

Number Theory

James Rickards

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Quadratic Residue Rules

Let a be an integer, and p an odd prime. Define

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \\ 0, & \text{if } p \mid a \\ -1, & \text{otherwise} \end{cases}$$

For $n = p_1 p_2 \cdots p_r$ an odd positive integer expressed as a product of distinct primes, we define

$$\left(\frac{a}{n}\right) = \prod_{i=1}^r \left(\frac{a}{p_i}\right)$$

Note that $\left(\frac{a}{n}\right) = 1$ does not necessarily mean a is a quadratic residue modulo n , merely that it is a non-residue modulo an even number of prime factors of n (with multiplicity).

To calculate quadratic residues, the following rules suffice (m, n are coprime odd positive integers):

$$\begin{aligned} \left(\frac{a}{n}\right) &= \left(\frac{b}{n}\right) \text{ if } a \equiv b \pmod{n} & \left(\frac{ab}{n}\right) &= \left(\frac{a}{n}\right) \left(\frac{b}{n}\right) \\ \left(\frac{-1}{n}\right) &= \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \\ -1 & \text{if } n \equiv 3 \pmod{4} \end{cases} & \left(\frac{2}{n}\right) &= \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{8} \\ -1 & \text{if } n \equiv \pm 3 \pmod{8} \end{cases} \\ \left(\frac{m}{n}\right) &= (-1)^{\frac{m-1}{2} \frac{n-1}{2}} \left(\frac{n}{m}\right) \end{aligned}$$

Quadratic Residue Problems

1. Let $p \equiv 1 \pmod{4}$. Prove that the sum of the quadratic residues modulo p in $[1, p-1]$ equals the sum of the non-residues in the interval. Does the same hold when $p \equiv 3 \pmod{4}$?

2. Let p be a prime. Find the value of $\sum_{a=1}^{p-1} \left(\frac{a^2 + a}{p}\right)$.

3. Let a be a positive integer which is not a square. Prove that there are infinitely many primes p such that $\left(\frac{a}{p}\right) = -1$.

4. Prove that for all $a \in \mathbb{Z}$, the number of solutions (x, y, z) of the congruence

$$x^2 + y^2 + z^2 = 2axyz \pmod{p}$$

equals $\left(p + \left(\frac{-1}{p}\right)\right)^2$.

5. Let a, b, c be positive integers. Prove that $\frac{a^2 + b^2 + c^2}{3(ab + bc + ca)}$ is not an integer.

6. (New Zealand, 2010) Show that there are infinitely many pairs of distinct primes (p, q) such that $p \mid 2^{q-1} - 1$ and $q \mid 2^{p-1} - 1$.

7. An odd *composite* positive integer n is called a *strong pseudoprime to base b* for an integer $b > 1$, coprime to n , if $b^{\frac{n-1}{2}} \equiv \left(\frac{b}{n}\right) \pmod{n}$. Prove that for every odd composite positive integer n , there exists a base $b > 1$ coprime to n such that n is *not* a strong pseudoprime to base b .

8. (China TST 2003) n is a positive integer which is not a multiple of 2 or 3, and there does not exist nonnegative integers a, b such that $|2^a - 3^b| = n$. Find the minimum possible value of n .

Other Generic Number Theory Problems

1. (Romania TST 2015) Given an integer $k \geq 2$, determine the largest number of divisors the binomial coefficient $\binom{n}{k}$ may have in the range $n - k + 1, \dots, n$, as n runs through the integers greater than or equal to k .

2. (Jacob's 2008 Summer Camp handout) Define a_n recursively by $\sum_{d|n} a_d = 2^n$. Prove that $n \mid a_n$.

3. (China TST 2008) Let $n > 1$ be an integer, and $n \mid 2^{\phi(n)} + 3^{\phi(n)} + \dots + n^{\phi(n)}$. Let p_1, p_2, \dots, p_k be the distinct prime factors of n . Prove that $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} + \frac{1}{p_1 p_2 \dots p_k}$ is an integer ($\phi(n)$ is Euler's totient function, the number of positive integers at most n which are relatively prime to n).

4. (2015 Iran MO) Let $n \geq 50$ be a natural number. Prove that we can express $n = x + y$, as a sum of two natural numbers x, y such that for every prime number p such that $p \mid x$ or $p \mid y$ we have $\sqrt{n} \geq p$. For example, for $n = 94$, we have $x = 80, y = 14$.

5. (2008 All-Russian) Given a finite set P of primes, prove that there exists a positive integer x such that it can be written in the form $a^p + b^p$ (for positive integers a, b) for each $p \in P$, and it *cannot* be written in that form for each $p \notin P$.