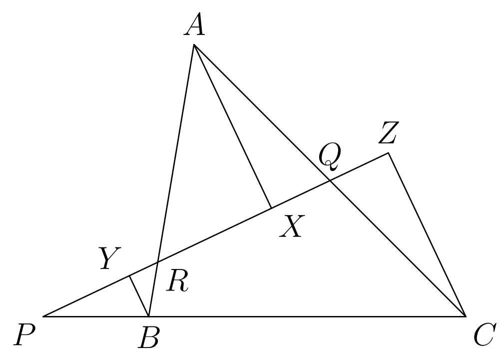
Concurrency & Collinearity

Solutions to Exercises

Section 1

1. Prove Menelaus's theorem.

   Solution. Assume that P, Q, and R are collinear. We take the case where Q and R lie on line segments AC and AB, respectively, and P lies on BC extended past B. (The other cases can be handled similarly.) Let X, Y, and Z be the projections of A, B, and C onto line PQR, respectively.



   Right triangles ARX and BRY are similar, so $AR/RB = AX/BY$. Right triangles BPY and CPZ are similar, so $BP/PC = -BY/CZ$. (This ratio is negative because segments BP and PC have opposite direction.) Right triangles CQZ and AQX are similar, so $CQ/QA = CZ/AX$. Hence,

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \frac{AX}{BY} \cdot \left(-\frac{BY}{CZ}\right) \cdot \frac{CZ}{AX} = -1.$$

   Conversely, assume that

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1.$$

   Let PQ intersect AB at R'. Then from our work above,

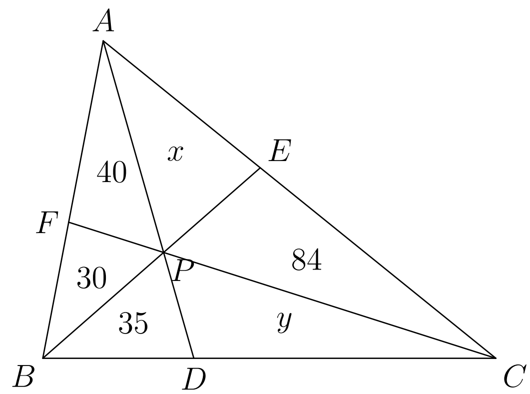
$$\frac{AR'}{R'B} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1.$$

   Hence, $AR/RB = AR'/R'B$, so points R and R' coincide. Therefore, P, Q, and R are collinear.

2. In triangle ABC, cevians AD, BE, and CF intersect at point P. The areas of triangles PAF, PFB, PBD, and PCE are 40, 30, 35, and 84, respectively. Find the area of triangle ABC. (AIME, 1985)

   Solution. Let $x = [AEP]$ and $y = [CDP]$. Triangles ACP and CDP have the same height with respect to line AD, so $[ACP]/[CDP] = AP/PD$. Similarly, triangles ABP and BDP have the same height with respect to line AD, so $[ABP]/[BDP] = AP/PD$. This gives us

$$\frac{x + 84}{y} = \frac{70}{35} = 2.$$



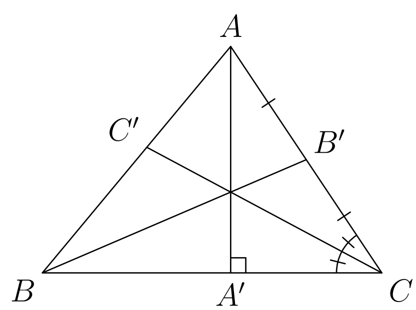
Similarly, $[ACP]/[AFP] = CP/PF = [BCP]/[BFP]$, which gives us

$$\frac{x+84}{40} = \frac{y+35}{30}.$$

Solving this system of equations, we find $x = 56$ and $y = 70$. Hence, $[ABC] = 40+30+35+70+84+56 = 315$.

3. Show that in triangle ABC, altitude AA', median BB', and angle bisector CC' concur if and only if $\sin A = \cos B \tan C$.

Solution. Note that BB' and CC' must intersect inside triangle ABC. Hence, altitude AA' must also lie inside triangle ABC.



From right triangles ABA' and ACA', $BA' = AB \cos \angle ABA' = c \cos B$ and $A'C = AC \cos \angle ACA' = b \cos C$. By the angle bisector theorem, $AC'/C'B = AC/BC = b/a$. Therefore, by Ceva's theorem, AA', BB', and CC' are concurrent if and only if

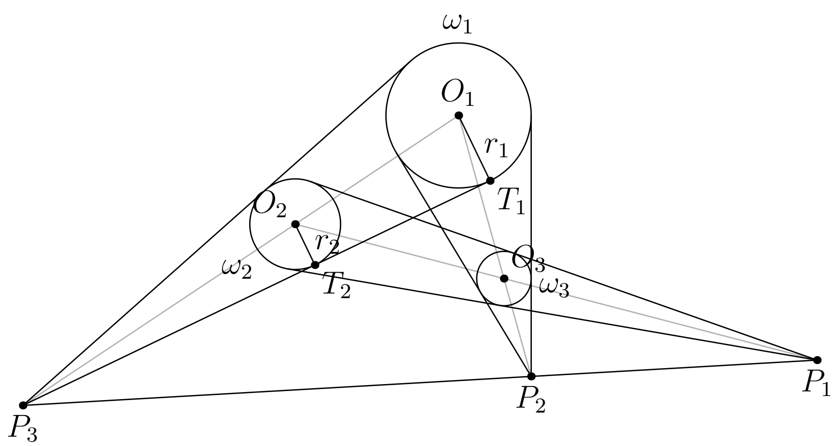
$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = \frac{b}{a} \cdot \frac{c \cos B}{b \cos C} \cdot 1 = 1,$$

which simplifies to $c \cos B = a \cos C$.

By the extended law of sines, $a = 2R \sin A$ and $c = 2R \sin C$. Substituting, our equation becomes $\sin C \cos B = \sin A \cos C$, or $\sin A = \cos B \tan C$.

4. (Monge's Theorem) Let ω_1, ω_2, and ω_3 be three circles, so that no circle contains another circle. Let P_1 be the intersection of the external common tangents of ω_2 and ω_3, and define points P_2 and P_3 similarly. Show that P_1, P_2, and P_3 are collinear.

   Solution. Let O_i and r_i denote the center and radius of circle ω_i, for $i = 1$, 2, and 3. Let a common external tangent of ω_1 and ω_2 touch ω_1 and ω_2 at T_1 and T_2, respectively.



   Then right triangles $P_3O_2T_2$ and $P_3O_1T_1$ are similar, so

$$\frac{O_1P_3}{P_3O_2} = -\frac{O_1T_1}{O_2T_2} = -\frac{r_1}{r_2}.$$

   Likewise, $O_2P_1/P_1O_3 = -r_2/r_3$ and $O_3P_2/P_2O_1 = -r_3/r_1$, so

$$\frac{O_1P_3}{P_3O_2} \cdot \frac{O_2P_1}{P_1O_3} \cdot \frac{O_3P_2}{P_2O_1} = \left(-\frac{r_1}{r_2}\right) \cdot \left(-\frac{r_2}{r_3}\right) \cdot \left(-\frac{r_3}{r_1}\right) = -1.$$

   Therefore, by Menelaus's theorem, P_1, P_2, and P_3 are collinear.

5. In triangle ABC, let D, E, and F be points on BC, AC, and AB, respectively. Prove that AD, BE, and CF are concurrent if and only if

$$\frac{\sin \angle BAD}{\sin \angle ABE} \cdot \frac{\sin \angle CBE}{\sin \angle BCF} \cdot \frac{\sin \angle ACF}{\sin \angle CAD} = 1.$$

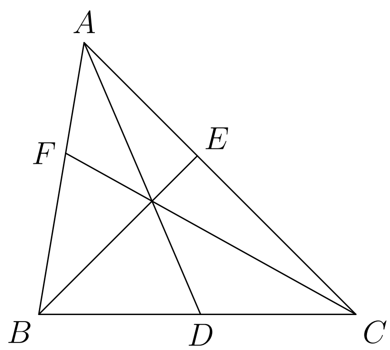
   (This result is known as the angle version of Ceva's theorem.)

   Solution. By the law of sines on triangles ABD and ACD,

$$\frac{BD}{\sin \angle BAD} = \frac{AD}{\sin \angle ABD} \quad \text{and} \quad \frac{CD}{\sin \angle CAD} = \frac{AD}{\sin \angle ACD}.$$

Dividing these equations, we get

$$\frac{BD}{DC} = \frac{\sin \angle BAD \sin \angle ACD}{\sin \angle CAD \sin \angle ABD} = \frac{\sin C \sin \angle BAD}{\sin B \sin \angle CAD}.$$



Similarly,

$$\frac{CE}{EA} = \frac{\sin A \sin \angle CBE}{\sin C \sin \angle ABE},$$
$$\frac{AF}{FB} = \frac{\sin B \sin \angle ACF}{\sin A \sin \angle BCF}.$$

Hence,

$$\begin{aligned}\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} &= \frac{\sin C \sin \angle BAD}{\sin B \sin \angle CAD} \cdot \frac{\sin A \sin \angle CBE}{\sin C \sin \angle ABE} \cdot \frac{\sin B \sin \angle ACF}{\sin A \sin \angle BCF} \\ &= \frac{\sin \angle BAD}{\sin \angle ABE} \cdot \frac{\sin \angle CBE}{\sin \angle BCF} \cdot \frac{\sin \angle ACF}{\sin \angle CAD}.\end{aligned}$$

Therefore, by Ceva's theorem, AD, BE, and CF are concurrent if and only if

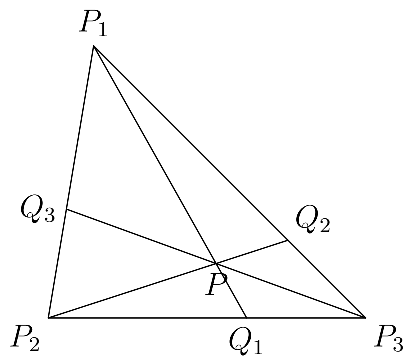
$$\frac{\sin \angle BAD}{\sin \angle ABE} \cdot \frac{\sin \angle CBE}{\sin \angle BCF} \cdot \frac{\sin \angle ACF}{\sin \angle CAD} = 1.$$

6. Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \quad \frac{P_2P}{PQ_2}, \quad \frac{P_3P}{PQ_3},$$

at least one is ≤ 2 and at least one is ≥ 2. (IMO, 1961)

Solution. Let $K_1 = [PP_2P_3]$, $K_2 = [PP_3P_1]$, and $K_3 = [PP_1P_2]$.



Then by the calculations in the handout,

$$\frac{P_1P}{PQ_1} = \frac{K_2+K_3}{K_1}, \quad \frac{P_2P}{PQ_2} = \frac{K_1+K_3}{K_2}, \quad \text{and} \quad \frac{P_3P}{PQ_3} = \frac{K_1+K_2}{K_3}.$$

Without loss of generality, assume that $K_1 \le K_2 \le K_3$. Then

$$\frac{P_3P}{PQ_3} = \frac{K_1+K_2}{K_3} \le \frac{K_3+K_3}{K_3} = 2,$$

and

$$\frac{P_1P}{PQ_1} = \frac{K_2+K_3}{K_1} \ge \frac{K_1+K_1}{K_1} = 2.$$

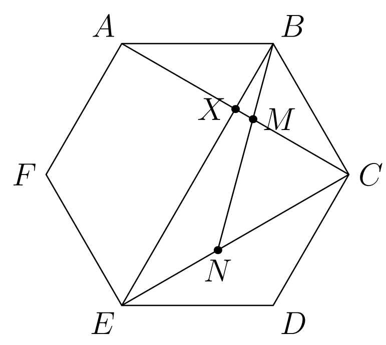
7. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear. (IMO, 1982)

Solution. Assume that the side length of regular hexagon $ABCDEF$ is 1. Let X be the intersection of AC and BE. By Menelaus's theorem on triangle CEX,

$$\frac{CN}{NE} \cdot \frac{EB}{BX} \cdot \frac{XM}{MC} = -1.$$



We see that

$$\frac{CN}{NE} = \frac{r}{1-r},$$
$$\frac{EB}{BX} = -4,$$
$$\frac{XM}{MC} = \frac{AM - AX}{MC} = \frac{r\sqrt{3} - \sqrt{3}/2}{(1-r)\sqrt{3}} = \frac{2r-1}{2(1-r)}.$$

Substituting, we get

$$\frac{r}{1-r} \cdot (-4) \cdot \frac{2r-1}{2(r-1)} = -1.$$

This equation simplifies to $3r^2 = 1$, so $r = 1/\sqrt{3}$.

8. In triangle ABC, A', B', and C' are on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Given that $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at the point O, and that

$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92,$$

find the value of

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}.$$

(AIME, 1992)

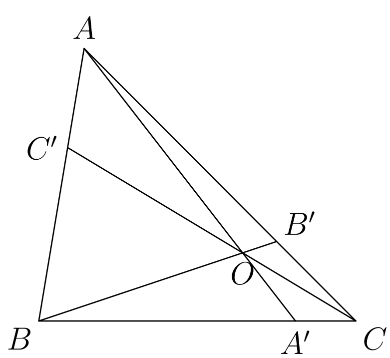
Solution. Let $K_1 = [OBC]$, $K_2 = [OCA]$, and $K_3 = [OAB]$. Then by the calculations in the handout,

$$\frac{AO}{OA'} = \frac{K_2 + K_3}{K_1}, \quad \frac{BO}{OB'} = \frac{K_1 + K_3}{K_2}, \quad \text{and} \quad \frac{CO}{OC'} = \frac{K_1 + K_2}{K_3}.$$

Substituting, we get

$$\frac{K_2 + K_3}{K_1} + \frac{K_1 + K_3}{K_2} + \frac{K_1 + K_2}{K_3} = 92.$$

This equation simplifies to $K_1^2 K_2 + K_1 K_2^2 + K_1^2 K_3 + K_1 K_3^2 + K_2^2 K_3 + K_2 K_3^2 = 92 K_1 K_2 K_3$.



Then

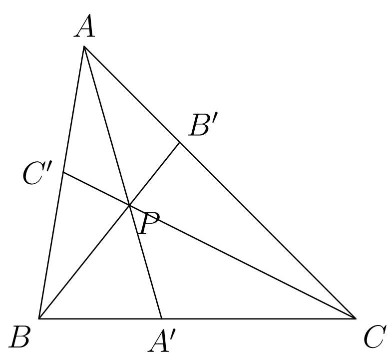
$$\begin{aligned}\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} &= \frac{K_2+K_3}{K_1} \cdot \frac{K_1+K_3}{K_2} \cdot \frac{K_1+K_2}{K_3} \\ &= \frac{K_1^2K_2 + K_1K_2^2 + K_1^2K_3 + K_1K_3^2 + K_2^2K_3 + K_2K_3^2 + 2K_1K_2K_3}{K_1K_2K_3} \\ &= \frac{92K_1K_2K_3 + 2K_1K_2K_3}{K_1K_2K_3} \\ &= \frac{94K_1K_2K_3}{K_1K_2K_3} \\ &= 94.\end{aligned}$$

9. Let ABC be a triangle with circumradius R. Let A', B', C' be points on sides BC, CA, AB, such that AA', BB', CC' are concurrent. Prove that

$$\frac{AB' \cdot BC' \cdot CA'}{[A'B'C']} = 2R.$$

(Ireland, 1992)

Solution. Let AA', BB', and CC' concur at P. Let $K_1 = [PBC]$, $K_2 = [PCA]$, and $K_3 = [PAB]$.



Then

$$\begin{aligned}[AB'C'] &= \frac{AC'}{AB} \cdot \frac{AB'}{AC} \cdot [ABC] \\ &= \frac{K_2}{K_1+K_2} \cdot \frac{K_3}{K_1+K_3} \cdot [ABC] \\ &= \frac{K_2K_3}{(K_1+K_2)(K_1+K_3)} \cdot [ABC].\end{aligned}$$

Similarly,

$$[A'BC'] = \frac{K_1K_3}{(K_1+K_2)(K_2+K_3)} \cdot [ABC],$$
$$[A'B'C] = \frac{K_2K_3}{(K_1+K_3)(K_2+K_3)} \cdot [ABC],$$

so

$$\begin{aligned}[A'B'C'] &= [ABC] - [AB'C'] - [A'BC'] - [A'B'C] \\ &= [ABC] - \frac{K_2K_3}{(K_1+K_2)(K_1+K_3)} \cdot [ABC] \\ &\quad - \frac{K_1K_3}{(K_1+K_2)(K_2+K_3)} \cdot [ABC] - \frac{K_2K_3}{(K_1+K_3)(K_2+K_3)} \cdot [ABC] \\ &= \frac{2K_1K_2K_3}{(K_1+K_2)(K_1+K_3)(K_2+K_3)} \cdot [ABC].\end{aligned}$$

We have that $AB'/AC = K_3/(K_1+K_3)$, so

$$AB' = \frac{K_3}{K_1+K_3} \cdot b.$$

Similarly,

$$BC' = \frac{K_1}{K_1+K_2} \cdot c \quad \text{and} \quad CA' = \frac{K_2}{K_2+K_3} \cdot a,$$

so

$$A'B \cdot BC' \cdot CA' = \frac{K_1K_2K_3}{(K_1+K_2)(K_1+K_3)(K_2+K_3)} \cdot abc.$$

Hence,

$$\frac{AB' \cdot BC' \cdot CA'}{[A'B'C']} = \frac{abc}{2[ABC]} = \frac{4[ABC]R}{2[ABC]} = 2R.$$

Section 2

1. Given triangle ABC, which points inside the triangle are their own isotomic conjugate? Which points inside the triangle are their own isogonal conjugate?

   Solution. Let P be a point inside the triangle, and let AP, BP, and CP intersect BC, AC, and AB at D, E, and F, respectively.

   Then P is its own isotomic conjugate if and only if AD, BE, and CF are medians, which means P is the centroid of triangle ABC. Similarly, P is its own isogonal conjugate if and only if AD, BE, and CF are angle bisectors, which means P is the incenter of triangle ABC.

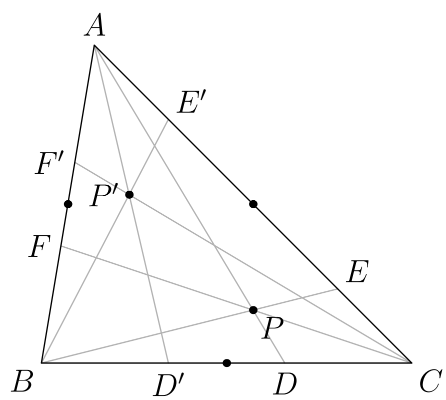
2. Given triangle ABC and points P and P', let $K_A = [PBC]$, $K_B = [PCA]$, $K_C = [PAB]$, $K'_A = [P'BC]$, $K'_B = [P'CA]$, and $K'_C = [P'AB]$.

   (a) Show that if P' is the isotomic conjugate of P, then
   $$K'_A : K'_B : K'_C = \frac{1}{K_A} : \frac{1}{K_B} : \frac{1}{K_C}.$$

   (b) Show that if P' is the isogonal conjugate of P, then
   $$K'_A : K'_B : K'_C = \frac{a^2}{K_A} : \frac{b^2}{K_B} : \frac{c^2}{K_C}.$$

Solution. Let AP and AP' intersect BC at D and D', respectively. Define points E, E', F, and F' similarly.

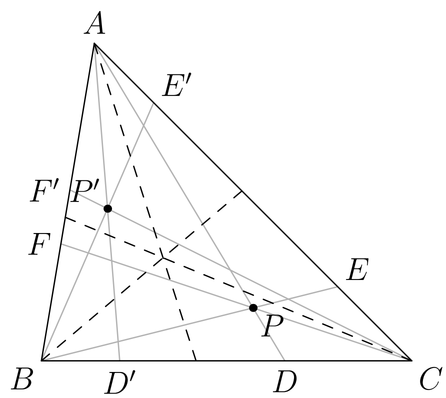
(a) We have that
$$\frac{K'_A}{K'_B} = \frac{BF'}{AF'}.$$
Since P' is the isotomic conjugate of P, $BF' = AF$ and $AF' = BF$, so
$$\frac{K'_A}{K'_B} = \frac{BF'}{AF'} = \frac{AF}{BF} = \frac{K_B}{K_A} = \frac{1/K_A}{1/K_B}.$$



Similarly,
$$\frac{K'_B}{K'_C} = \frac{K_C}{K_B} = \frac{1/K_B}{1/K_C}.$$

Hence,
$$K'_A : K'_B : K'_C = \frac{1}{K_A} : \frac{1}{K_B} : \frac{1}{K_C}.$$

(b) We have that
$$\frac{K'_A}{K'_B} = \frac{BF'}{AF'}.$$
By the law of sines on triangles BCF' and ACF',
$$\frac{BF'}{\sin\angle BCF'} = \frac{BC}{\sin\angle BF'C} \quad \text{and} \quad \frac{AF'}{\sin\angle ACF'} = \frac{AC}{\sin\angle AF'C}.$$
Dividing these equations, we get
$$\frac{K'_A}{K'_B} = \frac{BF'}{AF'} = \frac{BC\sin\angle BCF'\sin\angle AF'C}{AC\sin\angle ACF'\sin\angle BF'C}.$$
Since $\angle AF'C + \angle BF'C = 180^\circ$, $\sin\angle AF'C = \sin\angle BF'C$, so
$$\frac{K'_A}{K'_B} = \frac{a\sin\angle BCF'}{b\sin\angle ACF'}.$$



Since P' is the isogonal conjugate of P, $\angle BCF' = \angle ACF$ and $\angle ACF' = \angle BCF$, so
$$\frac{K'_A}{K'_B} = \frac{a\sin\angle ACF}{b\sin\angle BCF}.$$
By the same law of sines calculation on triangles BCF and ACF,
$$\frac{K_A}{K_B} = \frac{BF}{AF} = \frac{a\sin\angle BCF}{b\sin\angle ACF}.$$
Then
$$\frac{K'_A}{K'_B} = \frac{a\sin\angle ACF}{b\sin\angle BCF} = \frac{a}{b}\cdot\frac{aK_B}{bK_A} = \frac{a^2K_B}{b^2K_A} = \frac{a^2/K_A}{b^2/K_B}.$$

Similarly,

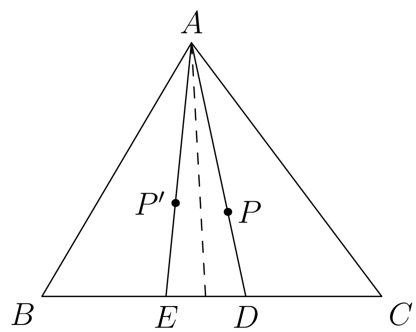
$$\frac{K'_B}{K'_C} = \frac{b^2 K_C}{c^2 K_B} = \frac{b^2/K_B}{c^2/K_C}.$$

Hence,

$$K'_A : K'_B : K'_C = \frac{a^2}{K_A} : \frac{b^2}{K_B} : \frac{c^2}{K_C}.$$

3. In triangle ABC, $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on side BC with $CD = 6$. Point E is on side BC such that $\angle BAE = \angle CAD$. Given that $BE = p/q$, where p and q are relatively prime positive integers, find q. (AIME II, 2005)

   Solution. Let P be an arbitrary point on AD, and let P' be the isogonal conjugate of P, so P lies on AE. Let $K_A = [PBC]$, $K_B = [PCA]$, $K_C = [PAB]$, $K_A = [P'BC]$, $K_B = [P'CA]$, and $K_C = [P'AB]$.



   Then by the previous exercise,

$$\frac{BE}{CE} = \frac{K'_C}{K'_B} = \frac{c^2 K_B}{b^2 K_C} = \frac{c^2}{b^2} \cdot \frac{CD}{BD}.$$

   Since $CD = 6$, $BD = BC - CD = 15 - 6 = 9$. Also, $CE = BC - BE = 15 - BE$, so

$$\frac{BE}{15 - BE} = \frac{13^2}{14^2} \cdot \frac{6}{9} = \frac{169}{294}.$$

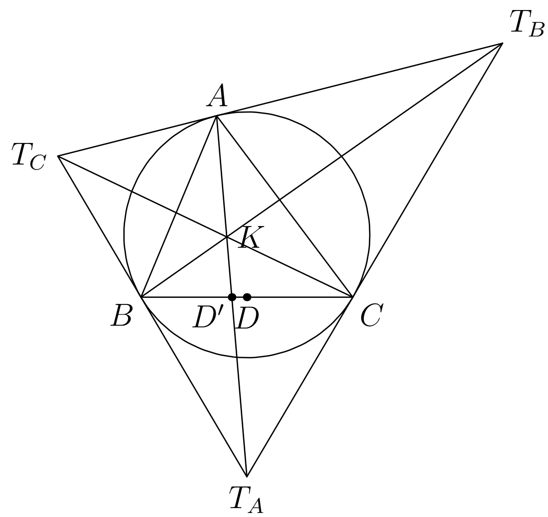
   Solving for BE, we find $BE = 2535/463$. The final answer is 463.

4. Given triangle ABC, the tangents to the circumcircle of triangle ABC at B and C intersect at T_A. Points T_B and T_C are defined similarly. Show that AT_A, BT_B, and CT_C concur at the isogonal conjugate of the centroid of triangle ABC. (This point of concurrence is known as the *Symmedian point* of triangle ABC, and the cevians AT_A, BT_B, and CT_C are known as the *symmedians* of triangle ABC.)

   Solution. First, we see that AT_A, BT_B, and CT_C concur at the Gergonne point of triangle $T_A T_B T_C$. Let K be this point of concurrency. Let AD, BE, and CF be the medians of triangle ABC, which concur at the centroid G.

Let AT_A, BT_B, and CT_C intersect BC, AC, and AB at D', E', and F', respectively. Let $K_A = [GBC]$, $K_B = [GCA]$, $K_C = [GAB]$, $K'_A = [KBC]$, $K'_B = [KCA]$, and $K'_C = [KAB]$. Then $K_A = K_B = K_C$, and

$$\frac{K'_B}{K'_C} = \frac{CD'}{BD'}.$$



By the law of sines on triangles ACD' and ABD',

$$\frac{CD'}{\sin\angle CAD'} = \frac{AC}{\sin\angle AD'C} \quad\text{and}\quad \frac{BD'}{\sin\angle BAD'} = \frac{AB}{\sin\angle AD'B}.$$

Dividing these equations, we get

$$\frac{K'_B}{K'_C} = \frac{CD'}{BD'} = \frac{AC\sin\angle CAD'\sin\angle AD'B}{AB\sin\angle BAD'\sin\angle AD'C}.$$

Since $\angle AD'B + \angle AD'C = 180^\circ$, $\sin\angle AD'B = \sin\angle AD'C$, so

$$\frac{K'_B}{K'_C} = \frac{b\sin\angle CAD'}{c\sin\angle BAD'} = \frac{b\sin\angle CAT_A}{c\sin\angle BAT_A}.$$

By the law of sines on triangles ACT_A and ABT_A,

$$\frac{CT_A}{\sin\angle CAT_A} = \frac{AT_A}{\sin\angle ACT_A} \quad\text{and}\quad \frac{BT_A}{\sin\angle BAT_A} = \frac{AT_A}{\sin\angle ABT_A}.$$

Dividing these equations, we get

$$\frac{CT_A}{BT_A} = \frac{\sin\angle CAT_A\sin\angle ABT_A}{\sin\angle BAT_A\sin\angle ACT_A}.$$

Since T_AB and T_AC are tangents from the same point to the same circle, they are equal in length. Also, $\sin\angle ABT_A = \sin\angle ABT_C = \sin\angle ACB = \sin C$ and $\sin\angle ACT_A = \sin\angle ACT_B = \sin\angle ABC = \sin B$, so

$$\frac{\sin\angle CAT_A}{\sin\angle BAT_A} = \frac{\sin\angle ACT_A}{\sin\angle ABT_A} = \frac{\sin B}{\sin C} = \frac{b}{c}.$$

Hence,

$$\frac{K'_B}{K'_C} = \frac{b\sin\angle CAT_A}{c\sin\angle BAT_A} = \frac{b^2}{c^2}.$$

Similarly,

$$\frac{K'_A}{K'_B} = \frac{a^2}{b^2}.$$

Therefore,

$$K'_A : K'_B : K'_C = a^2 : b^2 : c^2 = \frac{a^2}{K_A} : \frac{b^2}{K_B} : \frac{c^2}{K_C}.$$

To prove rigorously that K is the isogonal conjugate of G with respect to triangle ABC, we argue as follows: Let G' be the isogonal conjugate of G, and let AG', BG', and CG' intersect BC, AC, and AB, at D'', E'', and F'', respectively. Then by Exercise 2,

$$\frac{BD''}{CD''} = \frac{c^2K_B}{b^2K_C} = \frac{c^2}{b^2}.$$

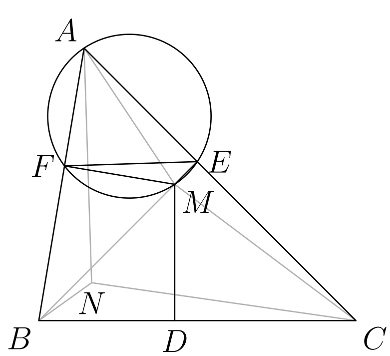
But $BD'/CD' = K'_C/K'_B = c^2/b^2$, so points D' and D'' coincide, which means K lies on AD''. Similarly, K lies on BE'', so K coincides with G'. In other words, K is the isogonal conjugate of G with respect to triangle ABC.

5. Let M and N be points inside triangle ABC that are isogonal conjugates. Prove that

$$\frac{AM\cdot AN}{AB\cdot AC} + \frac{BM\cdot BN}{AB\cdot BC} + \frac{CM\cdot CN}{AC\cdot BC} = 1.$$

(IMO Short List, 1998)

Solution 1. Let D, E, and F be the projections of M onto sides BC, AC, and AB, respectively. Then $\angle AEM = \angle AFM = 90^\circ$, so quadrilateral $AEMF$ is cyclic. In fact, E and F lie on the circle with diameter AM, so by the extended law of sines on triangle AEF, $EF = AM\sin A$.



Also, $\angle AFE + \angle FAN = \angle AME + \angle EAM = 90^\circ$, which means AN and EF are perpendicular. In other words, the diagonals of quadrilateral $AENF$ are perpendicular, so

$$[AENF] = \frac{1}{2} \cdot EF \cdot AN = \frac{1}{2} \cdot AM \cdot AN \cdot \sin A.$$

We also have that $[ABC] = \frac{1}{2} \cdot AB \cdot AC \cdot \sin A$. Dividing these equations, we get

$$\frac{AM \cdot AN}{AB \cdot AC} = \frac{[AENF]}{[ABC]}.$$

Similarly,

$$\frac{BM \cdot BN}{AB \cdot BC} = \frac{[BDNF]}{[ABC]} \quad \text{and} \quad \frac{CM \cdot CN}{AC \cdot BC} = \frac{[CDNE]}{[ABC]}.$$

Adding these equations, we get

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{AB \cdot BC} + \frac{CM \cdot CN}{AC \cdot BC} = \frac{[AENF] + [BDNF] + [CDNE]}{[ABC]} = \frac{[ABC]}{[ABC]} = 1.$$

Solution 2. Let K be the point on AN such that $\angle ACK = \angle AMB$. Since $\angle AMB > \angle ACB$, K lies outside triangle ABC.

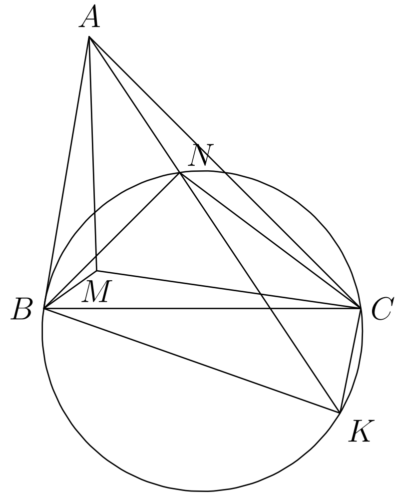
Since $\angle AMB = \angle ACK$ and $\angle BAM = \angle NAC = \angle KAC$, triangles AMB and ACK are similar. Hence,

$$\frac{AM}{AC} = \frac{AB}{AK} = \frac{BM}{CK}.$$

But $\angle MAC = \angle BAN = \angle BAK$, so triangles AMC and ABK are similar. This implies

$$\frac{AM}{AB} = \frac{CM}{BK} = \frac{AC}{AK},$$

and $\angle BKN = \angle BKA = \angle MCA = \angle BCN$, so quadrilateral $BNCK$ is cyclic.



Then by Ptolemy's theorem, $BC \cdot KN = BK \cdot CN + BN \cdot CK$. Since $KN = AK - AN$, we can write this as

$$BC \cdot (AK - AN) = BK \cdot CN + BN \cdot CK.$$

From the equations above, $AK = AB \cdot AC/AM$, $BK = AB \cdot CM/AM$, and $CK = AC \cdot BM/AM$, so

$$BC \cdot \left(\frac{AB \cdot AC}{AM} - AN\right) = \frac{AB \cdot CM}{AM} \cdot CN + BN \cdot \frac{AC \cdot BM}{AM},$$

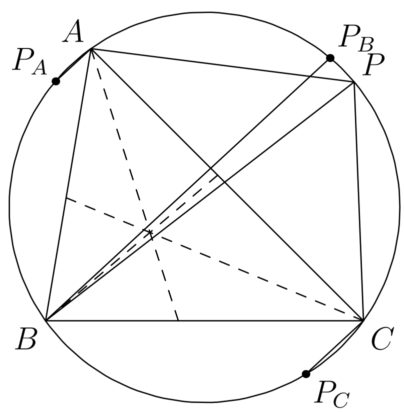
which implies

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{AB \cdot BC} + \frac{CM \cdot CN}{AC \cdot BC} = 1.$$

6. What can you say about the isogonal conjugate of a point on the circumcircle of triangle ABC?

Solution. Let P be a point on the circumcircle of triangle ABC, as shown. Let ℓ_A be the reflection of AP in the angle bisector of $\angle BAC$, and let P_A be the intersection of ℓ_A with the circumcircle. Define points P_B and P_C similarly. We claim that AP_A, BP_B, and CP_C are in fact parallel.

Let $\theta = \angle ABP = \angle CBP_B$. Then $\angle ABP_B = \angle CBP = B - \theta$. Also, $\angle BAP_A = \angle CAP = \angle CBP = B - \theta = \angle ABP_B$, so AP_A and BP_B are parallel. We also have that $\angle BCP_C = \angle ACP = \angle ABP = \theta = \angle CBP_B$, so BP_B and CP_C are parallel.



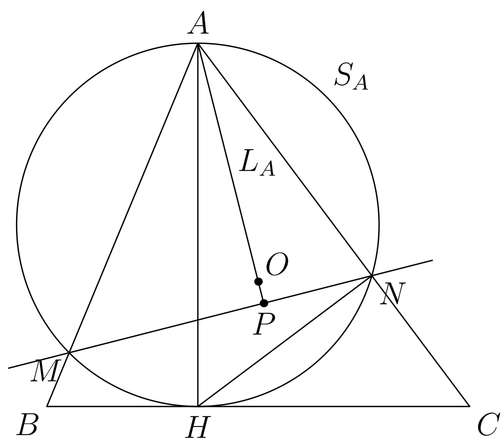
Therefore, AP_A, BP_B, and CP_C are parallel, so the isogonal conjugate of P does not exist. (We can also say that the isogonal conjugate of P is a point at infinity.) It can be shown that the points on the circumcircle are the only points that have this property.

Section 3

1. Let ABC be an acute-angled triangle. Three lines L_A, L_B, L_C are constructed through the vertices A, B, and C, respectively, according to the following prescription: let H be the foot of the altitude drawn

from the vertex A to the side BC; let S_A be the circle with diameter AH; let S_A meet the sides AB and AC at M and N, respectively, where M and N are distinct from A; then L_A is the line through A perpendicular to MN. The lines L_B and L_C are constructed similarly. Prove that L_A, L_B, and L_C are concurrent. (IMO Short List, 1988)

Solution. Let O be the circumcenter of triangle ABC, and let P be the projection of A onto MN, so P lies on L_A.



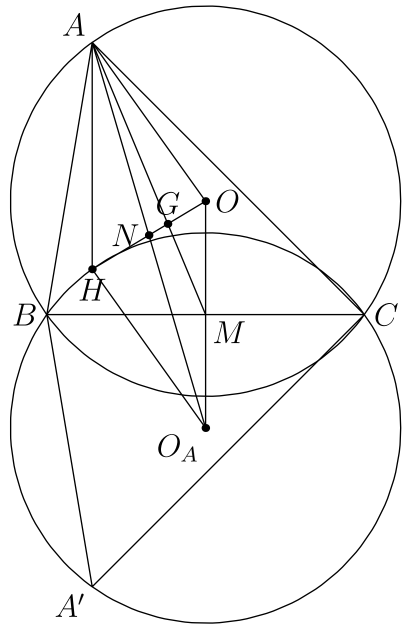
From right triangle APM, $\angle MAP = 90^\circ - \angle AMP$. Then $90^\circ - \angle AMP = 90^\circ - \angle AMN = 90^\circ - \angle AHN$. From right triangle ANH, $90^\circ - \angle AHN = \angle NAH = \angle CAH$. Finally, from right triangle ACH, $\angle CAH = 90^\circ - C$. Hence, $\angle MAP = 90^\circ - C$. But $\angle MAO = 90^\circ - C$, so O lies on L_A.

By symmetry, O lies on L_B and L_C, so L_A, L_B, and L_C concur at O.

2. For triangle ABC, let A' denote the reflection of A in side BC, and define points B' and C' similarly. Let O_A, O_B, and O_C be the circumcenters of triangles $A'BC$, $AB'C$, ABC', respectively. Show that AO_A, BO_B, and CO_C are concurrent.

   Solution. Let G, H, O, and N denote the centroid, orthocenter, circumcenter, and nine-point center of triangle ABC, respectively. Let M be the midpoint of BC.

   Since A' is the reflection of A in BC, O_A is the reflection of O in BC, so M is also the midpoint of OO_A. Hence, $OM = OO_A/2$.

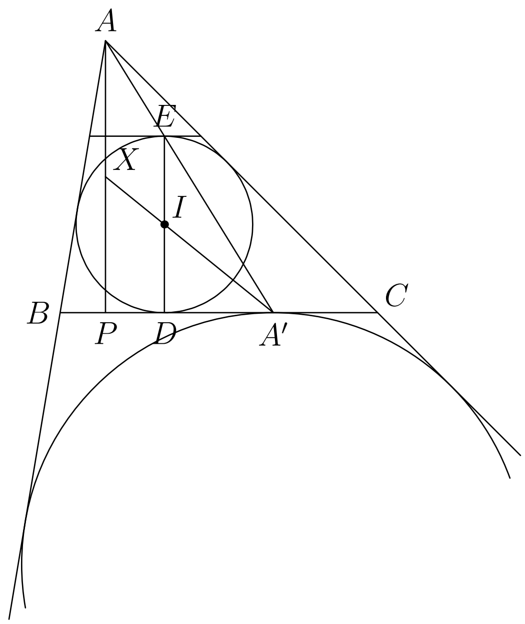


Let h be the homothety centered at G with scale factor $-1/2$. We know that $h(A) = M$ and $h(H) = O$, so $OM = AH/2$. Therefore, $AH = OO_A$. Furthermore, both AH and OO_A are perpendicular to BC, so AH and OO_A are parallel, which means that quadrilateral AOO_AH is a parallelogram.

Then AO_A passes through the midpoint of OH, which is N. By symmetry, BO_B and CO_C also pass through N, so AO_A, BO_B, and CO_C concur at N.

3. In triangle ABC, let X, Y, and Z be the midpoints of the altitudes from vertices A, B, and C, respectively. Let the excircle opposite vertex A touch side BC at A', and define points B' and C' similarly. Show that $A'X$, $B'Y$, and $C'Z$ are concurrent.

   Solution. Let P be the foot of the altitude from A. Let I be the incenter of triangle ABC. Let D be the point where the incircle touches side BC, and let E be the point diametrically opposite D.



There exists a homothety h centered at A that takes the incircle to the A-excircle. The tangent to the incircle at E and the tangent to the A-excircle at A' are parallel, so h takes one tangent to the other, which means $h(E) = A'$. In particular, E lies on AA'.

Since I is the midpoint of DE and X is the midpoint of AP, it follows that I lies on $A'X$. By symmetry, I lies on $B'Y$ and $C'Z$, so $A'X$, $B'Y$, and $C'Z$ concur at I.

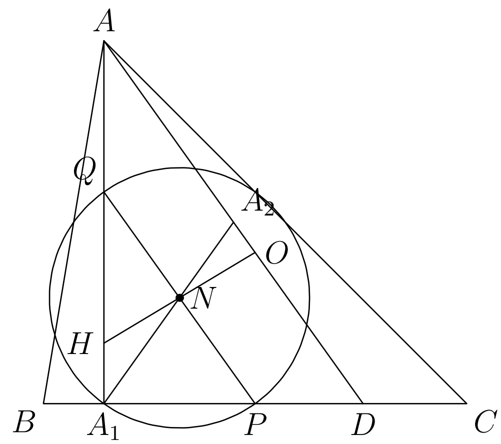
4. Consider triangle ABC with circumcenter O and orthocenter H. Let A_1 be the projection of A onto BC and let D be the intersection of AO with BC. Denote by A_2 the midpoint of AD. Similarly, we define B_1, B_2 and C_1, C_2. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent. (Mathematical Reflections)

   Solution. Let N be the nine-point center of triangle ABC. Let P and Q be the midpoints of BC and AH, respectively.

   We know that A_1, P, and Q lie on the nine-point circle. Furthermore, $\angle PA_1Q = 90^\circ$, so PQ is a diameter of the circle, which means that N is the midpoint of PQ.

   Since Q and N are the midpoints of AH and HO, respectively, NQ is parallel to AO, so PQ is parallel to AD. And since N and A_2 are the midpoints of PQ and AD, respectively, A_1A_2 passes through N.

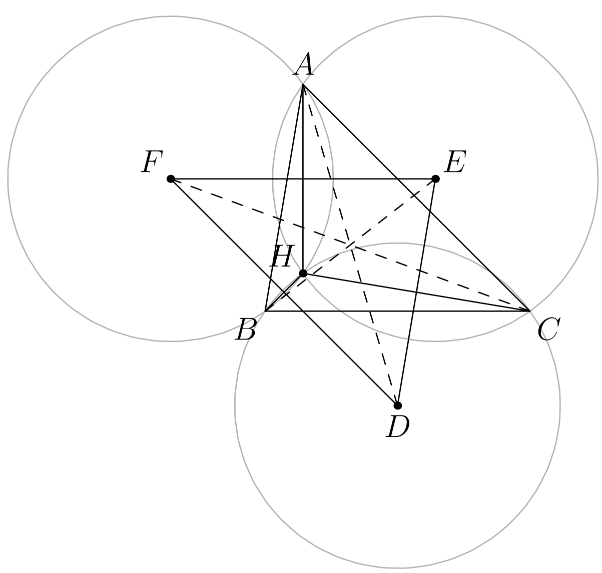
   By symmetry, B_1B_2 and C_1C_2 also pass through N, so A_1A_2, B_1B_2, and C_1C_2 concur at N.



Section 4

1. Let H be the orthocenter of triangle ABC, and let D, E, and F be the circumcenters of triangles BHC, CHA, and AHB, respectively. Prove that AD, BE, and CF are concurrent.

   Solution. We see that AH is a common chord of the circumcircles of triangles CHA and AHB, so AH is perpendicular to EF. But AH is also perpendicular to BC, so EF is parallel to BC



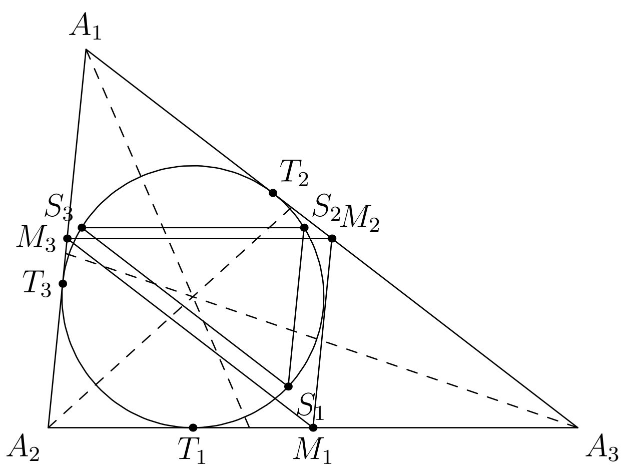
   Similarly, DF is parallel to AC and DE is parallel to AB, so the corresponding sides of triangles DEF

and ABC are parallel. Hence, triangles DEF and ABC are homothetic, which means AD, BE, and CF are concurrent. (This also follows from Exercise 2 of Section 3.)

2. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1$, 2, 3, M_i is the midpoint of side a_i, and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent. (IMO, 1982)

   Solution. We claim that triangles $S_1S_2S_3$ and $M_1M_2M_3$ are homothetic.

   First, we prove that S_2S_3 is parallel to M_2M_3. It suffices to prove that arcs S_3T_1 and S_2T_1 are equal.

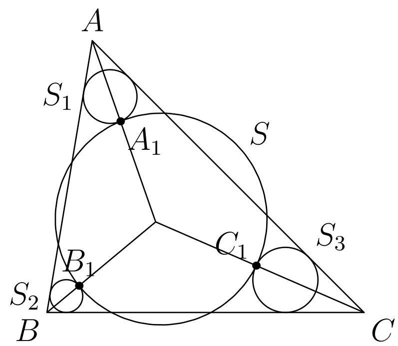


   Since T_3 and T_2 are the reflections of S_3 and T_1 in the angle bisector of $\angle A_1A_3A_2$, respectively, arcs S_3T_1 and T_2T_3 are equal. Since T_2 and T_3 are the reflections of S_2 and T_1 in the angle bisector of $\angle A_1A_2A_3$, respectively, arcs S_2T_1 and T_2T_3 are equal. Hence, arcs S_3T_1 and S_2T_1 are equal.

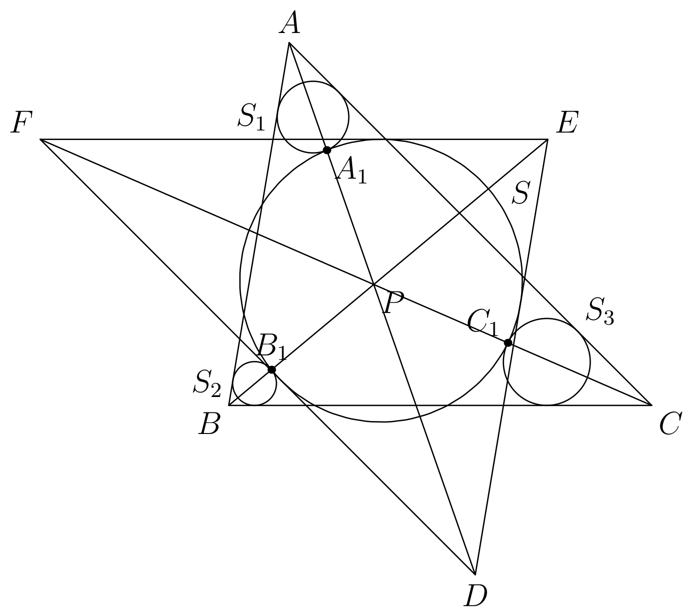
   It follows that S_2S_3 is parallel to M_2M_3. Similarly, S_1S_3 is parallel to M_1M_3, and S_1S_2 is parallel to M_1M_2. Therefore, triangles $S_1S_2S_3$ and $M_1M_2M_3$ are homothetic, which means M_1S_1, M_2S_2, and M_3S_3 are concurrent.

   In fact, the point of concurrency is the Feuerbach point of triangle ABC, the point where the nine-point circle and incircle are tangent. Can you prove it?

3. A triangle ABC and circle S is given in the plane. Circle S_1 is tangent to sides AB and AC, and to S at A_1. Circles S_2 and S_3 and points B_1 and C_1 are defined similarly. Prove that the lines AA_1, BB_1, and CC_1 are concurrent. (Russia, 1994)



Solution. Construct a triangle DEF, as shown, so that its sides are tangent to S, and the corresponding sides of triangles DEF and ABC are parallel. By our construction, there exists a homothety h, say centered at P, that takes triangle ABC to triangle DEF. In particular, P lies on AD.



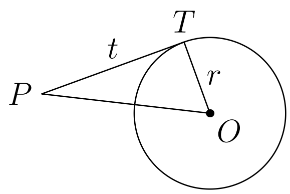
We see that AB and AC are tangents to S_1, and DE and DF are tangents to S. Furthermore, the corresponding tangents are parallel, and circles S_1 and S are externally tangent at A_1, so there exists a homothety h_1, centered at A_1, that takes S_1 to S. Then h_1 takes A to D, so A_1 lies on AD.

Therefore, AA_1 passes through P. Similarly, BB_1 and CC_1 also pass through P, so AA_1, BB_1, and CC_1 concur at P.

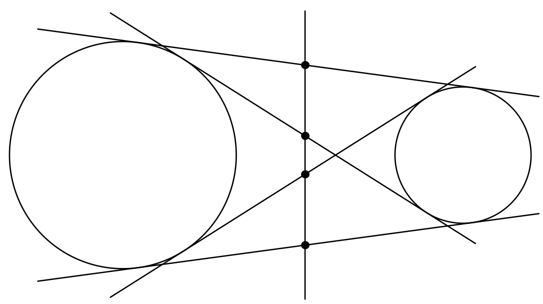
Section 5

1. Two circles in the plane are exterior to each other. Prove that the midpoints of the two common external tangents and two common internal tangents are collinear.

   Solution. Recall that the power of a point P with respect to a circle centered at O with radius r is $OP^2 - r^2$. If P lies outside the circle, then this is equal to t^2, where t is the length of the tangent from P to the circle.



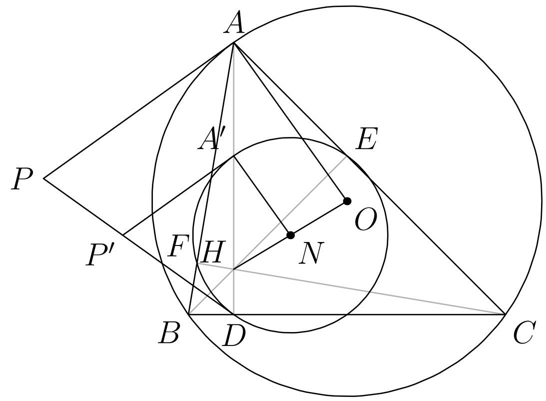
   Hence, if M is the midpoint of a common tangent (internal or external), then the length of the tangents from M to both circles is the same, so M has the same power with respect to both circles, which means that M lies on the radical axis of the two circles. Thus, all four midpoints lie on the radical axis of the two circles.



2. Let AD, BE, and CF be the altitudes of triangles ABC, and let Γ_1 and Γ_2 denote the circumcircles of triangles ABC and DEF, respectively. The tangent to Γ_1 at A and the tangent to Γ_2 at D intersect at P. Points Q and R are defined similarly. Show that P, Q, and R are collinear.

   Solution. Let H, O, and N denote the orthocenter, circumcenter, and nine-point center of triangle ABC, respectively. Let A' be the midpoint of AH, and let the tangents to the nine-point circle at A' and D intersect at P'.

   Consider the homothety h, centered at H, with scale factor $1/2$. We know that the h takes the circumcircle to the nine-point circle. We also see that $h(A) = A'$. Hence, h takes tangent AP to $A'P'$, which means they are parallel.



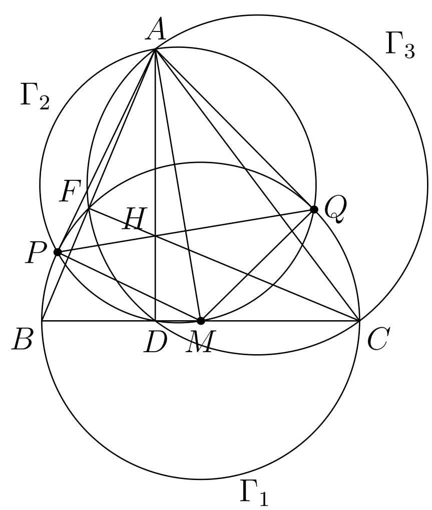
Since $P'A'$ and $P'D$ are tangents from the same point to the same circle, they are equal in length. In other words, triangle $P'A'D$ is isosceles. Triangles PAD and $P'A'D$ are similar, so triangle PAD is also isosceles, with $PA = PD$. Hence, P has the same power with respect to the circumcircle and the nine-point circle, which means that P lies on the radical axis of the two circles. (In other words, P lies on the orthic axis of triangle ABC.)

Similarly, Q and R also lie on the radical axis of the two circles, so P, Q, and R are collinear.

3. ABC is a triangle. The tangents from A touch the circle with diameter BC at P and Q. Show that the orthocenter of triangle ABC lies on PQ. (China, 1996)

Solution. Let AD and CF be altitudes of triangle ABC, let H be the orthocenter of triangle ABC, and let M be the midpoint of BC. Let Γ_1 be the circle with diameter BC, so F lies on Γ_1.

Since AP and AQ are tangents to Γ_1, which is centered at M, $\angle APM = \angle AQM = 90^\circ$. Hence, P and Q lie on Γ_2, the circle with diameter AM. Note that D also lies on Γ_2.



Finally, let Γ_3 be the circle with diameter AC, so D and F lie on Γ_3. Then the radical axis of Γ_1 and Γ_3 is CF, and the radical axis of Γ_2 and Γ_3 is AD. Therefore, the radical center of Γ_1, Γ_2, and Γ_3 is the intersection of CF and AD, namely the orthocenter H.

The radical axis of Γ_1 and Γ_2 is PQ. We conclude that PQ passes through H.

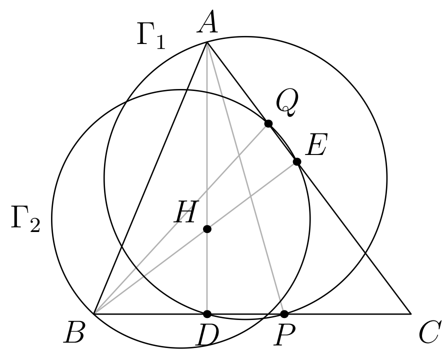
4. Let l_1, l_2, l_3, and l_4 be four lines in the plane. Let H_1 denote the orthocenter of the triangle determined by the lines l_2, l_3, and l_4, and define points H_2, H_3, and H_4 similarly. Show that H_1, H_2, H_3, and H_4 are collinear.

Solution. First, we prove the following lemma.

Lemma. Given triangle ABC, let P and Q be points on sides BC and AC, respectively. Let Γ_1 and Γ_2 be the circles with diameters AP and BQ, respectively. Then the orthocenter of triangle ABC lies on the radical axis of Γ_1 and Γ_2.

Proof. Let AD and BE be the altitudes of triangle ABC, and let H be the orthocenter.

We know that D and E lie on the circle with diameter AB, so by power of a point on H, $AH \cdot HD = BH \cdot HE$.

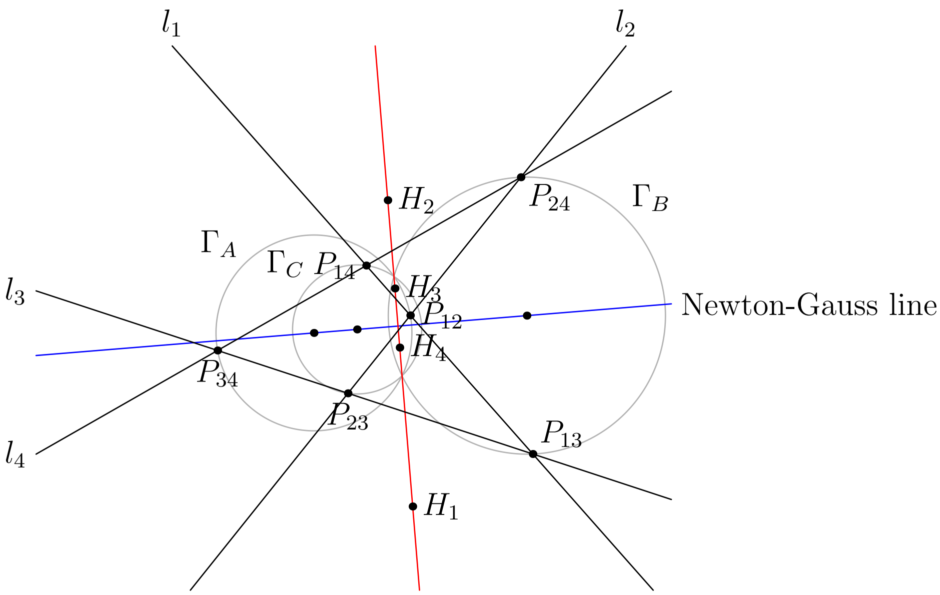


Since $\angle ADP = 90°$, D lies on Γ_1. Then the power of H with respect to Γ_1 is $AH \cdot HD$. Similarly, the power of H with respect to Γ_2 is $BH \cdot HE$. These powers are equal, so H lies on the radical axis of Γ_1 and Γ_2. ■

For all $1 \le i < j \le 4$, let P_{ij} denote the intersection of l_i and l_j. Let Γ_A, Γ_B, and Γ_C be the circles with diameters $P_{12}P_{34}$, $P_{13}P_{24}$, and $P_{14}P_{23}$, respectively.

First, we claim that orthocenters H_1, H_2, H_3, and H_4 are distinct.

Note that l_1 is the line passing through P_{12} that is perpendicular to H_2P_{34}, and l_2 is the line passing through P_{12} that is perpendicular to H_1P_{34}. Since lines l_1 and l_2 are distinct, points H_1 and H_2 are also distinct. (This argument proves the slightly stronger result that H_1, H_2, and P_{34} cannot be collinear.) By a similar argument, any two points H_i and H_j are distinct.



Consider triangle $P_{23}P_{24}P_{34}$, whose orthocenter is H_1. Since P_{12} lies on side $P_{23}P_{24}$, P_{13} lies on side $P_{23}P_{34}$, and P_{14} lies on side $P_{24}P_{34}$, by the lemma, H_1 lies on the radical axis of Γ_A and Γ_B, the radical axis of Γ_A and Γ_C, and the radical axis of Γ_B and Γ_C. Thus, either H_1 is the radical center of all three circles, or all three circles have a common radical axis.

By the same argument, H_2, H_3, and H_4 all have the same property. Since H_1, H_2, H_3, and H_4 are distinct, it must be the case that all three circles have a common radical axis. (We say that all three circles are *coaxal*.) Furthermore, H_1, H_2, H_3, and H_4 lie on this common radical axis.

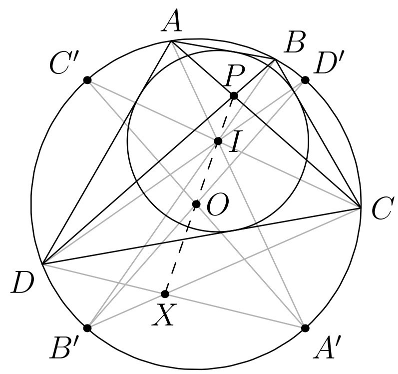
In this kind of diagram, the four lines l_1, l_2, l_3, and l_4 and their six points of intersection form a *complete quadrilateral*. We have shown that the orthocenters of the four triangles determined by the four lines are collinear. This line is called the *ortholine* of the complete quadrilateral.

Furthermore, since circles Γ_A, Γ_B, and Γ_C are coaxal, their centers must be collinear. In other words, the midpoints of $P_{12}P_{34}$, $P_{13}P_{24}$, and $P_{14}P_{23}$ are collinear. This line is called the *Newton-Gauss line* of the complete quadrilateral, and it is perpendicular to the ortholine.

Section 6

1. Let $ABCD$ be a quadrilateral that has both a circumcircle and an incircle. Let O be the circumcenter, let I be the incenter, and let P be the intersection of the diagonals AC and BD. Prove that O, I, and P are collinear. (IMO Short List, 1989)

   Solution. Let AI, the angle bisector of $\angle BAD$, intersect the circumcircle at A'. Define points B', C', and D' similarly.

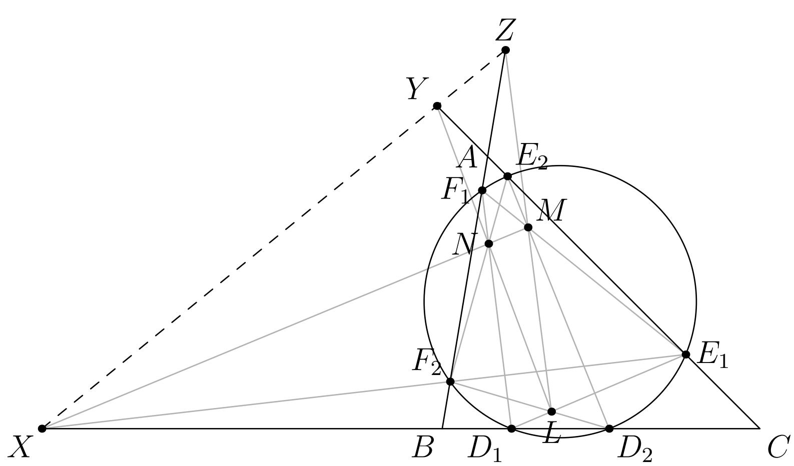


Then A' is the midpoint of arc BD containing C, and C' is the midpoint of arc BD containing A, so A' and C' are diametrically opposite, which means $A'C'$ passes through O. Similarly, $B'D'$ also passes through O.

Let X be the intersection of $A'D$ and $B'C$. Then by Pascal's theorem on hexagon $A'C'CB'D'D$, $O = A'C' \cap B'D'$, $I = C'C \cap D'D$, and $X = CB' \cap DA'$ are collinear. And by Pascal's theorem on hexagon $ACB'BDA'$, $P = AC \cap BD$, $X = CB' \cap DA'$, and $I = B'B \cap A'A$ are collinear. Therefore, O, I, and P are collinear.

2. A circle meets the three sides BC, CA, AB of triangle ABC at points D_1, D_2; E_1, E_2; and F_1, F_2, in turn. The line segments D_1E_1 and D_2F_2 intersect at point L, line segments E_1F_1 and E_2D_2 intersect at point M, and line segments F_1D_1 and F_2E_2 intersect at point N. Prove that the three lines AL, BM, and CN are concurrent. (China, 2005)

   Solution. Let $X = BC \cap MN$, $Y = AC \cap LN$, and $Z = AB \cap LM$. By Desargues's theorem, to prove that AL, BM, and CN are concurrent, it suffices to prove that X, Y, and Z are collinear.



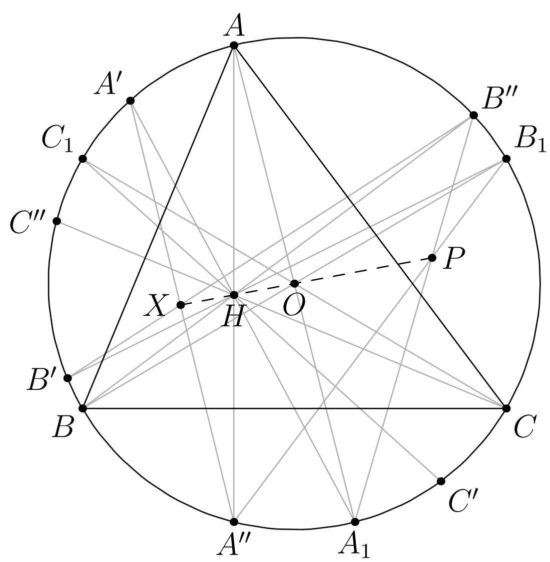
By Pascal's theorem on hexagon $F_1D_1D_2E_2F_2E_1$, $N = F_1D_1 \cap E_2F_2$, $D_1D_2 \cap F_2E_1$, and $M = D_2E_2 \cap E_1F_1$ are collinear. But X is the intersection of MN and D_1D_2, so X also lies on E_1F_2.

Similarly, Y lies on D_2F_1 and Z lies on D_1E_2. Then by Pascal's theorem on hexagon $D_1D_2F_1F_2E_1E_2$, $X = D_1D_2F_2E_1$, $Y = D_2F_1 \cap E_1E_2$, and $Z = F_1F_2 \cap E_2D_1$ are collinear, as desired.

3. Let H and O denote the orthocenter and circumcenter of triangle ABC, and let ω denote the circumcircle of triangle ABC. Let AO intersect ω at A_1, let A_1H intersect ω at A', and let AH intersect ω at A''. Define points B_1, C_1, B', C', B'', and C'', similarly.

   Prove that $A'A''$, $B'B''$, and $C'C''$ are concurrent at a point on the Euler line of triangle ABC. (Iran, 2005)

   Solution. Let P be the intersection of A_1B'' and $A''B_1$. Then by Pascal's theorem on hexagon $A_1B''BB_1A''A$, $P = A_1B'' \cap B_1A''$, $H = B''B \cap A''A$, and $O = BB_1 \cap AA_1$ are collinear.

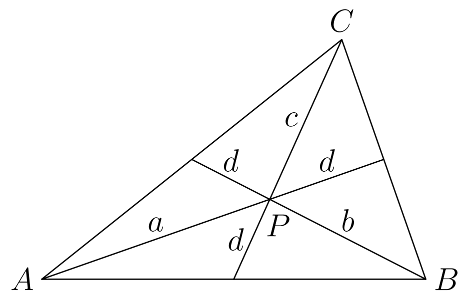


   Let X be the intersection of $A'A''$ and $B'B''$. Then by Pascal's theorem on hexagon $A'A''B_1B'B''A_1$, $X = A'A'' \cap B'B''$, $P = A''B_1 \cap B''A_1$, and $H = B_1B' \cap A_1A'$ are collinear. Therefore, X lies on OH, the Euler line of triangle ABC.

   By a similar argument, $A'A'' \cap C'C''$ also lies on the Euler line. But X is the intersection of $A'A''$ and the Euler line, so $A'A'' \cap C'C''$ coincides with X. We conclude that $A'A''$, $B'B''$, and $C'C''$ concur at X.

Section 7

1. Let P be an interior point of triangle ABC and extend the lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a+b+c=43$ and $d=3$. (AIME, 1988)



Solution. Let AP, BP, and CP intersect BC, AC, and AB at D, E, and F, respectively. Let $K_A = [PBC]$, $K_B = [PCA]$, and $K_C = [PAB]$. We know that $d/(a+d) = PD/AD = K_A/(K_A+K_B+K_C)$, $d/(b+d) = PE/BE = K_B/(K_A+K_B+K_C)$, and $d/(c+d) = PF/CF = K_C/(K_A+K_B+K_C)$. Adding these equations, we get

$$\frac{d}{a+d} + \frac{d}{b+d} + \frac{d}{c+d} = \frac{K_A+K_B+K_C}{K_A+K_B+K_C} = 1.$$

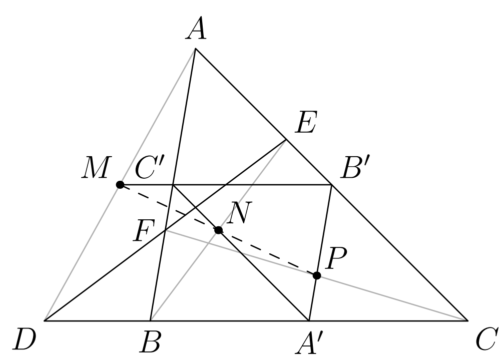
This equation simplifies to

$$abc = (a+b+c)d^2 + 2d^3.$$

Substituting $a+b+c=43$ and $d=3$, we find $abc = 43 \cdot 3^2 + 2 \cdot 3^3 = 441$.

2. Given triangle ABC, a straight line intersects the sides BC, AC, and AB at D, E, and F, respectively. Prove that the midpoints of the line segments AD, BE, and CF are collinear.

Solution 1. Let A', B', C', M, N, and P be the midpoints of BC, AC, AB, AD, BE, and CF, respectively.



Since B', C', and M are the midpoints of AC, AB, and AE, respectively, B', C', and M' are collinear, and $B'M/MC' = CD/DB$. Similarly, $C'N/NA' = AE/EC$ and $A'P/PB' = BF/FA$. Then

$$\frac{B'M}{MC'} \cdot \frac{C'N}{NA'} \cdot \frac{A'P}{PB'} = \frac{CD}{DB} \cdot \frac{AE}{EC} \cdot \frac{BF}{FA}.$$

By Menelaus's theorem,

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1,$$

so

$$\frac{B'M}{MC'} \cdot \frac{C'N}{NA'} \cdot \frac{A'P}{PB'} = -1.$$

Again by Menelaus's theorem, M, N, and P are collinear.

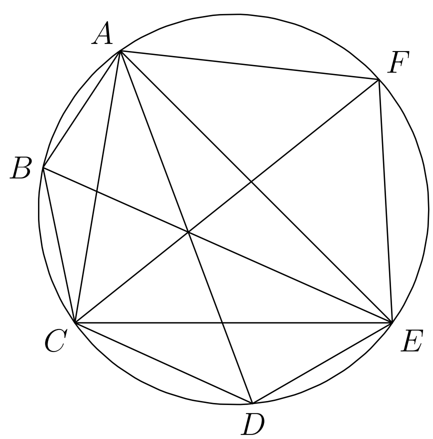
Solution 2. The points M, N, and P lie on the Newton-Gauss line of the complete quadrilateral formed by the four lines AFB, AEC, DFE, and DBC.

3. Let $ABCDEF$ be a convex, cyclic hexagon. Show that diagonals AD, BE, and CF are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

Solution. Let R be the circumradius of cyclic hexagon $ABCDEF$.

By the angle version of Ceva's theorem, AD, BE, and CF are concurrent if and only if

$$\frac{\sin \angle EAD}{\sin \angle CAD} \cdot \frac{\sin \angle ACF}{\sin \angle ECF} \cdot \frac{\sin \angle CEB}{\sin \angle AEB} = 1.$$



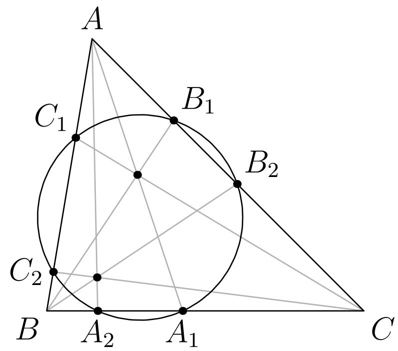
By the extended law of sines, $\sin \angle EAD = DE/(2R)$, $\sin \angle CAE = CD/(2R)$, and so on. Substituting into the equation above, we get

$$\frac{DE}{CD} \cdot \frac{AF}{EF} \cdot \frac{BC}{AB} = 1,$$

or $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

4. A circle intersects side AB of triangle ABC at C_1, C_2, side AC at B_1, B_2, and side BC at A_1, A_2. Prove that AA_1, BB_1, CC_1 are concurrent if and only if AA_2, BB_2, CC_2 are concurrent.

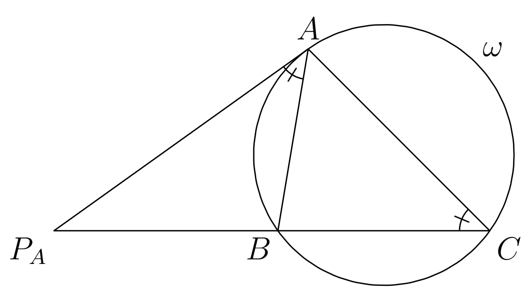
   Solution. Assume that AA_1, BB_1, and CC_1 are concurrent. Then by Ceva's Theorem,
$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = 1.$$



   By power of a point, $BA_1 \cdot BA_2 = BC_1 \cdot BC_2$, so $BA_1/C_1B = BC_2/A_2B$. Similarly, $CB_1/A_1C = CA_2/B_2C$ and $AC_1/B_1A = AB_2/C_2A$. Hence,
$$\frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A} \cdot \frac{AC_2}{C_2B} = 1.$$
   Again by Ceva's Theorem, AA_2, BB_2, and CC_2 are concurrent. The converse is clear.

5. Let ω be the circumcircle of triangle ABC. The tangent to ω at A meets BC at P_A, and the points P_B and P_C are defined similarly. Prove that P_A, P_B, and P_C are collinear.

   Solution 1. We see that $\angle P_AAB = \angle ACB$, so triangles P_AAB and P_ACA are similar. Then
$$\frac{P_AA}{P_AC} = \frac{P_AB}{P_AA} = \frac{AB}{BC}.$$
   Hence,
$$\frac{BP_A}{P_AC} = \frac{BP_A}{AP_A} \cdot \frac{AP_A}{P_AC} = \frac{AB}{AC} \cdot \frac{AB}{AC} = \frac{AB^2}{AC^2}.$$



Similarly, $CP_B/P_BA = BC^2/AB^2$ and $AP_C/P_CB = AC^2/BC^2$, so

$$\frac{BP_A}{P_AC} \cdot \frac{CP_B}{P_BA} \cdot \frac{AP_C}{P_CB} = \frac{AB^2}{AC^2} \cdot \frac{BC^2}{AB^2} \cdot \frac{AC^2}{BC^2} = 1.$$

Then by Menelaus's theorem (without signs), P_A, P_B, and P_C are collinear.

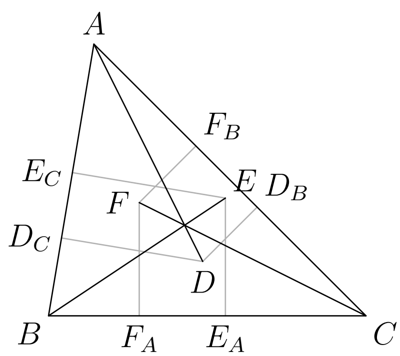
Solution 2. By Pascal's theorem on hexagon $AABBCC$, $AA \cap BC$, $AB \cap CC$, and $BB \cap AC$ are collinear. Since AA represents the tangent to the circumcircle at A, $AA \cap BC = P_A$. Similarly, $AB \cap CC = P_C$ and $BB \cap AC = P_B$, so P_A, P_B, and P_C are collinear.

6. Let D, E, and F be points in the same plane as triangle ABC. Let D_B, D_C be the projection of D onto sides AC and AB, respectively, and define E_A, E_C, F_A, and F_B similarly. Prove that AD, BE, and CF are concurrent if and only if

$$\frac{DD_B}{DD_C} \cdot \frac{EE_C}{EE_A} \cdot \frac{FF_A}{FF_B} = 1.$$

Solution. By the angle version of Ceva's Theorem, AD, BE, and CF are concurrent if and only if

$$\frac{\sin\angle DAC}{\sin\angle DAB} \cdot \frac{\sin\angle EBA}{\sin\angle EBC} \cdot \frac{\sin\angle FCB}{\sin\angle FCA} = 1.$$



From right triangle ADD_B, $\sin\angle DAC = \sin\angle DAD_B = DD_B/AD$. From right triangle ADD_C, $\sin\angle DAB = \sin\angle DAD_C = DD_C/AD$. Hence,

$$\frac{\sin\angle DAC}{\sin\angle DAB} = \frac{DD_B}{DD_C}.$$

Similarly, $\sin\angle EBA/\sin\angle EBC = EE_C/EE_A$ and $\sin\angle FCB/\sin\angle FCA = FF_A/FF_B$. Then

$$\frac{\sin\angle DAC}{\sin\angle DAB} \cdot \frac{\sin\angle EBA}{\sin\angle EBC} \cdot \frac{\sin\angle FCB}{\sin\angle FCA} = \frac{DD_B}{DD_C} \cdot \frac{EE_C}{EE_A} \cdot \frac{FF_A}{FF_B},$$

and the result follows.

7. Given a non-isosceles, non-right triangle ABC, let O denote the center of the circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that triangle OAA_1 is similar to OA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent. (USAMO, 1995)

Solution. Since triangles OAA_1 and OA_2A are similar,

$$\frac{OA}{OA_1} = \frac{OA_2}{OA}.$$

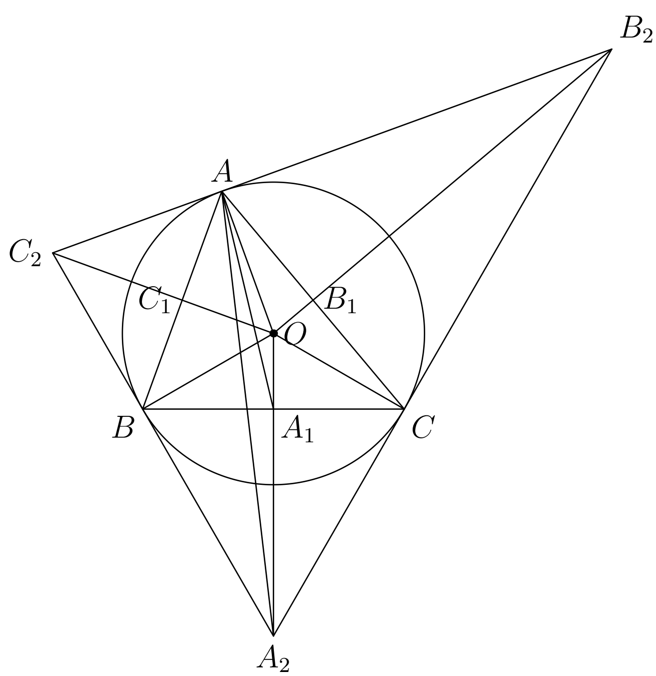
Since O is the circumcenter of triangle ABC, $OA = OB$, so

$$\frac{OB}{OA_1} = \frac{OA_2}{OB}.$$

Let A_3 be the point where the tangents to the circumcircle at B and C intersect. Then right triangles OBA_1 and OA_3B are similar, so

$$\frac{OB}{OA_1} = \frac{OA_3}{OB}.$$

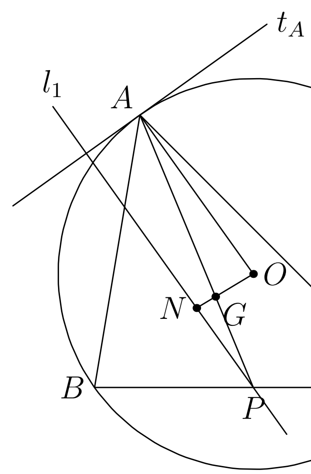
Since A_3 lies on OA_1, points A_2 and A_3 coincide. In other words, A_2 is the point where the tangents to the circumcircle at B and C intersect.



Similarly, B_2 is the point where the tangents to the circumcircle at A and C intersect, and C_2 is the point where the tangents to the circumcircle at A and B intersect. Hence, AA_2, BB_2, and CC_2 concur at the Gergonne point of triangle $A_2B_2C_2$.

8. Let Γ be the circumcircle of triangle ABC, and let P, Q, R be the midpoints of sides BC, AC, AB, respectively. Let l_1 be the perpendicular dropped from P to the tangent to Γ at A, and define l_2 and l_3 similarly. Show that l_1, l_2, and l_3 are concurrent.

   Solution. Let G, O, and N denote the centroid, circumcenter, and nine-point center of triangle ABC, respectively. Let t_A denote the tangent to the circumcircle at A.

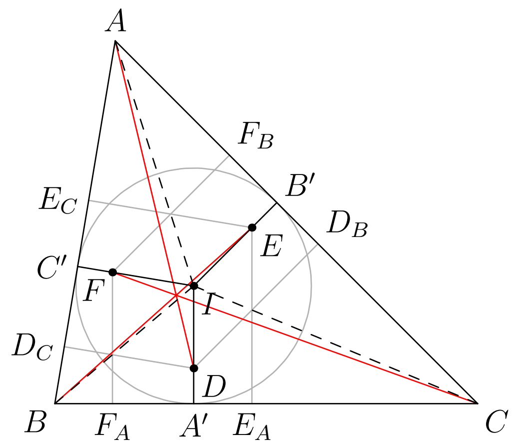


   Let h denote the homothety, centered at G, with scale factor $-1/2$. We know that $h(A) = P$ and $h(O) = N$. Therefore, the image of OA under the homothety h is NP. In particular, NP is parallel to OA. Since OA is perpendicular to t_A, NP is also perpendicular to t_A. Therefore, NP coincides with l_1. In other words, l_1 passes through N.

   By symmetry, l_2 and l_3 also pass through N, so l_1, l_2, and l_3 concur at N.

9. Let I be the incenter of triangle ABC, and let A', B', and C' be the points of tangency of the incircle with sides BC, AC, and AB, respectively. Let D, E, and F be points on IA', IB', and IC' such that $ID = IE = IF$. Show that AD, BE and CF are concurrent.

   Solution. We use the same notation as an exercise 6.



Since $IE = IF$, E and F are reflections of each other in AI, which means E_C and F_B are also reflections of each other in AI. Hence, $EE_C = FF_B$.

Similarly, $DD_C = FF_A$ and $DD_B = EE_A$, so

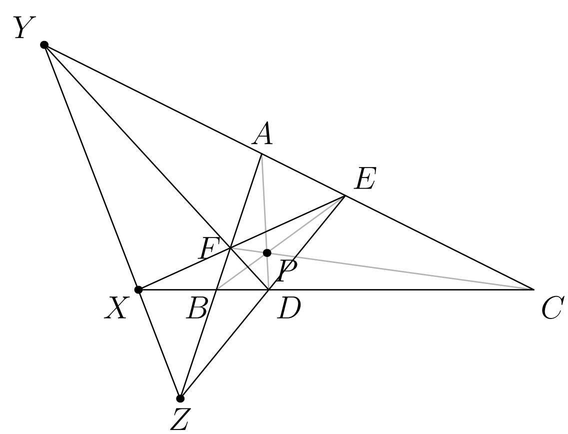
$$\frac{DD_B}{DD_C} \cdot \frac{EE_C}{EE_A} \cdot \frac{FF_A}{FF_B} = 1.$$

Therefore, AD, BE, and CF are concurrent.

10. (a) Given a triangle ABC and a point P, let AP, BP, and CP intersect BC, AC, and AC at D, E, and F, respectively. Let $X = EF \cap BC$, $Y = DF \cap AC$, and $Z = EF \cap AB$. Prove that X, Y, and Z are collinear. (Line XYZ is called the *trilinear polar* of P with respect to triangle ABC.)

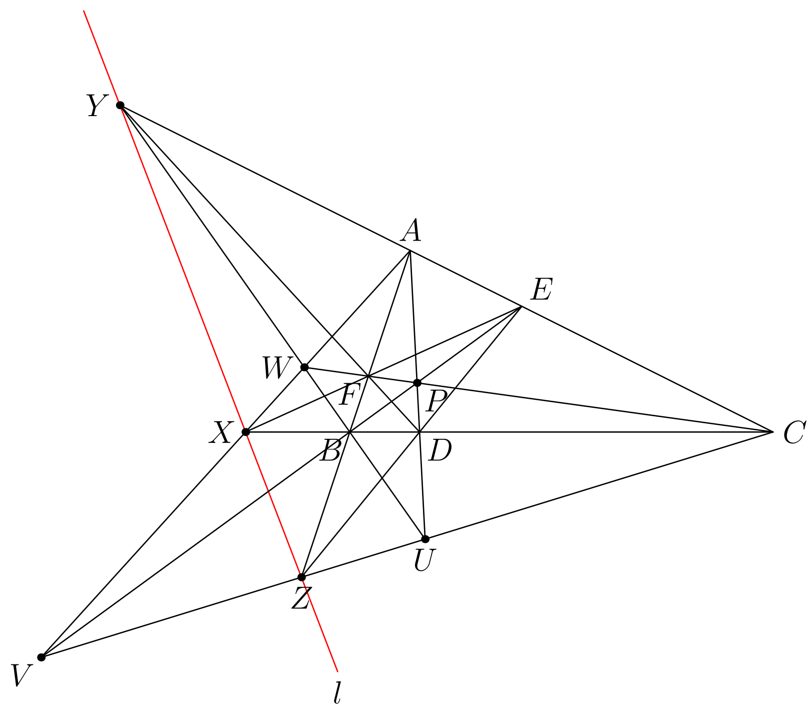
    (b) Given a triangle ABC and a line l, does there always exist a point P such that l is the trilinear polar of P? (In such a case, P is called the *trilinear pole* of l with respect to triangle ABC.)

Solution. (a) Consider triangles ABC and DEF. Since AD, BE, and CF are concurrent, by Desargues's theorem, $X = BC \cap EF$, $Y = AC \cap DF$, and $Z = BC \cap EF$ are collinear.



As an example, the orthic axis is the trilinear polar of the orthocenter.

(b) Given line l, we construct the point P as follows: Let $X = l \cap BC$, $Y = l \cap AC$, and $Z = l \cap AB$. Then let $U = BY \cap CZ$, $V = CZ \cap AX$, and $W = AX \cap BY$. Finally, let $D = AU \cap BC$, $E = BV \cap AC$, and $F = CW \cap AB$.



Consider triangles ABC and UVW. Since $X = BC \cap VW$, $Y = AC \cap UW$, and $Z = AB \cap UV$ are collinear, by Desargues's theorem, AU, BV, and CW are concurrent. Let P be the point of concurrence. Then P is also the point of concurrence of AD, BE, and CF.

Now, consider triangles ABC and DYZ. Since AD, BY, and CZ concur at U, by Desargues's theorem, $F = AB \cap DY$, $E = AC \cap DZ$, and $X = BC \cap YZ$ are collinear. In other words, X lies on EF.

Similarly, Y lies on DF, and Z lies on DE, so l is the trilinear polar of P.