GENEVA SCHOOL OF ECONOMICS AND MANAGEMENT — UNIVERSITÉ DE GENÈVE

Saddlepoint techniques for the statistical analysis of time series

Davide La Vecchia

results from joint work with E. Ronchetti, A. Moor, C. Jiang & O. Scaillet

June-2024, Erasmus University

Before losing the audience due to a boring talk...

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Maxine

Source: Maxine's drawing book, copyright fees paid by means of Swiss chocolate.

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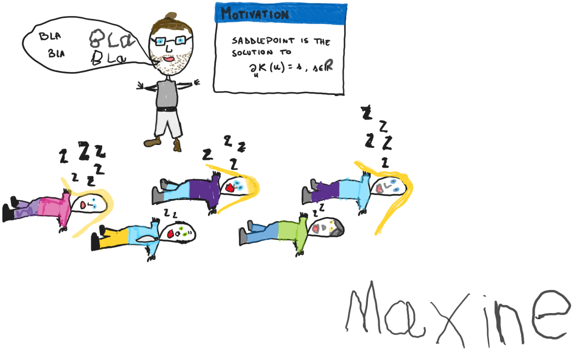

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... I state the aim of the talk ...

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- Illustrate that first-order asymptotic theory suffers from finite sample distortions
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- Illustrate numerically the performance of some saddlepoint techniques (testing)

First part: i.i.d. setting

Motivation from theoretical statistics

Typical statistical problem: For a given statistic/functional

$$T : \operatorname{dom} T \to \mathbb{R}^p, p \geq 1$$

(it can be an estimator $\hat{\theta}_n$), tail probabilities or quantiles at different levels are needed to carry out statistical inference (essentially, tests and confidence intervals).

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$\Rightarrow$ we have to rely on approximations.

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Analytical and resampling techniques can achieve higher order refinements over the first order asymptotic theory

Analytical techniques: Edgeworth expansion

Edgeworth/Charlier series is obtained as follows:

1. Assume we are given:

Random Variable	X	Y
Distr.	F_X	G
Measures	μ	ν
Charact. fct.	$\chi(u)$	$\xi(u)$
Cumulants	β_r	γ_r

where the two Fourier transforms are

$$\chi(u) = \int e^{iux} dF_X(x), \qquad \xi(u) = \int e^{iux} dG(x)$$

and the cumulants are (by definition)

$$\beta_r = (-i)^r \frac{d^r}{du^r} \ln \chi(u)\Big|_{u=0}, \qquad \gamma_r = (-i)^r \frac{d^r}{du^r} \ln \xi(u)\Big|_{u=0}$$

Analytical techniques: Edgeworth expansion

2 By **Taylor expansion of the cumulant generating function (c.g.f.) about** $u = 0$:

$$\ln \frac{\chi(u)}{\xi(u)} = \ln \chi(u) - \ln \xi(u) \overset{\text{Taylor at } u=0}{=} \sum_{r=1}^{\infty} (\beta_r - \gamma_r) \frac{(iu)^r}{r!},$$

thus,

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3 By Fourier inversion (under suitable assumptions on G, see Wallace (Ann. Math. Stats, 1958)):

$$F_X(x) = \exp\left\{ \sum_{r=1}^{\infty} (\beta_r - \gamma_r) \frac{(-D)^r}{r!} \right\} G(x),$$

D denotes a differentiation operator (with respect to x and $e^D = \sum_{j=0}^{\infty} D^j / j!$): it is such that the first term of the expansion of F_X is G.

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4 Differentiation yields an expansion for the density of X.

Motivation form theoretical statistics

Example (Sample mean)

Let $X \sim \mu$ absolutely continuous w.r.t. the Lebesgue measure and having density f_X.

We are given a random sample $\boldsymbol{X} = (X_1, ..., X_n)$ of i.i.d. copies of X, whose mgf and cgf exist and: $E_\mu[X] = 0$, $V_\mu(X) = \sigma^2 < \infty$.

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We make use of Step 1-3 to approximate the density f_n of the standardized mean via Edgeworth expansion...

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... using as $G(x)$ the standard normal, we obtain an expansion of f_n in powers of $n^{-1/2}$, where the leading term is the normal density and higher order terms correct for skewness, kurtosis:

$$g_{\text{Edg}}(s) = \phi(s)$$

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with λ_3 and λ_4 being the standardized cumulants of X of order three and four, while ϕ is the pdf of a standard normal.

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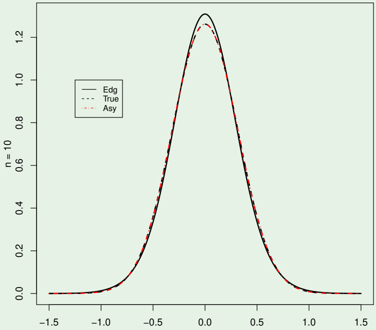
Motivation form theoretical statistics

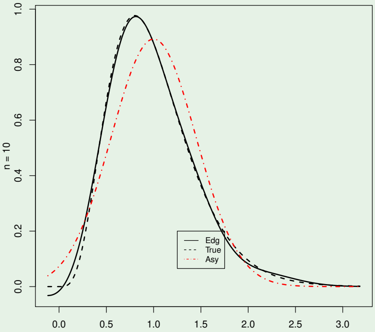
Example (cont'd)

For Asy and Edg, consider $\bar{X}_n$ for $n = 10, 50, 250$, for $X_i \sim \mathcal{N}(0,1)$ and $X_i \sim \exp(1)$

$\mathcal{N}(0,1)(n = 10)$



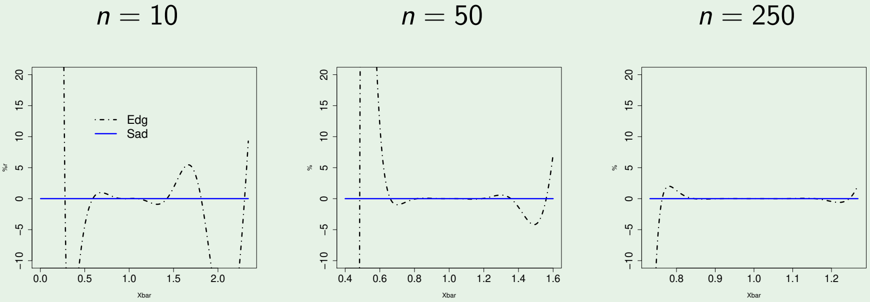
$\exp(1)(n = 10)$



Motivation form theoretical statistics

Example (cont'd)

for the exponential case, rel. err. $= 100 \cdot$ (true -approx)/true



Any other higher oder technique to cope with these issues? saddlepoint approx...

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In this example about $\bar{X}_n$, we know the c.g.f. and the saddlepoint density approx $g_n(s)$ is (Daniels (1954)):

$$g_n(s) = \left[\frac{n}{2\pi \mathcal{K}'' \{v(s)\}}\right]^{1/2} \exp\left(n\left[\mathcal{K}\{v(s)\} - v(s)s\right]\right) \qquad (1)$$

and $v(s)$ (saddlepoint) is the solution to

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namely, we look for $v(s)$ such that X has expected value equal to s.

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$$\mathcal{K}(\upsilon) = \frac{\nu^2}{2}, \quad \mathcal{K}'(\upsilon) = \upsilon \text{ and } \mathcal{K}''(\upsilon) = 1,$$

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- for $\sqrt{n}\bar{X}_n$, (by Jacobian formula) $g_n(s) = \left[\frac{1}{2\pi}\right]^{1/2} e^{-\frac{s^2}{2}}$

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- The density g_n is obtained by approximating the Fourier inversion of M^n, which yields f_n:

$$\begin{aligned} f_n(s) &= \frac{n}{2\pi}\int_{-\infty}^{\infty} e^{-i\upsilon ns} M^n(i\upsilon)d\upsilon \overset{(z=i\upsilon)}{=} \frac{n}{2\pi i}\int_{\mathcal{I}} e^{-nzs} M^n(z)dz \\ &= \frac{n}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{n(\mathcal{K}(z)-zs)}dz, \ \tau \in \mathbb{R}, \end{aligned}$$

which may be obtained using a Taylor expansion of $(\mathcal{K}(z) - zs)$ about $\upsilon(s)$.

Motivation form theoretical statistics

The sadd approx g_n is obtained via the method of the steepest descent: this is a general technique in complex analysis applied to compute asymptotic expansions of integrals

$$\int_{\mathcal{P}} e^{v\,w(z)} \xi(z) dz,$$

with $v \in \mathbb{R}^+$ is large, ξ and w being analytic functions of $z \in \mathbb{C}$.

Motivation form theoretical statistics

The sadd approx g_n is obtained via the method of the steepest descent: this is a general technique in complex analysis applied to compute asymptotic expansions of integrals

$$\int_{\mathcal{P}} e^{v\, w(z)} \xi(z) dz,$$

with $v \in \mathbb{R}^+$ is large, ξ and w being analytic functions of $z \in \mathbb{C}$.

Idea

Deform the path of integration (Cauchy's theorem) so that the new path of integration passes through the so-called saddlepoint, namely the zero of the derivative $w'(z)$. Then, we approximate the resulting integral using a series expansion (Watson's lemma). See Daniels (AoMS, 1954).

Loosely speaking, one dos a "Laplace-type approx" on $\mathbb{C}$. Jump to Laplace

Motivation form theoretical statistics

Alternative: derive g_n via convex analysis.

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- **repeat** *this procedure* **for every** $s \in \mathbb{R}$

$\Rightarrow$ *saddlepoint density approximation is a sequence of low-order local approximations; see Easton & Ronchetti (1986), JASA, Wang (1992) and La Vecchia et al. (2023).*

Motivations related to dependent data analysis

Many macroeconomic time series display a persistent time trend and contain only a few observations recorded at annual frequency. Much controversy in macroeconometrics has revolved around the suitability of ARIMA models; see the seminal paper of Nelson and Plosser (1982) and Gil-Alana and Robinson (1997) for a review of the literature.

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Within this setting, to model the slow decay of the autocorrelation function displayed by many macroeconomic time series, the use of (Gaussian) FARIMA models and first order Gaussian asymptotic theory (Wald-type test statistics) is routinely applied for confidence intervals and testing statistical hypotheses.

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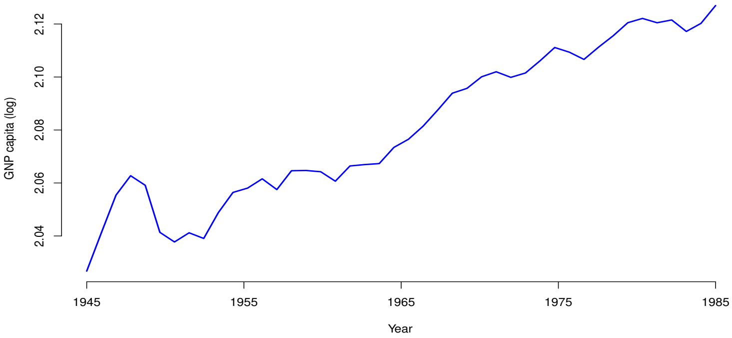
Saddlepoint approximations for short and long memory time series: A frequency domain approach

Davide La Vecchia, Elvezio Ronchetti

Motivations related to dependent data analysis

Focus on the extended Nelson and Plosser data set: plot log-GNP per capita (other time series available in the JoE paper)



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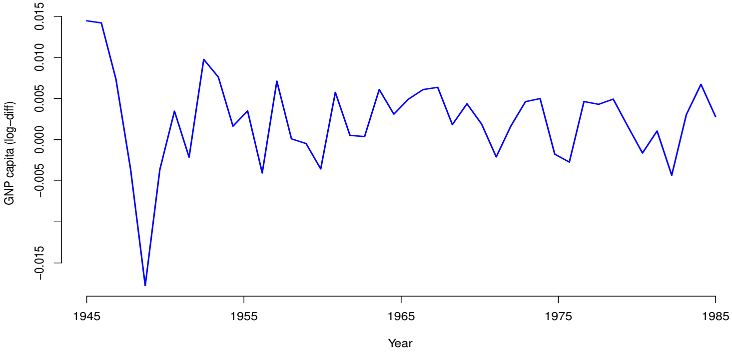

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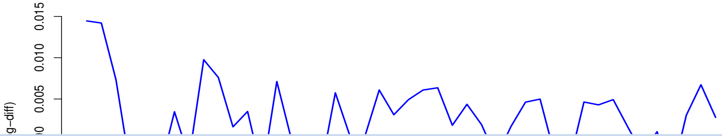


Remark

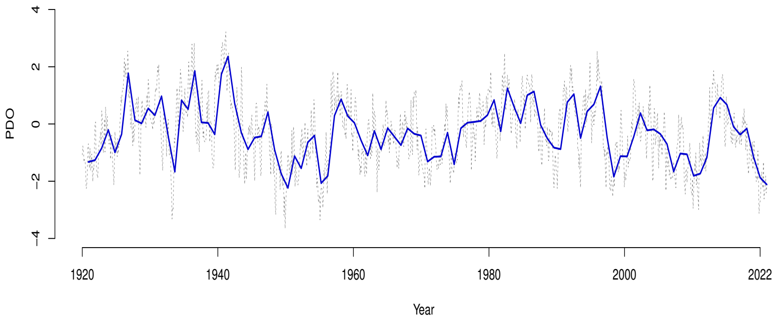
In the literature one is typically testing for the presence of long memory: ARFIMA models and

$$\mathcal{H}_0: \ d = 0 \quad vs \quad \mathcal{H}_1: \ d > 0$$

we resort on an M-estimator (Whittle), which is asymptotically χ^2 Wald-type test statistics (W_n) are applied when $n = 44$. Is this a sensible procedure? Is the asymptotics suffering from size distortion due to the small sample size?

Motivations related to dependent data analysis

The Pacific Decadal Oscillation (PDO) index measures the climatological situation of the Southern hemisphere: its extremes correspond to episodes of abnormal weather conditions.



Motivations related to dependent data analysis

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Remark

Whiting et al. (2003) model the time series by an ARFIMA$(0, d, 0)$. Data analysis and inference is conducted using **annual data**, *from 1920 to 2022, so $n = 122$, relying on M-estimator (Whittle), which yields Wald-type statistic W_n from first order asymptotic theory to test*

$$\mathcal{H}_0 : \ d = 0 \quad vs \quad \mathcal{H}_1 : \ d > 0.$$

Motivations related to dependent data analysis

Example (ARFIMA synthetic data)

Let $\{Y_t,\ t \in \mathbb{Z}\}$ be an ARFIMA(p, d, q), having dynamics

$$\theta(L)(1-L)^d Y_t = \phi(L)\epsilon_t, \tag{3}$$

where $\forall t$, the $\{\epsilon_t\}$ are i.i.d. with zero mean and known $\sigma_\epsilon^2 = 1$.

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- We consider increasing values of the sample size $n = 250, 2500, 5000$.
- We estimate θ via the routinely applied Whittle's M-estimator, as implemented in the routine `WhittleEst` available in the R package `longmemo`.

Motivations related to dependent data analysis

Example (cont'd)

The goal of our inference is to test

$$\mathcal{H}_0 : d = 0 \text{ vs. } \mathcal{H}_1 : d > 0,$$

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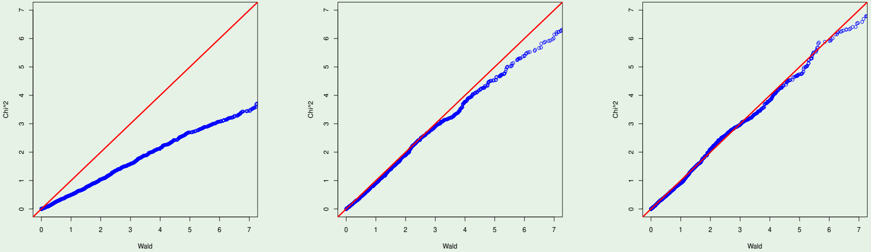

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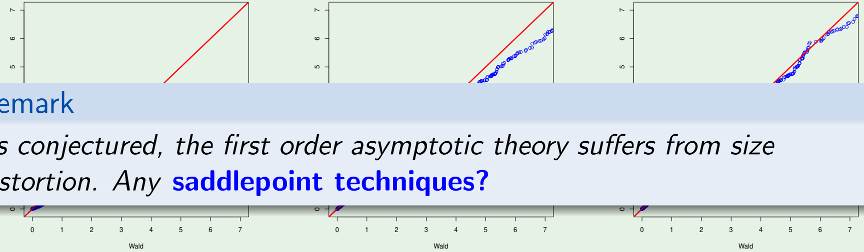

Remark

As conjectured, the first order asymptotic theory suffers from size distortion. Any **saddlepoint techniques?**

Motivations related to dependent data analysis

Another example comes from the literature on Spatial Autoregressive processes (random fields):

Second part: time series setting

Menu

- Literature: a bird's-eye view

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Literature: a bird's-eye view

(i) Most of the results on **saddlepoint techniques** are available for the **iid setting:** see Field & Ronchetti (1990), Jensen (1995), Kolassa (2006), Butler (2007), or Brazzale et al. (2007) for book-length presentation.

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(ii) In time series, some results for **saddlepoint density and tail area approximations for $M-$estimators** in the noncircular **Gaussian AR(1) process**: see Daniels (Annals, 1956), Phillips (Biometrika, 1978), Cox & Solomon (Biometrika, 1988), Wang (Biometrika, 1992), Butler & Paolella (Bernoulli, 1998), Pereira et al. (Stats & Prob Letters, 2008), Lozada-Can & Davison (American Statistician, 2010),and Field & Robinson (Annals, 2013); for a book-length discussion see Taniguchi & Kakitzawa (2001).

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(iii) **Higher order techniques in frequency domain (spectral analysis) for time series** are available: see Taniguchi (JMA, 1987, Edgeworth for Whittle under SRD), Franke & Härdle (Annals, 1992, FDB), Dahlhaus & Janas (Annals, 1996, FDB), Andrews & Lieberman (Econometric Theory, 2005, Edgeworth for Whittle under LRD).

Some elements of spectral analysis

To start with, let me recall the autocovariance function

$$\gamma_Y(h) = cov(Y_{t+h}, Y_t) = E[(Y_{t+h} - \mu)(Y_t - \mu)]$$

for $h \in \mathbb{Z}$ and with $E(Y_t) = \mu$, $\forall t$.

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for $h \in \mathbb{Z}$ and with $E(Y_t) = \mu$, $\forall t$.
Under suitable assumptions, we have (for $i \in \mathbb{C}$)

$$\gamma_Y(h) = \int_{-1/2}^{1/2} \exp\{2\pi i \lambda h\} f(\lambda) d\lambda, \quad h = 0, \pm 1, \pm 2 \ldots$$

as the inverse Fourier transform of the spectral density $f(\cdot)$:

$$f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma_Y(h) \exp\{-i 2\pi \lambda h\}, \quad -1/2 \leq \lambda \leq 1/2.$$

Some elements of spectral analysis

Definition

Given time series data $Y_1, \ldots, Y_n$, the discrete Fourier transform (DFT) is

$$d(\lambda_j) = n^{-1/2} \sum_{t=1}^{n} Y_t \exp\{-2\pi i \lambda_j t\},$$

for $j = 0, 1, \ldots, n-1$, where the frequencies $\lambda_j = j/n$ are called Fourier or fundamental frequencies.

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The periodogram at λ_j is $I(\lambda_j) = |d(\lambda_j)|^2$, and we have that

$$I(\lambda_j) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}_Y(h) \exp\{-2i\pi\lambda_j h\},$$

where $\hat{\gamma}_Y(h)$ is the empirical covariance.

Some elements of spectral analysis

Property 1. The periodogram is an asymptotically unbiased (nonparametric) estimator of the spectral density $f(\lambda)$.

Some elements of spectral analysis

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Property 2. The periodogram ordinates are such that

$$I(\lambda) \xrightarrow{\mathcal{D}} i.d.\ \xi f(\lambda), \quad \xi \sim \exp(1) \tag{4}$$

Remark

The asymptotic iid-ness of the standardized periodogram ordinates allows to transform problems for dependent data into problems for iid data.

Some elements of spectral analysis

Property 2 allows to derive a frequency domain likelihood and parameter estimation is obtained maximazing this likelihood.

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This idea goes back to Whittle (1951): if there is a parametric model for $f(\lambda, \theta)$, then we may work on:

$$\mathcal{L}_W(\theta) = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \ln f(\lambda, \theta) d\lambda + \int_{-\pi}^{\pi} \frac{I(\lambda)}{f(\lambda, \theta)} d\lambda\right], \tag{5}$$

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The optimization of $L_W(\theta)$ (the Riemann-discretized version of $\mathcal{L}_W$):

$$\hat{\theta}_n = \arg\max_{\theta} L_W(\theta)$$

(or $\nabla_\theta L_W(\hat{\theta}_n) = 0$) defines an **M-estimator in the frequency domain**. Then,

$$\mathcal{V}_n = \sqrt{n}(\hat{\theta}_n - \theta^0)$$

and we want an approximation to its density $f_{\hat{\theta}_n}$.

Incidentally....

At the other edge of the spectrum, the frequency domain approach can be suitably adapted and exploited for the statistical analysis of high-dimensional (large time, spatial, cross-sectional dimensions) random fields over a regular network:

Statistics > Methodology

[Submitted on 5 Dec 2023]

General Spatio-Temporal Factor Models for High-Dimensional Random Fields on a Lattice

Matteo Barigozzi, Davide La Vecchia, Hang Liu

Motivated by the need for analysing large spatio-temporal panel data, we introduce a novel dimensionality reduction methodology for n-dimensional random fields observed across a number S spatial locations and T time periods. We call it General Spatio-Temporal Factor Model (GSTFM). First, we provide the probabilistic and mathematical underpinning needed for the representation of a random field as the sum of two components: the common component (driven by a small number q of latent factors) and the idiosyncratic component (mildly cross-correlated). We show that the two components are identified as $n \to \infty$. Second, we propose an estimator of the common component and derive its statistical guarantees (consistency and rate of convergence) as $\min(n, S, T) \to \infty$. Third, we propose an information criterion to determine the number of factors. Estimation makes use of Fourier analysis in the frequency domain and thus we fully exploit the information on the spatio-temporal covariance structure of the whole panel. Synthetic data examples illustrate the applicability of GSTFM and its advantages over the extant generalized dynamic factor model that ignores the spatial correlations.

Setting: SRD and LRD

Suppose that $\{Y_t\}$ is a **linear process**, second order stationary process,

$$Y_t = \sum_{r=0}^{\infty} a_r \varepsilon_{t-r},$$

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for $t \in \mathbb{Z}$, with spectral density function

$$f(\lambda, \theta) = |\lambda|^{-2d} L(\lambda, \vartheta), \quad \lambda \in \Pi = (-\pi, \pi] \tag{6}$$

where $d \in [0, 0.5)$, $\vartheta \in \mathbb{R}^p$ with $p \geq 1$ and $\theta = (d, \vartheta)$.

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where $d \in [0, 0.5)$, $\vartheta \in \mathbb{R}^p$ with $p \geq 1$ and $\theta = (d, \vartheta)$.

Definition

We classify the process $\{Y_t\}$ as short-range dependent (SRD) or long-range dependent (LRD)

- when $d = 0$ and the function $L(\cdot, \vartheta)$ is bounded with $L(0, \vartheta) \neq 0$, then the process $\{Y_t\}$ features SRD
- Otherwise, the process $\{Y_t\}$ features LRD—f has a pole at $\lambda = 0$.

Saddlepoint approximation (exponential-based)

First order asymptotic theory implies

$$\mathcal{V}_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, V).$$

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To have a better density approximation to $f_{\hat{\theta}_n}$, we may derive the saddlepoint density approximation $g_{\hat{\theta}_n}$ treating the periodogram ordinates as independently and exponentially distributed r.v.'s: we use it to approximate the c.g.f. $\mathcal{K}_{\mathcal{V}_n}$ ...

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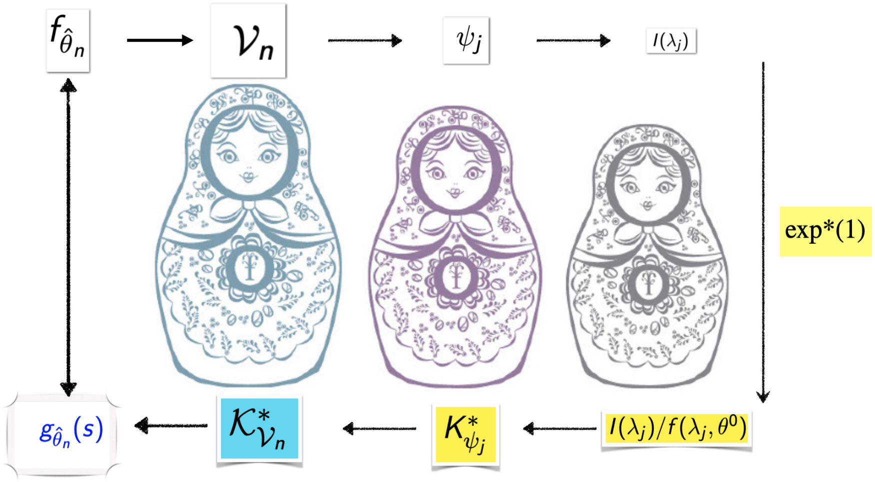
... let's visualize the idea ...

Visualisation



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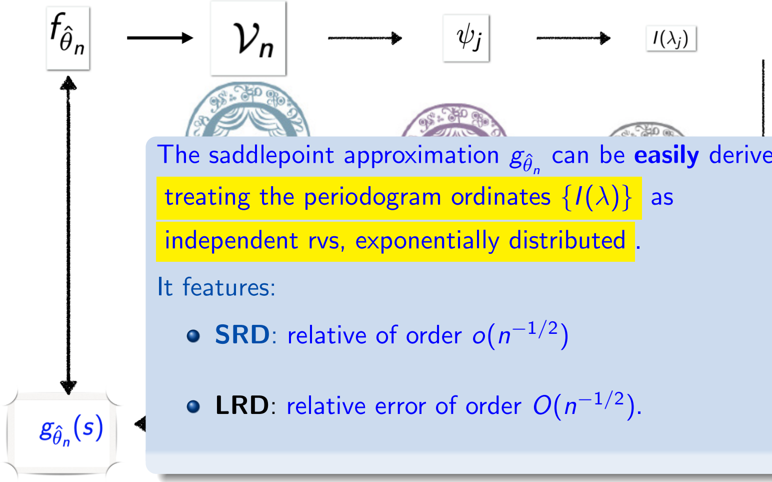

The saddlepoint approximation $g_{\hat{\theta}_n}$ can be **easily** derived treating the periodogram ordinates $\{I(\lambda)\}$ as independent rvs, exponentially distributed.

It features:

- **SRD**: relative of order $o(n^{-1/2})$
- **LRD**: relative error of order $O(n^{-1/2})$.

Saddlepoint approximation (exponential-based)

Specifically:

- Whittle's estimating function is

$$\psi_j\left(I(\lambda_j),\theta\right) = \left(\boxed{\frac{I(\lambda_j)}{f(\lambda_j,\theta)}} - 1\right)\nabla_\theta \ln f(\lambda_j,\theta),$$

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- define $\mathcal{K}^*_{\mathcal{V}_n}(\upsilon,s) = \sum_j K^*_{\psi_j}(\upsilon,s)$, where

$$K^*_{\psi_j}(\upsilon,s) = \ln\left(E^*\left[\exp\{\upsilon\psi_j(I(\lambda_j),s)\}\right]\right),$$

with E^* computed treating $I(\lambda_j)/f(\lambda_j,\theta^0) \sim \exp(1)$.

Saddlepoint approximation (exponential-based)

The saddlepoint density approximation is:

$$g_{\hat{\theta}_n}(s) = \left[\frac{n}{2\pi \boxed{\mathcal{K}^*_{\mathcal{V}_n}}''(\upsilon_0, s)}\right]^{1/2} e^{\boxed{\mathcal{K}^*_{\mathcal{V}_n}}(\upsilon_0,s)}, \tag{7}$$

and the saddlepoint $\upsilon_0 = \upsilon_0(s)$ solves

$$\boxed{\mathcal{K}^*_{\mathcal{V}_n}}'(\upsilon, s) = 0.$$

Remark

The advantage of using $I(\lambda)/f(\lambda, \theta) \sim \exp(1)$ *is that* $\boxed{\mathcal{K}^*_{\mathcal{V}_n}}$ *is strictly convex, thus the saddlepoint equation admits a unique solution—which can be computed using standard methods, like the one based on the secant.*

Saddlepoint approximation (exponential-based)

Example (ASY and frequency domain bootstrap (FDB))

Let us consider the AR(1):

$$Y_t = \theta_0 Y_{t-1} + \epsilon_t, \quad \epsilon_t \sim t_6 \quad \theta^0 = 0.4.$$

and the Whittle's estimator $\hat{\theta}_n$. Goal: approximate $P_{\theta^0}(\hat{\theta}_n > t_0)$ (e.g. for p-value).

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	12.5%	10%	5%	2.5%

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$$Y_t = \theta_0 Y_{t-1} + \epsilon_t, \quad \epsilon_t \sim t_6 \quad \theta^0 = 0.4.$$

and the Whittle's estimator $\hat{\theta}_n$. Goal: approximate $P_{\theta^0}(\hat{\theta}_n > t_0)$ (e.g. for p-value).

	12.5%	10%	5%	2.5%
		$n = 36$		
SAD	12.2%	9.1%	4.4%	2.0%
ASY	15.0%	11.8%	6.4%	3.2%
FDB	—	—	—	—

Saddlepoint approximation (exponential-based)

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FDB	—	—	—	—
		$n = 150$		
SAD	12.7%	9.9%	4.9%	2.3%
ASY	12.1%	9.2%	4.4%	2.0%
FDB	13.5%	10.8%	5.6%	2.9%
$(q_1; q_3)$	(10.5%; 15.7%)	(8.0%; 12.7%)	(4.0%; 6.6%)	(2.0%; 3.5%)

Saddlepoint approximation (exponential-based)

More generally, let $\theta = (\theta^{(1)}, \theta^{(2)})$, where $\theta^{(2)} \in \mathbb{R}^{p_2}, 1 < p_2 < p$ and consider testing

$$\mathcal{H}_0 : \theta^{(2)} = 0 \ \text{ vs } \ \mathcal{H}_1 : \theta^{(2)} > 0$$

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- $g_{\hat{\theta}_n}$ is available: construct the test using analytical marginalization techniques
- adapt the univariate saddlepoint test statistic of Robinson et al (2003. AoS):

$$\tilde{\mathcal{K}}^{\dagger}(\hat{\theta}_n^{(2)}) = 2 \inf_{\theta^{(1)}} \left[\sup_{\upsilon} \{ - \sum_j K^*_{\psi_j}(\upsilon; (\theta^{(1)}, \hat{\theta}_n^{(2)})) \} \right],$$

where υ solves the saddlepoint equation. The distribution of $\tilde{\mathcal{K}}^{\dagger}(\hat{\theta}_n^{(2)})$ under the null, can be approximated by a $\chi^2_{p_2}$ and it is asymptotically first order equivalent to the Wald test.

Saddlepoint approximation (exponential-based)

Example (Gaussian ARFIMA $(0, d, 0)$)

Testing about the long-memory (simple hypothesis) for $n = 100, 250$:

$$\mathcal{H}_0 : d = d^0 \quad \text{vs} \quad \mathcal{H}_1 : d > d^0.$$

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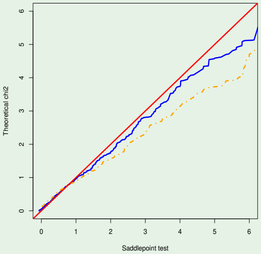
$d^0 = 0.1$ | $d^0 = 0.35$ | Power



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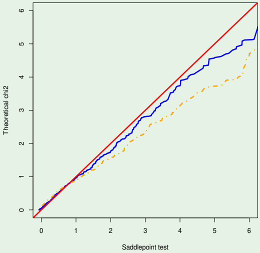

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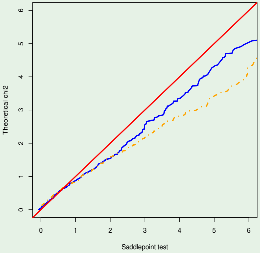

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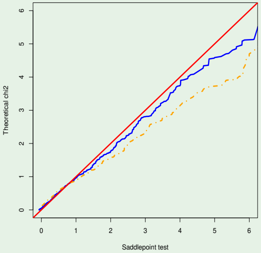

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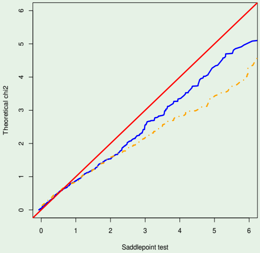

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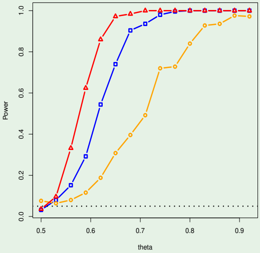

Saddlepoint approximation (empirical version)

Remark

In the time series literature, some papers made use of empirical distribution of the periodogram ordinates: keep their independence but not rely on the exponential distribution.

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- *Dahlhaus & Janas (1996. AoS) (FDB)*
- *Monti (1997, Biom.) (FDEL) and Nordman & Lahiri (2006, AoS)*
- *Kakizawa (2013, JTSA) (FDGEL)*

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This suggests that the c.g.f. may be approximated using the empirical distribution of the periodogram ordinates and obtain novel saddlepoint techniques!

Saddlepoint approximation (empirical version)

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In the time series literature, some papers made use of empirical distribution of the periodogram ordinates: keep their independence but not rely on the exponential distribution.

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[Submitted on 19 Mar 2024]

On the use of the cumulant generating function for inference on time series

Alban Moor, Davide La Vecchia, Elvezio Ronchetti

We introduce innovative inference procedures for analyzing time series data. Our methodology enables density approximation and composite hypothesis testing based on Whittle's estimator, a widely applied M-estimator in the frequency domain. Its core feature involves the (general Legendre transform of the) cumulant generating function of the Whittle likelihood score, as obtained using an approximated distribution of the periodogram ordinates. We present a testing algorithm that significantly expands the applicability of the state-of-the-art saddlepoint test, while maintaining the numerical accuracy of the saddlepoint approximation. Additionally, we demonstrate connections between our findings and three other prevalent frequency domain approaches: the bootstrap, empirical likelihood, and exponential tilting. Numerical examples using both simulated and real data illustrate the advantages and accuracy of our methodology.

Subjects: **Methodology (stat.ME)**; Computation (stat.CO)

Saddlepoint approximation (empirical version)

The empirical saddlepoint density approximation is

$$\hat{g}_{\hat{\theta}_n}(s) = \left(\frac{m}{2\pi}\right)^{p/2} \left|\det \hat{M}(s)\right| \left|\det \hat{\Sigma}(s)\right|^{-1/2} \exp\{m\, \hat{K}(s)\}, \tag{8}$$

where

$$\hat{K}(s) = \hat{K}(\hat{v}, s) = \ln\left[\frac{1}{m}\sum_{j=1}^{m} \exp\{\hat{v}^T \psi_j(I_j, s)\}\right], \tag{9}$$

$$\hat{M}(s) = \frac{1}{m}\exp\{-\hat{K}(s)\}\sum_{j=1}^{m} \nabla_w \psi_j(I_j, w)|_{w=s} \exp\{\hat{v}^T \psi_j(I_j, s)\},$$

$$\hat{\Sigma}(s) = \frac{1}{m}\exp\{-\hat{K}(s)\}\sum_{j=1}^{m} \psi_j(I_j, s)\psi_j(I_j, s)^T \exp\{\hat{v}^T \psi_j(I_j, s)\}$$

and the empirical saddlepoint $\hat{v}$ satisfies:

$$\sum_{j=1}^{m} \psi_j(I_j, s) \exp\{\hat{v}^T \psi_j(I_j, s)\} = 0. \tag{10}$$

Saddlepoint approximation (empirical version)

The empirical saddlepoint is based on the c.g.f. $\hat{K}$ as an approximation to the true c.g.f.: it is the key tool needed to compute $\hat{g}_{\hat{\theta}_n}$ and it unveils important connection with the FDEL.

Saddlepoint approximation (empirical version)

The empirical saddlepoint is based on the c.g.f. $\hat{K}$ as an approximation to the true c.g.f.: it is the key tool needed to compute $\hat{g}_{\hat{\theta}_n}$ and it unveils important connection with the FDEL.
Indeed, FDEL solves the system of (tilted) estimating equations

$$\sum_{j=1}^{m} \psi_j(I_j, s)[1 + \hat{\xi}^T \psi_j(I_j; s)]^{-1} = 0, \tag{11}$$

where we use the shorthand notation $\hat{\xi} = \hat{\xi}(s)$. Then, Monti defines a FD version of Owen's statistics as

$$\hat{W}(s) = 2 \sum_{j=1}^{m} \ln\{1 + \hat{\xi}^T \psi_j(I_j; s)\},$$

and $\hat{W}$ is first-order equivalent to W_n.

Saddlepoint approximation (empirical version)

Now notice that

- the saddlepoint satisfies (Taylor expansion of the exp) the equation

$$\sum_{j=1}^{m} \psi_j(I_j; s)[1 + \hat{\upsilon}^T \psi_j(I_j; s)] = O_P(n^{-1}),$$

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- a Taylor expansion of the equation defining the FDEL yields
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Remark

The empirical saddlepoint and the empirical likelihood solve at the order $O_P(n^{-1})$ the same equation.

Saddlepoint approximation (empirical version)

Building on this remark, we prove that:

$$-2n \underbrace{\boxed{\hat{K}(s)}}_{\text{Emp Sadd Test}} = 2 \underbrace{\hat{W}(s)}_{\text{Owen stat}} - \frac{2m^{-1/2}}{3} \sum_{j=1}^{m} \left\{ u^T \hat{M}^T \hat{\Sigma}^{-1} \psi_j(I_j; \hat{\theta}_n) \right\}^3 + R_n$$

where, under some conditions, $R_n = O_P(n^{-1})$, $\hat{\Sigma} = \hat{\Sigma}(\hat{\theta}_n)$ and $\hat{M} = \hat{M}(\hat{\theta}_n)$.

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(iii) *it yields a nonparametric approximation to the density of Whittle's estimator based on the FDEL*

Saddlepoint approximation (empirical version)

On the practical side: use the empirical saddlepoint under $\mathcal{H}_0$ to approximate the distribution of Wald-type (or EL, ET) test statistics, where

$$\mathcal{H}_0 : \theta = \theta^0 \text{ vs. } \mathcal{H}_1 : \theta \neq \theta^0.$$

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To this end,

- We define the Wald-type statistic, with $\hat{V} = \hat{M}^{-1}\hat{\Sigma}\hat{M}^{-1}$ (estimate of asym var of Whittle estim.),

$$W_n(\theta) = n(\hat{\theta}_n - \theta)^T \hat{V}^{-1}(\hat{\theta}_n - \theta).$$

Typically, the distribution of W_n is approximated by a χ^2.

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- In contrast, we make use of $\hat{g}_{\hat{\theta}_n}$ to obtain

$$P[W_n(\theta^0) > \tilde{w}(\theta^0) \mid \mathcal{H}_0] \approx 1 - \int_{\mathcal{B}} \hat{g}_{\hat{\theta}_n}(\theta) d\theta, \tag{12}$$

  where $\tilde{w}(\theta^0)$ is the observed value of the test statistic and

$$\mathcal{B} = \left\{ \theta \in \mathbb{R}^d \mid W_n(\theta) \geq \tilde{w}(\theta^0) \right\}.$$

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- **To compute the integral in (12), we suggest to use an importance sampling scheme** based on an instrumental Gaussian distribution.

Saddlepoint approximation (empirical version)

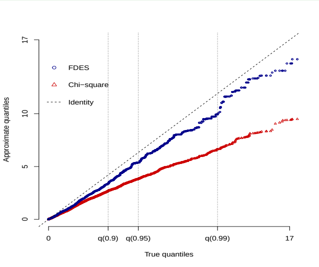
Example

We consider **ARFIMA**$(0, d, 0)$ model test for

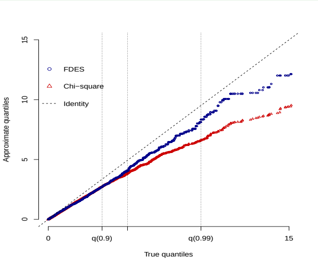
$$\mathcal{H}_0: \; d^0 = 0 \text{ vs. } \mathcal{H}_1: \; d \neq 0$$

using the empirical saddlepoint. We compare the approx quantiles to true quantiles (as obtained by MC simulations), for the saddlepoint technique and first-order asymptotic theory (χ^2_1).

$n = 30$



$n = 250$



Saddlepoint approximation (empirical version)

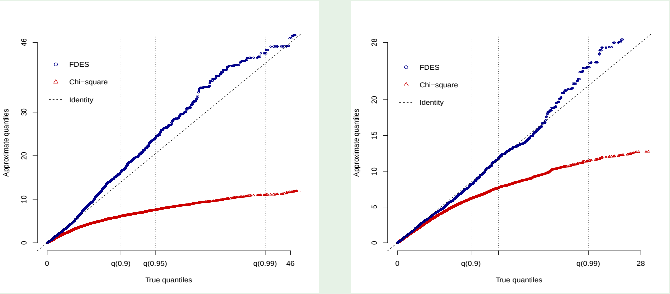
Example

We consider an **ARFIMA(1,d,1)** with $\theta^0 = (0.5, 0.25, 0.5)$ and test

$$\mathcal{H}_0 : \theta = \theta^0 \text{ vs. } \mathcal{H}_1 : \theta \neq \theta^0$$

using the empirical saddlepoint. We compare the approx quantiles to true quantiles (as obtained by MC simulations), for the saddlepoint technique and first-order asymptotic theory (χ^2_3).

$n = 100$ $\qquad$ $n = 500$



Saddlepoint approximation (empirical version)

Remark

- **Testing in the presence of nuisance.** *There are cases where only certain components of* θ *have to be tested. Namely, taking the partition* $(\theta_{(1)}\ \theta_{(2)})$, *we test*

$$\mathcal{H}_0 : \theta_{(1)} = \theta^0_{(1)}$$

w.l.o.g. To perform this type of test, we can simply modify our procedure, redefining the integration set as

$$\mathcal{B} \leftarrow \left\{ \theta \in \Theta \mid \tilde{w}\left(\theta_{(1)}, \hat{\theta}_{(2),n}\right) > \tilde{w}\left(\theta^0_{(1)}, \hat{\theta}_{(2),n}\right) \right\},$$

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- **Numerical accuracy.** *Also using the empirical distribution of the periodogram ordinates, the saddlepoint technique yields an improvement on the first order asymptotic theory.*

Thank you

For questions: davide.lavecchia@unige.ch

Laplace in brief

The Laplace method is typically applied to approximate integrals of type:

$$\int_a^b e^{v k(x)} dx,$$

where $k(\cdot)$ has unique maximum at $x_0 \in (a, b) \subset \mathbb{R}$ and $v \in \mathbb{R}^+$ is large.

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where $k(\cdot)$ has unique maximum at $x_0 \in (a, b) \subset \mathbb{R}$ and $v \in \mathbb{R}^+$ is large. A second-order Taylor expansion for $k(\cdot)$ yields

$$\int_a^b e^{v\, \boxed{k}(x)} dx \sim e^{v\, \boxed{k}(x_0)} \int_{x_0-\epsilon}^{x_0+\epsilon} e^{v\, \boxed{k^{('')}}(x_0)\frac{x^2}{2}} dx$$

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where (i) for $\epsilon > 0$, we deform the path of integration $\int_a^b \mapsto \int_{x_0-\epsilon}^{x_0+\epsilon}$

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$$\int_a^b e^{v\, k(x)} dx,$$

where $k(\cdot)$ has unique maximum at $x_0 \in (a,b) \subset \mathbb{R}$ and $v \in \mathbb{R}^+$ is large. A second-order Taylor expansion for $k(\cdot)$ yields

$$\int_a^b e^{v\, k(x)} dx \sim e^{v\, k(x_0)} \int_{x_0-\epsilon}^{x_0+\epsilon} e^{v\, k^{('')}(x_0)\frac{x^2}{2}} dx \sim e^{v\, k(x_0)} \sqrt{\frac{2\pi}{-v\, k^{('')}(x_0)}},$$

where (i) for $\epsilon > 0$, we deform the path of integration $\int_a^b \mapsto \int_{x_0-\epsilon}^{x_0+\epsilon}$ and (ii) we solve the Gaussian integral—getting an approx featuring relative error, under suitable assumptions.

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