Art of Problem Solving

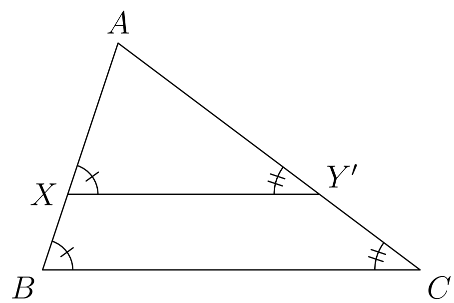
WOOT 2012–13

Similar Triangles/ Power of a Point

Solutions to Exercises

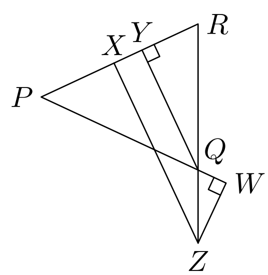
1. Let X be on $\overline{AB}$ and Y be on $\overline{AC}$ such that $AX/AB = AY/AC$. Show that $\overline{XY} \parallel \overline{BC}$.

   Solution. Let Y' be the point on line AC such that $XY' \parallel BC$.



   Then $\angle AXY' = \angle ABC$ and $\angle AY'X = \angle ACB$, so triangles AXY' and ABC are similar. Hence, $AX/AB = AY'/AC$. But we are given that $AX/AB = AY/AC$, so $AY'/AC = AY/AC$. This means points Y and Y' coincide, so $XY \parallel BC$.

2. In the diagram below, $PQ = PR$, $\overline{ZX} \parallel \overline{QY}$, $\overline{QY} \perp \overline{PR}$, and $\overline{PQ}$ is extended to W such that $\overline{WZ} \perp \overline{PW}$.



   (a) Show that $\triangle QWZ \sim \triangle RXZ$.
   (b) Show that $YQ = ZX - ZW$.

   Solution. (a) Since $PR = PQ$, we have $\angle R = \angle PQR = \angle WQZ$. Since $\overline{ZX} \parallel \overline{QY}$, we have $\angle ZXR = \angle QYR = 90^\circ$. So, $\angle ZXR = \angle ZWQ$, and we have $\triangle QWZ \sim \triangle RXZ$ by AA Similarity.

   (b) First, we note that $RQ = RZ - QZ$, which looks a lot like the expression we want to prove. Since $\overline{XZ} \parallel \overline{YQ}$, we have $\angle RYQ = \angle RXZ$ and $\angle RQY = \angle RZX$, so $\triangle RYQ \sim \triangle RXZ$. This similarity gives us $RZ/ZX = RQ/YQ$, so $RQ = (RZ/ZX)(YQ)$.

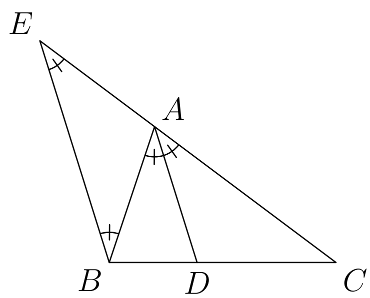
   From $\triangle QWZ \sim \triangle RXZ$ in the last part, we have $RZ/ZX = QZ/ZW$, so $QZ = (RZ/ZX)(ZW)$. Substituting these into $RQ = RZ - QZ$ gives

$$\frac{(RZ)(YQ)}{ZX} = RZ - \frac{(RZ)(ZW)}{ZX}.$$

Multiplying this equation by ZX/RZ gives the desired $YQ = ZX - ZW$.

3. **The Angle Bisector Theorem**. If D is on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle BAC$, then $AB/BD = AC/CD$. Take a look here for hints.

   Solution. Let E be the point on line AC so that BE is parallel to AD. Then triangle $\angle AEB = \angle CAD$ and $\angle ABE = \angle BAD$. But AD is the angle bisector of $\angle BAC$, so $\angle CAD = \angle BAD$, which means $\angle AEB = \angle ABE$. Hence, triangle ABE is isosceles, with $AB = AE$.

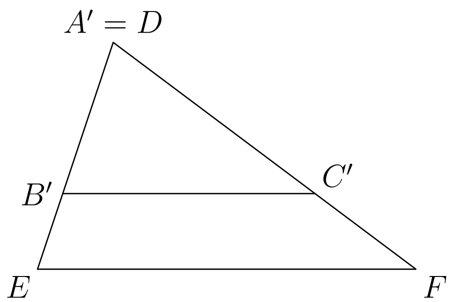
   

   Furthermore, since BE is parallel to AD, triangles EBC and ADC are similar, which means
   $$\frac{CE}{AC} = \frac{BC}{CD}.$$
   We can re-write this equation as
   $$\frac{AE + AC}{AC} = \frac{BD + CD}{CD},$$
   which simplifies to
   $$\frac{AE}{AC} + 1 = \frac{BD}{CD} + 1.$$
   Then $AE/AC = BD/CD$. But $AE = AB$, so $AB/AC = BD/CD$, which implies $AB/BD = AC/CD$.

4. Prove that SAS Similarity works without using the law of sines.

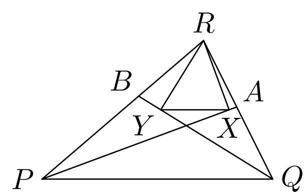
   Solution. To prove SAS similarity, we must prove that if we have two triangles ABC and DEF such that $AB/DE = AC/DF$ and $\angle BAC = \angle EDF$, then triangles ABC and DEF are similar.

   Since $\angle BAC = \angle EDF$, we can shift triangle ABC to triangle $A'B'C'$ in the plane, so that points A' and D coincide, B' lies on DE, and C' lies on DF.

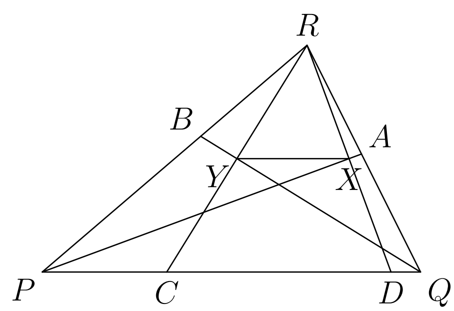


Since $DB' = AB$ and $DC' = AC$, we have that $DB'/DE = DC'/DF$. Hence, by Exercise 1, $B'C'$ is parallel to EF. Then $\angle DB'C' = \angle DEF$ and $\angle DC'B' = \angle DFE$, which means triangles $DB'C'$ and DEF are similar, so triangles ABC and DEF are similar.

5. $\overline{PA}$ and $\overline{BQ}$ bisect angles $\angle RPQ$ and $\angle RQP$, respectively. Given that $\overline{RX} \perp \overline{PA}$ and $\overline{RY} \perp \overline{BQ}$, prove that $\overline{XY} \parallel \overline{PQ}$.



Solution. We extend $\overrightarrow{RY}$ and $\overrightarrow{RX}$ to meet $\overline{PQ}$ at C and D, respectively, as shown.

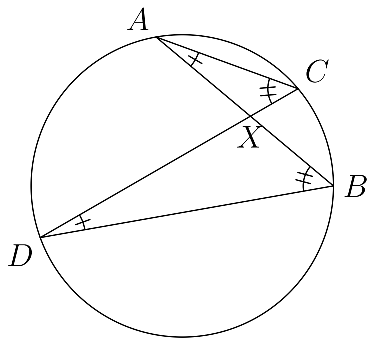


Since $\angle RPA = \angle APQ$, $PX = PX$, and $\angle RXP = \angle DXP$, we have $\triangle RXP \cong \triangle DXP$ by ASA Congruence. Similarly, we have $\triangle CYQ \cong \triangle RYQ$. Therefore, $RX = XD$ and $RY = YC$, so X and Y are midpoints of $\overline{RD}$ and $\overline{RC}$, respectively.

So, we have $RY/RC = RX/RD = 1/2$, which gives us $\triangle RYX \sim \triangle RCD$ by SAS Similarity. Therefore, we have $\angle RYX = \angle RCD$, so $\overline{XY} \parallel \overline{DC}$. Since $\overline{DC}$ is on the same line as $\overline{PQ}$, we have $\overline{XY} \parallel \overline{PQ}$.

6. Chords $\overline{AB}$ and $\overline{CD}$ meet at X. Prove that $(XA)(XB) = (XC)(XD)$. (No citing the Power of a Point Theorem! You're being asked here to prove it!)

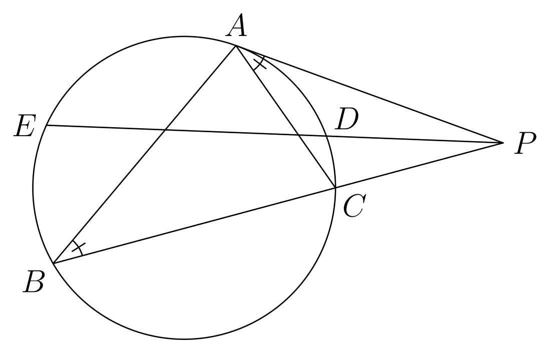
Solution. We take the case where X lies inside the circle. (The case where X lies outside the circle is handled similarly.)



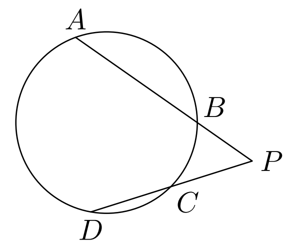
Both $\angle BAC$ and $\angle BDC$ subtend arc BC, so $\angle BAC = \angle BDC$. Both $\angle ACD$ and $\angle ABD$ subtend arc AD, so $\angle ACD = \angle ABD$. Hence, triangles XAC and XDB are similar, which means $XA/XD = XC/XB$. This implies $XA \cdot XB = XC \cdot XD$.

7. Point P is outside a circle and A is on the circle such that $\overline{PA}$ is tangent to the circle. A line through P meets the circle at B and C. Prove that $PA^2 = (PB)(PC)$. Use this to show that if a second line through P meets the circle at D and E, then $PA^2 = (PD)(PE)$. (Again, you can't just cite Power of a Point!)

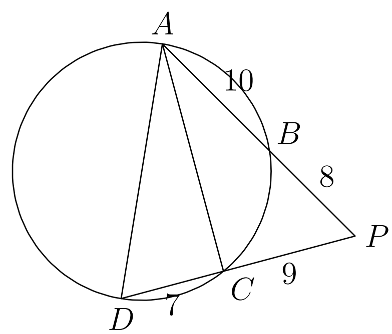
Solution. Since AP is tangent to the circle at A, $\angle PAC = \angle ABC = \angle PBA$. Hence, triangles PAC and PBA are similar. It follows that $PA/PB = PC/PA$, so $PA^2 = PB \cdot PC$. By the same argument, $PA^2 = PD \cdot PE$.



8. In the diagram below, we have $BP = 8$, $AB = 10$, $CD = 7$, and $\angle APC = 60^\circ$. Find the area of the circle. *Source: AHSME*



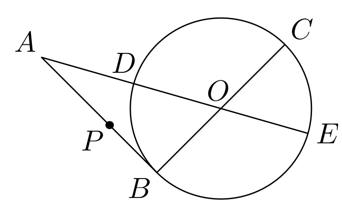
Solution. Let $x = PC$. Then by power of a point on P, $PA \cdot PB = PC \cdot PD$, or $18 \cdot 8 = x(x+7)$, which simplifies to $x^2 + 7x - 144 = 0$. This equation factors as $(x-9)(x+16) = 0$, so $x = 9$.



Since $PC/PA = 9/18 = 1/2$ and $\angle APC = 60^\circ$, triangle APC is a 30°-60°-90° triangle. In particular, $AC = CP\sqrt{3} = 9\sqrt{3}$, and $\angle ACP = 90^\circ$, so $\angle ACD = 90^\circ$. Then by Pythagoras on right triangle ACD, $AD^2 = AC^2 + CD^2 = (9\sqrt{3})^2 + 7^2 = 292$. Since $\angle ACD = 90^\circ$, AD is a diameter of the circle, so the area of the circle is

$$\pi \cdot \left(\frac{AD}{2}\right)^2 = \frac{AD^2}{4} \cdot \pi = 73\pi.$$

9. Point O is the center of the circle, $\overline{AB} \perp \overline{BC}$, $AP = AD$, and $\overline{AB}$ has length twice the radius of the circle. Prove that $AP^2 = (PB)(AB)$.



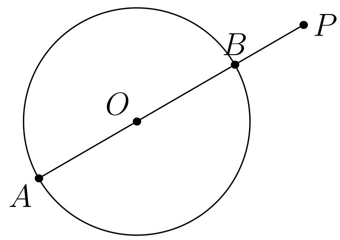
Solution. Let r be the radius of the circle. Since AB is perpendicular to BC, and BC is a diameter of the circle, AB is tangent to the circle at B. Then by power of a point on A, $AD \cdot AE = AB^2 = (2r)^2 = 4r^2$. But $AE = AD + DE = AD + 2r$, so $AD(AD + 2r) = 4r^2$. Expanding, we get $AD^2 + 2rAD = 4r^2$, so

$$AD^2 = 4r^2 - 2rAD = (2r - AD)(2r).$$

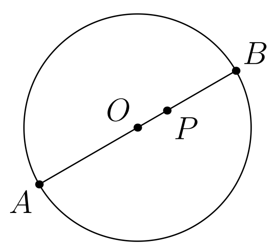
Since $AD = AP$, $AD^2 = AP^2$. Also, $PB = AB - AP = 2r - AD$, and $AB = 2r$, so $AD^2 = PB \cdot AB$, as desired.

10. (This one can be particularly useful.) Show that if P is outside circle $\odot O$, and that the radius of $\odot O$ is r, then the power of point P is $OP^2 - r^2$. What if P is inside the circle?

Solution. Let the line through O and P intersect the circle at A and B. Then by power of a point on P, $PA \cdot PB = (OP + AO)(OP - BO) = (OP + r)(OP - r) = OP^2 - r^2$.



Now, let P be inside the circle. As above, let the line through O and P intersect the circle at A and B. Then $PA \cdot PB = (AO + OP)(OB - OP) = (r + OP)(r - OP) = r^2 - OP^2$.

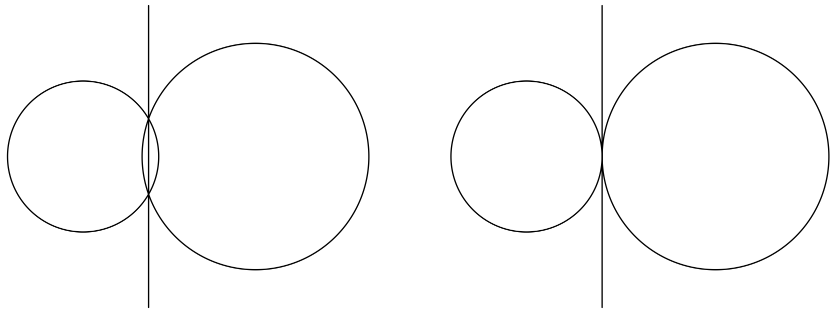


However, it is more conventional to define the power of a point P as $OP^2 - r^2$. Note that under this definition, the power is positive if P lies outside the circle, and is negative if P lies inside the circle. (This simplifies the formula for the power of a point. The other idea behind this definition is the use of *directed line segments*. For example, in the first diagram, PA and PB point in the same direction, so their product is considered positive. However, in the second diagram, PA and PB point in opposite directions, so their product is considered negative.)

11. Describe the radical axis of two intersecting circles.

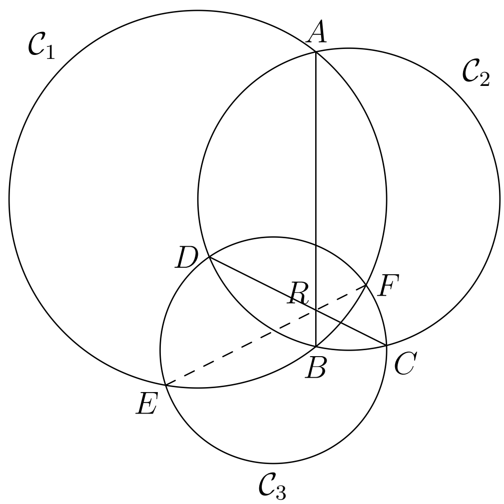
Solution. Consider one of the points of intersection. Since this point lies on both circles, the power of this point with respect to both circles is 0, which means that it lies on the radical axis of both circles. The same holds for the other intersection.

We know that the radical axis is a line, so the radical axis of two intersecting circles is the line passing through their points of intersection. Note that if the two circles are tangent, then the radical axis is their common internal tangent.



12. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ meet at A and B. Circles $\mathcal{C}_2$ and $\mathcal{C}_3$ meet at C and D. Circles $\mathcal{C}_1$ and $\mathcal{C}_3$ meet at E and F. Show that $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$ are concurrent.

Solution. Let R be the intersection of AB and CD.



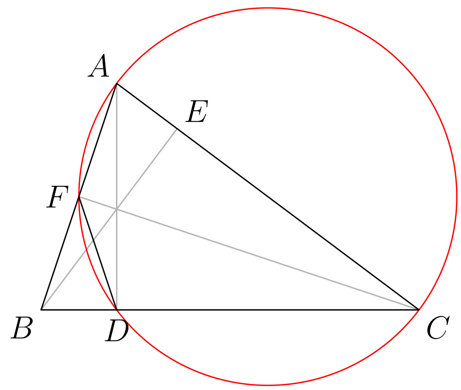
By Exercise 11, AB is the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$, so R has the same power with respect to circles $\mathcal{C}_1$ and $\mathcal{C}_2$. Similarly, CD is the radical axis of $\mathcal{C}_2$ and $\mathcal{C}_3$, so R has the same power with respect to circles $\mathcal{C}_2$ and $\mathcal{C}_3$. Hence, R has the same power with respect to circles $\mathcal{C}_1$ and $\mathcal{C}_3$.

Therefore, R lies on CD, the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_3$. This proves that AB, CD, and EF are concurrent at R.

(We can use the same argument to prove that given any three circles, the radical axes of the circles taken pairwise are concurrent. The point of concurrence is called the *radical center* of the three circles.)

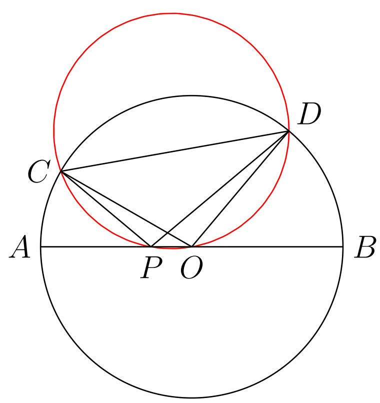
13. Altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ of acute triangle $\triangle ABC$ intersect at H. Show that $\angle BFD = \angle ACB$.

Solution. Since $\angle ADC = \angle AFC = 90^\circ$, quadrilateral $ACDF$ is cyclic. (In fact, D and F lie on the circle with diameter AC.) Hence, $\angle BFD = 180^\circ - \angle AFD = \angle ACD = \angle ACB$.



14. Points A, C, D, and B are in that order on a circle with center O such that $\overline{AB}$ is a diameter of the circle and $\angle DOB = 50^\circ$. Point P is on $\overline{AB}$ such that $\angle PCO = \angle PDO = 10^\circ$. Find the measure in degrees of arc CD.

Solution. Since $\angle PCO = \angle PDO = 10^\circ$, quadrilateral $OPCD$ is cyclic. Then $\angle PCD = 180^\circ - \angle POD = \angle DOB = 50^\circ$, so $\angle OCD = \angle PCD - \angle PCO = 50^\circ - 10^\circ = 40^\circ$.

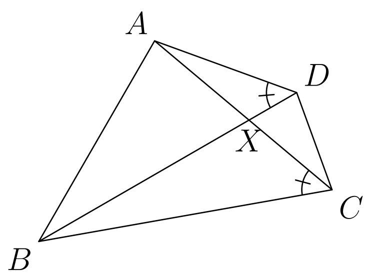


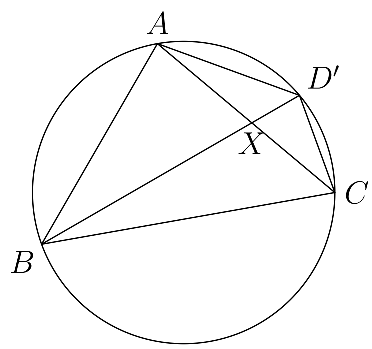
Since $OC = OD$, triangle OCD is isosceles, so $\angle ODC = \angle OCD = 40^\circ$. Then $\angle COD = 180^\circ - \angle ODC - \angle OCD = 180^\circ - 40^\circ - 40^\circ = 100^\circ$, so arc CD is 100°.

15. Prove that each of the methods above does indeed show that a quadrilateral is cyclic.

- If $\angle ACB = \angle ADB$ in convex quadrilateral $ABCD$, then $ABCD$ is cyclic.

Let X be the intersection of diagonals AC and BD. Since $\angle BCX = \angle ADX$ and $\angle BXC = \angle AXD$, triangles BCX and ADX are similar. Hence, $BX/AX = CX/DX$, so $BX \cdot DX = AX \cdot CX$.

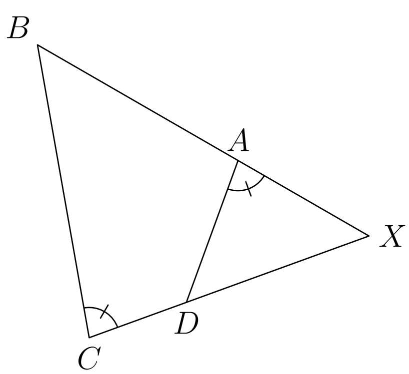


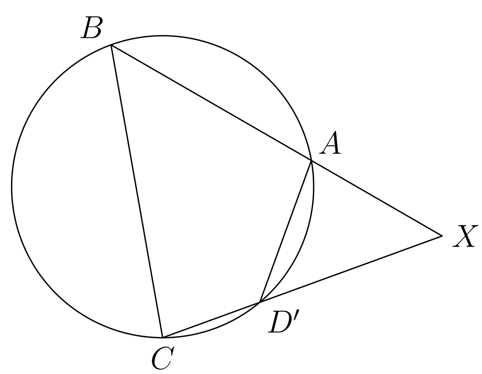


Let the circumcircle of triangle ABC intersect BD at D'. Then by power of a point on X, $BX \cdot D'X = AX \cdot CX$. Hence, points D and D' coincide, which means quadrilateral $ABCD$ is cyclic.

- If $\angle BAD + \angle BCD = 180^\circ$ in convex quadrilateral $ABCD$, then $ABCD$ is cyclic.

Let X be the intersection of AB and CD. Note that $\angle XAD = 180^\circ - \angle BAD = \angle BCD = \angle BCX$. Also, $\angle AXD = \angle CXB$. Hence, triangles AXD and CXB are similar. It follows that $AX/CX = DX/BX$, so $AX \cdot BX = CX \cdot DX$.

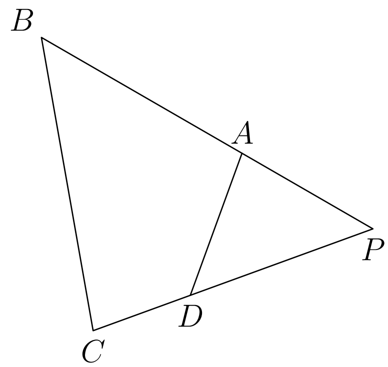




Let the circumcircle of triangle ABC intersect CX at D'. Then by power of a point on X, $AX \cdot BX = CX \cdot D'X$. Then points D and D' coincide, so quadrilateral $ABCD$ is cyclic.

- If P is the intersection of AB and CD, and $PA \cdot PB = PC \cdot PD$, then $ABCD$ is a cyclic quadrilateral.

From $PA \cdot PB = PC \cdot PD$, we get $PA/PC = PD/PB$. Since $\angle APD = \angle DPB$, triangles APD and CPB are similar. Then $\angle BCD = \angle BCP = \angle DAP = 180^\circ - \angle BAD$, so quadrilateral $ABCD$ is cyclic.

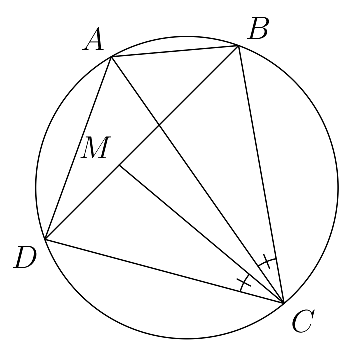


16. Prove Ptolemy's Theorem. This is pretty tough to do if you've never seen the proof before, so I'll give you the first step. Ptolemy's Theorem involves a bunch of side lengths and a circle, so we might think of Power of a Point. Unfortunately, most of the side lengths are chords of the circle and are not well situated to use Power of a Point. However, circles give us equal angles, which can help find similar triangles. So, here's the first step: take a point M on diagonal $\overline{BD}$ such that $\angle ACB = \angle MCD$. Now, find some similar triangles. Then mark equal angles and find more similar triangles.

Solution. Take point M on BD so that $\angle ACB = \angle MCD$. Also, $\angle BAC = \angle BDC = \angle MDC$, so triangles ABC and DMC are similar. Hence,

$$\frac{DM}{AB} = \frac{CD}{AC},$$

so $DM = AB \cdot CD/AC$.



Subtracting equal angles $\angle ACB$ and $\angle MCD$ from $\angle BCD$, we find $\angle ACD = \angle BCM$. Also, $\angle DAC = \angle DBC = \angle MBC$, so triangles ACD and BCM are similar. Hence,

$$\frac{BM}{AD} = \frac{BC}{AC},$$

so $BM = AD \cdot BC/AC$.

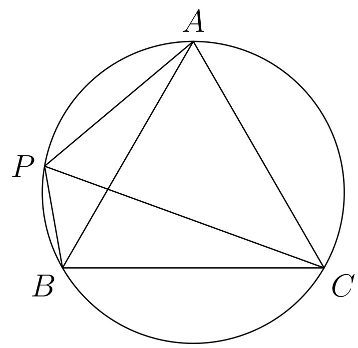
From our two equations,

$$\frac{AB \cdot CD}{AC} + \frac{AD \cdot BC}{AC} = BM + DM = BD,$$

so $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

17. Let ABC be an equilateral triangle and let P be a point on its circumference such that PC is greater than both PA and PB. Prove that $PA + PB = PC$.

Solution. For PC to be greater than both PA and PB, P must lie on arc AB.



By Ptolemy's theorem on quadrilateral $APBC$,

$$PA \cdot BC + PB \cdot AC = PC \cdot AB.$$

Since $AB = AC = BC$, it follows that $PA + PB = PC$.