

Now, we prove that if $AB+CD = BC+AD$, then convex quadrilateral $ABCD$ is circumscribed. Without loss of generality, assume that AB is the shortest side of quadrilateral $ABCD$. Then $CD = BC + (AD - AB) \geq BC$, and $CD = AD + (BC - AB) \geq AD$, so CD is the longest side of quadrilateral $ABCD$.

Let E be an arbitrary point on side AB. Then there exist points F and H on sides BC and AD, respectively, such that $BF = BE$ and $AH = AE$. There also exists a point G on side CD such that $CG = CF$.



Then

$$\begin{aligned} DG &= CD - CG \\ &= BC + AD - AB - CF \\ &= AD - AB + (BC - CF) \\ &= AD - AB + BF \\ &= AD - AB + BE \\ &= AD - (AB - BE) \\ &= AD - AE \\ &= AD - AH \\ &= DH. \end{aligned}$$

Hence, triangles AHE, BEF, CFG, and DGH are all isosceles. Then $\angle AEH = (180^\circ - \angle A)/2$ and $\angle BEF = (180^\circ - \angle B)/2$, so

$$\begin{aligned} \angle HEF &= 180^\circ - \angle AEH - \angle BEF \\ &= 180^\circ - \frac{180^\circ - \angle A}{2} - \frac{180^\circ - \angle B}{2} \\ &= \frac{\angle A + \angle B}{2}. \end{aligned}$$

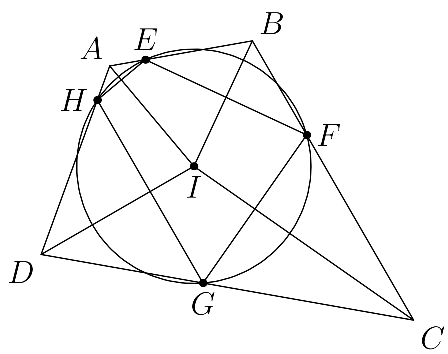
Similarly,

$$\begin{aligned} \angle FGH &= 180^\circ - \angle CGF - \angle DGH \\ &= 180^\circ - \frac{180^\circ - \angle C}{2} - \frac{180^\circ - \angle D}{2} \\ &= \frac{\angle C + \angle D}{2}, \end{aligned}$$

so

$$\angle HEF + \angle FGH = \frac{\angle A + \angle B + \angle C + \angle D}{2} = \frac{360^\circ}{2} = 180^\circ.$$

Therefore, quadrilateral $EFGH$ is cyclic. Let I be the circumcenter of quadrilateral $EFGH$.



Then I lies on the perpendicular bisector of EH. But triangle AEH is isosceles, so A also lies on the perpendicular bisector of EH, which means that AI is the angle bisector of $\angle DAB$. Similarly, BI, CI, and DI are the angle bisectors of $\angle ABC$, $\angle BCD$, and $\angle CDA$, respectively.

Since I lies on the angle bisector of $\angle DAB$, I is equidistant from AB and AD. Similarly, I is equidistant from AB and BC, BC and CD, and CD and DA, which means that I is equidistant from all four sides of quadrilateral $ABCD$. Therefore, I is the center of a circle that is tangent to all four sides. ■

8 Some General Strategies

These are general guidelines for approaching geometry problems. As we noted earlier, one of the largest challenges in olympiad geometry problems is that there are usually zillions of different things you can try on any given problem. The trouble is managing all the information you have, and all the information you can find.

- **Draw large precise diagrams**. A great many solutions hinge on discovering one or two surprising facts. By drawing a large, precise diagram, you might identify three lines that look concurrent, or lines that look parallel, or segments that look equal, or angles that look right, etc. Moreover, in trying to figure out how to construct a diagram with straightedge and compass, you might discover some facts you can use to solve the problem.
- **Draw more than one diagram**. Just because your first diagram makes two lines look parallel doesn't mean a second will. Before you spend 20 minutes trying to show that two lines are parallel, draw a second diagram and see if whatever your first diagram suggests is true still looks true in the second diagram.
- **Don't just stare**. Mark equal angles and equal segments as you find them. Assign variables to lengths of segments or to measures of angles, and use your geometry toolchest to find other lengths or measures in terms of those of those variables. Then, write this information on your diagram. Having all this information on your diagram will help you see relationships you wouldn't see if you just hold this information in your head.
- **Use the clues**. Use the general information you have in the problem to determine what general tools are likely to be useful. Don't just forget about all other tools, but focus on those that are most likely to be fruitful first.
- **Work backwards**. In many geometry problems, particularly proofs, you can work backwards from what you want until you hit something you know how to find (or prove). Don't just work forwards from the given information.
- **Keep all your observations organized**. On most hard problems, you'll work both forwards and backwards. So, you'll have a list of 'What I Know' and a list of 'What I Need'. When something pops up on both lists, then you're done. However, if you mix up your lists, you're almost certain to mess up

(usually, you'll end up with circular reasoning somewhere). So, keep these lists separate — on different papers for more complicated problems.

- **Use all your information**. If you're stuck on a problem, look back at the given information. If there's anything there you haven't used to make more observations, that's where you should look. Also, look at your 'What I Know' list; anything there that looks fruitful, but is yet unused, is something you should investigate further. 'What haven't I used yet?' is the first question that comes into my mind whenever I am stuck.

- **Don't fall for the fancy stuff**. You know inversion. You know complex numbers. You know vectors. You know projective geometry. Big deal. Most olympiad problems are specifically chosen so that the fancy-shmancy tools are no more helpful than the basic tools of Euclidean geometry. You shouldn't spend more than a few minutes on any of your fancy stuff — if the fancy stuff helps, it will help almost immediately. But if you spend 30 minutes playing with inversion, then you're probably barking up the wrong tree.

- **Don't spend too much time on algebra**. You really love analytic geometry and pounding away with equations. I won't say you should never use analytic geometry, but you should use it very judiciously. Set up the problem so that your expressions are as simple as possible. If it isn't obvious within 5-10 minutes that analytic geometry will provide a solution (even if it will take you 10 minutes to wade through the algebra), then you should probably stop and try other approaches. Don't just blindly push equations around and hope something good happens. Only slog through algebra on a geometry problem when you have a clear plan to get the answer.

9 Exercises

1. Equilateral triangle ABC is inscribed in a circle. Let E and F be the midpoints of AB and AC, respectively. Line segment EF is extended past F to meet the circle at G. Find the ratio EF/FG.

2. Let $ABCD$ be a quadrilateral with perpendicular diagonals, and let M, N, P, and Q be the midpoints of sides AB, BC, CD, and DA, respectively. Let M' be the projection of M onto the opposite side CD, and define N', P', and Q' similarly. Prove that the points M, N, P, Q, M', N', P', and Q' all lie on the same circle.

3. Let ABC be an equilateral triangle, and let P be a point on side BC. The line AP intersects the circumcircle of triangle ABC at Q. Show that
$$\frac{1}{PQ} = \frac{1}{BQ} + \frac{1}{CQ}.$$

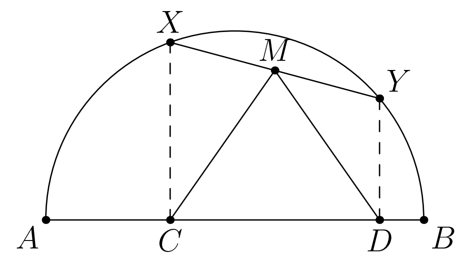
4. A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by AB, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A. (AIME, 1991)

5. Two circles, Γ_1 and Γ_2, are internally tangent at P. A line intersects Γ_1 at A and D, and Γ_2 at B and C. Prove that $\angle APB = \angle CPD$.

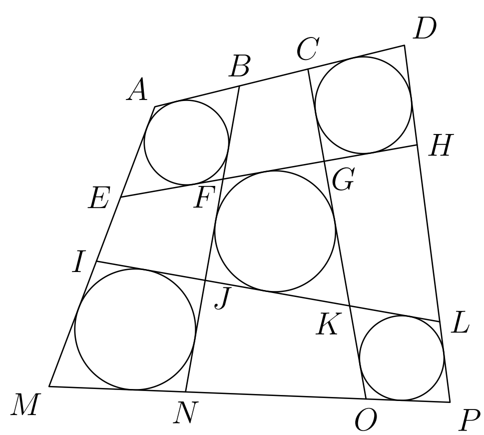
6. Let ABC be a triangle. A circle intersects side BC at A_1 and A_2, side AC at B_1 and B_2, and side AB at C_1 and C_2. The line through A_1 perpendicular to BC, the line through B_1 perpendicular to AC, and the line through C_1 perpendicular to AB are concurrent.

   Show that the line through A_2 perpendicular to BC, the line through B_2 perpendicular to AC, and the line through C_2 perpendicular to AB are concurrent.

7. Let XY be a chord of constant length that slides around the semicircle with diameter AB. Let M be the midpoint of AB, and let C and D be the projections of X and Y onto AB, respectively. Show that $MC = MD$, and that $\angle CMD$ is constant for all positions of chord XY.



8. In the diagram below, quadrilaterals $ABFE$, $CDHG$, $FGKJ$, $IJNM$, and $KLPO$ are all circumscribed. Show that quadrilateral $ADPM$ is also circumscribed.



9. Let Γ_1 and Γ_2 be two circles, with centers O_1 and O_2, respectively. Let KL be a common external tangent, and let MN be a common internal tangent, with K and M on Γ_1, and L and N on Γ_2. Prove that KM, LN, and O_1O_2 are concurrent.

10. Let H be the orthocenter of triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, and H are collinear. (China, 1996)