Geometry: Length Chasing

David Arthur
darthur@gmail.com

1 Facts

This talk is about using lengths (and Sine law, which has some similarities) to solve geometry problems. This is *not* the same thing as true trig-bashing, where you start doing lengthy algebraic computations involving trigonometric identities. It's just that you look at lengths in addition to angles. Often it will show you things you could not find in other ways.

You probably know most or all of the following facts already, but they are the bread-and-butter techniques for length chasing:

1. *Sine Law*
2. *Similar Triangles*
3. *Power of a Point:* In addition to the normal definition, don't forget that the power of P with respect to a circle with center O and radius r is $OP^2 - r^2$.
4. *Angle Bisector Theorem:* Let ABC be a triangle. Suppose the internal and external angle bisectors of $\angle BAC$ hit BC at D and E. Then
$$\frac{BD}{CD} = \frac{BE}{CE} = \frac{BA}{CA}.$$
5. *Ceva / Sine Ceva and Menelaos*
6. *Concurrent Perpendiculars:*
   - Let A, B, C, D be points in the plane. Then $CD \perp AB$ if and only if $AC^2 - AD^2 = BC^2 - BD^2$.
   - Let A, B, C, D, E, F be points in the plane. Let ℓ_A be the line through D perpendicular to BC, ℓ_B be the line through E perpendicular to CA, and ℓ_C be the line through F perpendicular to AB. Then ℓ_A, ℓ_B, and ℓ_C meet at a point if and only if $AF^2 - BF^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0$.
7. *Apollonius and Stewart:*
   - (Apollonius) Let D be the midpoint of BC. Then $AB^2 + AC^2 = 2(AD^2 + BD^2)$.
   - (Stewart) Let D be a point on segment BC. Then $AB^2 \cdot DC + AC^2 \cdot DB = BC \cdot (AD^2 + BD \cdot DC)$.
8. *Ptolemy* Let $ABCD$ be a quadrilateral. Then $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$, with equality if and only if $ABCD$ is cyclic.

2 Examples

We start with an example that you have seen recently that is very easy with length chasing, but not super nice.

Example 1. *Let ABC be an acute triangle. The points M and N are taken on the sides AB and AC respectively. The circles with diameters BN and CM intersect at points P and Q. Prove that P, Q, and the orthocenter H are collinear.*

Solution. Let ω_1 and ω_2 be the circles with diameters BN and CM. We need to show H has equal power with respect to these circles. This can be solved immediately by looking for a radical center. However, length chasing gives a slightly longer but completely brain-dead solution.

Applying Apollonius's theorem, the difference between the powers is

$$\begin{aligned} & \left(\frac{HB^2 + HN^2 - 0.5 \cdot BN^2}{2} - \frac{BN^2}{4}\right) - \left(\frac{HC^2 + HM^2 - 0.5 \cdot CM^2}{2} - \frac{CM^2}{4}\right) \\ = \; & \left(\frac{HB^2 + HN^2 - BN^2}{2}\right) - \left(\frac{HC^2 + HM^2 - CM^2}{2}\right). \end{aligned}$$

Let E and F be the foots of the perpendiculars from B and C to the opposite sides. Then, the above simplifies to:

$$\begin{aligned} = \; & \left(\frac{HB^2 + HE^2 - BE^2}{2}\right) - \left(\frac{HC^2 + HF^2 - CF^2}{2}\right) \\ = \; & -HB \cdot HE + HC \cdot HF, \end{aligned}$$

which is 0 between $BCEF$ is cyclic. □

The next example is from the China Girls Math Olympiad 2012, and it shows how length chasing can actually be quite pretty.

Example 2. *Circles Γ_1 and Γ_2 are externally tangent to each other at point T. Choose points A and E on Γ_1, and choose B, D on Γ_2 so that AB and DE are tangent to Γ_2. Let AE and BD meet at point P.*

1. *Prove that $\frac{AB}{AT} = \frac{ED}{ET}$.*

2. *Prove that $\angle ATP + \angle ETP = 180°$.*

Solution. By similar triangles, $\frac{AA'}{AB} = \frac{AB}{AT}$, so $\frac{AA'}{AT} = \left(\frac{AB}{AT}\right)^2$. However, the homothety about T taking Γ_1 to Γ_2 also takes A to A'. Therefore, if r_1 and r_2 denote the radii of Γ_1 and Γ_2, we have $\frac{AB}{AT} = \sqrt{\frac{r_1+r_2}{r_2}}$. This is independent of A, so it also holds for E, which finishes the proof of the first part.

For the second part, we need to prove $\frac{PE}{PA} = \frac{TE}{TA}$ by the angle bisector theorem. By the first part, this is equivalent to showing $\frac{PE}{PA} = \frac{ED}{AB} \Leftrightarrow \frac{PE}{ED} = \frac{PA}{AB}$.

However, this is easy to get a handle on. Based on the sine law or drawing D' on BD such that AD' is parallel to ED, we can see that this condition in turn is implied by $\angle ABD = \angle EDB$. These are both equal to the arc BD of circle ω_2, so we are done! □

Finally, here is one more example that uses Menelaos:

Example 3. *Let ABC be a triangle. Circle ω passes through points B and C. Circle ω_1 is tangent internally to ω and also to sides AB and AC at T, P, and Q, respectively. Let M be midpoint of arc BC (containing T) of ω. Prove that lines PQ, BC, and MT are concurrent.*

Solution. Let PQ intersect BC at K and let MT intersect BC at K'.

Applying Menelaos to $\triangle ABC$, we have $\frac{KB}{KC} \cdot \frac{QC}{QA} \cdot \frac{PA}{PB} = 1$. Since $PA = QA$, this implies $\frac{KB}{KC} = \frac{BP}{CQ}$. On the other hand, $\angle MTB = \angle MCB = \angle MBC = 180° - \angle MTC$, so TM is the external bisector of $\angle BTC$. Therefore, $\frac{K'B}{K'C} = \frac{TB}{TC}$. Thus, it suffices to prove $\frac{BP}{CQ} = \frac{TB}{TC}$.

Let E and F be the intersections of BT and CT with ω_1. Since a homothety about T takes ω_1 to ω, we must have that EF is parallel to BC. Therefore, $\frac{BE \cdot BT}{BT^2} = \frac{CF \cdot CT}{CT^2}$. By Power of a Point, we also have $\frac{BE \cdot BT}{BP^2} = 1 = \frac{CF \cdot CT}{CQ^2}$. Combining these two results gives $\frac{BP}{BT} = \frac{CQ}{CT} \implies \frac{BP}{CQ} = \frac{TB}{TC}$, as required. □

3 Problems

1. Let AP, AQ and AR be chords in a circle such that $\angle PAQ = \angle QAR = \angle RAS$. Show that $(AP + AR) \cdot AR = (AQ + AS) \cdot AQ$.

2. Let Γ_1 and Γ_2 be concentric circles, with Γ_2 in the interior of Γ_1. From a point A on Γ_1, one draws the tangent AB to Γ_2 ($B \in \Gamma_2$). Let C be the second point of intersection of AB and Γ_1, and let D be the midpoint of AB. A line passing through A intersects Γ_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find with proof the ratio $\frac{AM}{MC}$.

3. Let D and E be points on sides BC and AC of a triangle ABC, respectively. The circumscribed circle of triangle CED and the line through C parallel to AB meet again at point F ($F \neq C$). Suppose that the line FD meets segment AB at point G, and let H be the point on line AB such that $\angle HDA = \angle GEB$ and A is between H and B. Given that $DG = EH$, prove that the segments AD and BE intersect on the bisector of angle ACB.

4. The quadrilateral $ABCD$ is inscribed in a circle which has diameter BD. Points A' and B' are symmetric to A and B with respect to the lines BD and AC respectively. If the lines $A'C$ and BD intersect at P, and the lines AC and $B'D$ intersect at Q, prove that PQ is perpendicular to AC.

5. Let M be the midpoint of the side BC of acute triangle ABC and let H be the orthocenter of ABC. Prove that if D is the base of the perpendicular dropped from A to the line HM, then the bisectors of $\angle DBH$ and $\angle DCH$ meet on line HM.

6. The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at $A_4 (\neq A_2)$. Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.

7. Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way.

Prove that
$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

8. Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

9. The circle ω_1 with diameter AB and the circle ω_2 with center A intersect at points C and D. Let E be a point on the circle ω_2 which is outside ω_1 and on the same side of AB as C. Let BE intersect ω_2 again at F. Choose a point $K \in \omega_1$ so that (a) $2 \cdot CK \cdot AC = CE \cdot AB$, and (b) K and A are on the same side of the diameter of ω_1 passing through C. Let KF intersect ω_1 again at L. Show that the reflection of D about line BE lies on the circumcircle of $\triangle LFC$.

10. Let O be a point inside acute-angled $\triangle ABC$. Denote by A_1, B_1, and C_1 its projections on the sides BC, CA, and AB. Let P be the point where the line through A orthogonal to B_1C_1 intersects the line through B orthogonal to C_1A_1. If H is the projection of P onto AB, prove that $A_1B_1C_1H$ is cyclic.

11. Let $ABCD$ be a circumscribed quadrilateral and let P be the orthogonal projection of its incenter on AC. Prove that $\angle APB = \angle APD$.

12. In $\triangle ABC$, let AA_0, BB_0, CC_0 be altitudes. Let A_1 be a point inside $\triangle ABC$ such that $\angle A_1BC = \angle A_1AB$ and $\angle A_1CB = \angle A_1AC$. Define B_1 and C_1 symmetrically. Let A_2, B_2, C_2 be the midpoints of AA_1, BB_1, CC_1 respectively. Prove that the lines A_2A_0, B_2B_0, and C_2C_0 are concurrent.

13. In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies
$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$
Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

14. In convex quadrilateral $ABCD$, CB, DA are external angle bisectors of $\angle DCA, \angle CDB$, respectively. Points E, F lie on the rays AC, BD respectively such that $CEFD$ is a cyclic quadrilateral. Point P lies in the plane of quadrilateral $ABCD$ such that DA, CB are external angle bisectors of $\angle PDE, \angle PCF$ respectively. AD intersects BC at Q. Prove that P lies on AB if and only if Q lies on segment EF.

15. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and
$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$
Prove that line BD is tangent to the circumcircle of triangle TSH.

16. Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show the I is the incenter of triangle KAB.

17. Let $ABCD$ be a convex quadrilateral with $AB \neq BC$. Denote by ω_1 and ω_2 the incircles of triangles ABC and ADC. Suppose that there exists a circle ω inscribed in angle ABC, tangent to the extensions of line segments AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

18. Let ABC be a triangle inscribed in a circle of radius R, and let P be a point in the interior of ABC. Prove that
$$\frac{PA}{BC^2} + \frac{PB}{CA^2} + \frac{PC}{AB^2} \geq \frac{1}{R}.$$

4 Hints

1. Use Ptolemy's theorem.
2. This problem is all Power of a Point. Show CDEF is cyclic. (USAMO 1998, #2)
3. Prove $AEDG$ and $HEDB$ are cyclic. Use Ceva's theorem and angle bisector theorem. (Serbia 2014, #2)
4. AB and AD are internal and external angle bisectors of $\angle PAD$. You need to show PQ is an external angle bisector of $\angle BQD$. (Singapore 2014, #1)
5. Let F be the foot of the perpendicular from C to AB. Prove that $DFMC$ is cyclic, and calculate $\frac{CD}{CH}$. (Mongolia 2013, TST 3.1)
6. Sine Ceva tells you what you need to prove. You just have to calculate everything. (North Korea 2012 TST, #1)
7. By Power of a Point, $O_1A_1^2 - r_1^2 = A_1A_3 \cdot A_1B_1$ where B_1 is the second intersection of ω_1 with A_1A_3. Let M be the intersection of the diagonals. Write the expression in terms of MA_1, MA_2, MA_3, MA_4. (IMO Shortlist 2011, G2)
8. You want to evaluate $\frac{AQ}{AD_2}$ and $\frac{AP}{AD_2}$. For the former, remember Yufei's lemmas! For the latter, use Menelaos. (USAMO 2001, #2)
9. Let X be the intersection of BE and ω_1. Prove $XE = XF$ and XCD' is collinear. Next show $\angle CLK = \angle CFE$ from the length condition, and then show $\angle DEX = \angle ECX$. Finally, use Power of a Point to complete the problem. (Turkey 2013, #1)
10. Use Power of a Point to show that HC_1A_1M is cyclic. Where is its center? (Bulgaria 2011, #4)
11. Let I be the incenter, and let E, F, G, H be the points of tangency on AB, BC, CD, DA. Prove $\frac{\sin \angle HPD}{\sin \angle GPD} = \frac{\sin \angle EPB}{\sin \angle FPB}$. (Bulgaria 2003 TST, #5)
12. Prove that $A_0B_0C_0A_2B_2C_2$ all lie on the 9-point circle, and use trig Ceva on $\triangle A_0B_0C_0$. Where does AA_2 hit BC? (Ukraine 2008, Grade 11 #8)
13. Using $\angle CPD = \angle PBA$ and $\angle PDC = \angle ADB$, prove $\frac{AP}{CP} = \frac{\sin \angle BCD}{\sin \angle BAD} \cdot \frac{\sin \angle CPD}{\sin \angle APB} = \frac{\sin \angle BCD}{\sin \angle BPC} \cdot \frac{\sin \angle CPD}{\sin \angle DPA}$. Now prove $\frac{\sin \angle CPD}{\sin \angle APB} = 1$. (IMO 2004, #5)
14. Lots and lots of sine law! (China 2009 TST #2)
15. Let X, Y be the centers of the circles CHS and CHT. Prove that AHC is the Apollonius circle for segment XY, then $\frac{XS}{XA} = \frac{YT}{YA}$, and finally that the perpendicular bisectors of HS and HT intersect on AH. (IMO 2014, #3)
16. Working forward, prove $AD = DF$ and use Menelaos on $\triangle ADF$ with respect to line CIK. Working backward, we want to prove $\angle IAB = \angle KAI$, which can come from triangles ADK and BDA being similar. (IMO Shortlist 2011, G6)

17. To get started, prove $AP = CQ$ and look for collinearities with Yufei's lemma. (IMO Shortlist 2008, G7)

18. Let X, Y, Z be the feet of the perpendiculars from P to BC, CA, AB. First prove $BC \cdot PA \geq AB \cdot PY + AC \cdot PZ$ with basic trig. Apply this to the original expression and use $R = \frac{4S}{abc}$. (China 1993)