Transformations, Ceva and Menelaus Theorems, and harmonic points

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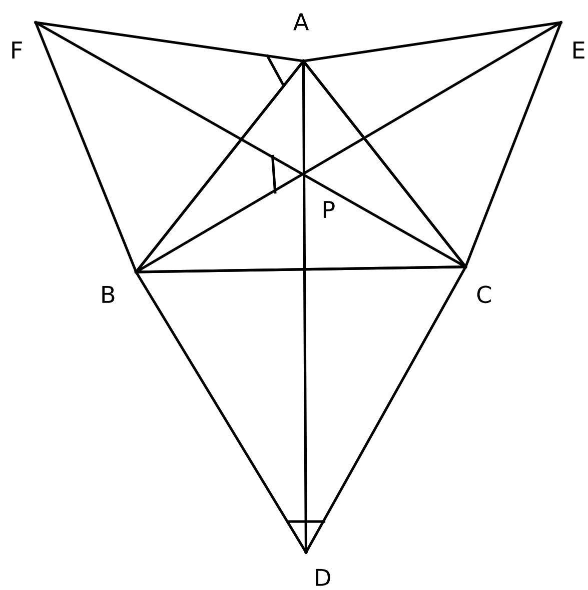
Transformations: rotations and the Fermat point

The following lemmas demonstrate a very nice use of rotations:

Lemma 1. Let ABC be a triangle with no angle greater than 120^o. Construct equilateral triangles BCD, ACE and ABF on the sides of $\triangle ABC$. Then AD, BE, CF are equal in length and concurrent at a point P, called the *Fermat point.*



Proof. Suppose BE and FC intersect at a point P. Since $\triangle FAC$ is congruent to $\triangle BAE$, and $\angle FAB = 60^o$, $\triangle FAC$ is taken to $\triangle BAE$ by a 60^o rotation about A. Thus $FC = BE$, and FC and BE are at a 60^o angle to each other, so $\angle FPB = 60^o$.

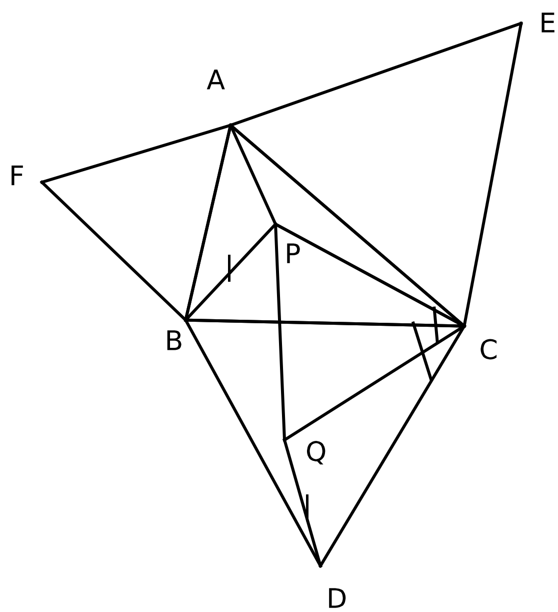
Since $\angle FAB = \angle BDC = 60^o$, $FAPB$ and $BPCD$ are concyclic. Then, $\angle FPA = \angle FBA = 60^o$, and $\angle BPD = \angle BCD = 60^o$. Thus, $\angle FPA + \angle FPB + \angle BPD = 180^o$, so P lies on AD. □

Lemma 2. Let P be a point inside $\triangle ABC$. Prove that $AP + BP + CP$ is minimized when P is the Fermat point.

Proof. Construct point Q such that $\triangle PCQ$ is equilateral. Since $\angle PCQ = \angle BCD = 60^o$, $PC = QC$ and $BC = CD$, the triangles BPC and DQC are congruent, and related by a 60^o rotation. Thus, $PB = QC$, so

$$AP + BP + CP = AP + DQ + PQ$$

This is the length of the path $APQD$, whose minimum value is the length of AD. This minimum is attained when P and Q both lie on AD, which implies that $\angle APC = 180^o - 60^0 = 120^o$, and $\angle BPC = \angle DQC = 180^o - 60^0 = 120^o$. Thus, $APBF$, $APCE$ and $BPCD$ are concyclic, and it follows that P must be the Fermat point. □



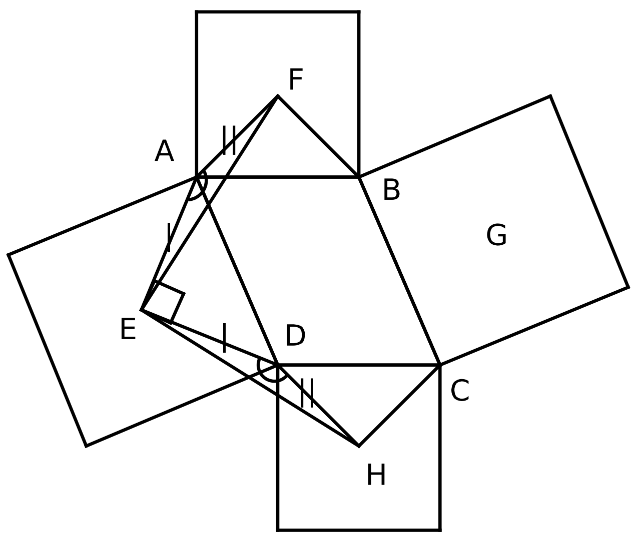
Example 1. Given a parallelogram $ABCD$, construct squares externally on its four sides. Prove that the centres of these squares form a square.

Proof. Let the centres of the squares be E, F, G, and H. We show that $\triangle EDH \sim \triangle EAF$.

From the square constructed on side AD, we get that $AE = DE$. Since $AB = CD$, the squares constructed on these sides have the same side length, and so $AF = DH$. Also,

$$\begin{aligned}\angle EAF &= 360^o - \angle EAD - \angle FAB - \angle DAB \\ &= 270^o - \angle DAB \\ &= 90^o + \angle ADC \\ &= \angle EDH\end{aligned}$$

Thus, a rotation about E by 90^o takes $\triangle EDH$ to $\triangle EAF$. In particular, EH is taken to EF, so $EH = EF$ and $\angle HEF = 90^o$. Similarly, we show this for the other sides of $EFGH$, so it must be a square. $\square$



More problems on rotations

1. Let ABC be a triangle with $\angle BAC = 90^o$. Let D be the foot of the perpendicular from A onto BC. Let I, J be the incentres of triangles BAD, CAD respectively. Prove the angle bisector from A in $\triangle ABC$ is perpendicular to IJ.

2. $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are two squares such that $A_1 = A_2$ but $B_1 \neq B_2$, $C_1 \neq C_2$, $D_1 \neq D_2$. Show that the lines B_1B_2, C_1C_2, D_1D_2 are concurrent.

3. Let ABC and BCD be equilateral triangles that share a side (A and D are distinct). A line passes through D, intersecting AC extended at M and AB at N. Let CN intersect BM at P. Prove that $\angle BPC = 60^o$. ("Mathematical Olympiad Challenges")

4. A point N is chosen on the longest side AC of triangle ABC. Perpendicular bisectors of AN and NC intersect AB and BC at K and M respectively. Prove that the circumcentre O of triangle ABC lies on the circumcircle of $\triangle KBM$. (Hint: you can use that the composition of two rotations is a rotation plus a translation.)

Miscellaneous problems on transformations and constructions

1. A knight is riding from city A to city B on the same side of a straight river. It's a long way, so he needs to stop by the river to give his horse a drink. What path should he take to spare his horse and minimize the total distance covered? In other words, if he reaches the river at point X, choose X so that $AX + XB$ is minimized.

2. P is a point inside rectangle $ABCD$ such that $\angle APD + \angle BPC = 180^o$. Find $\angle BAP + \angle DCP$. (MOSP 1995)

3. An equilateral triangle ABC of side length 1 is stacked on top of a square $BCDE$ of side length 1. Find the circumradius of $\triangle ADE$.

4. Let C_1 and C_2 be circles whose centers are 10 units apart and whose radii are 1 and 3. Find, with proof, the locus of points M for which there exist points X on C_1 and Y on C_2 such that M is the midpoint of the segment XY. (Putnam 1996, A2)

5. Let AC be a chord in a circle, and D be the midpoint of the minor arc AC. Let B be a point on the minor arc DC. Drop a perpendicular from D to AB at E. Prove that $AE = BE + BC$. (Archimedes' broken chord theorem)

Ceva and Menelaus Theorems, and harmonic points

The following results are useful in many geometry problems:

Ceva's Theorem. In $\triangle ABC$, let D, E, F be points on sides BC, AC, AB respectively. Then AD, BE, CF are concurrent iff

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1$$

Menelaus Theorem. In $\triangle ABC$, let E and F be points on sides AC and AB respectively, and let D be a point on BC extended. Then D, E and F are collinear iff

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1$$

The following lemma is a nice application of Ceva and Menelaus:

Definition. Four points A, B, C and D that are on a line (in the given order) are called *harmonic* if

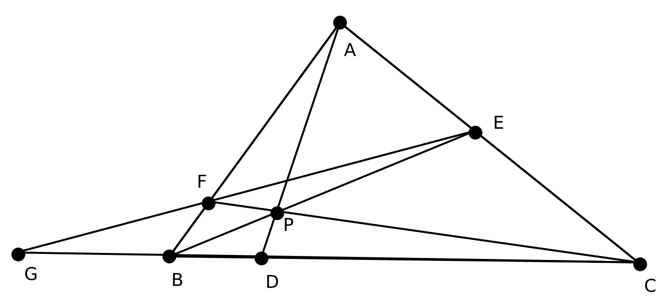
$$\frac{AB}{AD} = \frac{BC}{CD}$$

Lemma 3. In $\triangle ABC$, let D, E, F be points on sides BC, AC, AB respectively. Let G be the intersection of lines BC and EF, and K be the intersection of AD and EF. Then,

a) AD, BE, CF are concurrent iff G, B, D and C are harmonic (Pappus' Harmonic Theorem).

b) If AD, BE, CF are concurrent, then G, F, K and E are harmonic.

c) Also, for any collinear points G, B, D, C (in that order), and for M the midpoint of BC $\frac{BD}{DC} = \frac{BG}{GC} \Leftrightarrow GD \cdot GM = GB \cdot GC \Leftrightarrow DM \cdot DG = DB \cdot DC$.



Proof. a) By Menelaus' Theorem in $\triangle ABC$,

$$\frac{BG}{GC} \cdot \frac{CG}{AG} \cdot \frac{AF}{BF} = 1$$

Thus, $\frac{BD}{DC} \cdot \frac{CG}{AG} \cdot \frac{AF}{BF} = 1$ (AD, BE, CF are concurrent by Ceva's Theorem) iff $\frac{BD}{DC} = \frac{BG}{GC}$.

b) Proven using 3 applications of Menelaus - try it :).

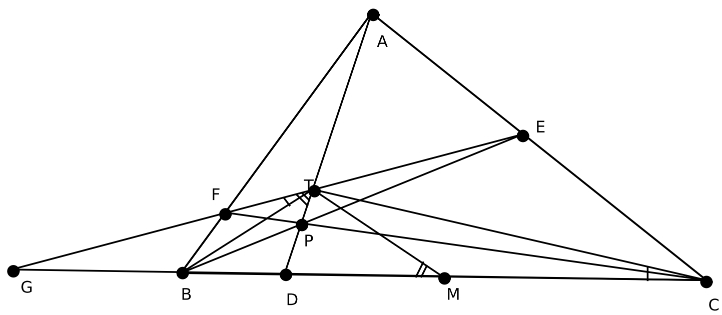
c) Since G, D, B and C are harmonic, $BD \cdot GC = GB \cdot DC$.

$$\begin{aligned}
GD \cdot GM &= (GB + BD) \cdot \frac{GB + GC}{2} \\
&= \frac{1}{2}(GB^2 + GB \cdot BD + GB \cdot GC + BD \cdot GC) \\
&= \frac{1}{2}(GB^2 + GB \cdot BD + GB \cdot GC + GB \cdot DC) \\
&= \frac{1}{2}GB \cdot (GB + BD + DC + GC) \\
&= \frac{1}{2}GB \cdot 2GC \\
&= GB \cdot GC \\
DG \cdot DM &= (GB + DB) \cdot \frac{DC - DB}{2} \\
&= \frac{1}{2}(GB \cdot DC + DB \cdot DC - GB \cdot DB - DB^2) \\
&= \frac{1}{2}(DB \cdot GC + DB \cdot DC - GB \cdot DB - DB^2) \\
&= \frac{1}{2}DB \cdot (GC + DC - GB - DB) \\
&= \frac{1}{2}DB \cdot 2DC \\
&= DB \cdot DC
\end{aligned}$$

The converses are proven similarly.

□

Example 2. Consider a point P inside a triangle ABC. Let AD, BE, CF be cevians through P. The midpoint M of BC different from D, and T is the intersection of AD and EF. Prove that if the circumcircle of $\triangle BTC$ is tangent to the line EF, then $\angle BTD = \angle MTC$. ("Mathematical Reflections", 2009, Issue 2, S119)



Proof. Let lines EF and BC intersect at G. By part b) of the Lemma, we have $GD \cdot GM = GB \cdot GC = GT^2$ (by Power of a Point). Thus, the circumcircle of $\triangle DTM$ is tangent to the line EF at T. Then,

$$\angle BTD = \angle GTD - \angle GTB = \angle DMT - \angle BCT = \angle MTC$$

□

More problems on harmonic points

1. In a convex quadrilateral $ABCD$, AC and BD intersect at E, AB and CD intersect at F. Line EF intersects AD and BC at X and Y. Let M and N be the midpoints of AD and BC, respectively. Prove that quadrilateral $BCMX$ is cyclic iff $AYND$ is cyclic. ("Mathematical Reflections", 2009, Issue 5, O135)

2. Consider triangle ABC with altitudes AD, BE, CF, and orthocentre H. Let lines EF and BC intersect at G, and let M be the midpoint of BC. Prove that $AM \perp GH$.

3. Let ABC be a triangle, and let D, E, F be the points of tangency of its incircle with the sides BC, AC and AB respectively. Let X be in the interior of $\triangle ABC$ such that the incircle of XBC touches XB, XC and BC in Z, Y and D respectively. Prove that $EFZY$ is cyclic. (IMO Shortlist 1995)

4. Let ABC be a right triangle with $\angle A = 90^o$. Let BN be the angle bisector of $\angle ABC$, and let D be a point on side AC between N and C. Denote by E the reflection of A across the line BD and F the intersection point of CE with the perpendicular to BC through D. Prove that AF, DE and BC are concurrent. (Junior Balkan TST 2007)

5. In $\triangle ABC$, I is the incentre, and E is the excentre opposite A. Suppose the excircle opposite A touches BC at F. Let AD be the altitude from A, with midpoint M. Prove that F, I, and M are collinear.

References

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[4] Titu Andreescu, Razvan Gelca, *Mathematical Olympiad Challenges*. Birkhauser, Boston, 2005.