Some different geometry problems

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I was going to name it "Combinatorial Geometry", but I realized that many of those problems don't require Combinatorics at all!

1 The "Where's Waldo" method

If you don't know the exact location of a point, find all possibilities, by delimiting a region. Of course, you can delimit a set of *forbidden points* as well.

Some examples:

- If A and B are fixed points, then the locus of points P such that $\angle APB < \alpha$ is the union of the interior of two arcs.
- If A and B are fixed points, then the locus of points P such that $\angle PAB < \angle PBA \iff PB < PA$ is the semiplane determined by the perpendicular bisector of AB that contains B.
- If $\angle P$ is the largest angle of the triangle ABP then P must be inside the intersection of the disks with centers A and B and both radii equal to AB.

2 Some hints

- You might want to consider the **convex hull** of a set of points to pur some order in them.
- **Extremal arguments** like considering the maximum or minimum distance might be useful.
- **Induction**, if applicable, never gets out of fashion!

3 Problems

1. Let ABC be a triangle with $\angle C > 10°$ and $\angle B = \angle C + 10°$. Let E and D be points on sides AB and AC, respectively, such that $\angle ACE = 10°$ and $\angle ABD = 15°$. Let $Z \neq A$ the intersection of circumcircles of triangles ABD and AEC. Prove that $\angle ZBA > \angle ZCA$.
2. (IMOSL 1990) Let G, I and H respectively be the centroid, incenter and ortocenter of triangle ABC. Prove that $\angle GIH > 90°$.
3. (Hungary 1998) Let P be a convex polygon with integer sidelenghts and odd perimeter. Prove that its area is not less than $\sqrt{3}/4$.
4. (Brazil 2005) A square is contained in a cube when all of the points of the square are on the surface of the cube or inside the cube. Find the biggest ℓ such that there exists a square contained in a unit cube.

5. (Iberoamerican 2007) Two teams, A and B, fight for a territory limited by a circumference.

   A has n blue flags and B has n white flags ($n \geq 2$ fixed). They play alternatively and A begins the game. Each team, in its turn, places one of his flags in a point of the circumference that has not been used in a previous play. Each flag, once placed, cannot be moved.

   Once all $2n$ flags have been placed, territory is divided between the two teams. A point of the territory belongs to A if the closest flag to it is blue, and it belongs to B if the closest flag to it is white. If the closest blue flag to a point is at the same distance than the closest white flag to that point, the point is neutral (not from A nor from B). A team wins the game is their points cover a greater area that that covered by the points of the other team. There is a draw if both cover equal areas.

   Prove that, for every n, team B has a winning strategy.

6. (Iberoamerican 2007) Let $\mathcal{F}$ be a family of all hexagons H satisfying simultaneously the following conditions:

   - the opposite sides of H are parallel;
   - any three vertices of H can be covered by a strip of height 1.

   Determine the least real number ℓ such that it is possible to cover each hexagon from $\mathcal{F}$ with a strip of height ℓ.

   A strip of height ℓ is the region of the plane between two parallel lines whose distance is ℓ (the lines are included in the strip).

7. Let S be a set of n points and M be the biggest distance between two points from S. Prove that there exists at most n pairs of points whose distance is M.

8. (Brazil 2002) Given a finite set of squares whose areas add up to 4, prove that one can cover a unit square with them.

9. (Brazil 2007) Consider n points in a plane which are vertices of a convex polygon. Prove that the set of the lengths of the sides and the diagonals of the polygon has at least $\lfloor n/2 \rfloor$ elements.

10. (Cono Sur 2001) Three acute triangles are inscribed in the same circle, determining nine distinct vertices. Prove that it is possible to choose one vertex from each triangle which determine a triangle whose angles do not exceed $90°$.

11. (IMO 2002) Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be unit disks in the plane, $n \geq 3$. Let $O_1, O_2, \ldots, O_n$ be its respective centers.

    Suppose that no line in the plane intersects three or more of the disks. Prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}$$

12. (IMO 2003) Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\frac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

13. (Iberoamerican 1996) Let $A_1, A_2, \ldots, A_n$ be n distincts points in the plane. Assign to each point A_i the real number $\lambda_i \neq 0$ such that $A_i A_j^2 = \lambda_i + \lambda_j$ for all $i \neq j$. Prove that $n \leq 4$ and that if $n = 4$ then $\sum_{i=1}^{4} \frac{1}{\lambda_i} = 0$.

14. (Cono Sur 1998) A city has some bus lines such that

    - each line has exactly three stops;

- each pair of distinct lines has exactly one common stop;
- each pair of distinct stops are in exactly one same line.

Find the total number of stops in this city.

15. (Brazil 2002) Show that we cannot form more than 4096 binary sequences of length 24 so that any two differ in at least 8 positions.

16. (IMO 2006) Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

17. (Olympic Revenge 2007) [1] Let $A_1A_2B_1B_2$ be a convex quadrilateral. Vertices A_1 and A_2 are adjacent and represent two Argentine cities. Vertices B_1 and B_2 are also adjacent and represent two Brazilian cities. There are a Argentine cities and b Brazilian cities, no three cities, regardless of its country, collinear. Determine if it is possible, regardless of the cities positions, to build straight roads, each one connecting either two Argentine cities or two Brazilian cities, such that:

    - two roads never intersect at a point that is not a city;
    - it is possible to reach, by using the roads, any Argentine city from any other Argentine city;
    - it is possible to reach, by using the roads, any Brazilian city from any other Brazilian city.

    If it is indeed possible to do so, find an algorithm to build such roads.

18. (IMOSL 2008) Let k and n be integers with $0 \leq k \leq n-2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersections of lines in L. Let O be a point in the plane not lying on any line of L. A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

19. A polygon is *orthogonal* if every consecutive sides are perpendicular and all its sides are integer. Prove that if one can cover an orthogonal polygon with dominoes then at least one of its sides is odd.

20. Prove that the number of unit distances in a set with n points does not exceed $n^{3/2}$.

21. Prove that n points in the plane determines at least $\sqrt{n-2}$ distinct distances.

[1] *Olympic Revenge* is a competition for Brazilian teachers. The students pose the problems for the teachers during a training camp, held every year in January!