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Modelling and Calibration of Multi-Name Credit Product

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Contents

Introduction

The main goal of our project is to price the tranches of an MBS which has a reference portfolio composed of $I = 1\,000$ mortgages, considered homogeneous with an average notional of 2 MIO€, an average recovery π of 65% for each mortgage and a default probability of 7% for the period T. We assume for simplicity that mortgages provide a single payment at the end of the interest period T and we neglect the effect due to discount factors. In order to price tranches we have to calibrate the parameters of two different models, double t-Student and t-Student, given the implied correlations for the cumulated tranches:

Subordinator (K_u)	Correlation (Vasicek)
3%	22.8%
6%	26.0%
9%	29.1%
12%	32.1%
22%	37.9%

Table 1: Tranche Correlation

In this document we first calibrate the parameters for the double t-Student model under the assumption of the Large Homogeneous Portfolio (LHP) and then we do the same for the t-Student model. Using these optimal parameters we price the tranches under LHP and HP (Homogeneous Portoflio) assumptions and KL (Kullback-Leibler) approximation and compare the results. Finally we calibrate the parmeters of our models considering the KL approximation.

1 Framework

There are two main approaches to describe the default probability of each obligor in the reference portfolio: intensity based and firm value models. In this project we consider only multi-name firm-value models, i.e. models which relate the default probability to the value of each obligor (in our case to the value of each mortgage).

Once the mortgages are issued, the special purpose vehicle (SPV) transfers the credit risk exposure of the sponsor by issuing debt obligations called tranches or credit-linked notes. Each one of these tranches is characterized by two subordinators: the attachment and detachment points (respectively K_d and K_u). When the cumulative percentage loss of the portfolio reaches the lower subordinator, investors in the tranche of interest start to lose their notional up to the upper subordinator where the investors lose all their principal (waterfall structure). From here on another tranche will be affected by the loss of the reference portfolio.

When analyzing the reference portfolio one can make two main basic assumptions: Homogeneous and Large Homogeneous Portfolio (respectively HP and LHP).

Definition 1.1 (**HP hypothesis**)**.** The HP assumption sets that for every obligor $i = 1, ..., I$, the probability of default is constant with respect to i ($p_i = p$) as is the recovery rate ($\pi_i = \pi$).

Definition 1.2 (**LHP hypothesis**)**.** Under the LHP assumption the probability of default and the recovery rate are constant and equal for every obligor, but an 'infinite' number of obligors is considered, i.e. we consider $I \to +\infty$ and instead of studying the exact number of obligors that default, we study the fraction of defaulted ones.

Under the HP hypothesis with a large number of obligors ($I = 1\,000$ is already enough) the prices of the tranches coincide (up to an non significant error) with the ones obtained under the LHP hypothesis.

Since HP hypothesis is numerically exact only for small values of I and the LHP one is an asymptotic result, a solution is to use the Kullback - Leibler approximation (KL) which is very precise for intermediate values of I (in particular $I \in (10, 10^4)$).

Definition 1.3 (**KL approximation**)**.** When computing the price of a tranche under HP hypothesis, as presented below, one can also use the Stirling approximation for the binomial coefficient $\binom{I}{m}$

$$\binom{I}{m} \simeq \sqrt{\frac{I}{m}} \frac{I^I e^{-I}}{m^m e^{-m}}$$

and use in the pricing formula the Kullback - Leibler entropy function:

$$K(z, p(y)) = z \ln\left(\frac{z}{p(y)}\right) + (1 - z) \ln\left(\frac{1 - z}{1 - p(y)}\right) \tag{1}$$

1.1 Vasicek model

In the literature the standard model used to price tranches is Vasicek's, where the value of the firm (in our case the mortgage i) is described (according to [1]) as follows:

$$v_i = \sqrt{\rho}y + \sqrt{1-\rho}\,\epsilon_i \tag{2}$$

where y and ϵ_i are standard normal independent random variables and ρ represents the correlation (assumed to be constant) among all obligors. Each obligor defaults iff

$$v_i \leq k$$

where $k = N^{-1}(p)$ with N the cumulative distribution function of a standard normal random variable and p is the default probability. This implies that, given y:

$$\epsilon_i \leq \frac{1}{\sqrt{1-\rho}}(k - \sqrt{\rho}y) \Longleftrightarrow p(y) := \mathbb{P}(v_i \leq k|y) = N\left(\frac{k-\sqrt{\rho}y}{\sqrt{1-\rho}}\right) \tag{3}$$

Given y, the v_i's are independent so that under the HP assumption the probability of having m defaults $\mathbb{P}(m|y)$ follows a binomial distribution:

$$\begin{aligned} \mathbb{P}(m|y) &= \binom{I}{m} p(y)^m (1-p(y))^{I-m} \\ \mathbb{P}(m) &= \int_{-\infty}^{\infty} \mathbb{P}(m|y)\Phi(y)dy \end{aligned} \tag{4}$$

where $\Phi(y)$ is the density function of a standard normal random variable. From here one can compute the expected loss of each tranche and therefore, subracting this value to the tranche's notional, its price.

Defining $u = \frac{K_u}{1-\pi}$, $d = \frac{K_d}{1-\pi}$ and $N_t = (K_u - K_d)M$ as the notional of each tranche (where M represents the notional of the reference portfolio) the loss of the tranche is given by:

$$L_t\left(\frac{m}{I}\right) = N_t \frac{min(max(\frac{m}{I} - d, 0), u-d)}{u-d} \tag{5}$$

so that its price will be given by:

$$P_t = N_t(1 - \mathbb{E}[L_t]) = N_t\left(1 - \sum_{m=0}^{I} L_t\left(\frac{m}{I}\right)\mathbb{P}(m)\right) \tag{6}$$

Notice that, apart from the expression of $\mathbb{P}(m)$, this formula holds for every model, so in order to use this in our pricing evaluation we will just need to specify this distribution.

If one wants to use the KL approximation (1) in (6) and defines:

$$C^{(1)}(z) = \sqrt{\frac{I}{2\pi z(1-z)}}$$

the following holds true:

$$\begin{aligned} P_t &\simeq N_t\left(1-\sum_{m=0}^{I} L_t\left(\frac{m}{I}\right)\int_{-\infty}^{+\infty}\frac{C^{(1)}\left(\frac{m}{I}\right)}{I}e^{-IK(\frac{m}{I},p(y))}\Phi(y)dy\right) \\ &= N_t\left(1-\int_0^1 L_t(z)dz\int_{-\infty}^{+\infty}\Phi(y)C^{(1)}(z)e^{-IK(z,p(y))}dy\right) \end{aligned} \tag{7}$$

where the second equality holds when assuming I sufficiently large, so that $\frac{m}{I}\to z\in[0,1]$.

Under the LHP assumption, setting $z=\frac{m}{I}$ as fractions of defaults and $x=\frac{n}{I}$ as the maximum fractions of defaults that can occur in our tranche, and $p(y)$ being a decreasing function of y, $\exists\hat{y}$ such that:

$$p(y)=z\leq x=p(\hat{y})\iff y\geq\hat{y}$$

whose expression is derived imposing $x=p(\hat{y})$:

$$\hat{y}=\frac{k-N^{-1}(x)\sqrt{1-\rho}}{\sqrt{\rho}}$$

The condition $y\geq\hat{y}$ can be written as $-y\leq-\hat{y}=y^*$. Since y follows a standard normal distribution, $-y$ follows a standard normal as well. Therefore we consider the following cumulative distribution function:

$$\mathbb{P}(z\leq x)=N(y^*) \tag{8}$$

At this point, by differentiating the previous formula with respect to x, one obtains the probability density function of the fraction of defaults $f(z)$ and as before, one can compute the expected loss of the tranche and its price.

$$P_t=N_t\left(1-\int_{-\infty}^{+\infty}L_t(z)f(z)dz\right) \tag{9}$$

An explicit formula of $f(z)$ can be found in [3].

1.2 Alternative models

The market however is not well represented by a normal distribution since it is skewed and has fat-tails. Therefore other models are needed: double t-Student and t-Student are generalizations, obtained via a mixing variable, of the Vasicek model. In this section in particular we analyze the LHP case for these two models.

1.2.1 Double t-Student

For the double t-Student distribution the firm value v_i is described by:

$$v_i=\sqrt{\rho}y+\sqrt{1-\rho}\,\epsilon_i \tag{10}$$

where y and ϵ_i are independent t-Student random variables with mean 0, unitary variance and same number of degrees of freedom ν (this assumption is made in order to have a parsimonious model and have a limited number of parameters to be chosen and/or calibrated). From a practical point of view in order to use standardized distributions both terms must be multiplied by $\sqrt{\frac{\nu-2}{\nu}}$.

Following the same idea of Vasicek one has that each obligor defaults iff $v_i \leq k$, being v_i given by (10). This condition can be translated, given y, in a condition on ϵ_i:

$$\epsilon_i \leq \frac{k - \sqrt{\rho}y}{\sqrt{1-\rho}}$$

which implies that:

$$p(y) := \mathbb{P}(v_i \leq k) = t_\nu \left(\frac{k - \sqrt{\rho}y}{\sqrt{1-\rho}} \right) \tag{11}$$

where t_ν denotes the cumulative distribution function of the t-Student with ν degrees of freedom. Considering fractions of defaulted obligors, given y all defaults are independent and being $p(y)$ a decreasing function of y, $\exists \hat{y}$ such that:

$$p(y) = z \leq x = p(\hat{y}) \Longleftrightarrow y \geq \hat{y}$$

whose value using (11) is given by:

$$\hat{y} = -\frac{\sqrt{1-\rho}\, t_\nu^{-1}(x) - k}{\sqrt{\rho}} \tag{12}$$

The condition $y \geq \hat{y}$ can be written as $-y \leq -\hat{y} = y^*$. Since y follows a t-Student distribution with ν degrees of freedom, $-y$ follows a t-Sudent distribution with ν degrees of freedom as well. In this way the probability of having a given fraction of defaults is $\mathbb{P}(z \leq x) = t_\nu(y^*)$ and from here one can find the density function by differentiating with respect to x, obtaining:

$$f(x) = \Phi_\nu \left(\frac{\sqrt{1-\rho}\, t_\nu^{-1}(x) - k}{\sqrt{\rho}} \right) \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \frac{1}{\Phi_\nu(t_\nu^{-1}(x))} \tag{13}$$

where Φ_ν is the t-Student density function. Then by using in (9) this density function one can compute the price of the tranche.

The main issue in this case is the value of k which, since it has no analytical solution, has to be computed numerically. In order to find this number (effective parameter of the market) we impose that $\mathbb{P}(v_i \leq k) = p$ and determine k by numerical inversion, as presented in Calibration section. This operation is necessary since we do not have a closed analytical formula for the distribution of the sum of two t-Student random variables.

1.2.2 t-Student

On the other hand for the 'single' t-Student model one has:

$$v_i = \frac{\sqrt{\rho}y + \sqrt{1-\rho}\,\epsilon_i}{\sqrt{\frac{W}{\nu}}} \tag{14}$$

where y and ϵ_i are standard normal independent random variables and $W \sim \chi^2$ with ν degrees of freedom.

From Vasicek approach one has that each obligor defaults iff $v_i \leq k$ where v_i is given by (14). The condition on v_i is translated, given W and y on a condition on ϵ_i as follows:

$$\epsilon_i \leq \frac{k\sqrt{\frac{W}{\nu}} - \sqrt{\rho}y}{\sqrt{1-\rho}} = \frac{\eta}{\sqrt{1-\rho}}$$

where $k = t_\nu^{-1}(p)$ and $\eta = k\sqrt{\frac{W}{\nu}} - \sqrt{\rho}y$ is the mixing variable. In this way given η all defaults are independent and one gets:

$$p(\eta) := \mathbb{P}(v_i \leq k) = N\left(\frac{\eta}{\sqrt{1-\rho}}\right) \tag{15}$$

So now considering the LHP assumption and the fractions of defaulted obligors, being $p(\eta)$ and increasing function of η, $\exists \eta^*$ such that:

$$p(\eta) = z \leq x = p(\eta^*) \Longleftrightarrow \eta \leq \eta^*$$

whose value is derived by (15):

$$\eta^* = \sqrt{1-\rho}N^{-1}(x) \tag{16}$$

In this way the probability of having a certain fraction of defaults will be simply given by $\mathbb{P}(z) = F(\eta^*)$ with $F(\eta)$ the cumulative distribution function of the mixing variable. From here by differentiating this quantity we can use (9) in order to get the price of the tranche we are interested in.

The main issue in this case is the distribution of η which is not easy to compute. Moreover due to its complexity, it slows down the computation. The cumulative distribution and the density functions of η have the following forms (given by [2]):

$$\begin{aligned} F(\eta) &= N\left(\frac{\eta}{\sqrt{\rho}}\right) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{\eta}{\sqrt{\rho}}} \Gamma\left(\frac{\nu}{2}, \frac{\nu(\eta + \sqrt{\rho}u)^2}{2k^2}\right) e^{-\frac{u^2}{2}} du \\ f(\eta) &= \frac{1}{\sqrt{\rho\pi}2^{\frac{\nu+1}{2}}\Gamma(\frac{\nu}{2})} \int_0^{+\infty} e^{-\frac{1}{2\rho}(\eta - k\sqrt{\frac{\omega}{\nu}})^2} \omega^{\frac{\nu}{2}-1} e^{-\frac{\omega}{2}} d\omega \end{aligned} \tag{17}$$

1.3 Implied correlation

In our models we consider just one value of correlation ρ which is the same for all tranches, whereas we have different implied correlations according to different tranches. Market implied correlations are the Vasicek-like correlations that allow us to obtain market prices via Vasicek model. On the similar note, model implied correlations allow us to match the prices of the model through Vasicek. There are two types of implied correlations: compound and base correlations ([5]).

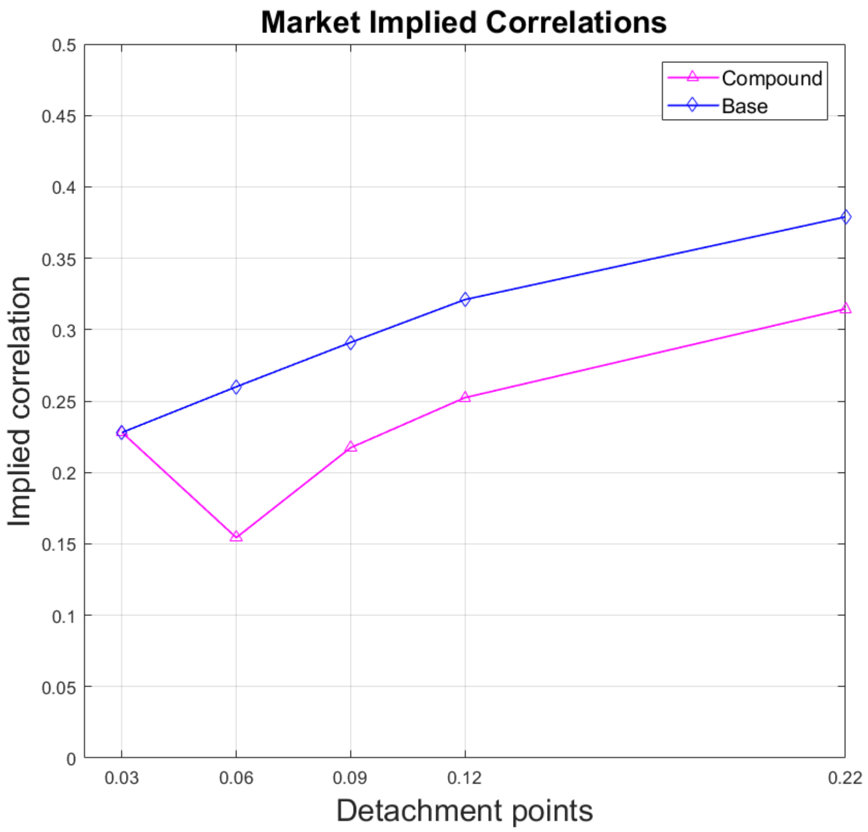
Compound correlation This type of implied correlation is computed starting with market prices for mezzanines and the main characteristic is that each tranche is considered separately. After selecting a model and pricing the tranches, the compound correlation can be determined as the correlation that, when used in the standard model, returns the same price for the considered mezzanine.

What one gets in this case is a typical shape called correlation smile. This phenomenon is due to several reasons, among which our assumptions on correlation, probability to default and recovery that we make in the gaussian model, as well as the difference between supply and demand between mezzanine and equity tranches. The flat implied correlation curve indeed is achieved if we consider the Vasicek model (since by assumption the implied correlations we are given are Vasicek-like).

The main drawback of compound correlation is that the function is not monotonic. Therefore one can find two distinct values of implied correlation for the same tranche.

Base correlation This kind of correlation was proposed to overcome the limitations of the compound correlation. Instead of being computed on quoted tranches, it is computed on 'fictive' cumulative equity tranches. The price of the fictive tranche can be calculated as the sum of the equity and mezzanine tranches up to the detachment point. Then, as for the compound correlation, the price is inverted through the Vasicek model obtaining the implied correlation.

Even if less intuitive than the compound correlation, this is an improvement with respect to it since it doesn't have problems of uniqueness and moreover it gives the possibility to be computed also for non quoted tranches (by using interpolation on the implied correlations' curve). However it is more sensitive to price variation for senior tranches.



Compound	Base
22.80%	22.80%
15.43%	26.00%
21.73%	29.10%
25.23%	32.10%
31.45%	37.90%

Table 2: Market Implied Correlations

In this document we have used both types of implied correlations to calibrate the parameters ν and ρ of the two models (double t and t-Student) in order to check the consistency of our results.

2 Calibration

For both models the calibration of ν and ρ is done by minimizing the mean squared error. The algorithm we use is the following:

a. Fix a real number of degrees of freedom $\nu \in [2, +\infty)$

b. Calculate k and compute the correlation ρ such that the equity tranche price matches the market one.

$$\text{Vasicek\_price(Mkt\_imp,0\%,3\%)=model\_price}(\rho\text{,0\%,3\%})$$

c. For all tranches find the implied correlation ρ_imp such that

$$\text{Vasicek\_price}(\rho\text{\_imp},K_d,K_u)\text{=model\_price}(\rho,K_d,K_u)$$

where, in the case of of compound correlation one uses the mezzanine subordinators, and in the case of base correlation one uses the equity subordinators.

d. Compute the MSE as the mean of the square of the difference between the implied correlation and the corresponding market one

$$\text{MSE}=\frac{1}{5}\sum_{i=1}^{5}(\rho\text{\_imp}_i\text{-Mkt\_imp}_i)^2$$

Notice that by construction ρ_imp$_1$=Mkt_imp$_1$ therefore the sum starts with i=2. In particular when no ρ or ρ_imp are found (i.e. when one of these two quantities or both are NaN) the code automatically sets them to 0 since the MSE in these cases would be not the minimum one.

e. Iterate over different values of ν

f. Choose ν and the corresponding ρ which minimize the MSE. We do this using a minimization method 'derivative-free' (fminbnd) on the above defined range of values for ν with a level of tolerance of 10^{-4} (according to the level of precision of the given data of implied correlations) and a maximum number of allowed iterations of 50.

2.1 Double t-Student

In this section we calibrate the parameters of the double t-Student distribution under LHP assumptions. These are: ρ i.e. the correlation of the defaults among mortgages, ν i.e. the number of degrees of freedom of the distribution and within these calculations also the effective parameter of the market k.

2.1.1 Calibration through compound correlation

The algorithm we use for the calibration is the one described above and for this specific model some specifications are needed.

Fixing ν After performing an exploratory analysis for different values of ν we decided to restrict the search interval to $[2, 20]$.

Choice of k Since as stated above we have no analytical formula for the distribution of the sum of two t-Student random variables, the value of k such that $\mathbb{P}(v_i \leq k)$ has to be found numerically. In particular we have used a semi-analytic approach (as explained in [4]). The cumulative distribution function of v_i given y is:

$$\mathbb{P}(v_i \leq k|y) = t_\nu \left(\frac{k - \sqrt{\rho}y}{\sqrt{1-\rho}} \right)$$

Therefore by the law of iterated expectation:

$$\mathbb{P}(v_i \leq k) = \int_{-\infty}^{+\infty} \mathbb{P}(v_i \leq k|y)\Phi_\nu(y)dy$$

Since the above formula is equal to the default probability p, k can be computed through a roots finding algorithm.

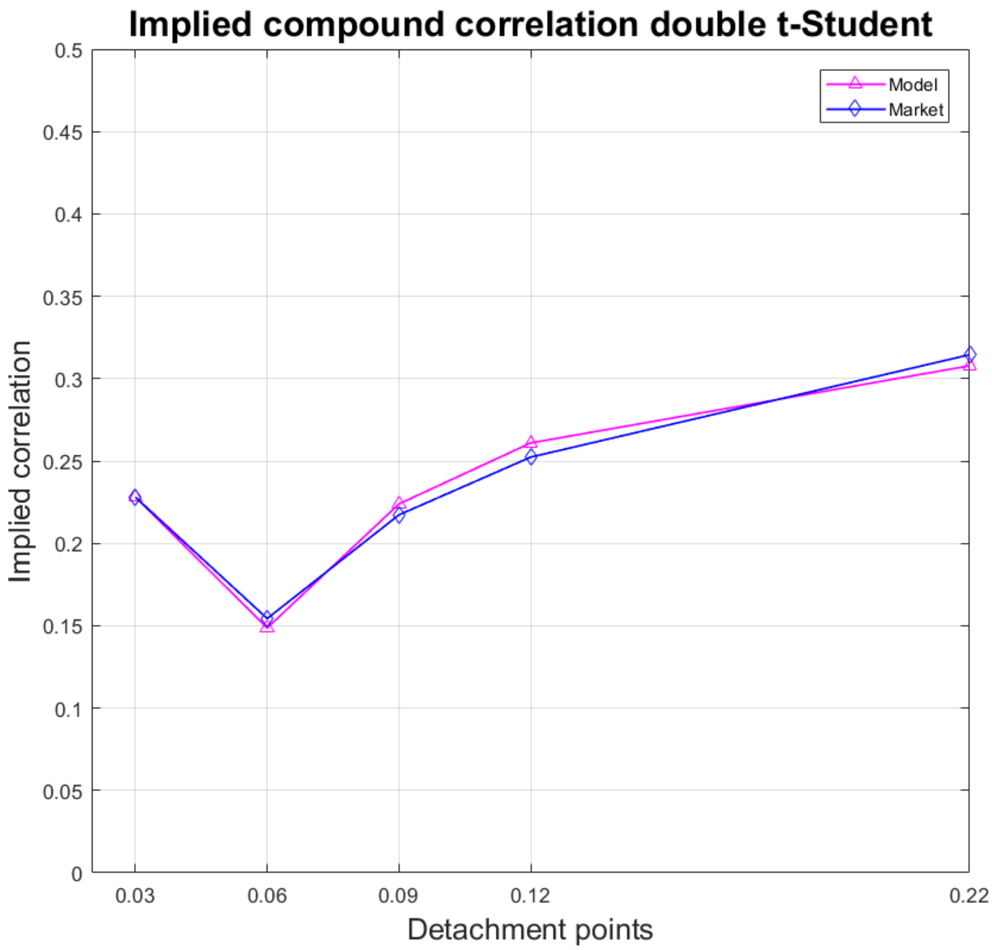
After this calibration of ν (within the function Calibration.m) we compute the corresponding implied correlations in the function ImpliedComputation.m.

Using this algorithm we obtain $\nu = 9.4756$, $\rho = 0.2531$ and the following compound correlations:

Market	Model
22.80%	22.80%
15.43%	14.87%
21.73%	22.39%
25.23%	26.09%
31.45%	30.78%

Table 3: Tranche Compound Correlation

By plotting these values with respect to the market ones, we can observe that with these calibrated parameters, market implied compound correlations are very well fitted by the ones of the model. We can conclude that this model is a good choice in order to reproduce market data.



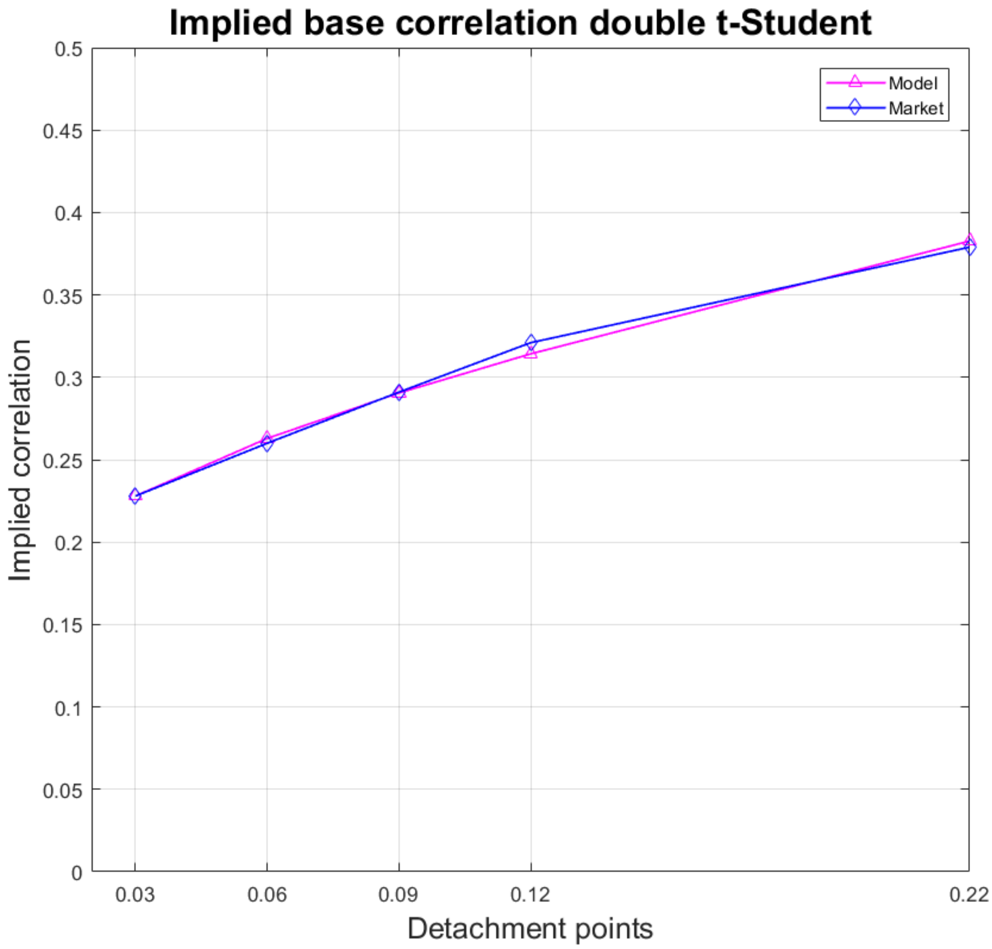
2.1.2 Calibration through base correlation

Using the same algorithm one can also calibrate ν and ρ using base implied correlations. In this case we get $\nu = 9.6559$, $\rho = 0.2527$ and base implied correlations:

Market	Model
22.80%	22.80%
26.00%	26.29%
29.10%	29.05%
32.10%	31.42%
37.90%	38.27%

Table 4: Tranche Base Correlation

As expected these implied correlations fit well market ones. One can also notice that both ν and ρ computed in the case of base and compound correlation are really close.


Implied base correlation double t-Student
Model
Market
Implied correlation
Detachment points
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.03
0.06
0.09
0.12
0.22

2.2 t-Student

In this section the main goal is to calibrate the parameters ν and ρ for the t-Student distribution.

2.2.1 Calibration through compound correlation

In order to perform this calibration we use the previously defined algorithm where the only significant difference is in the range of values which ν can assume.

Fixing ν For this model, after a first analysis on the code, we noticed that the MSE decreases asymptotically as ν tends to ∞ (i.e. the t-Student distribution tends to a normal distribution). Hence, we decided to limit the research interval to [7,12] as suggested in [2], since Mashal and Zevi found the maximum likelihood estimator to be in this interval.

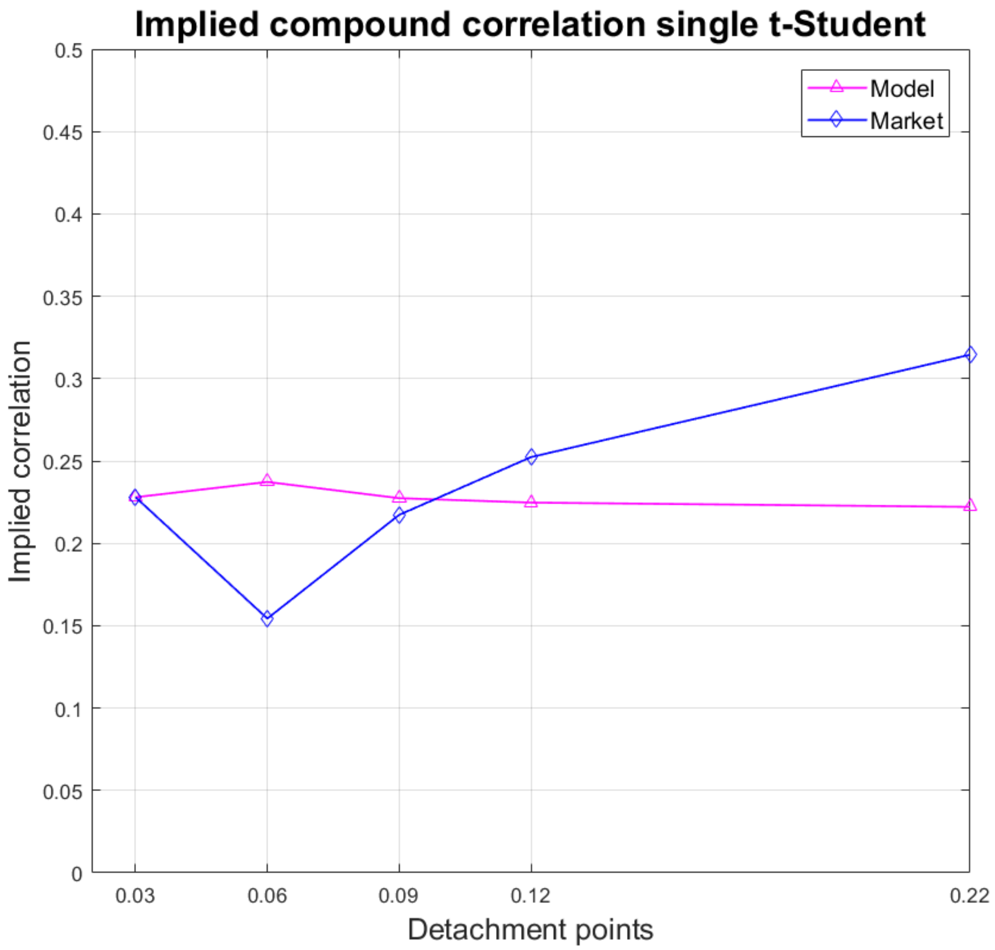
In this case due to (17) the calibration is slower than the other model: every time an fzero is called on the pricing functions the integrals above have to be computed.

As a result we get (as expected) $\nu = 12$, $\rho = 0.1540$ and the following compound correlations:

Market	Model
22.80%	22.80%
15.43%	23.73%
21.73%	22.74%
25.23%	22.47%
31.45%	22.21%

Table 5: Tranche Compound Correlation

It is clear that these results are not as good as the ones of the double t-Student model, but we expected that as the implied correlation curve is similar to the one seen in class and in [6].



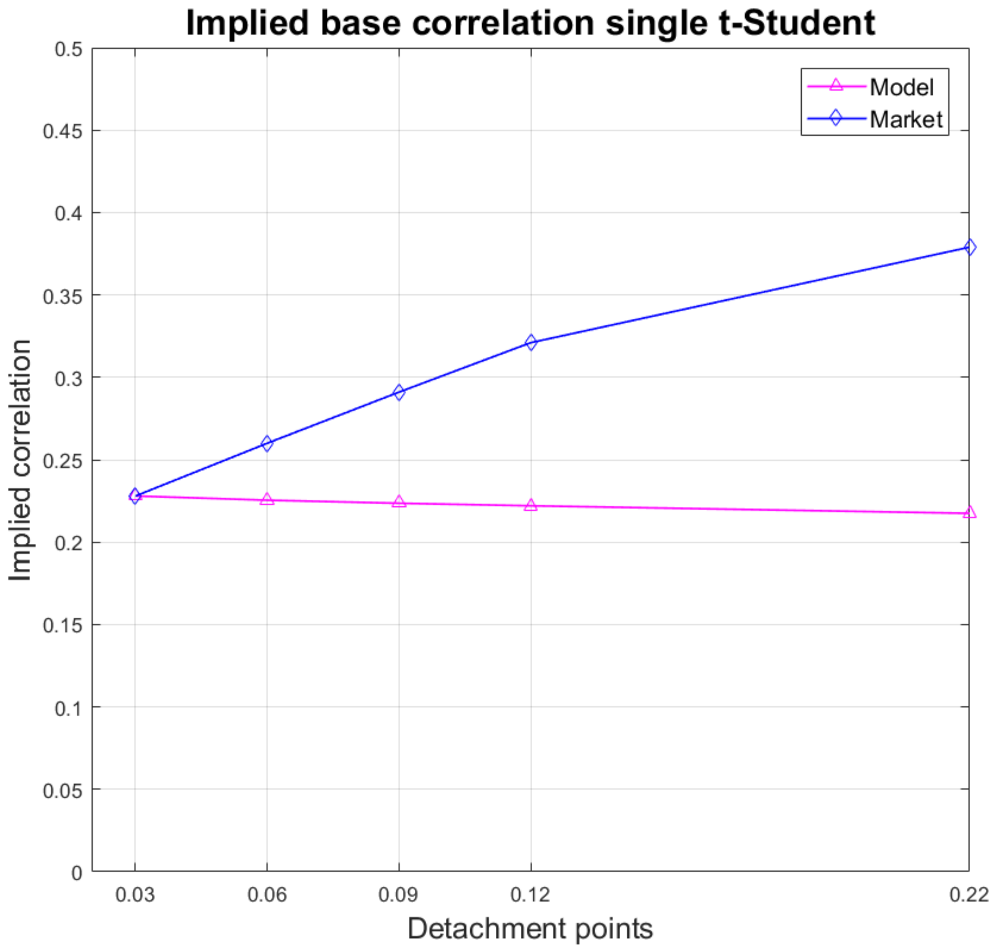
2.2.2 Calibration through base correlation

Also in this case it is possible to calibrate ν and ρ using base correlations. The results in this case are: $\nu = 12$, $\rho = 0.1540$ and implied base correlations:

Market	Model
22.80%	22.80%
26.00%	22.54%
29.10%	22.36%
32.10%	22.21%
37.90%	21.75%

Table 6: Tranche Base Correlation

As already noticed this model doesn't describe well market data. Instead it is quite close to the Vasicek model, as we would expect since the t-Student has a similar distribution to the gaussian. In Vasicek case in fact the implied correlation would be just a straight horizontal line with value equal to the first market implied correlation (by construction). We can therefore conclude that this model is not the best choice to describe the market properly.



In both cases (t-Student, and double t-Student) it is possible to perform the calibration of the parameters using market and model prices. This calibration is discussed in Appendix.

3 Comparison between the models

After using the calibration algorithm we know we have the best possible fit these models can give, but the results are very different.

In the case of double t-Student we can notice that both the compound and base implied correlations follow almost perfectly the market curve. On the other hand the t-Student implied correlations (both compound and base) do not fit market implied ones at all. Moreover from a computational point of view the t-Student model is more time consuming than the double t-Student one: in this case a longer computational time doesn't imply a better approximation, instead the approximation becomes weaker.

It is therefore reasonable to assume that the double t-Student generalization presented in (10) is a good choice to fit market data. We can say instead that the t-Student is not a big step forward from the Vasicek model because the fits are very close and the computational time is extremely longer.

4 Impact of LHP assumption on pricing

In this section we analyze the impact of the LHP hypothesis on pricing the equity tranche 0-3% and the mezzanines 3-6% and 6-9%. In particular we use three different methods for pricing: HP, LHP assumptions and KL approximation. For a large number of mortgages the three methods shall give approximately the same prices for all the tranches. The difference between what we did above when pricing tranches and what we explain below is that in HP and KL approximation we have a dependence on I, while LHP is an asymptotic result. In this part, we use those values of ν and ρ found via compound implied correlation in the previous sections, since we didn't experience any issue of uniqueness and since compound correlation is easier to relate to the mezzanine prices.

KL pricing When pricing via KL approximation we need to distinguish between equity and mezzanine tranches.
For mezzanines one can simply use (7) using instead of $\Phi(y)$ (density function of the standard normal distribution) $\Phi_\nu(y)$ for the double t-Student model and $f(\eta)$ for the t-Student one (whose expression can be found in (17)).
For equity tranches, with upper subordinator K_u and lower subordinator $K_d = 0$ instead, in order to compute the price we use the following scheme:

- Compute the price (and the corresponding loss L_m) of the mezzanine tranche K_u-100%
- Compute the loss of the reference portfolio as:

$$L_{rp} = M\,(1-\pi)\int_{-\infty}^{+\infty} p(y)\Phi(y)dy$$

  where $\Phi(y)$ designates the probability density function of the model we consider.

- Find the price of the equity tranche as:

$$P_t = MK_u - (L_{rp} - L_m)$$

We have to do this distinction between the two cases (equity and mezzanines) because the Stirling approximation of the binomial coefficient is inaccurate as $I \to 0$.

4.1 Double t-Student pricing

We start analyzing the impact of the previous assumption in case of the double t-Student model.

Pricing LHP The prices of the different tranches under LHP assumption are obtained using (13) in (9). As already noticed these values are constant and independent of the number of obligors I. In this case we obtain:

Results in MIO€

	Equity (0-3%)	Mezzanine (3-6%)	Mezzanine (6-9%)
Model	25.758	51.233	56.838
Market	25.758	51.106	56.949

As shown in the table, the error committed in the pricing is of the order 10^5 for an asset of the value of 10^7.

Pricing HP On the contrary in case of HP assumption given (11) one has that the probability of having m defaults is given by:

$$\mathbb{P}(m) = \int_{-\infty}^{+\infty} \Phi_\nu(y) \binom{I}{m} p(y)^m (1 - p(y))^{I-m} dy \tag{18}$$

Given this and the density function of the t-Student distribution we can find the price of the different tranches using (6).

Notice that as $I \to +\infty$ the computational cost for the binomial coefficient present in the formula becomes significant and for $I \simeq 50$ it starts to give an approximation of the correct value of the binomial coefficient. We have therefore decided to stop as this approximation arises (i.e. when a 'warning' is detected in the loop of the pricing function HP_double_t.m) and that is why in the plot we just have a part of the curve.

Results in MIO€

I	Equity (0-3%)	Mezzanine (3-6%)	Mezzanine (6-9%)
10	34.973	47.709	54.103
20	30.957	48.671	55.407
50	27.882	50.090	56.301

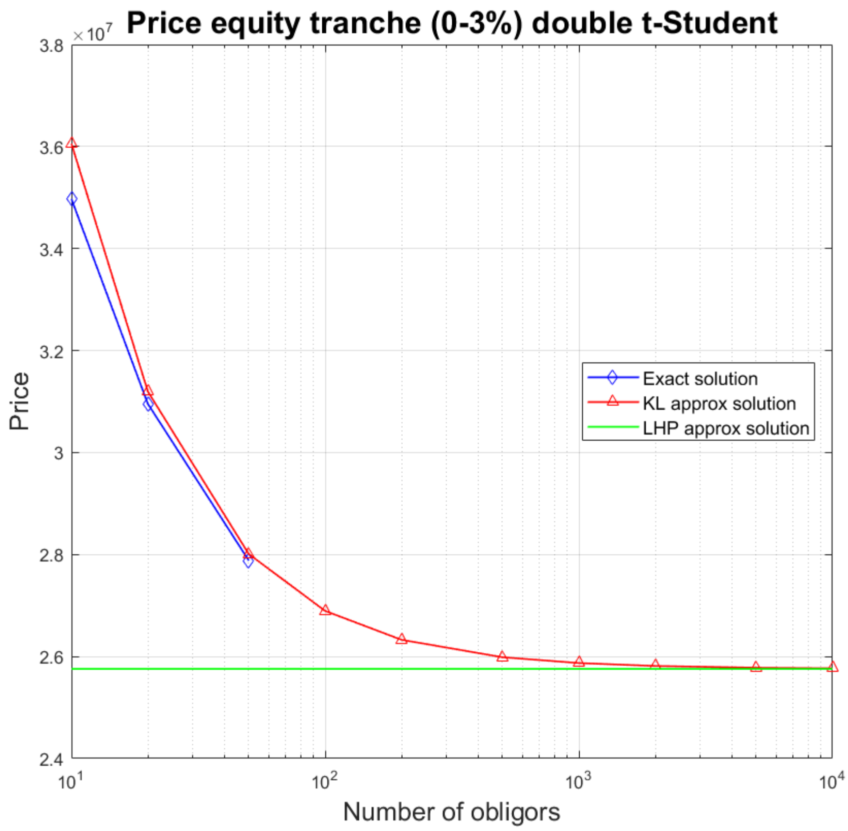
Pricing KL Using the pricing algorithm above, one can price the equity and mezzanine tranches according to the number of obligors.

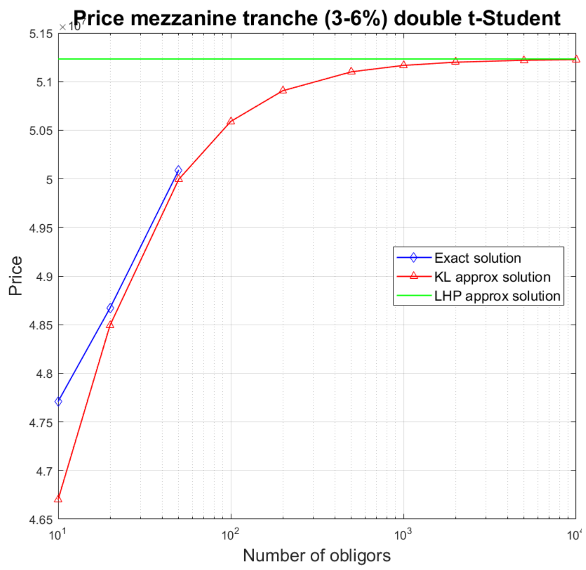
Results in MIO€

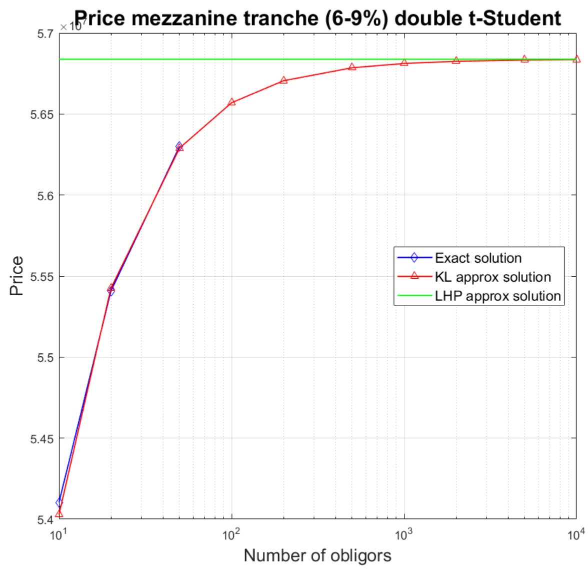
I	Equity (0-3%)	Mezzanine (3-6%)	Mezzanine (6-9%)
10	36.057	46.699	54.028
20	31.198	48.494	55.425
50	28.010	49.995	56.290
100	26.894	50.590	56.569
200	26.328	50.906	56.705
500	25.986	51.101	56.785
1000	25.872	51.166	56.812
2000	25.815	51.199	56.825
5000	25.781	51.219	56.833
10000	25.769	51.226	56.835

We notice the same variations of the price of the equity and mezzanines according to I as in the HP hypothesis.

The following plots represent our results.







We can notice that when I is small the LHP hypothesis doesn't give an accurate estimation of the exact solution but, if we focus on our specific case ($I = 1000$) all the methods work well. We can also assume that the Kullback-Leibler approximation works as an exact formula from the number of obligors $\overline{I}$ (i.e. the number of obligors from which the exact solution stops working) up to large values of I.

4.2 t-Student pricing

In this section we analyze the impact of LHP in case of t-Student model using the previuosly calibrated parameters ν and ρ.

Pricing LHP The prices of all tranches under LHP assumption are computed using (17) in (9). In this way we obtain:

Results in MIO€

	Equity (0-3%)	Mezzanine (3-6%)	Mezzanine (6-9%)
Model	25.758	49.908	56.780
Market	25.758	51.106	56.949

In comparison with the double t-Student, the results here are bad. Here the error on the first mezzanine is just one order of magnitude smaller than the price. We could actually expect such error by looking at the gap in the compound correlation curve.

Pricing HP Under HP assumption given (15) we have that the probability of having m defaults is given by:

$$\mathbb{P}(m) = \int_{-\infty}^{+\infty} f(\eta) \binom{I}{m} p(y)^m (1 - p(y))^{I-m} d\eta \tag{19}$$

where $f(\eta)$ is the density function of η given in (17). In this way using this in (6) we can obtain the price of each tranche. Again this solutions gives a result only up to $I \simeq 50$ since from this point on the binomial coefficient faces an approximation.

Results in MIO€

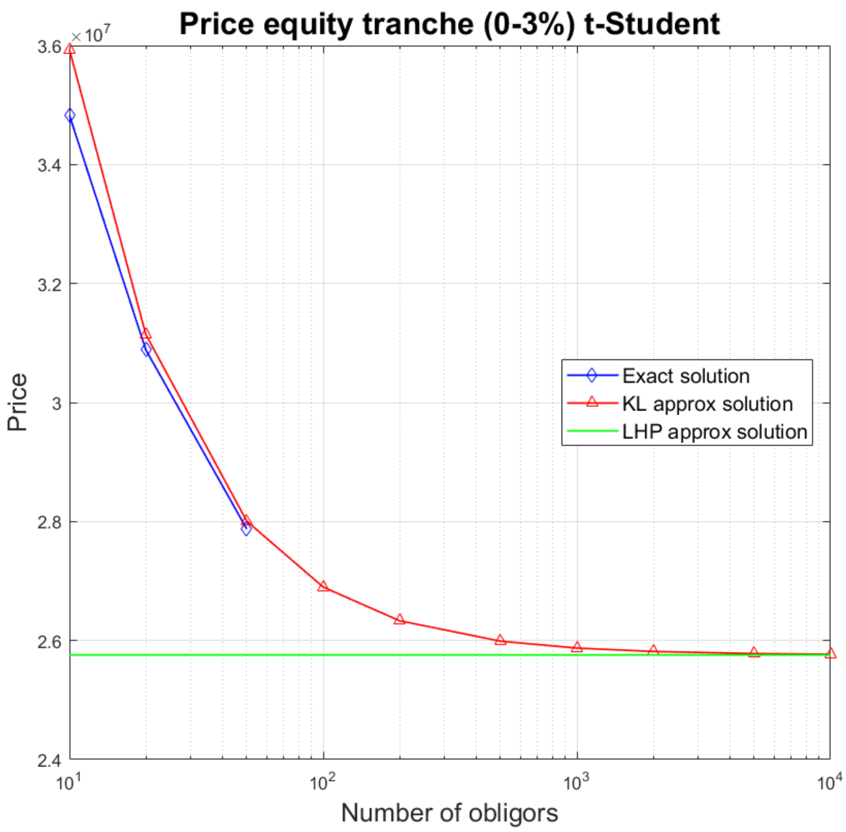
I	Equity (0-3%)	Mezzanine (3-6%)	Mezzanine (6-9%)
10	34.829	47.255	53.838
20	30.887	47.907	55.121
50	27.874	49.007	56.114

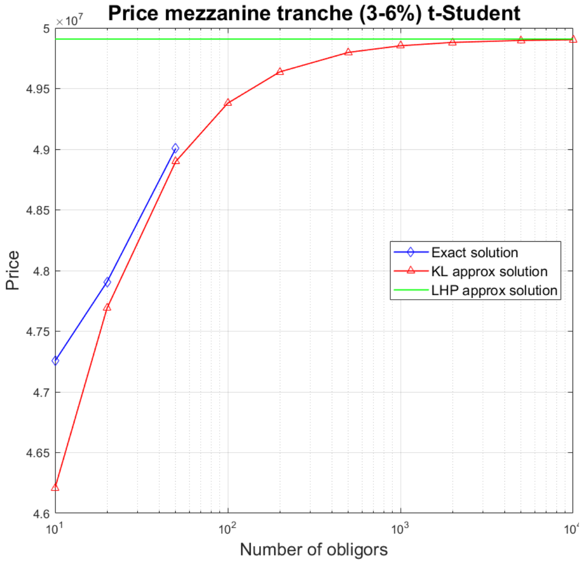
Pricing KL The KL approximation works exactly in the same way as the double t-Student case, giving the following prices:

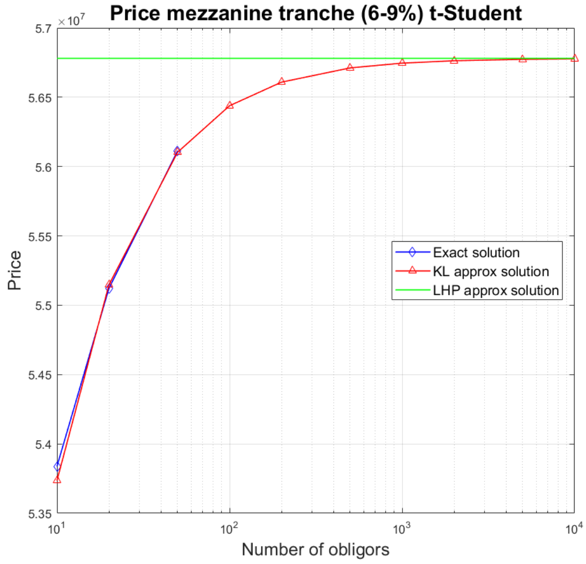
Results in MIO€

I	Equity (0-3%)	Mezzanine (3-6%)	Mezzanine (6-9%)
10	35.932	46.209	53.736
20	31.145	47.690	55.147
50	28.006	48.899	56.103
100	26.899	49.380	56.439
200	26.333	49.637	56.609
500	25.989	49.798	56.712
1000	25.874	49.853	56.746
2000	25.816	49.880	56.763
5000	25.781	49.897	56.773
10000	25.769	49.902	56.777

The following plot represent our results.







The same remarks and observations made for the double t-Student model hold also for this model.

5 Calibration Kullback-Leibler approximation

From the previous points it is clear that Kullback-Leibler (with $I \in [10, 10^4]$) works well to approximate prices. That is why we can also perform the calibration of the parameters using this approximation. In particular we performed the calibration only using base implied correlations.

5.1 KL double t-Student

The idea behind this calibration is exactly the same as the one discussed above, the only difference now is that we have to consider $I = 1000$ and the pricing functions for equity and mezzanine tranches under KL approximation.

Also in this case we obtain a good fit of market data, in particular we get values for degrees of freedom, correlation of the model and implied base correlations very close to the ones computed under the LHP assumptions in previous sections.

The minimization algorithm has to take into account the number of obligors at every function call and compute the integral, therefore the process is extremely slow. We can conclude that this kind of calibration has to be done if strictly necessary and if the number of obligors is small (since for I sufficiently large LHP assumption works well).

Using this approximation with our 1 000 obligors we obtain: $\nu = 9.6175$, $\rho = 0.2528$ and the following base implied correlations:

Market	Model
22.80%	22.80%
26.00%	26.28%
29.10%	29.04%
32.10%	31.42%
37.90%	38.28%

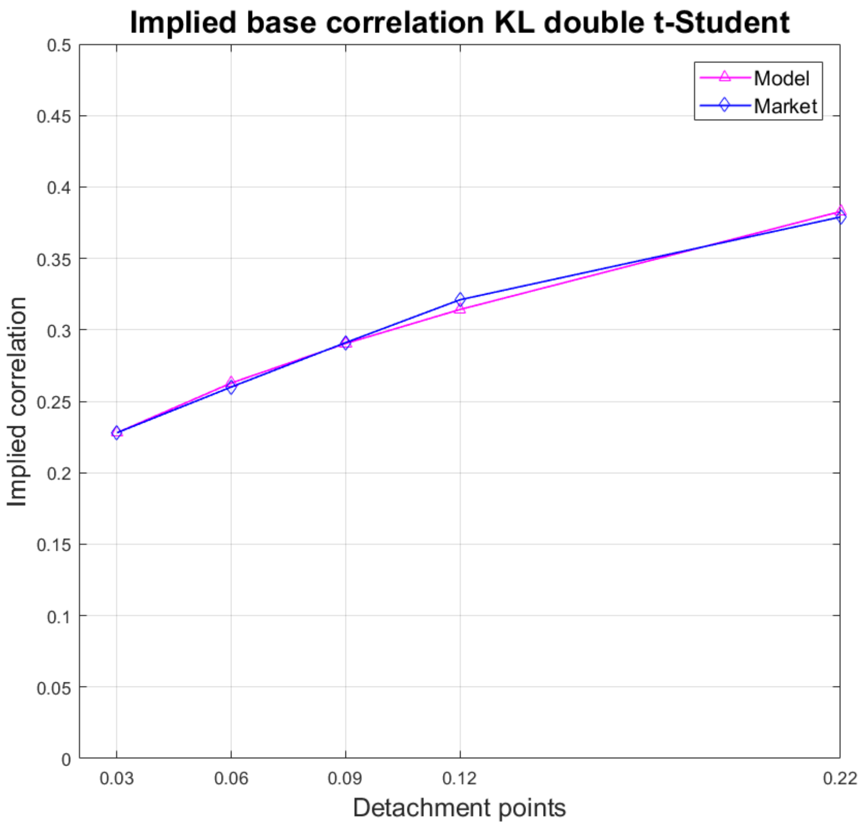
Table 7: Tranche Base Correlation

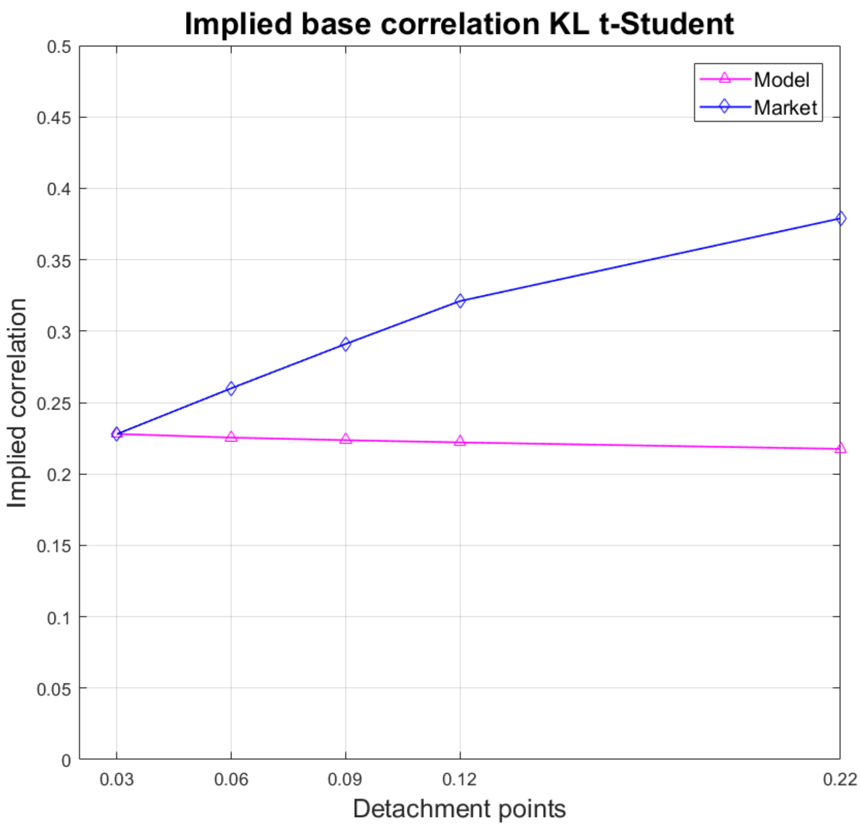
5.2 KL t-Student

Using exactly the same argument as above one gets the following results: $\nu = 12$, $\rho = 0.1539$ and the following base implied correlations:

Market	Model
22.80%	22.80%
26.00%	22.55%
29.10%	22.36%
32.10%	22.21%
37.90%	21.75%

Table 8: Tranche Base Correlation


Implied base correlation KL double t-Student
Model
Market
Implied correlation
Detachment points
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.03
0.06
0.09
0.12
0.22


Implied base correlation KL t-Student
Model
Market
Implied correlation
Detachment points
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.03
0.06
0.09
0.12
0.22

5.3 Comments on KL calibration

As seen in the previous sections, the results obtained with $I = 1\,000$ for the KL calibration are extremely similar to the ones of the LHP calibration. If we consider the significative difference in computational time, the KL calibration is in general not suggested for a high number of obligors. Instead, in case of MBS composed by a low number of obligors, the KL calibration will provide more appropriate results than the LHP calibration, considering that the latter is independent of the number of obligors.

Approach	Time
LHP	16s
KL	747s

Table 9: Calibration time for Double-t for 1 000 obligors

I	10	100	500	1000
ν	6.1450	9.2625	9.5788	9.6175

Table 10: Calibrated ν for Double-t

Conclusions

In this document we have analyzed two different alternative models to price an MBS: the double t-Student and the t-Student. In both cases we had to perform the parameters calibration under LHP hypothesis choosing as optimal value of degrees of freedom ν (and the corresponding value of correlation ρ) the one for which the MSE was minimized.

As a conclusion one can say that the double t-Student represents well the market. The Vasicek model might save some computational time, since there's no calibration of ν, but the double t-Student model fits the market data better. On the other hand the 'single' t-Student model, as $\nu \to \infty$, is close to a normal one, and in practice the calibration of the model gives us high values of ν. Therefore this model is time-consuming, and since it provides similar results to Vasicek's, it is not to be considered.

In any case with both models we could compute prices for different tranches using three different methods: under LHP and HP assumptions and using KL approximation.

For a high number of obligors (around 1 000 and above), the LHP hypothesis provides a satisfying computational time and error. For smaller numbers of obligors KL is the best solution, as it is close to the exact values of the HP hypothesis, and it is less time-consuming. However when it comes to calibration, the computational time for the KL approximation gets way larger than the LHP hypothesis. As a consequence, since the impact on the calibration is significant only for low numbers of obligors ($I \approx 100$), the LHP calibration is probably the most valuable option for standard portfolios.

Appendices

In this document we have chosen the optimal parameter for the number of degrees of freedom ν minimizing the MSE using differences between implied market correlation and compound (or base) ones.

Another possibility is to choose the optimal parameters for these models minimizing on market prices. This practice is not suggested as market prices are more volatile, and practitioners prefer to calibrate on correlation.

However, a calibration on price can be used as a plausibility check for ν and ρ.

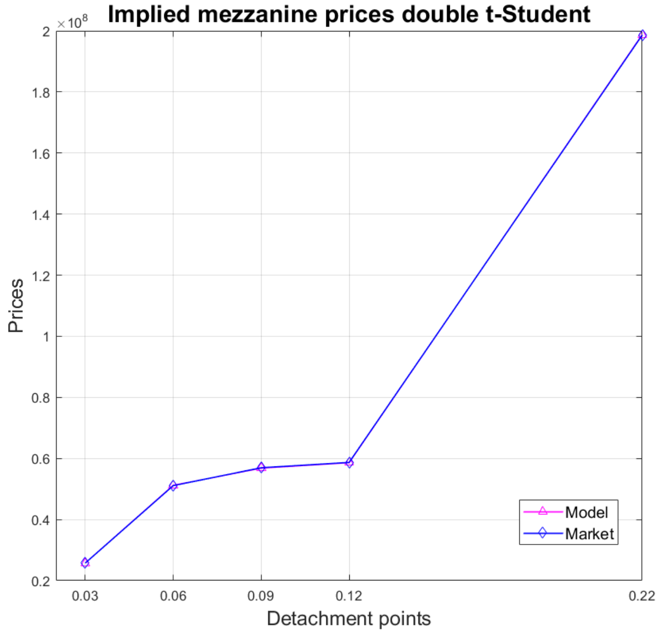
Double t-Student

The algorithm we use to calibrate parameters is the same as the one of implied correlations but considering implied prices. In particular the MSE is computed as the mean of the square of the difference between market and model prices. To reproduce market compound implied correlations we use mezzanines prices, instead for market base implied correlations we use equities.

For the mezzanine case we obtain: $\nu = 9.8273$, $\rho = 0.2523$ and the following implied prices in MIO €:

Market	Model
25.758	25.758
51.106	51.185
56.949	56.833
58.692	58.612
198.630	198.750

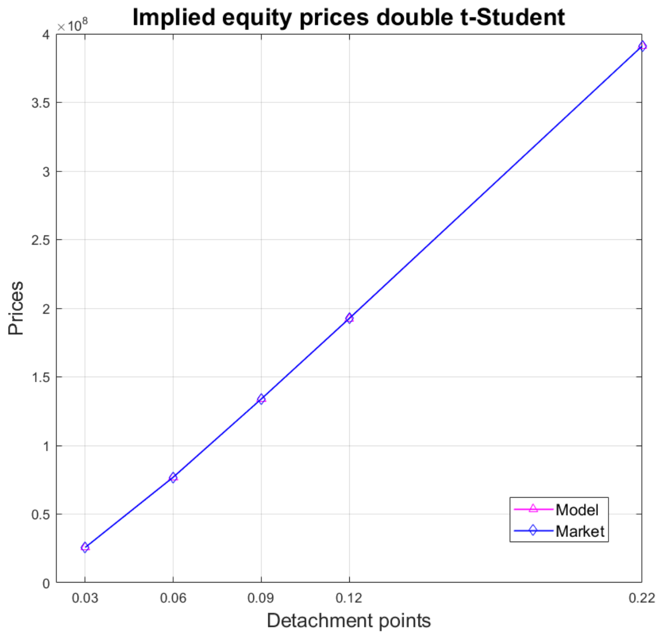
Table 11: Implied Mezzanine Prices



For the equities case instead we get: $\nu = 9.6647$, $\rho = 0.2527$ and the following implied prices:

Market	Model
25.758	25.758
76.863	76.964
133.810	133.800
192.500	192.410
391.130	391.140

Table 12: Implied Equity Prices



In both cases we can see that the implied prices' curve is very close to the market's one. This justifies our preference for this model: in fact it reproduces almost perfectly market prices.

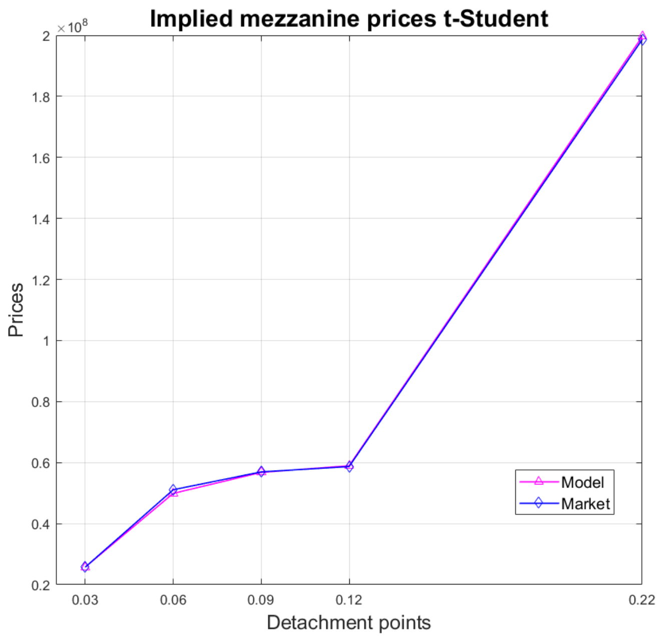
t-Student

Also in this other model using the usual algorithm we can perform the calibration both using mezzanine and equity prices.

For the mezzanine case we obtain: $\nu = 12$, $\rho = 0.1540$ and the following implied prices in MIO €:

Market	Model
25.758	25.758
51.106	49.908
56.949	56.780
58.692	58.975
198.63	199.580

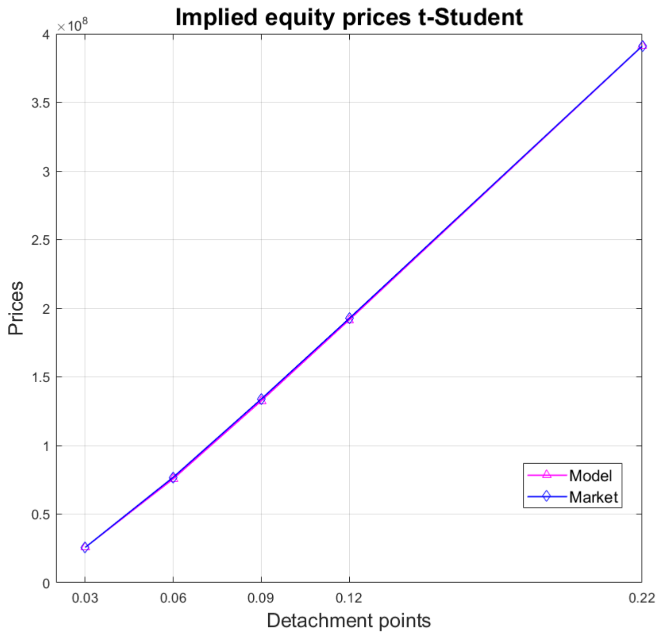
Table 13: Implied Mezzanine Prices



For the equities case instead we get: $\nu = 12$, $\rho = 0.1540$ and the following implied prices:

Market	Model
25.758	25.758
76.863	75.666
133.810	132.450
192.500	191.420
391.130	391.000

Table 14: Implied Equity Prices



In this case the two implied prices curves present a light difference with the market ones as expected from the previous analysis on implied correlations.

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[2] L. Schoegl, D. O'Kane *A note on the large homogeneous portfolio approximation with Student - t copula* [page 579-580]

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[6] X. Burtshcell, J. Gregory ans J.-P. Laurent *A comparative of CDO pricing models* (2008)