Completeness

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for each $x, y \in \mathbb{R}$, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.

2. Prove **Kronecker's Theorem**: Let α be an irrational real. Prove that for any interval $I \subseteq [0, 1)$, there is a positive integer n with $\{n\alpha\} \in I$.

3. Let $x_1, x_2, \ldots, x_k$ be real numbers and let $\epsilon > 0$. Prove that there exists a positive integer n such that $\{nx_i\} < \epsilon$ for all i.

4. Consider the sequence defined by $a_1 = 1$, and $a_{n+1} = a_n + 1/a_n^2$ for $n \geq 1$.

   (a) Is the sequence $(a_n)_{n=1}^{\infty}$ bounded?

   (b) Prove that $a_{9000} > 30$.

5. Let $x_1, x_2, \ldots, x_{2n+1}$ be real numbers with the property that for any $1 \leq i \leq 2n + 1$ one can make two groups of n numbers from the x_j, $j \neq i$, in such a way that the two groups each have the same sum. Prove that all the numbers must be equal.

6. Let k, m, n be positive integers with $k, m < n$ and $(k, m) = 1$. Suppose that $a_1, a_2, \ldots, a_n$ are real numbers such that for any indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ there exist indices $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ with

$$\frac{a_{i_1} + a_{i_2} + \cdots + a_{i_k}}{k} = \frac{a_{j_1} + a_{j_2} + \cdots + a_{j_m}}{m}.$$

   Prove that $a_1 = a_2 = \cdots = a_n$.

7. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $x_0 = 1$ and $x_{i+1} \leq x_i$ for $i = 0, 1, 2, \ldots$.

   (a) Prove that for every such sequence there is an $n > 0$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

   (b) Find such a sequence in which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4$$

   for all n.

8. Define the sequence of rational numbers $\{t_n\}$ as follows. Let c_1 be a given positive integer, and let $t_1 = \frac{1}{c_1}$. For a positive integer n, let $t_{n+1} = t_n$ if $t_n = 1$. Otherwise, let c_{n+1} be the least integer such that $c_{n+1} > c_n$ and

$$t_{n+1} = t_n + \frac{1}{c_{n+1}} \leq 1.$$

Show that the sequence$\{t_n\}$ is eventually constant.

9. Two sequences of positive real numbers $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=1}^{\infty}$, satisfy $x_{n+2} = x_n + x_{n+1}^2$ and $y_{n+2} = y_n^2 + y_{n+1}$. for all $n > 0$. Prove that if $x_1, x_2, y_1, y_2 > 1$, then $x_k > y_k$ for some k.

10. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that $a_{n+1} = 2b_n - a_n$ and $b_{n+1} = 2a_n - b_n$ for every positive integer n. Prove that if $a_n > 0$ for all n, then $a_1 = b_1$.

11. Find all functions $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ such that for all $x, y \in \mathbb{R}^{>0}$,

$$f(x)^2 \geq f(x+y)(f(x) + y).$$

12. An infinite set S of points on the plane has the property that no 1×1 square of the plane contains infinitely many points from S. Prove that there exist two points A and B from S such that $\min\{XA, XB\} \geq 0.999AB$ for any other point X in S.

13. Determine whether there exists a polynomial $P(x)$ with real coefficients, not identically zero, for which we can find a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the relation

$$f(x) - \frac{x^3}{3} \cdot f\left(\frac{3x-3}{3+x}\right) = P\left(\frac{3x+3}{3-x}\right)$$

for all irrational numbers x.

14. Find all strictly increasing $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + f^{-1}(x) = 2x$.

15. Find all $f : \mathbb{R} \to \mathbb{R}$ for which $f(xy) = f(x)f(y) - f(x+y) + 1$.

16. The sequence $f_1, f_2, \ldots, f_n, \ldots$ is defined for $x > 0$ recursively by

$$f_1(x) = x, \qquad f_{n+1}(x) = f_n(x)\left(f_n(x) + \frac{1}{n}\right).$$

Prove that there exists one and only one positive number a such that $0 < f_n(a) < f_{n+1}(a) < 1$ for all integers $n \geq 1$.

17. Let $\mathcal{S}$ be the set of all polygonal areas in the plane. Prove that there is a function $f : \mathcal{S} \to (0, 1)$ which satisfies

$$f(S_1 \cup S_2) = f(S_1) + f(S_2)$$

for any $S_1, S_2 \in \mathcal{S}$ which have common points only on their borders.

18. The infinite sequence of 2's and 3's

$$\begin{gathered} 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, \\ 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots \end{gathered}$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the n^{th} term of the sequence if 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m.