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Estimating Sums

The objective of this lecture is to give some level of organization to an area of study on the border of algebra and combinatorics that currently is not generally recognized, and to survey the techniques that are useful in this area. I hereby denote this field of study "sum-estimation." In the typical sum-estimation scenario, you have a bunch of very loosely constrained real numbers (or vectors) and you want to bound the magnitude of some linear combination of them.

The most general sum-estimating technique is, of course, the pigeonhole principle. In particular, if one wants small sums or differences, a standard technique is to construct a bunch of sums, pigeonhole to find two that are close together, and subtract them.

1. (a) Given $\alpha > 0$ and integer $n > 0$, prove that there exists an integer $0 < k < n$ such that either $\{k\alpha\} \leq 1/n$ or $\{k\alpha\} \geq 1 - 1/n$. (The braces denote fractional part.)

   (b) Given positive real numbers $x_1, x_2, \ldots, x_n$ whose sum is an integer, prove that one can choose a nonempty proper sublist of the x_i such that the fractional part of the sum of this sublist is at most $1/n$.

2. (IMO, 1987) Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i| \leq k - 1$ for all i, and $|a_1x_1 + a_2x_2 + \cdots + a_nx_n| \leq (k-1)\sqrt{n}/(k^n - 1)$.

3. Given a set of n nonnegative numbers whose sum is 1, prove that there exist two disjoint subsets, not both empty, whose sums differ by at most $1/(2^n - 1)$.

4. Let n be an odd positive integer and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be nonnegative real numbers satisfying $x_1 + \cdots + x_n = y_1 + \cdots + y_n$. Show that there exists a proper, nonempty subet $J \subseteq \{1, \ldots, n\}$ such that

$$\frac{n-1}{n+1} \sum_{j \in J} x_j \leq \sum_{j \in J} y_j \leq \frac{n+1}{n-1} \sum_{j \in J} x_j.$$

5. Fix c with $1 < c < 2$ and, for $x_1 < x_2 < \cdots < x_n$, call the (unordered) set $\{x_1, x_2, \ldots, x_n\}$ "biased" if there exist $1 \leq i, j \leq n-1$ such that $x_{i+1} - x_i > c(x_{j+1} - x_j)$. Suppose $s_1, s_2, \ldots$ are distinct real numbers and $0 \leq s_i \leq 1$ for all i. Prove that there are infinitely many n such that the set $\{s_1, s_2, \ldots, s_n\}$ is biased.

   When the objects in question clearly exist in more than one dimension, drawing a picture is often helpful in deciding how to apply the pigeonhole principle.

6. (Poland, 1998) For $i = 1, 2, \ldots, 7$, a_i and b_i are nonnegative numbers such that $a_i + b_i \leq 2$. Prove that there exist distinct indices i, j such that $|a_i - a_j| + |b_i - b_j| \leq 1$.

7. Let $n \geq 3$ be odd. Given numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ from the interval $[0, 1]$, show that there exist distinct indices i, j such that $0 \leq a_i b_j - b_i a_j \leq 2/(n-1)$.

Another technique, which might be used in conjunction with the pigeonhole, is "crossing a line": one constructs a sequence of sums which change gradually and shows that some such sum lies within a specified interval (or has some other desired property).

8. (Hungary, 1997) We are given 111 unit vectors in the plane whose sum is zero. Show that there exist 55 of the vectors whose sum has length less than 1.

9. (IMO, 1997) Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $|x_1 + \cdots + x_n| = 1$ and $|x_i| \leq (n+1)/2$ for all i. Show that there exists a permutation (y_i) of (x_i) such that $|y_1 + 2y_2 + \cdots + ny_n| \leq (n+1)/2$.

Induction is also an extremely useful tool for constructing sums or differences of objects, since you can often just add or subtract some objects and then work with the new, smaller set.

10. (Spain, 1997, adapted) The real numbers $x_1, \ldots, x_n$ have a sum of 0. Prove that there exists an index i such that $x_i + x_{i+1} + \cdots + x_j \geq 0$ for all $i \leq j < i + n$, where the indices are defined modulo n.

11. (Austria-Poland, 1995) Let $v_1, v_2, \ldots, v_{95}$ be three-dimensional vectors with all coordinates in the interval $[-1, 1]$. Show that among all vectors of the form $s_1 v_1 + s_2 v_2 + \cdots + s_{95} v_{95}$, where $s_i \in \{-1, 1\}$ for each i, there exists a vector (a, b, c) satisfying $a^2 + b^2 + c^2 \leq 48$. Can the bound of 48 be improved?

12. Let $n \geq 1$, and let a_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n+2)$ be $n(n+2)$ arbitrary real numbers. Prove that there exist distinct j, j' such that

$$a_{1j}a_{1j'} + a_{2j}a_{2j'} + \cdots + a_{nj}a_{nj'} \geq 0.$$

Sometimes it is necessary to discriminate among the numbers or vectors given – on the basis of size, sign, proximity to other numbers, etc – in order to establish an explicit method for constructing the desired sums. Deciding how to distinguish these objects can be tricky.

13. (Iran, 1999) Suppose that $r_1, r_2, \ldots, r_n$ are real numbers. Prove that there exists $S \subseteq \{1, 2, \ldots, n\}$ such that $1 \leq |S \cap \{i, i+1, i+2\}| \leq 2$ for $1 \leq i \leq n-2$, and

$$\left|\sum_{i \in S} r_i\right| \geq \frac{1}{6} \sum_{i=1}^{n} |r_i|.$$

14. (USA, 1996) For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive numbers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that, in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

15. (Russia, 1997) 300 apples are given, no one of which weighs more than 3 times any other. Show that the apples may be divided into groups of 4 such that no group weighs more than 3/2 times any other group.

16. (Iran, 1999) Suppose that $-1 \leq x_1, \ldots, x_n \leq 1$ are real numbers such that $x_1 + \cdots + x_n = 0$. Prove that there exists a permutation σ such that, for every $1 \leq p \leq q \leq n$,

$$|x_{\sigma(p)} + \cdots + x_{\sigma(q)}| \leq 2 - \frac{1}{n}.$$

Finally, sometimes one may want to estimate the frequency with which sums lie in an interval, rather than simply showing that some such sums exist. Here too, there are a variety of techniques, often – though not always – similar to those used in the existence situation.

17. (Iran, 1996) For $S = \{x_1, \ldots, x_n\}$ a set of n real numbers, all at least 1, we count the number of reals of the form $\sum_{i=1}^{n} \epsilon_i x_i, \epsilon_i \in \{0, 1\}$ lying in an open interval I of length 1. Find the maximum value of this count over all I and S.

18. (Beatty's Theorem) If α and β are positive irrationals satisfying $1/\alpha + 1/\beta = 1$, show that every interval $(n, n+1)$, where n is a positive integer, contains exactly one integer multiple of either α or β.

19. (Putnam, 1994) Let (r_n) be a sequence of positive reals with limit 0. Let S be the set of all numbers expressible in the form $r_{i_1} + \cdots + r_{i_{1994}}$ for positive integers $i_1 < i_2 < \cdots < i_{1994}$. Prove that every interval (a, b) contains a subinterval (c, d) whose intersection with S is empty.

20. $x_1, \ldots, x_n$ are arbitrary real numbers. Prove that the number of pairs $\{i, j\}$ satisfying $1 < |x_i - x_j| < 2$ does not exceed $n^2/4$.