Saddlepoint techniques for the statistical analysis of time series

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$\Rightarrow$ **The need for saddlepoint techniques is rooted in both the theory and practice of statistics and other disciplines.**

First part

Motivation form theoretical statistics

Theorem (Karlin-Rubin, as stated in Casella-Berger)

Consider testing

$$\mathcal{H}_0 : \theta \leq \theta^0 \quad \textit{versus} \quad \mathcal{H}_1 : \theta > \theta^0.$$

Suppose that T *is a sufficient statistic for* θ *and the family of pdfs or pmfs* $\{g(t \mid \theta) : \theta \in \Theta\}$ *of* T *has a Monotone Likelihood Ratio. Then for any* t_0*, the test that rejects* $\mathcal{H}_0$ *if and only if* $T > t_0$ *is a UMP level* α *test, where*

$$\alpha = P_{\theta^0}(T > t_0).$$

Motivation from financial econometrics

Diffusions-type processes

$$dY(t) = \mu(Y_t)dt + \sigma(Y_t)dW_t + J_t dN_t$$

where N_t is a Poisson process, J_t is the jump size, W_t is a BM.

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2. transition density for time interval $\Delta > 0$ and for $\tau \in \mathbb{R}$ (by Fourier inversion, $i^2 = -1$)

$$p(y|x, \Delta) = \frac{1}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} \exp\left\{ \boxed{K_{y|x}}(\Delta, z; x) - zy \right\} dz$$

needed for inference on the model parameter; see e.g. Bibby et al. (Handbook of Fin. Econ., 2010), La Vecchia & Trojani, (JASA, 2012)

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4. ...

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Typical statistical problem: For a given statistic $T : \operatorname{dom} T \to \mathbb{R}$ or an estimator $\hat{\theta}_n$, tail probabilities or quantiles at different levels are needed to carry out statistical inference (essentially, tests and confidence intervals).

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$\Rightarrow$ we have to rely on approximations

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Analytical and resampling techniques can achieve higher order refinements over the first order asymptotic theory

Analytical techniques: Edgeworth expansion

Edgeworth/Charlier series is obtained as follows:

1. Assume we are given:

Random Variable	X	Y
Distr.	$H(x)$	$G(x)$
Charact. fct.	$\chi(u)$	$\xi(u) = \int e^{iux} dG(x)$
Cumulants	β_r	γ_r

where the cumulants are (by definition)

$$\beta_r = (-i)^r \frac{d^r}{du^r} \ln \chi(u) \Big|_{u=0} \qquad \gamma_r = (-i)^r \frac{d^r}{du^r} \ln \xi(u) \Big|_{u=0}$$

Analytical techniques: Edgeworth expansion

2 By Taylor expansion of the cumulant generating function (c.g.f.) about $u = 0$:

$$\ln \frac{\chi(u)}{\xi(u)} = \ln \chi(u) - \ln \xi(u) = \sum_{r=1}^{\infty} (\beta_r - \gamma_r) \frac{(iu)^r}{r!},$$

thus,

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3 By Fourier inversion:

$$H(x) = \exp \left\{ \sum_{r=1}^{\infty} (\beta_r - \gamma_r) \frac{(-D)^r}{r!} \right\} G(x),$$

D denotes differentiation with respect to x and $e^D = \sum_{j=0}^{\infty} D^j / j!$

Motivation form theoretical statistics

Let $X \sim \mu$ with measure absolutely continuous w.r.t. the Lebesgue measure and having density f_X.

We are given a random sample $\boldsymbol{X} = (X_1, ..., X_n)$ of i.i.d. copies of X, whose mgf and cgf are:

$$M(v) = E_\mu[\exp(vX)] \quad \text{and} \quad \mathcal{K}(v) = \ln E_\mu[\exp(vX)], \; v \in \mathbb{R}$$

and $E_\mu[X] = 0$.

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We make use of Step 1-3 to approximate the density f_n of the standardized mean via Edgeworth expansion...

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with λ_3 and λ_4 being the standardized cumulants of X of order three and four, while ϕ is the pdf of a standard normal.

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- *they can even become negative in the tails.*

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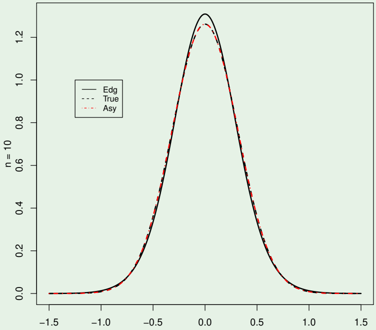
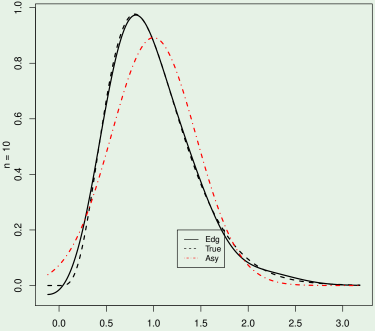
Example (Sample mean)

For Asy and Edg, consider $\bar{X}_n$ for $n = 10, 50, 250$, for $X_i \sim \mathcal{N}(0, 1)$ and $X_i \sim \exp(1)$

Gauss $(n = 10)$



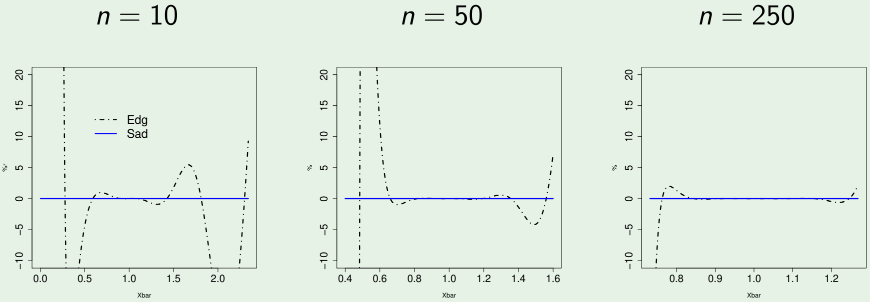
Exp $(n = 10)$



Motivation form theoretical statistics

Example (cont'd)

for the exponential case, rel. err. $= 100 \cdot$ (true -approx)/true



Any other higher oder technique to cope with these issues? saddlepoint approx...

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Example (cont'd)

In this example about $\bar{X}_n$, we know the c.g.f. and the saddlepoint density approx $g_n(s)$ is (Daniels (1954)):

$$g_n(s) = \left[\frac{n}{2\pi \mathcal{K}''\{v(s)\}}\right]^{1/2} \exp\left(n\left[\mathcal{K}\{v(s)\} - v(s)s\right]\right) \quad (1)$$

and $v(s)$ (saddlepoint) is the solution to

$$\mathcal{K}'(v) - s = 0,$$

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namely, we look for $v(s)$ such that X has expected value equal to s.

Motivation form theoretical statistics

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- To find the saddlepoint we need to solve

$$\mathcal{K}'(v) - s = 0 \iff \sup_{v \in \mathbb{R}} [\mathcal{K}\{v(s)\} - v(s)s] = -\mathcal{K}^{\dagger}(s),$$

with $\mathcal{K}^{\dagger}$ being the Legendre transform of $\mathcal{K}$

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- for $\sqrt{n}\bar{X}_n$, (by Jacobian formula) $g_n(s) = \left(\frac{1}{2\pi}\right)^{1/2} e^{-\frac{s^2}{2}}$ pdf of $\mathcal{N}(0, 1)$

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Example (cont'd)

- The saddlepoint density approximation g_n features relative error of order $O(n^{-1})$ over the whole $\mathbb{R}$

$$f_n(s) = g_n(s)\left\{1 + O\left(n^{-1}\right)\right\} \tag{2}$$

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- The density g_n is obtained by approximating the Fourier inversion of M^n, which yields f_n:

$$\begin{aligned} f_n(s) &= \frac{n}{2\pi}\int_{-\infty}^{\infty} e^{-i\upsilon ns} M^n(i\upsilon)d\upsilon \overset{(z=i\upsilon)}{=} \frac{n}{2\pi i}\int_{\mathcal{I}} e^{-nzs} M^n(z)dz \\ &= \frac{n}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{n(\mathcal{K}(z)-zs)}dz,\ \tau \in \mathbb{R}, \end{aligned}$$

which may be obtained using a Taylor expansion of $(\mathcal{K}(z) - zs)$ about $\upsilon(s)$.

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The sadd approx is obtained via the method of the steepest descent: this is a general technique to compute asymptotic expansions of integrals

$$\int_{\mathcal{P}} e^{v\, w(z)} \xi(z) dz,$$

with $v \in \mathbb{R}^+$ is large, ξ and w being analytic functions of $z \in \mathbb{C}$.

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Idea

Deform the path of integration (Cauchy's theorem) so that the new path of integration passes through the so-called saddlepoint, namely the zero of the derivative $w'(z)$. Then, we approximate the resulting integral using a series expansion (Watson's lemma). See Daniels (AoMS, 1954).

Loosely speaking, one dos a "Laplace-type approx" on $\mathbb{C}$. Jump to Laplace

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- **by means of $\upsilon(s)$, recenter/Esscher tilt** *the density of X: we embed the original density f_X into an exponential family, and then define the (conjugate) density h_s such that it centers at s the density of the rv ($f_X \mapsto h_s$ via $\upsilon(s)$)*

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- *compute a low-order* **Edgeworth expansion** *for the tilted density (centered at s, so it works well!) to obtain $g_n(s)$*

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- **by means of $\upsilon(s)$, recenter/Esscher tilt** *the density of X: we embed the original density f_X into an exponential family, and then define the (conjugate) density h_s such that it centers at s the density of the rv ($f_X \mapsto h_s$ via $\upsilon(s)$)*
- *compute a low-order* **Edgeworth expansion** *for the tilted density (centered at s, so it works well!) to obtain $g_n(s)$*
- **repeat** *this procedure* **for every** $s \in \mathbb{R}$

Motivation form theoretical statistics

Alternative: derive g_n via convex analysis.

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- **repeat** *this procedure* **for every** $s \in \mathbb{R}$

$\Rightarrow$ *saddlepoint density approximation is a sequence of low-order local approximations; see Easton & Ronchetti (1986), JASA and Wang (1992).*

Motivations related to dependent data analysis

Many macroeconomic time series display a persistent time trend and contain only a few observations recorded at annual frequency. Much controversy in macroeconometrics has revolved around the suitability of ARIMA models; see the seminal paper of Nelson and Plosser (1982) and Gil-Alana and Robinson (1997) for a review of the literature.

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Within this setting, to model the slow decay of the autocorrelation function displayed by many macroeconomic time series, the use of (Gaussian) FARIMA models and first order Gaussian asymptotic theory (Wald-type test statistics) is routinely applied for confidence intervals and testing statistical hypotheses.

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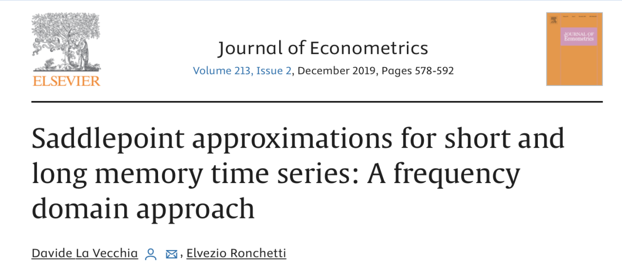

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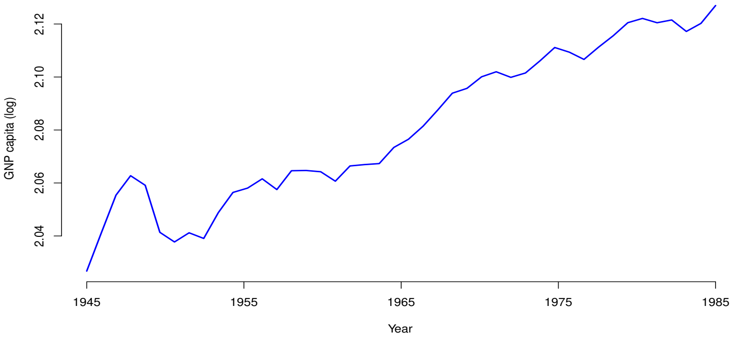
Saddlepoint approximations for short and long memory time series: A frequency domain approach

Davide La Vecchia, Elvezio Ronchetti

Motivations related to dependent data analysis

Focus on the extended Nelson and Plosser data set: plot log-GNP per capita (other time series available in the JoE paper)



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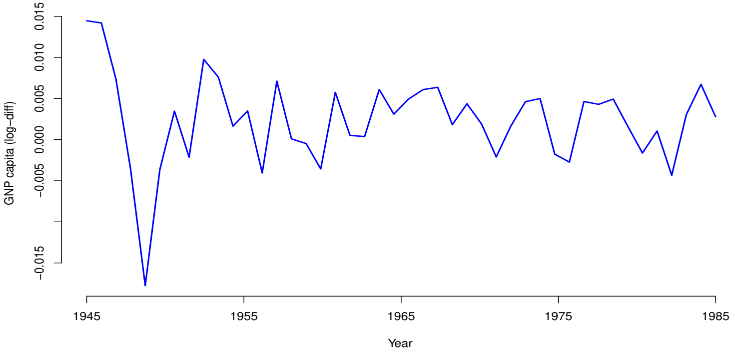

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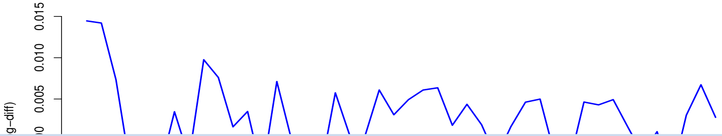


Remark

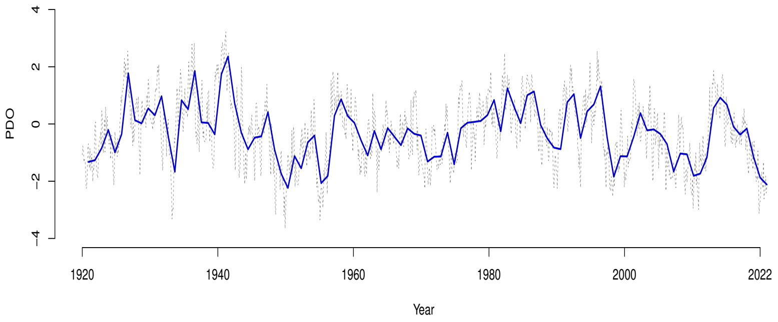
In the literature one is typically testing for the presence of long memory: ARFIMA models and

$$\mathcal{H}_0: \; d = 0 \quad vs \quad \mathcal{H}_1: \; d > 0$$

we resort on an M-estimator (Whittle), which is asymptotically χ^2 Wald-type test statistics (W_n) are applied when $n = 44$. Is this a sensible procedure? Is the asymptotics suffering from size distortion due to the small sample size?

Motivations related to dependent data analysis

The Pacific Decadal Oscillation (PDO) index measures the climatological situation of the Southern hemisphere: its extremes correspond to episodes of abnormal weather conditions.



Motivations related to dependent data analysis

The Pacific Decadal Oscillation (PDO) index measures the climatological situation of the Southern hemisphere: its extremes correspond to episodes of abnormal weather conditions.

Remark

Whiting et al. (2003) model the time series by an ARFIMA$(0, d, 0)$. *Data analysis and inference is conducted using* **annual data**, *from 1920 to 2022, so* $n = 122$, *relying on M-estimator (Whittle), which yields Wald-type statistic* W_n *from first order asymptotic theory to test*

$$\mathcal{H}_0 : \ d = 0 \quad vs \quad \mathcal{H}_1 : \ d > 0.$$

Motivations related to dependent data analysis

Example (ARFIMA synthetic data)

Let $\{Y_t,\ t \in \mathbb{Z}\}$ be an ARFIMA(p, d, q), having dynamics

$$\theta(L)(1-L)^d Y_t = \phi(L)\epsilon_t, \tag{3}$$

where $\forall t$, the $\{\epsilon_t\}$ are i.i.d. with zero mean and known $\sigma_\epsilon^2 = 1$.

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- We focus on an ARFIMA of order $p = 2$, $d = 0$ and $q = 0$ process, with AR coefficients $\theta_1 = 0.1$, $\theta_2 = 0.2$ and Gaussian errors.

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- We consider different increasing values of the sample size $n = 250, 2500, 5000$.
- We estimate θ via the routinely applied Whittle's M-estimator, as implemented in the routine `WhittleEst` available in the R package `longmemo`.

Motivations related to dependent data analysis

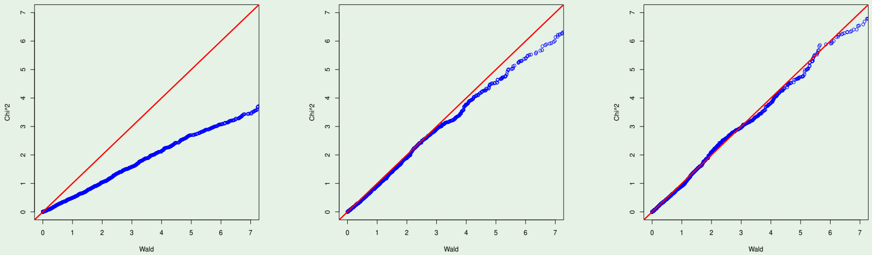
Example (cont'd)

The goal of our inference is to test

$$\mathcal{H}_0 : d = 0 \text{ vs. } \mathcal{H}_1 : d > 0,$$

and we resort on the Wald test statistic W_n for Whittle's estimator, as available in the statistical software, comparing χ^2 quantiles to the true (as obtained by MC simulation).

$n = 250$ $n = 2500$ $n = 5000$



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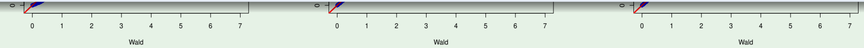
$n = 250$ $n = 2500$ $n = 5000$

Remark

As conjectured, the first order asymptotic theory suffers from size distortion. Any **saddlepoint techniques?**



Motivations related to dependent data analysis

Another example comes from the literature on Spatial Autoregressive processes:

Second part

Menu

- Literature: a bird's-eye view

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- Time series
    - Some elements of spectral analysis
    - Setting: SRD & LRD

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- Conclusion: take home message

Literature: a bird's-eye view

(i) Most of the results on **saddlepoint techniques** are available for the **iid setting:** see Field & Ronchetti (1990), Jensen (1995), Kolassa (2006), Butler (2007), or Brazzale et al. (2007) for book-length presentation.

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(iii) **Higher order techniques in frequency domain (spectral analysis) for time series** are available: see Taniguchi (JMA, 1987, Edgeworth for Whittle under SRD), Franke & Härdle (Annals, 1992, FDB), Dahlhaus & Janas (Annals, 1996, FDB), Andrews & Lieberman (Econometrica, 2005, Edgeworth for Whittle under LRD).

Some elements of spectral analysis

Let us start from a peculiar function of time series data: the autocovariance function

$$\gamma_Y(h) = cov(Y_{t+h}, Y_t) = E[(Y_{t+h} - \mu)(Y_t - \mu)]$$

for all h and with $E(Y_t) = \mu$, $\forall t$.

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for all h and with $E(Y_t) = \mu$, $\forall t$.
Under suitable assumptions, we have (for $i \in \mathbb{C}$)

$$\gamma_Y(h) = \int_{-1/2}^{1/2} \exp\{2\pi i \lambda h\} f(\lambda) d\lambda, \quad h = 0, \pm 1, \pm 2 \ldots$$

as the inverse Fourier transform of the spectral density $f(\cdot)$:

$$f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma_Y(h) \exp\{-i 2\pi \lambda h\}, \quad -1/2 \leq \lambda \leq 1/2.$$

Some elements of spectral analysis

Definition

Given time series data $Y_1, \ldots, Y_n$, the discrete Fourier transform (DFT) is

$$d(\lambda_j) = n^{-1/2} \sum_{t=1}^{n} Y_t \exp\{-2\pi i \lambda_j t\},$$

for $j = 0, 1, \ldots, n-1$, where the frequencies $\lambda_j = j/n$ are called Fourier or fundamental frequencies.

The periodogram at λ_j is $I(\lambda_j) = |d(\lambda_j)|^2$, and we have that

$$I(\lambda_j) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}_Y(h) \exp\{-2i\pi\lambda_j h\},$$

where $\hat{\gamma}_Y(h)$ is the empirical covariance and $\bar{Y}$ is the sample average.

Some elements of spectral analysis

Property 1. The periodogram is an asymptotically unbiased (nonparametric) estimator of the spectral density $f(\lambda)$.

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Property 2. The periodogram ordinates are such that

$$I(\lambda) \stackrel{\mathcal{D}}{\to} i.d.\ \xi f(\lambda), \quad \xi \sim \exp(1) \tag{4}$$

Remark

The asymptotic iid-ness of the standardized periodogram ordinates allows to transform problems for dependent data into problems for iid data.

Some elements of spectral analysis

Property 2 allows to derive a frequency domain likelihood and parameter estimation is obtained maximazing this likelihood.

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This idea goes back to Whittle (1951): if there is a parametric model for $f(\lambda, \theta)$, then we may work on:

$$\mathcal{L}_W(\theta) = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \ln f(\lambda, \theta) d\lambda + \int_{-\pi}^{\pi} \frac{I(\lambda)}{f(\lambda, \theta)} d\lambda \right], \tag{5}$$

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The optimization of $L_W(\theta)$ (the Riemann-discretized version of $\mathcal{L}_W$):

$$\hat{\theta}_n = \arg\max_{\theta} L_W(\theta)$$

(or $\nabla_\theta L_W(\hat{\theta}_n) = 0$) defines an **M-estimator in the frequency domain**. Then,

$$\mathcal{V}_n = \sqrt{n}(\hat{\theta}_n - \theta^0)$$

and we want an approximation to its density $f_{\hat{\theta}_n}$.

Setting: SRD and LRD

Suppose that $\{Y_t\}$ is a linear process, second order stationary process,

$$Y_t = \sum_{r=0}^{\infty} a_r \varepsilon_{t-r},$$

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$$f(\lambda, \theta) = |\lambda|^{-2d} L(\lambda, \vartheta), \quad \lambda \in \Pi = (-\pi, \pi] \tag{6}$$

where $d \in [0, 0.5)$, $\vartheta \in \mathbb{R}^p$ with $p \geq 1$ and $\theta = (d, \vartheta)$.

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Definition

We classify the process $\{Y_t\}$ as short-range dependent (SRD) or long-range dependent (LRD)

- when $d = 0$ and the function $L(\cdot, \vartheta)$ is bounded with $L(0, \vartheta) \neq 0$, then the process $\{Y_t\}$ features SRD
- Otherwise, the process $\{Y_t\}$ features LRD—f has a pole at $\lambda = 0$.

Saddlepoint approximation (exponential-based)

First order asymptotic theory implies

$$\mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, V).$$

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To have a better density approximation to $f_{\hat{\theta}_n}$, we may derive the saddlepoint density approximation $g_{\hat{\theta}_n}$ treating the periodogram ordinates as independently and exponentially distributed r.v.'s: we use it to approximate the c.g.f. $\mathcal{K}_{\mathcal{V}_n}$ and its general Legendre transform …

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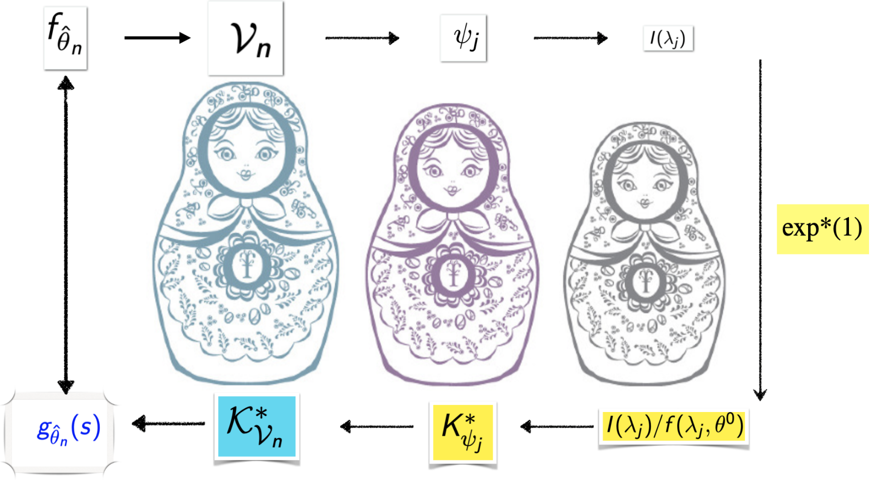
... let's visualize the idea ...

Visualisation

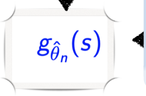


Visualisation

The saddlepoint approximation $g_{\hat\theta_n}$ can be **easily** derived treating the periodogram ordinates $\{I(\lambda)\}$ as independent rvs, exponentially distributed.

It features:

- **SRD**: relative of order $o(n^{-1/2})$
- **LRD**: relative error of order $O(n^{-1/2})$.



Saddlepoint approximation (exponential-based)

Specifically:

- Whittle's estimating function is

$$\psi_j\left(I(\lambda_j), \theta\right) = \left(\frac{I(\lambda_j)}{f(\lambda_j, \theta)} - 1 \right) \nabla_\theta \ln f(\lambda_j, \theta),$$

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- define $\mathcal{K}^*_{\mathcal{V}_n}(\upsilon, s) = \sum_j K^*_{\psi_j}(\upsilon, s)$, where

$$K^*_{\psi_j}(\upsilon, s) = \ln\left(E^*\left[\exp\{\upsilon\psi_j(I(\lambda_j), s)\}\right]\right),$$

with E^* computed treating $I(\lambda_j)/f(\lambda_j,\theta^0) \sim \exp(1)$.

Saddlepoint approximation (exponential-based)

The saddlepoint density approximation is:

$$g_{\hat{\theta}_n}(s) = \left[\frac{n}{2\pi \, \mathcal{K}^{*\,\prime\prime}_{\mathcal{V}_n}(\upsilon_0, s)} \right]^{1/2} e^{\mathcal{K}^*_{\mathcal{V}_n}(\upsilon_0, s)}, \tag{7}$$

and the saddlepoint $\upsilon_0 = \upsilon_0(s)$ solves

$$\mathcal{K}^{*\,\prime}_{\mathcal{V}_n}(\upsilon, s) = 0.$$

Remark

The advantage of using $I(\lambda)/f(\lambda, \theta) \sim \exp(1)$ is that $\mathcal{K}^*_{\mathcal{V}_n}$ is strictly convex, thus the saddlepoint equation admits a unique solution—which can be computed using standard methods, like the one based on the secant.

Saddlepoint approximation (exponential-based)

Example (ASY and frequency domain bootstrap (FDB))

Let us consider the AR(1):

$$Y_t = \theta_0 Y_{t-1} + \epsilon_t, \quad \epsilon_t \sim t_6 \quad \theta^0 = 0.4.$$

and the Whittle's estimator $\hat{\theta}_n$. Goal: approximate $P_{\theta^0}(\hat{\theta}_n > t_0)$ (e.g. for p-value).

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SAD	12.2%	9.1%	4.4%	2.0%
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FDB	—	—	—	—

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		$n = 150$		
SAD	12.7%	9.9%	4.9%	2.3%
ASY	12.1%	9.2%	4.4%	2.0%
FDB	13.5%	10.8%	5.6%	2.9%
$(q_1; q_3)$	(10.5%; 15.7%)	(8.0%; 12.7%)	(4.0%; 6.6%)	(2.0%; 3.5%)

Saddlepoint approximation (exponential-based)

More generally, let $\theta = (\theta^{(1)}, \theta^{(2)})$, where $\theta^{(2)} \in \mathbb{R}^{p_2}, 1 < p_2 < p$ and consider testing

$$\mathcal{H}_0 : \theta^{(2)} = 0 \;\; \text{vs} \;\; \mathcal{H}_1 : \theta^{(2)} > 0$$

with $\theta^{(1)}$ being the nuisance parameter (care is needed for σ^2).

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with $\theta^{(1)}$ being the nuisance parameter (care is needed for σ^2). Two options:

- $g_{\hat{\theta}_n}$ is available: construct the test using analytical marginalization techniques

Saddlepoint approximation (exponential-based)

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with $\theta^{(1)}$ being the nuisance parameter (care is needed for σ^2). Two options:

- $g_{\hat{\theta}_n}$ is available: construct the test using analytical marginalization techniques
- adapt the univariate saddlepoint test statistic of Robinson et al (2003. AoS):

$$\tilde{\mathcal{K}}^{\dagger}(\hat{\theta}_n^{(2)}) = 2 \inf_{\theta^{(1)}} \left[\sup_{\upsilon} \{ - \sum_j \boxed{K^*_{\psi_j}} (\upsilon; (\theta^{(1)}, \hat{\theta}_n^{(2)})) \} \right],$$

where υ solves the saddlepoint equation. The distribution of $\tilde{\mathcal{K}}^{\dagger}(\hat{\theta}_n^{(2)})$ under the null, can be approximated by a $\chi^2_{p_2}$ and it is asymptotically first order equivalent to the Wald test.

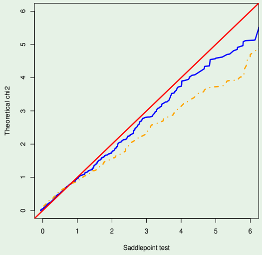
Saddlepoint approximation (exponential-based)

Example (Gaussian ARFIMA $(0, d, 0)$)

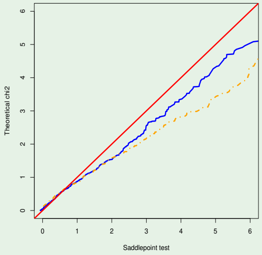
Testing about the long-memory (no nuisance, no need for the inf) for $n = 100, 250$:

$$\mathcal{H}_0 : d = d^0 \quad \text{vs} \quad \mathcal{H}_1 : d > d^0.$$

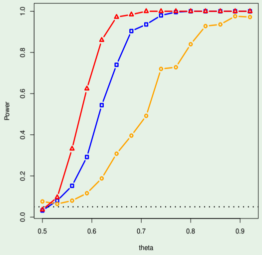
$d^0 = 0.1$



$d^0 = 0.35$



Power



Saddlepoint approximation (empirical version)

Remark

The c.g.f. may be approximated using the empirical distribution of the periodogram ordinates, keeping their independence but not relying on the exponential distribution.

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- *Dahlhaus & Janas (1996. AoS) (FDB)*
- *Monti (1997, Biom.) (FDEL) and Nordman & Lahiri (2006, AoS)*
- *Kakizawa (2013, JTSA) (FDGEL)*

Saddlepoint approximation (empirical version)

[Submitted on 19 Mar 2024]

On the use of the cumulant generating function for inference on time series

Alban Moor, Davide La Vecchia, Elvezio Ronchetti

We introduce innovative inference procedures for analyzing time series data. Our methodology enables density approximation and composite hypothesis testing based on Whittle's estimator, a widely applied M-estimator in the frequency domain. Its core feature involves the (general Legendre transform of the) cumulant generating function of the Whittle likelihood score, as obtained using an approximated distribution of the periodogram ordinates. We present a testing algorithm that significantly expands the applicability of the state-of-the-art saddlepoint test, while maintaining the numerical accuracy of the saddlepoint approximation. Additionally, we demonstrate connections between our findings and three other prevalent frequency domain approaches: the bootstrap, empirical likelihood, and exponential tilting. Numerical examples using both simulated and real data illustrate the advantages and accuracy of our methodology.

Subjects: **Methodology (stat.ME)**; Computation (stat.CO)

Saddlepoint approximation (empirical version)

The empirical saddlepoint density approximation is

$$\hat{g}_{\hat{\theta}_n}(s) = \left(\frac{m}{2\pi}\right)^{p/2} \left|\det \hat{M}(s)\right| \left|\det \hat{\Sigma}(s)\right|^{-1/2} \exp\{m\, \boxed{\hat{K}(s)}\}, \tag{8}$$

where

$$\boxed{\hat{K}(s)} = \boxed{\hat{K}(\hat{\upsilon}, s)} = \ln\left[\frac{1}{m}\sum_{j=1}^{m} \exp\{\hat{\upsilon}^T \psi_j(I_j, s)\}\right], \tag{9}$$

$$\hat{M}(s) = \frac{1}{m}\exp\{-\hat{K}(s)\}\sum_{j=1}^{m} \nabla_w \psi_j(I_j, w)|_{w=s} \exp\{\hat{\upsilon}^T \psi_j(I_j, s)\},$$

$$\hat{\Sigma}(s) = \frac{1}{m}\exp\{-\hat{K}(s)\}\sum_{j=1}^{m} \psi_j(I_j, s)\psi_j(I_j, s)^T \exp\{\hat{\upsilon}^T \psi_j(I_j, s)\}$$

and the empirical saddlepoint $\hat{\upsilon}$ satisfies:

$$\sum_{j=1}^{m} \psi_j(I_j, s) \exp\{\hat{\upsilon}^T \psi_j(I_j, s)\} = 0. \tag{10}$$

Saddlepoint approximation (empirical version)

The empirical saddlepoint is based on the c.g.f. $\hat{K}$ as an approximation to the true c.g.f.: it is the key tool needed to compute $\hat{g}_{\hat{\theta}_n}$ and it unveils important connection with the FDEL.

Saddlepoint approximation (empirical version)

The empirical saddlepoint is based on the c.g.f. $\hat{K}$ as an approximation to the true c.g.f.: it is the key tool needed to compute $\hat{g}_{\hat{\theta}_n}$ and it unveils important connection with the FDEL.
Indeed, FDEL solves the system of (tilted) estimating equations

$$\sum_{j=1}^{m} \psi_j(I_j, s)[1 + \hat{\xi}^T \psi_j(I_j; s)]^{-1} = 0, \tag{11}$$

where we use the shorthand notation $\hat{\xi} = \hat{\xi}(s)$. Then, Monti defines a FD version of Owen's statistics as

$$\hat{W}(s) = 2 \sum_{j=1}^{m} \ln\{1 + \hat{\xi}^T \psi_j(I_j; s)\},$$

and $\hat{W}$ is first-order equivalent to W_n.

Saddlepoint approximation (empirical version)

Now notice that

- the saddlepoint satisfies (Taylor expansion of the exp) the equation

$$\sum_{j=1}^{m} \psi_j(I_j; s)[1 + \hat{\upsilon}^T \psi_j(I_j; s)] = O_P(n^{-1}),$$

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Remark

The empirical saddlepoint and the empirical likelihood solve at the order $O_P(n^{-1})$ the same equation.

Saddlepoint approximation (empirical version)

Building on this remark, we prove that:

$$-2n \underbrace{\boxed{\hat{K}(s)}}_{\text{Emp Sadd Test}} = 2 \underbrace{\hat{W}(s)}_{\text{Owen stat}} - \frac{2m^{-1/2}}{3} \sum_{j=1}^{m} \left\{ u^T \hat{M}^T \hat{\Sigma}^{-1} \psi_j(I_j; \hat{\theta}_n) \right\}^3 + R_n$$

where, under some conditions, $R_n = O_P(n^{-1})$, $\hat{\Sigma} = \hat{\Sigma}(\hat{\theta}_n)$ and $\hat{M} = \hat{M}(\hat{\theta}_n)$.

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The latter result has a three-fold importance:

(i) *it connects our FDES to the FDEL*

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(ii) *it illustrates that the difference between* $\boxed{\hat{K}}$ *and* $\hat{W}$ *depends on the third moment of the Whittle's score: both correct* W_n *for the skewness but in a different way*

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(iii) *it yields a nonparametric approximation of the density of Whittle's estimator based on the FDEL*

Saddlepoint approximation (empirical version)

On the practical side: use the empirical saddlepoint under $\mathcal{H}_0$ to approximate the distribution of Wald-type (or EL, ET) test statistics, where

$$\mathcal{H}_0 : \theta = \theta^0 \text{ vs. } \mathcal{H}_1 : \theta \neq \theta^0.$$

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To this end,

- We define the Wald-type statistic, with $\hat{V} = \hat{M}^{-1}\hat{\Sigma}\hat{M}^{-1}$ (estimate of asym var of Whittle estim.),

$$W_n(\theta) = n(\hat{\theta}_n - \theta)^T \hat{V}^{-1}(\hat{\theta}_n - \theta).$$

Typically, the distribution of W_n is approximated by a χ^2.

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- In contrast, we make use of $\hat{g}_{\hat{\theta}_n}$ to obtain

$$P[W_n(\theta^0) > \tilde{w}(\theta^0) \mid \mathcal{H}_0] \approx 1 - \int_{\mathcal{B}} \hat{g}_{\hat{\theta}_n}(\theta) d\theta, \tag{12}$$

  where $\tilde{w}(\theta^0)$ is the observed value of the test statistic and

$$\mathcal{B} = \left\{ \theta \in \mathbb{R}^d \mid W_n(\theta) \geq \tilde{w}(\theta^0) \right\}.$$

- To compute the integral in (12), we suggest to use an importance sampling scheme based on an instrumental Gaussian distribution.

Saddlepoint approximation (empirical version)

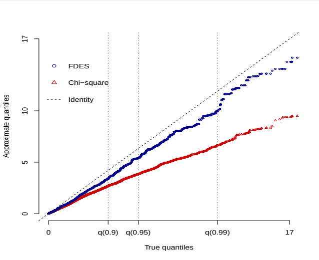
Example

ARFIMA$(0, d, 0)$ model test for

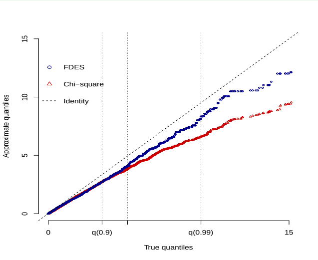
$$\mathcal{H}_0 : \ d^0 = 0 \text{ vs. } \mathcal{H}_1 : \ d \neq 0$$

using the empirical saddlepoint. We compare the approx quantiles to true quantiles (as obtained by MC simulations), for the saddlepoint technique and first-order asymptotic theory (χ^2_1).

$n = 30$



$n = 250$



Saddlepoint approximation (empirical version)

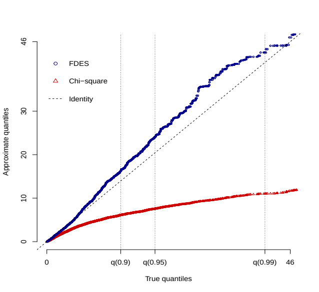
Example

We consider an ARFIMA(1,d,1) with $\theta^0 = (0.5, 0.25, 0.5)$ and test

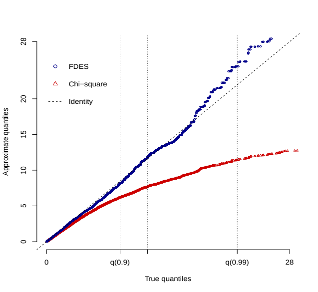
$$\mathcal{H}_0 : \theta = \theta^0 \text{ vs. } \mathcal{H}_1 : \theta \neq \theta^0$$

using the empirical saddlepoint. We compare the approx quantiles to true quantiles (as obtained by MC simulations), for the saddlepoint technique and first-order asymptotic theory (χ^2_3).

$n = 100$



$n = 500$



Saddlepoint approximation (empirical version)

Remark

- **Testing in the presence of nuisance.** *There are cases where only certain components of* θ *have to be tested. Namely, taking the partition* $(\theta_{(1)}\ \theta_{(2)})$, *we test*

$$\mathcal{H}_0 : \theta_{(1)} = \theta^0_{(1)}$$

w.l.o.g. To perform this type of test, we can simply modify our procedure, redefining the integration set as

$$\mathcal{B} \leftarrow \left\{ \theta \in \Theta \mid \tilde{w}\left(\theta_{(1)}, \hat{\theta}_{(2),n}\right) > \tilde{w}\left(\theta^0_{(1)}, \hat{\theta}_{(2),n}\right) \right\},$$

instead of

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- **Numerical accuracy.** *Also using the empirical distribution of the periodogram ordinates, the saddlepoint technique yields an improvement on the first order asymptotic theory.*

Take home message

- First-order asymptotics and Edgeworth expansions may deliver poor inference in the setting of dependent data in small samples since they exhibit severe absolute and relative distortions in the tail areas.

Take home message

- First-order asymptotics and Edgeworth expansions may deliver poor inference in the setting of dependent data in small samples since they exhibit severe absolute and relative distortions in the tail areas.
- Saddlepoint techniques are fast (no resampling) and accurate, and provide a better alternative than first-order asymptotics, Edgeworth expansions.

Thank you

For questions: davide.lavecchia@unige.ch

Laplace in brief

The Laplace method is typically applied to approximate integrals of type:

$$\int_a^b e^{v \, k(x)} dx,$$

where $k(\cdot)$ has unique maximum at $x_0 \in (a, b) \subset \mathbb{R}$ and $v \in \mathbb{R}^+$ is large.

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where (i) for $\epsilon > 0$, we deform the path of integration $\int_a^b \mapsto \int_{x_0-\epsilon}^{x_0+\epsilon}$ and (ii) we solve the Gaussian integral—getting an approx featuring relative error, under suitable assumptions.

Jump Back