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On Writing Good Math Problems

I am a professional writer of mathematics problems. Strictly speaking, this is true: I compose virtually every problem that appears on the Mandelbrot Competition, and this undertaking does result in enough revenue each fall to permit me to continue producing the contest for another year. However, my primary purpose in writing these problems is to generate interest in mathematics, not to generate income. I enjoy writing math problems and have achieved a fair measure of success in doing so: aside from creating thirteen tests worth of problems each year for Mandelbrot my problems have appeared on the USAMO for the past two years, among other places.

I believe that writing and, more to the point, seeking out math problems is a vitally important component of one's mathematical well-being. You may wonder if perhaps I am overstating the case for problem writing, but I don't think so. Choosing to be a consumer only, rather than a producer also, of mathematics is comparable to taking an interest in the outdoors but never venturing off wide, well-marked paths or attempting to explore seemingly inaccessible peaks. True, there are still plenty of nice views to be had, but this approach lacks the heightened level of engagement with the terrain that accompanies a foray without a clear route or destination. Ultimately we must experience uncharted mathematics in order to appreciate the compelling nature of the subject to the degree necessary for it to have an impact upon us of true significance. It is this sort of experience that transforms math enthusiasts and decent problem-solvers into mathematicians.

Of course, just as with wandering off the trail, looking for nice problems can be a risky proposition. You might create problems that you cannot answer or that you find unappealing. Worse yet, you may feel that you are unable to write an original problem at all. (But this is almost always an illusion.) To be honest, I feel a twinge of dread every spring when the time comes to put together the next batch of Mandelbrot contests. It seems inconceivable that there are enough interesting, elementary problems remaining to populate five rounds worth of tests. But each year I take the step of faith necessary to begin writing, and each August I am amazed, frankly, to discover that I have a set of polished tests ready for the printers, full of nice problems. (Well, at least I think they are nice.)

So what happens during May and June? Where do all those problems come from? How exactly does one go about writing interesting math problems? This is a rather deep inquiry, akin to asking an artist to explain (in fifteen minutes or less) how to create a nice painting. There are quite a few elements that play an important role in the process. Let's take a look at some of them now.

1. Mathematical Exposure

All good problem writers have spent countless hours reading about clever ideas, working on interesting problems, attending talks—whether interesting and clever or not—and discussing all of the above with their peers. (If you are reading this essay then you probably already fall into this category of people, so I won't belabor the point.) Of course, one shouldn't engage in these activities solely as a means to problem-writing ends. But it is safe to assume that more mathematical exposure can only lead to a greater number and wider variety of original ideas. So go ahead and learn what Conway's surreal numbers are all about. You won't necessarily write problems on precisely this topic; but who knows, perhaps you will mentally file away some particularly fascinating aspect of the theory, which will then play a key role in the development of your own ideas at a later date.

2. Focused Creative Efforts

It won't do to stare at a blank sheet of paper hoping that an intriguing math problem will materialize in your head. It is usually much more productive to approach problem-writing with a plan in mind, such as "Let's see if I can create a slick system of two equations in two variables," or "I wonder what can be said about the distances from the vertices to the incenter of a triangle," or maybe "What sorts of counting problems involve coloring the edges of a cube?" You will be amazed at how quickly you find a direction of inquiry that is appealing to you. (Incidentally, when I run problem writing workshops I am equally amazed at how a roomful of twenty individuals can take topics such as these and go in twenty completely different directions.)

For example, when I proposed the first topic to myself a few years back, I decided to make linear equations with cute coefficients. The result was a popular Mandelbrot problem that featured Fibonacci numbers in a system of two equations. The second inquiry led to the discover that the product of the distances from the vertices of a triangle to its incenter equals $4Rr^2$, where r is the inradius and R is the circumradius. And the final idea turned into a USAMTS problem that later decorated the T-shirt distributed to participants that year.

Perhaps a word on originality is in order. It is inconceivable that any of the results just mentioned is brand new, in the sense that nobody has ever explored them before. In reality, I'm sure that a significant percentage of elementary mathematics is discovered and rediscovered many times over. But why should the delight of developing a nice mathematical idea be limited to a single person? Granted, there is a certain thrill that accompanies the unearthing of a truly original result. On the flip side of that coin, there is also a responsibility to find out how much is already known before claiming a result as one's own. However, these matters alone should never deter one from delving deeper into an appealing topic in search of nice problems.

3. Good Questioning Ability

If I had to pinpoint one aspect of our educational system (or of our culture in general, for that matter) that inhibits individuals interested in math from becoming mathematicians, I would respond that we do not teach students how to ask questions. *Mathematicians know how to ask questions.* By the way, there is a difference between problems and questions, in my mind. A problem is a request to perform a mathematical task, or provide a mathematical explanation, which you are unsure how to accomplish at the outset. (By comparison, an exercise is a mathematical task which you do already know how to achieve.) But a question is more open-ended; it invites exploration in a certain direction; it is a specific wondering.

As an illustration, a mathematician might ask any one of the following questions upon learning of Hero's formula[1] $K = \sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle: Does it give the correct result for an equilateral triangle? What about a 3-4-5 triangle? More generally, how does the formula reduce algebraically to give the correct result for any right triangle? Does a similar formula exist for quadrilaterals? Is there a clever geometric proof of the formula? Can it be extended to tetrahedra? When does a triangle with integer side lengths have an integral area? What if the side lengths are the roots of a cubic polynomial—can we predict the area given the polynomial? This final thought led to one of my all-time favorite Mandelbrot questions; namely, let the cubic $x^3 - 4x^2 + 5x - 1.9$ have real roots a, b, and c. Find the area of the triangle with sides of length a, b, and c. (Answer later.)

[1]Here a, b, and c are the side lengths and $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter.

I believe that asking good questions is more of a discipline than an innate talent. In other words, most of us are capable of asking promising questions, but we tend not to in the absence of sufficient prodding. Not to mention that most people are woefully out of practice to boot. So the next time that you come across an interesting idea, ask yourself (or the person who shared the idea with you) lots of related questions. Don't be satisfied with the first three that come to mind; the most obvious questions are usually not the ones that lead in really intriguing or unexpected directions. And finally, jot down your best questions in a notebook dedicated to promising ideas, so that you can come back to them later when you have time for serious investigation. Because if you actually get into the habit, you will discover that you are capable of generating far more interesting questions than you have time to follow up on!

4. Problem-Solving Skills

Obviously you can only go as far with a question as your problem-solving skills will carry you. There is more at stake, however, than not being able to find a solution to a problem that you pose to yourself. As I will discuss in detail momentarily, it happens more frequently than one might guess that on the way to unraveling one problem, another more inspired problem comes to light. It's as if on the way to one mountain peak, another more majestic one comes into view after making enough progress to round a bend. But that second gem is not accessible unless sufficient headway is made on the original problem. Since much has been written on mathematical problem-solving, I will not pursue the topic further here, other than to say that this skill is an important component of discovering and writing nice problems.

5. Intuition and Insight

These are the most elusive of the problem-writing skills. Intuition is the art of being able to follow one's nose; the ability to detect a nice problem or make a clever conjecture from seemingly inadequate clues. Insight is the act of realizing how to properly understand a mathematical structure; of suddenly seeing the correct perspective from which to approach a problem. Rather than attempting to offer advice on how to develop these skills (I'm not sure that I have any words of wisdom to share, to be honest), let me instead illustrate them by briefly relating the story of how problem six on the 2008 USAMO came into being.

More than two years earlier I had asked myself whether I could construct an interesting polynomial based on a simple graph. I had recently learned a bit about Alexander polynomials and Reidemeister moves at a talk on knot theory and was inspired to play around with these ideas myself in the context of simple graphs, with the ulterior motive of finding a nice problem for use on the Mandelbrot Team Play. I did come up with such a polynomial in two variables, having the property that its partial derivative was equal to the sum of the polynomials associated with all of the vertex-deleted subgraphs. (By this point I was also interested in the Reconstruction Conjecture, a notoriously difficult open problem in graph theory.) Ultimately my polynomial did not lead to any graph theoretical breakthroughs, but it did spawn quite a few other interesting ideas.

The most tantalizing of these observations concerned the fact that the values of my polynomial for $x = -1$ appeared to always be of the form $\pm 2^k$, or else equal to 0. By this point I had realized that the problem could be restated in terms of a sum over all possible subgraphs of a given simple graph. I figured the explanation would involve an easy induction, but several days later I still didn't have a proof. Days became weeks, other projects occupied my attention, and it was nearly a year before I applied myself in earnest to understanding this curious phenomenon and finally found my first, fairly ugly proof. In the process I had the insight to rephrase the problem to include a broader class of graphs for which the result still held. (The insight was analogous, for instance, to realizing that some problem involving real numbers is better understood in the context of complex numbers.)

It wasn't long before I had generalized this phenomenon, found more appealing proofs of my conjectures, brought matrices and roots of unity into the picture, and generally "followed my nose" to discover that my original observation fit into a much larger mathematical structure than I had previously imagined. At one point I realized that a fact about a certain 4×4 matrix was equivalent to the following statement:

> At a certain math conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two.

I managed to find an elementary proof of this fact and proposed it as a problem for the USAMO. The committee enjoyed the problem and selected it to appear as the final problem on the 2008 exam.

6. Patience and Persistence

In relating the above story I am struck anew by how long that problem was in the making. The proof I now have of my original observation (the one that took a year to find) reads so naturally, is so "obvious," that I figure I just must have been stupid not to have found it sooner. But this sort of story (and accompanying feelings of inadequacy, I suspect) is ubiquitous among mathematicians. Euler spent years finding and making rigorous his proof that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Gauss spent just as long working on his theory of quadratic reciprocity. These great mathematicians certainly were not short on insight, intuition, or problem-solving skills. But these qualities are not always enough when it comes to significant mathematical undertakings. The quest for a nice problem (and its accompanying solution) always benefits from a healthy dose of patience and persistence; most nice mathematical results cannot be found without them.

I like to distinguish between patience and persistence in the following manner: patience is the ability to set a task down; persistence is the habit of resuming it again. In other words, if you find a promising idea but are not satisfied an hour later with any of the ensuing problems you have written, then exercise patience and leave the topic be for a while. The temptation is to assume that if you can't capitalize on a promising avenue of inquiry right away, then you might as well give up. Patience is believing that you will be able to generate fresh ideas towards creative problems (and their solutions) after taking a break. And persistence, as already noted, is that inner determination and discipline to return to them as often as it takes to come up with a problem and solution with which you are pleased. Don't settle for a problem that you think isn't as clever or appealing as it could be, or a solution which is less than crisp and clean. Persist until you have created mathematics of which you are proud.

7. Motivation

It's no fun to labor over a polished problem if nobody is going to appreciate it afterwards. In that sense I'm lucky; over four thousand high school students tackle problems on the Mandelbrot Competition each year. (Let's not dwell on what happens when a typo creeps in or a legitimate alternate solution is overlooked, though.) Since nice math problems are perennially in short supply and high demand, it is usually possible to find a good home for them. For example, consider posting a problem to one of the forums at `artofproblemsolving.com`, where hundreds of your peers can try it out and reply with their solutions. College students are also welcome to submit original problems for use on the Mandelbrot Competition by sending a message to `<info@mandelbrot.org>`. More than a few Mandelbrot problems arise in this manner, including a nice counting problem on the upcoming round that was inspired by a student at St. Lawrence University, where I teach. Regardless, once you discover a nice problem, be sure to share it with as many people as you can.

All that remains is for you to take that step of faith, set aside some time, and begin the process of looking for nice math problems. What are you waiting for? Oh, maybe the solution to that triangle area problem from earlier. The answer is $\frac{1}{\sqrt{5}}$, believe it or not. Here's why. If a, b and c are the roots of the given polynomial then it may be factored as

$$x^3 - 4x^2 + 5x - 1.9 = (x-a)(x-b)(x-c).$$

By comparing the coefficients of x^2, it follows that $a + b + c = 4$, so the semi-perimeter is $s = 2$. We may now plug in $x = s$ to discover that

$$(s-a)(s-b)(s-c) = s^3 - 4s^2 + 5s - 1.9 = 2^3 - 4(2^2) + 5(2) - 1.9 = \tfrac{1}{10}.$$

Hence the area is $K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(2)(\frac{1}{10})} = \frac{1}{\sqrt{5}}$, as claimed. Now it's your turn. Happy problem hunting!