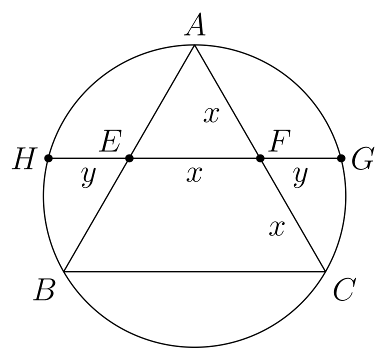
Solutions to Exercises

1. Equilateral triangle ABC is inscribed in a circle. Let E and F be the midpoints of AB and AC, respectively. Line segment EF is extended past F to meet the circle at G. Find the ratio EF/FG.

   Solution. Let $x = AF = CF = EF$ and $y = FG$. Extend EF past E to intersect the circle at H. Then by symmetry, $EH = FG = y$.

   By power of a point on F, $x^2 = y(x + y) = xy + y^2$, so

   $$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0.$$

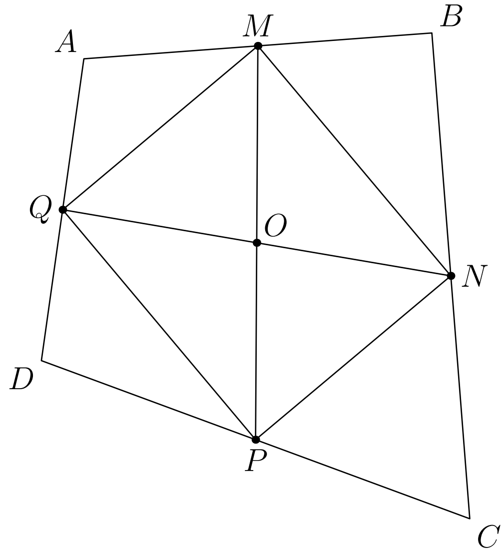
   Let $r = EF/FG = x/y$, so $r^2 - r - 1 = 0$. By the quadratic formula,

   $$r = \frac{1 \pm \sqrt{5}}{2}.$$

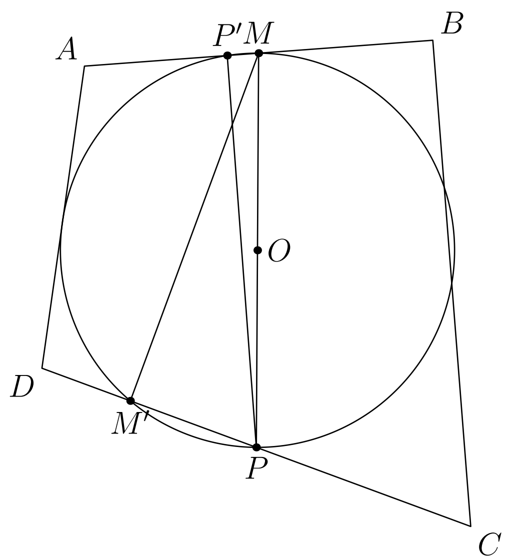
   Since r is positive, $r = (1 + \sqrt{5})/2$.

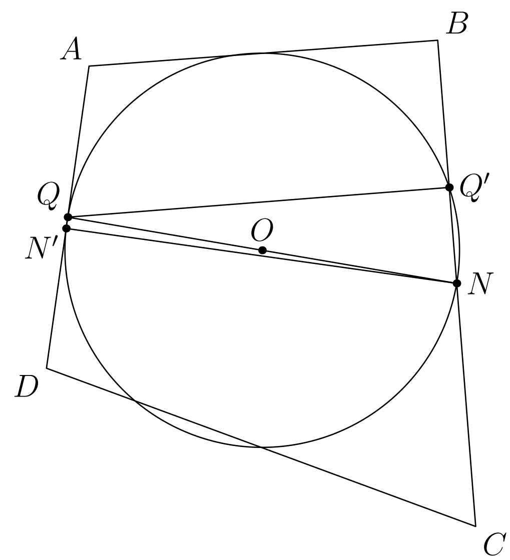
2. Let $ABCD$ be a quadrilateral with perpendicular diagonals, and let M, N, P, and Q be the midpoints of sides AB, BC, CD, and DA, respectively. Let M' be the projection of M onto the opposite side CD, and define N', P', and Q' similarly. Prove that the points M, N, P, Q, M', N', P', and Q' all lie on the same circle.

   Solution. We know that MN and PQ are parallel to AC, and that $MN = PQ = AC/2$. Also, NP and QM are parallel to BD, and $NP = QM = BD/2$. Since AC and BD are perpendicular, quadrilateral $MNPQ$ is a rectangle. Let O be the center of rectangle $ABCD$.

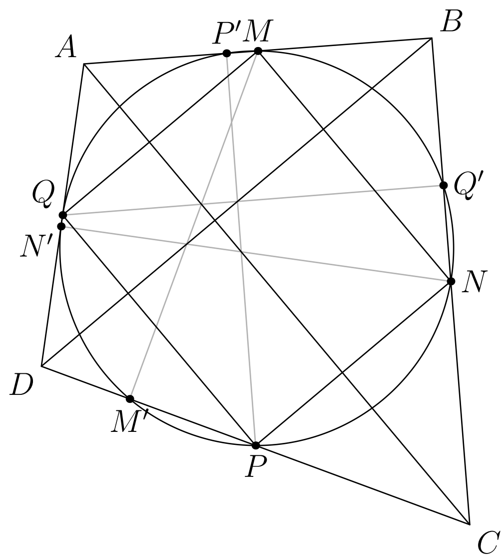


Since $\angle MM'P = \angle MP'P = 90^\circ$, M' and P' lie on the circle with diameter MP. Similarly, $\angle NN'Q = \angle NQ'Q = 90^\circ$, so N' and Q' lie on the circle with diameter NQ.





But quadrilateral $MNPQ$ is a rectangle, so the circles with diameters MP and NQ coincide. Thus, M, N, P, Q, M', N', P', and Q' all lie on the same circle, namely the circumcircle of rectangle $MNPQ$.

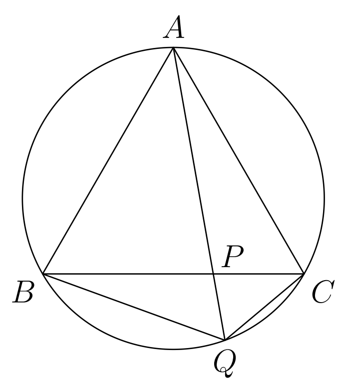


Comment. The diagram seems to indicate that NN', PP', and AC are concurrent. Can you prove this?

3. Let ABC be an equilateral triangle, and let P be a point on side BC. The line AP intersects the circumcircle of triangle ABC at Q. Show that

$$\frac{1}{PQ} = \frac{1}{BQ} + \frac{1}{CQ}.$$

Solution. Triangles PBQ and PAC are similar, so $PQ/BQ = PC/AC$. Triangles PCQ and PAB are similar, so $PQ/CQ = PB/AB$.



Hence,

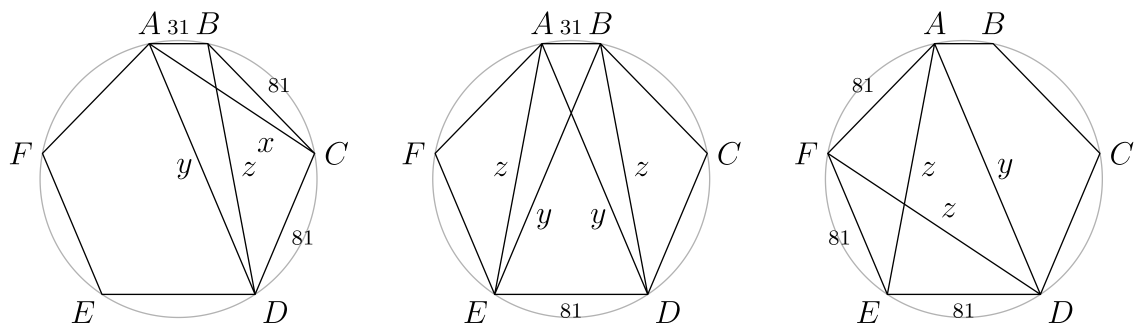
$$\frac{PQ}{BQ} + \frac{PQ}{CQ} = \frac{PC}{AC} + \frac{PB}{AB} = \frac{PC}{BC} + \frac{PB}{BC} = \frac{BC}{BC} = 1,$$

which means

$$\frac{1}{BQ} + \frac{1}{CQ} = \frac{1}{PQ}.$$

4. A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by AB, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A. (AIME, 1991)

Solution. Let the vertices of the hexagon be A, B, C, D, E, and F. Let $x = AC$, $y = AD$, and $z = AE$. Since equal chords subtend equal angles, $BD = z$, $BE = y$, and $DF = z$.



By Ptolemy's theorem on quadrilaterals $ABCD$, $ABDE$, and $ADEF$, respectively, we obtain the system of equations

$$\begin{aligned} xz &= 31 \cdot 81 + 81y, \\ y^2 &= 31 \cdot 81 + z^2, \\ z^2 &= 81^2 + 81y. \end{aligned}$$

Then from the second and third equations,

$$y^2 = 31 \cdot 81 + z^2 = 31 \cdot 81 + 81^2 + 81y,$$

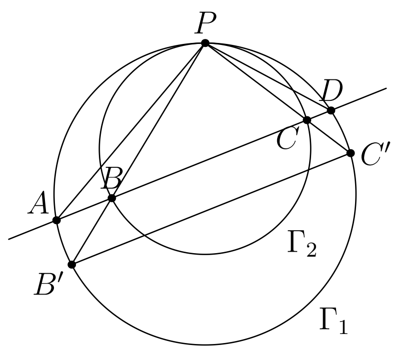
which simplifies as $y^2 - 81y - 9072 = (y - 144)(y + 63) = 0$. Therefore, $y = 144$. Then from the third equation, $z^2 = 81^2 + 81 \cdot 144 = 81 \cdot (81 + 144) = 81 \cdot 225 = 9^2 \cdot 15^2$, so $z = 9 \cdot 15 = 135$. Finally, from the first equation,

$$x = \frac{31 \cdot 81 + 81y}{z} = \frac{31 \cdot 81 + 81 \cdot 144}{135} = \frac{81 \cdot 175}{135} = 105.$$

Therefore, the sum of the diagonals from A is equal to $AC + AD + AE = x + y + z = 384$.

5. Two circles, Γ_1 and Γ_2, are internally tangent at P. A line intersects Γ_1 at A and D, and Γ_2 at B and C. Prove that $\angle APB = \angle CPD$.

Solution. Since circles Γ_1 and Γ_2 are internally tangent at P, there exists a homothety centered at P that takes Γ_2 to Γ_1. Let B' and C' be the images of B and C under this homothety, respectively.



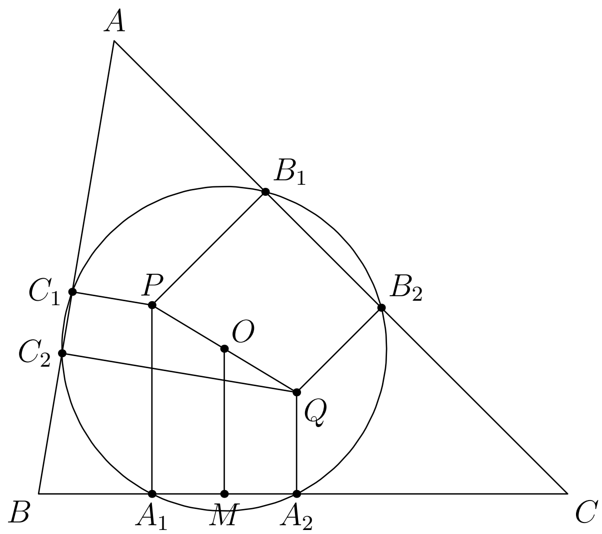
Then $B'C'$ is parallel to line $ABCD$, so arcs AB' and $C'D$ are equal. Hence, $\angle APB = \angle CPD$.

6. Let ABC be a triangle. A circle intersects side BC at A_1 and A_2, side AC at B_1 and B_2, and side AB at C_1 and C_2. The line through A_1 perpendicular to BC, the line through B_1 perpendicular to AC, and the line through C_1 perpendicular to AB are concurrent.

Show that the line through A_2 perpendicular to BC, the line through B_2 perpendicular to AC, and the line through C_2 perpendicular to AB are concurrent.

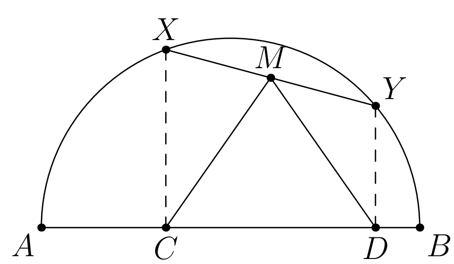
Solution. Let O be the center of the circle. Let the line through A_1 perpendicular to BC, the line through B_1 perpendicular to AC, and the line through C_1 perpendicular to AB concur at P. Let Q be the reflection of P through O.

Let M be the midpoint of chord A_1A_2. Then OM is perpendicular to A_1A_2. Also, O is the midpoint of PQ, and PA_1 is perpendicular to A_1A_2, so QA_2 is perpendicular to A_1A_2.

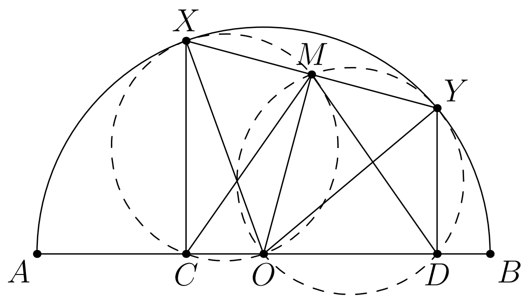


Similarly, QB_2 is perpendicular to B_1B_2, and QC_2 is perpendicular to C_1C_2. Thus, the line through A_2 perpendicular to BC, the line through B_2 perpendicular to AC, and the line through C_2 perpendicular to AB concur at Q.

7. Let XY be a chord of constant length that slides around the semicircle with diameter AB. Let M be the midpoint of AB, and let C and D be the projections of X and Y onto AB, respectively. Show that $MC = MD$, and that $\angle CMD$ is constant for all positions of chord XY.

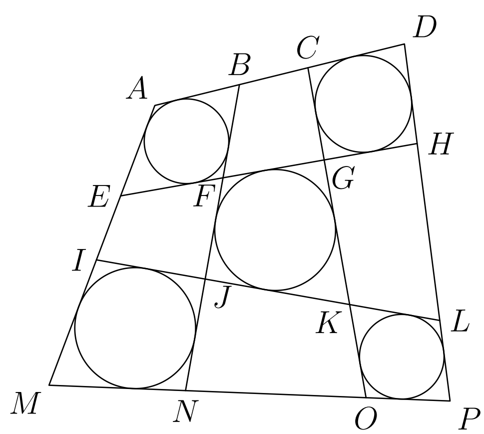


Solution. Let O be the center of the semicircle. Since M is the midpoint of chord XY, $\angle OMX = 90°$. But $\angle OCX = 90°$, so quadrilateral $OCXM$ is cyclic. Similarly, $\angle OMY = \angle ODY = 90°$, so quadrilateral $ODYM$ is cyclic.

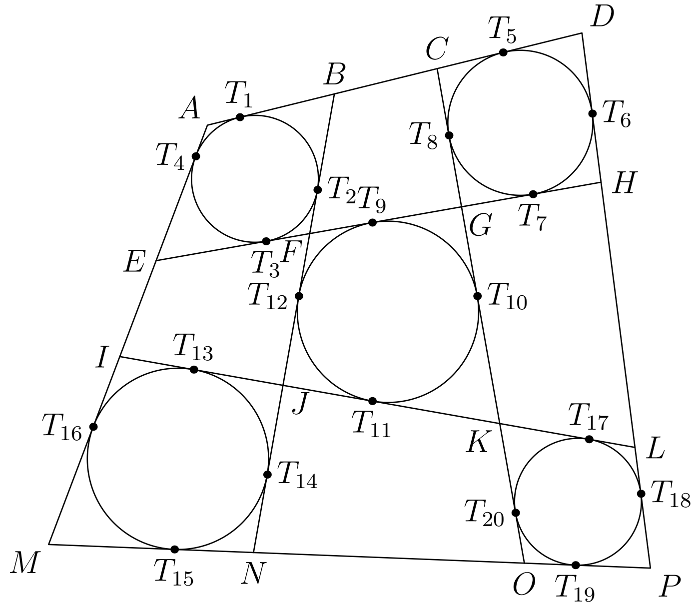


Then $\angle MCO = \angle MXO$ and $\angle MDO = \angle MYO$. But $\angle MXO = \angle MYO$, so $\angle MCO = \angle MDO$, which means that triangle CMD is isosceles with $MC = MD$. And since $\angle MXO$ is constant, $\angle MCD$ is constant, which means $\angle CMD$ is constant.

8. In the diagram below, quadrilaterals $ABFE$, $CDHG$, $FGKJ$, $IJNM$, and $KLPO$ are all circumscribed. Show that quadrilateral $ADPM$ is also circumscribed.



Solution. Let T_1, T_2, $\ldots$, T_{20} be the points of tangency, as shown. By the Pitot theorem, to prove that quadrilateral $ADPM$ is circumscribed, it suffices to prove that $AD + PM = DP + MA$.



We see that $AD = AT_1 + T_1T_5 + T_5D$. Note that $AT_1 = AT_4$ and $DT_5 = DT_6$. Also, $T_1T_5 = T_3T_7$, since these are both external common tangents of the same two circles.

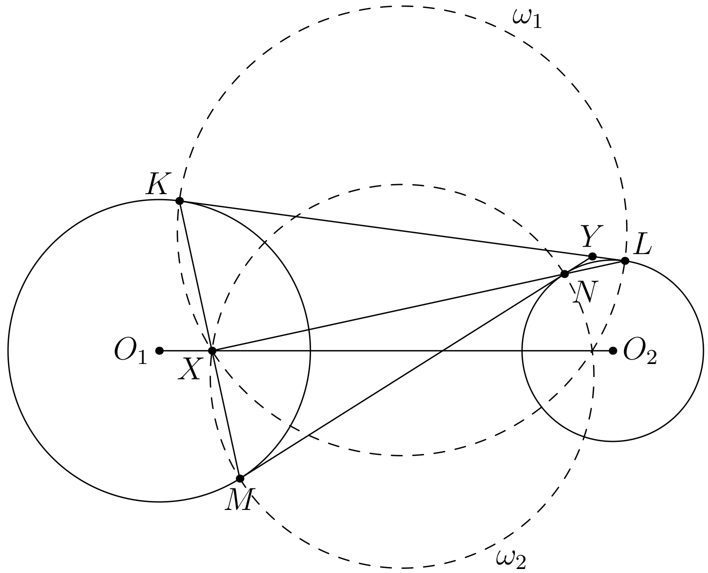
We can write $T_3T_7 = T_3T_9 + T_9T_7$. We see that $T_3T_9 = T_2T_{12}$, since these are both internal common tangents of the same two circles. Similarly, $T_7T_9 = T_8T_{10}$.

By the same reasoning, we can write $MP = MT_{15} + T_{15}T_{19} + T_{19}P$, and $T_{15}T_{19} = T_{13}T_{17} = T_{13}T_{11} + T_{11}T_{17} = T_{12}T_{14} + T_{10}T_{20}$. Therefore,

$$\begin{aligned}
AD + PM &= (AT_1 + T_1T_5 + T_5D) + (MT_{15} + T_{15}T_{19} + T_{19}P) \\
&= AT_1 + T_3T_7 + T_5D + MT_{15} + T_{13}T_{17} + T_{19}P \\
&= AT_1 + T_3T_9 + T_9T_7 + T_5D + MT_{15} + T_{13}T_{11} + T_{11}T_{17} + T_{19}P \\
&= AT_4 + T_2T_{12} + T_8T_{10} + T_6D + MT_{16} + T_{12}T_{14} + T_{10}T_{20} + T_{18}P \\
&= (AT_4 + T_2T_{12} + T_{12}T_{14} + T_{16}M) + (DT_6 + T_8T_{10} + T_{10}T_{20} + T_{18}P) \\
&= (AT_4 + T_2T_{14} + T_{16}M) + (DT_6 + T_8T_{20} + T_{18}P) \\
&= (AT_4 + T_4T_{16} + T_{16}M) + (DT_6 + T_6T_{18} + T_{18}P) \\
&= AM + DP.
\end{aligned}$$

9. Let Γ_1 and Γ_2 be two circles, with centers O_1 and O_2, respectively. Let KL be a common external tangent, and let MN be a common internal tangent, with K and M on Γ_1, and L and N on Γ_2. Prove that KM, LN, and O_1O_2 are concurrent.

   Solution. Let X be the intersection of KM and LN, and let Y be the intersection of KL and MN. Let ω_1 and ω_2 denote the circles with diameters KL and MN, respectively.



We see that O_1Y is perpendicular to KM, O_2Y is perpendicular to LN, and O_1Y is perpendicular to O_2Y. Hence, KM is perpendicular to LN, which means that X lies on both ω_1 and ω_2.

Since O_1K is tangent to ω_1, O_1M is tangent to ω_2, and $O_1K = O_1M$, O_1 has the same power with respect to both ω_1 and ω_2. Similarly, O_2L is tangent to ω_1, O_2N is tangent to ω_2, and $O_2L = O_2N$, so O_2 has the same power with respect to both ω_1 and ω_2. Therefore, O_1O_2 is the radical axis of ω_1 and ω_2.

Since X lies on both ω_1 and ω_2, it follows that X lies on O_1O_2.

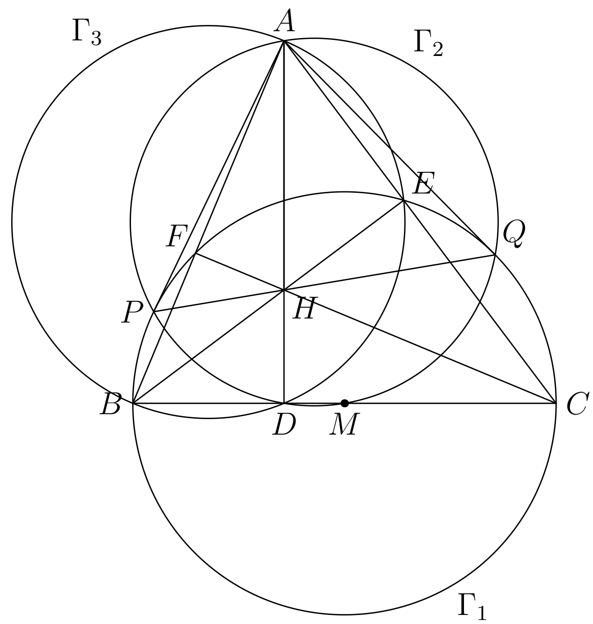
Comment. What is the significance of the other intersection of ω_1 and ω_2?

10. Let H be the orthocenter of triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, and H are collinear. (China, 1996)

    Solution. Let AD, BE, and CF be the altitudes of triangle ABC. Let Γ_1 be the circle with diameter BC. Since $\angle BEC = \angle BFC = 90^\circ$, E and F lie on Γ_1.

    Let M be the midpoint of BC, and let Γ_2 be the circle with diameter AM. Since M is the center of Γ_1, P and Q are the intersections of Γ_1 and Γ_2.

    Finally, let Γ_3 be the circle with diameter AB. Since $\angle AEB = \angle ADB = 90^\circ$, D and E lie on Γ_3.



The radical axis of Γ_1 and Γ_3 is BE, and the radical axis of Γ_2 and Γ_3 is AD. Then the radical center of Γ_1, Γ_2, and Γ_3 is the intersection of AD and BE, namely H. Hence, H lies on the radical axis of Γ_1 and Γ_2, namely PQ.