GENEVA SCHOOL OF ECONOMICS AND MANAGEMENT

UNIVERSITÉ DE GENÈVE

Statistical analysis of network data: learning from small samples (with a couple of detours)

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University of Geneva

June 30th, 2022

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This talk is based on

- *Saddlepoint approximations for spatial panel data models,* **JASA, 2022**, with C.Jiang, E.Ronchetti and O.Scaillet
- *On some connections between Esscher's titlting, saddlepoint approximations, and optimal transportation: a statistical perspective*, **Stat. Science, 2022** with E.Ronchetti and A.Ilievski

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for $i = 1, .., n$ (cross-sectional dimension) and $t = 1, ... T$ (time series dimension).

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Aim

Investigate the spillover: test for the presence of spatial autocorrelation (between country i and country j, $i \neq j$) in the investment-saving relationship, e.g. using p-value, as in the finance/economics literature.

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$$\boxed{\{\mathrm{Inv}_{it}\} \overset{?}{\Longleftrightarrow} \{\mathrm{Inv}_{kt}\} \overset{?}{\Longleftrightarrow} \{\mathrm{Inv}_{jt}\} \overset{?}{\Longleftrightarrow} \{\mathrm{Inv}_{it}\} \Longleftrightarrow \{\mathrm{Sav}_{it}\}}$$

for $i, j, k \in \mathbb{N}$ and $t = 1, \ldots T$.

Aim (rephrased)

Test if a change in the saving rate in one country, i say, affects the investment rate of that country, which in turn affects the investment rates of other countries (j, k say), which then feed back to the investment rate of country i.

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We consider a Gaussian random field described by the SARAR(1,1) model

$$\begin{aligned} \mathrm{Inv}_{nt} &= \lambda_0 W_n \,\mathrm{Inv}_{nt} + \beta_0 \mathrm{Sav}_{nt} + c_{n0} + E_{nt}, \\ E_{nt} &= \rho_0 M_n E_{nt} + V_{nt}, \qquad\qquad t = 1, 2, \cdots, T \end{aligned}$$

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- Inv_{nt} is the $n \times 1$ vector of investment rates for all countries
- Sav_{nt} is the $n \times 1$ vector of saving rates
- V_{nt} is an $n \times 1$ vector and each element v_{it} in it is i.i.d across i and t, having Gaussian distribution with zero mean and variance σ_0^2.
- c_{n0} is an $n \times 1$ vector of fixed effects
- we assume $W_n = M_n$ and these matrices control for the spatial relationships (e.g. Rook, Queen, k-nearest neighbours.)
- while the scalars λ_0 and ρ_0 control the spatial dependency.

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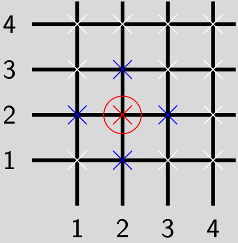

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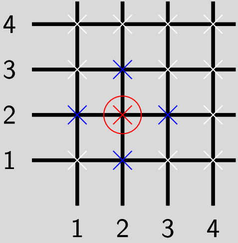

$$W_n = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

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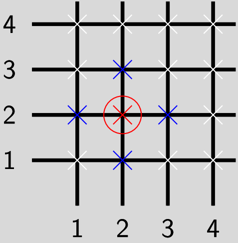

$$\mathrm{W}_n = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where

$$w_{it} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}.$$

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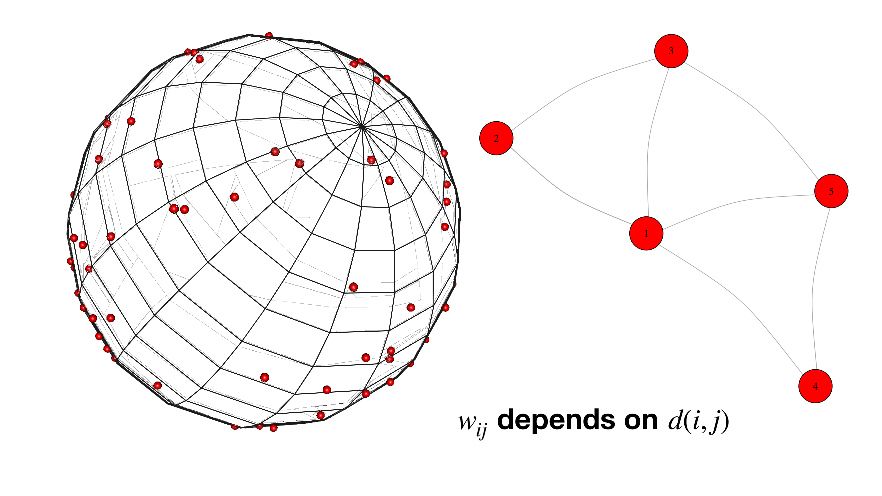

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	Inverse distance			7 nearest neighbours		
	1960-1970	1971-1985	1986-2000	1960-1970	1971-1985	1986-2000
β_0	0.935	0.638	0.356	0.932	0.633	0.368
λ_0	0.004	0.381	0.430	-0.016	0.340	0.437
ρ_0	-0.305	0.334	0.222	-0.219	0.258	0.025

The problem in spatial econometrics terms

The goal is to test for spatial dependency in the investments, thus:

$$\underbrace{\mathcal{H}_0 : \ \lambda_0 = 0}_{\text{no spatial dep.}} \quad \text{vs} \quad \underbrace{\mathcal{H}_1 : \ \lambda_0 > 0}_{\text{pos. spatial dep.}}.$$

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		1960-1970	1971-1985	1986-2000	1960-1970	1971-1985	1986-2000
$\lambda_0 = 0$	ASY	1.0000	0.0116	<u>0.5679</u>	0.9987	0.0130	<u>0.1123</u>
$\rho_0 = 0$	ASY	0.5890	0.2261	0.9578	0.7101	0.3898	0.9998
$\lambda_0 = \rho_0 = 0$	ASY	0.4615	0.0000	0.0000	0.5042	0.0000	0.0000

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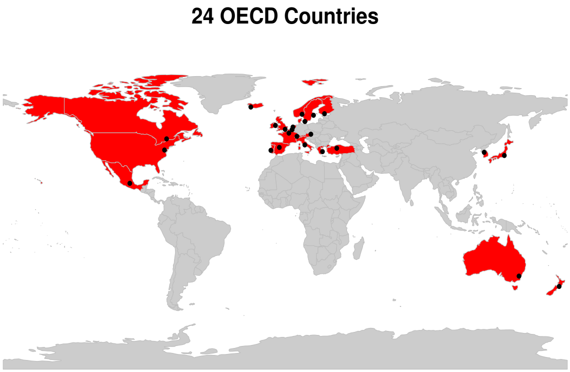
- *Is the use of asymptotics legitimate? Is the asymptotics suffering from size distortion due to the small samples, in both the cross section and in time?*
- *If this is the case, can we rely on small sample techniques?*

The problem in spatial econometrics terms

The problem can be reformulated in terms of network analysis and use graph theory jargon to describe it...

The problem in network analysis terms



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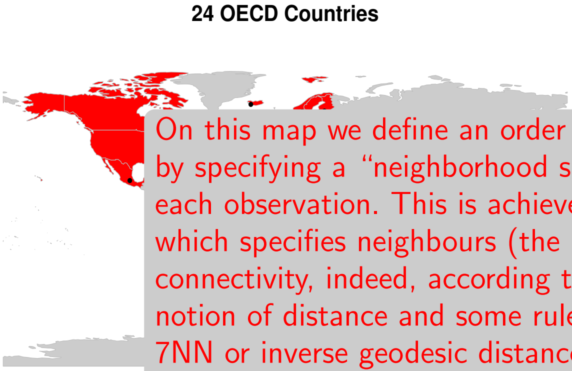

On this map we define an order structure by specifying a “neighborhood set” for each observation. This is achieved by W_n, which specifies neighbours (the connectivity, indeed, according to some notion of distance and some rule, e.g. 7NN or inverse geodesic distance) for each capital city.

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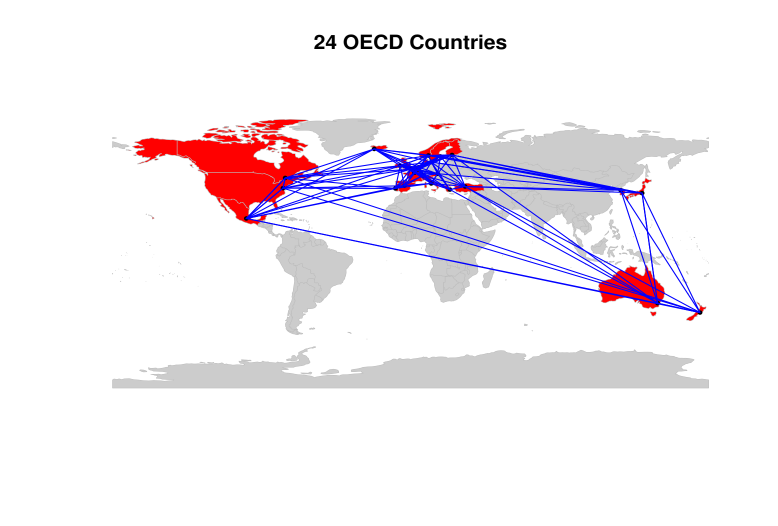
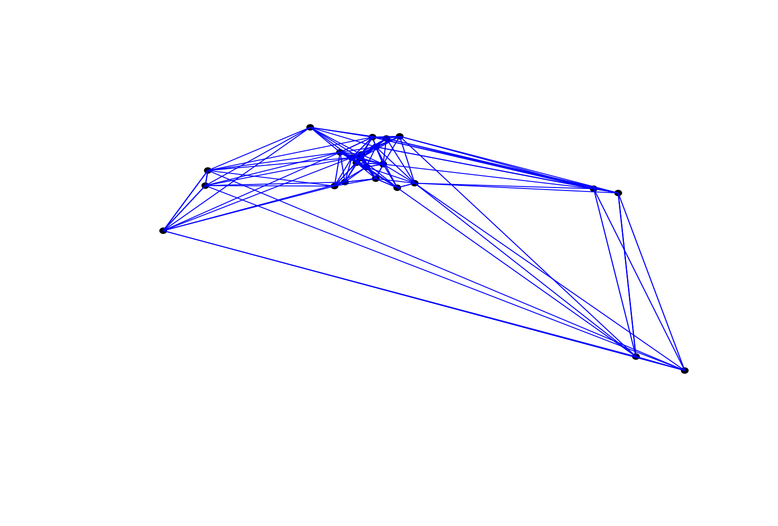


The problem in network analysis terms

For each t, an undirected graph $\mathcal{G}$ describes the network:



The problem in network analysis terms

- W_n is a weighted adjacency matrix. We have a graph $\mathcal{G} = (V, E)$, where V is the set of vertices (capital cities) and E is the set of edges (connections between capital cities). We have $|V| = n$ and W_n is such that if there is an edge $e = (i, j)$ connecting vertices i and j, the entry w_{ij} of W_n represents the weight of the edge—we assume no self-edges, thus $w_{ii} = 0$.

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- we have repeated observations (one for each time point t) of the random field on the graph.

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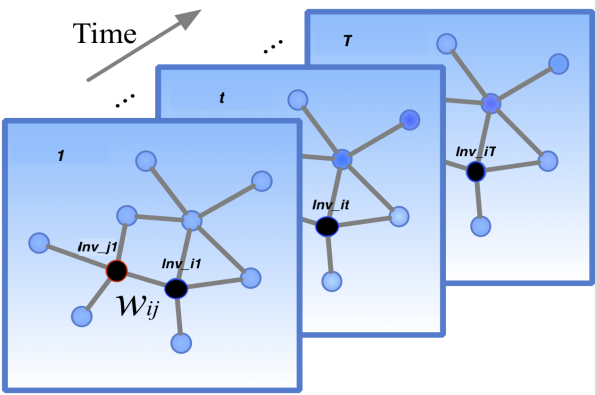

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- Conclusion: take home message

Testing in small samples

General problem: For a given statistics $\hat{\theta}_{n,T}$, tail probabilities

$$P[\hat{\theta}_{n,T} > x]$$

are needed to carry out statistical inference (essentially, tests and confidence intervals).

Unless the (test) statistic $\hat{\theta}_{n,T}$ has a simple form (e.g. linear) and/or the underlying distribution of data has a particular form (e.g. normal), tail probabilities (more generally the whole distribution) cannot be computed exactly.

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$\Rightarrow$ we have to rely on asymptotic approximations

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- *Asymptotic theory*: use of Central Limit Theorem to get a Gaussian approximation in large samples,
- *Resampling techniques*: use of resampling (bootstrap, subsampling) to get an approximation in small samples,
- *Analytical techniques*: use of expansions (Edgeworth, saddlepoint) to get an approximation in small samples.

Edgeworth expansion

Expansions of the density in powers of $n^{-1/2}$, where the leading term is the normal density with absolute error of order $O(n^{-1/2})$—Maclaurin series for the cumulant generating function. Typically, the expansion is computed up to order $O(n^{-1})$

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- they can even become negative in the tails.

$\Rightarrow$ To solve these problems, saddlepoint approximations have been introduced.

Saddlepoint density approx for the mean

As in Daniels (1954), let $X_1, \ldots, X_n$ be **iid** random variables from a distribution F with $M(\eta) = E[e^{\eta} X]$ being the moment generating function such that

$$\mathcal{K}(\eta) = \log M(\eta)$$

is the cumulant generating function. Assume that $E[X_1] = 0$.

Inferential problem

Approximate, for every n, the density f_n (say) of the

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

which is an M-estimator of location.

Saddlepoint techniques for the mean (cont.)

By standard Fourier inversion, the density f_n, at a point α, is obtained as

$$f_n(\alpha) = \frac{n}{2\pi i} \int_{-i\infty}^{i\infty} \exp\{n(\mathcal{K}(\eta) - \eta\alpha)\} d\eta.$$

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The integral is typically not available in a closed form and an approximation is obtained by the method of the steepest descent, which yields the saddlepoint approx $p_n(\alpha)$:

$$f_n(\alpha) = p_n(\alpha)\left\{1 + O\left(n^{-1}\right)\right\},$$

$$p_n(\alpha) = \left[\frac{n}{2\pi \mathcal{K}^{('')}\{\eta(\alpha)\}}\right]^{1/2} \exp\left(n\left[\mathcal{K}\{\eta(\alpha)\} - \eta(\alpha)\alpha\right]\right),$$

and $\eta(\alpha)$ (saddlepoint) is the solution to $\boxed{\mathcal{K}^{(')}(\eta) - \alpha = 0}$.

Remark

The approximation p_n features relative error of order $O(n^{-1})$ over the whole $\mathbb{R}$

Spatial Autoregressive models with covariates (SARAR)

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Q. How about MLE for the SARAR model?

For user-specified spatial matrices W_n and M_n, let's re-consider the model (1) in more general terms:

$$\begin{aligned} Y_{nt} &= \lambda_0 W_n Y_{nt} + X_{nt}\beta_0 + c_{n0} + E_{nt}, \\ E_{nt} &= \rho_0 M_n E_{nt} + V_{nt}, \quad t = 1, ..., T. \end{aligned}$$

where $Y_{nt} = (y_{1t}, y_{2t}, ..., y_{nt})$, X_{nt} is an matrix of non stochastic time-varying regressors, c_{n0} is an $n \times 1$ vector of fixed effects and $V_{nt} = (v_{1t}, v_{2t}, .., v_{nt})'$ are $n \times 1$ vectors and $v_{it} \sim \mathcal{N}(0, \sigma_0^2)$, i.i.d. across i and t (Lee and Yu (JoE, 2010)).

We label this model SARAR(1,1) to emphasize the spatial dependence in both the response variable (Y_{nt}) and in the error term (E_{nt}). The model parameter is

$$\theta_0 := (\lambda_0, \beta_0, \rho_0, \sigma_0^2)$$

and we estimate it using the Gaussian MLE ($\hat{\theta}_{n,T}$).

SARAR (first-order asymptotics)

Let's set $m := n(T-1)$. Moreover, we define $S_n(\lambda) = I_n - \lambda W_n$, and analogously $R_n(\rho) = I_n - \rho M_n$. Then, we introduce

Assumption A. (Lee & Yu)

(i) *The elements $\omega_{n,ij}$ of W_n and the elements $m_{n,ij}$ of M_n are at most of order $\tilde{h}_n^{-1}$, denoted by $O(1/\tilde{h}_n)$, uniformly in all i,j, where the rate sequence $\{\tilde{h}_n\}$ is bounded, and $\tilde{h}_n$ is bounded away from zero for all n. As a normalization, we have $\omega_{n,ii} = m_{n,ii} = 0$, for all i.*

(ii) *n diverges, while $T \geq 2$ and it is finite.*

(iii) *Denote $C_n = \ddot{G}_n - n^{-1}tr(\ddot{G}_n)I_n$ and $D_n = H_n - n^{-1}tr(H_n)I_n$ where*

$$\ddot{G}_n = R_n G_n R_n^{-1} \text{ and } H_n = M_n R_n^{-1}.$$

Then $C_n^s = C_n + C_n^{'}$ and $D_n^s = D_n + D_n^{'}$. The limit of $n^{-2}\left[tr(C_n^s C_n^s)tr(D_n^s D_n^s) - tr^2(C_n^s D_n^s)\right]$ is strictly positive as $n \to \infty$.

SARAR (first-order asymptotics)

Let $\hat{\theta}_{n,T}$ be the Gaussian MLE. Under Assumption A(i)-A(iv), Theorem 1 part(ii) in Lee & Yu shows that

$$\lim_{n\to\infty} \hat{\theta}_{n,T} = \theta_0.$$

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Furthermore, as $n \to \infty$, the MLE $\hat{\theta}_{n,T}$ satisfies

$$\sqrt{m}(\hat{\theta}_{n,T} - \theta_0) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \Sigma_{0,T}^{-1}\right),$$

and $\Sigma_{0,T} = \text{plim}_{n\to\infty}\Sigma_{0,n,T}$. The expression of $\Sigma_{0,n,T}$ is available in Lee & Yu.

Our contribution: Main theoretical result

We derive the saddlepoint density approximation for $q(\hat{\theta}_{n,T})$ at point α, where

$$q : \Theta \to \mathbb{R}$$

and $\theta \in \Theta \subset \mathbb{R}^p$, $p \geq 1$:

$$p_{n,T}(\alpha) = \left[\frac{n}{2\pi \tilde{\mathcal{K}}^{('')}_{n,T}(\eta)} \right]^{1/2} \exp\left\{ n \left[\tilde{\mathcal{K}}_{n,T}(\nu) - \eta\alpha \right] \right\},$$

with (under additional assumptions) relative error of order $O(m^{-1})$, where $\eta := \eta_\alpha$ is the saddlepoint defined by

$$\boxed{\tilde{\mathcal{K}}^{(')}_{n,T}(\eta) - \alpha = 0.}$$

See paper for the explicit expression of $\tilde{\mathcal{K}}_{n,T}$ (approximate c.g.f. of $q(\hat{\theta}_{n,T})$).

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More details are available in Proposition 3 and Proposition 4 of our paper.

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Intuition for tilted-Edgeworth (BNS & Cox, 89, Ch 4) *Let us consider again $\bar{X}$ in the iid. The Edgeworth expansion for the pdf at a point $\alpha \in \mathbb{R}$ is*

$$p^{Edg}(\alpha) = \phi(\alpha)\left\{1 + \frac{1}{\sqrt{n}}\kappa_3(\alpha^3 - 3\alpha) + O(n^{-1})\right\},$$

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Exploit it: **We perform an Esscher's tilting (move the probability mass)** of the underling distribution in such a way that each X_i has expectation equal to α: we compute an accurate Edgeworth! The resulting **tilted-Edgeworth** is the **saddlepoint density approx.**

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We apply this idea to approximate the distribution of $q(\hat{\theta}_{n,T})$.

The problem in optimal transportation terms

The Esscher's tilting can be reformulated in terms of optimal transportation theory...

Measure transportation

Looking at the issue of finding the best way to move given piles of sand to fill up given holes of the same total volume, **Gaspard Monge** (1746-1818) formulated a **mathematical problem** that in modern jargon reads as:

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Let μ and ν denote two probability measures over (for simplicity) $(\mathbb{R}^d, \mathcal{B}^d)$, for $d \geq 1$. Let $c : \mathbb{R}^{2d} \to \mathbb{R}$ be a Borel-measurable cost function such that $c(\mathsf{x}, \mathsf{y})$ represents the cost of transporting x to y. Then, find a measurable transport map $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ that achieves

$$\inf_{\mathcal{T} \in M} \int_{\mathbb{R}^d} c[\mathsf{x}, \mathcal{T}(\mathsf{x})] \mathrm{d}\mu \tag{2}$$

where

$$M := \{\mathcal{T} : \mathsf{X} \to \mathsf{Y}\},$$

with $\mathsf{X} \sim \mu$, $\mathsf{Y} \sim \nu$. The map $\mathcal{T} \# \mu = \nu$ does the push forward of μ to ν.

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$\Rightarrow$ The map solution to (2) is called the optimal transportation map.

Measure transportation

Monge's problem remained open until the 1940s, when it was revisited by **Leonid Vitaliyevitch Kantorovich** (1912-1986; Nobel Prize in Economics in 1975) for the economic problem of optimal allocation of resources; see e.g. Villani (2008), Santambrogio (2015), Galichon (2016).

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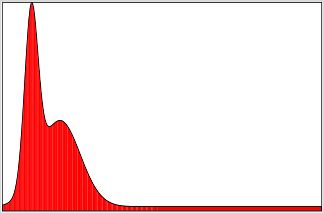
In the Kantorovich primal problem, the objective is to find the optimal transportation plan γ, which solves

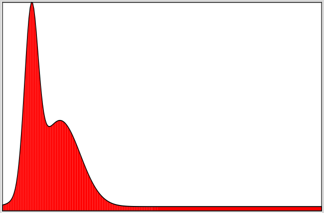
$$\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(\mathrm{x}, \mathrm{y}) \mathrm{d}\gamma(\mathrm{x}, \mathrm{y}), \tag{3}$$

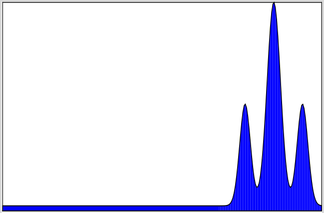
where the infimum is over all coupling $(\mathbf{X}, \mathbf{Y})$ of (μ, ν), belonging to $\Gamma(\mu, \nu)$, the set of probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$, satisfying

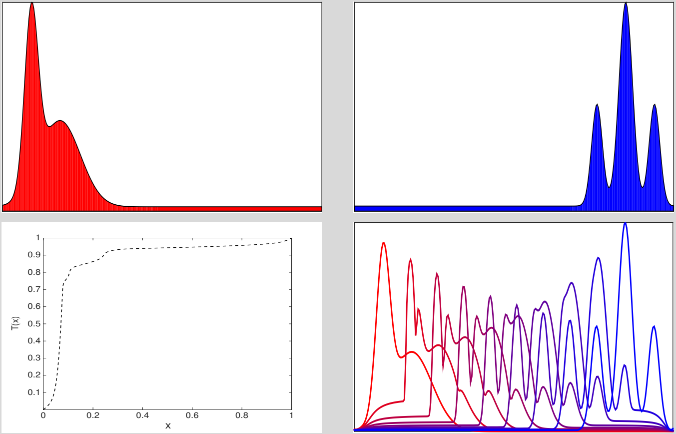
$$\gamma(A \times \mathbb{R}^d) = \mu(A) \text{ and } \gamma(\mathbb{R}^d \times B) = \nu(B),$$

for measurable sets $A, B \subset \mathbb{R}^d$.





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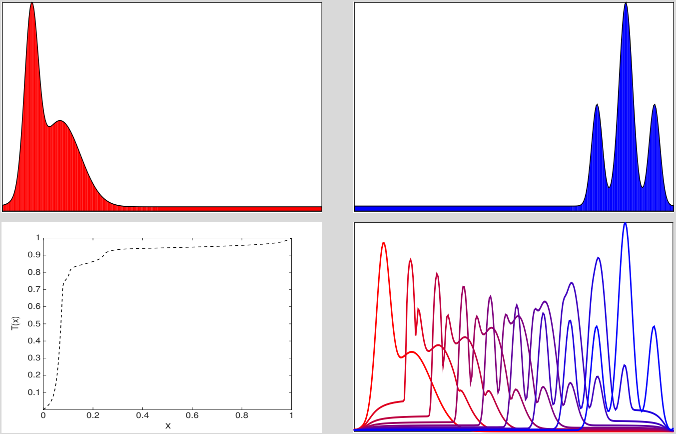
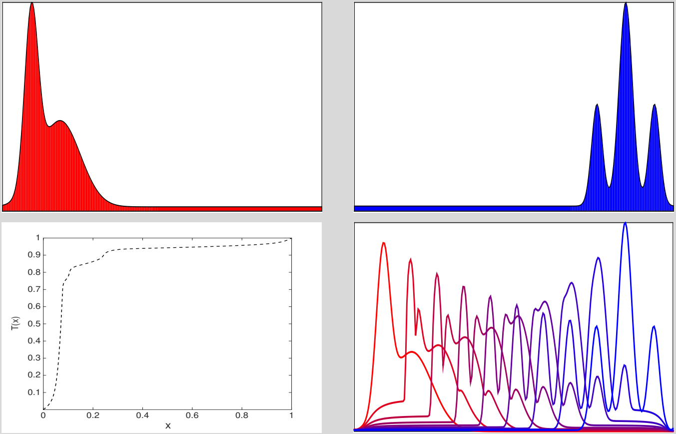


Connection: the saddlepoint density approx hinges on the **Esscher's tilting**, which is a form of **optimal transportation** from μ to ν: in the case of $\bar{X}$, $E_{\nu}[X_i] = \alpha$.
The push-forward $\mathcal{T}$ depends on the Legendre transform of $\mathcal{K}$. Details available in Proposition 2.1 in La Vecchia et al. (2022).



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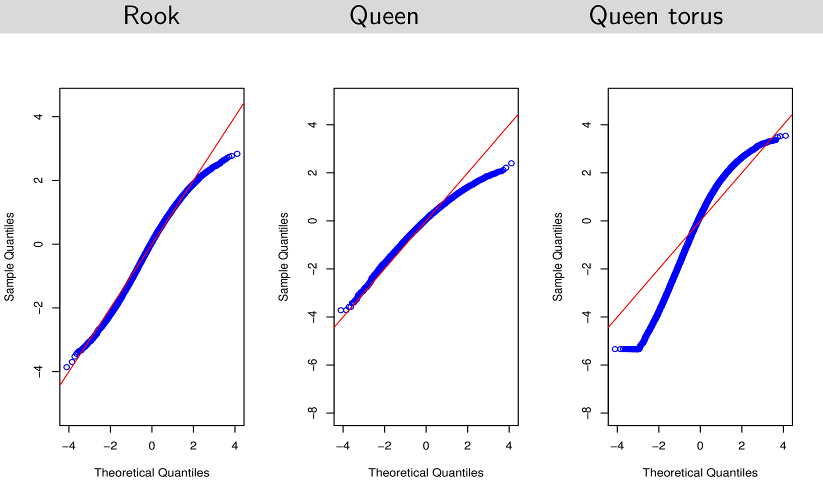
Comparison with asymptotic theory

SAR(1): distribution of $\hat{\lambda}$, for $n = 24$, $T = 2$ and $\lambda_0 = 0.2$



Comparison with asymptotic theory (cont'd)

Is the saddlepoint density approximation doing any better?

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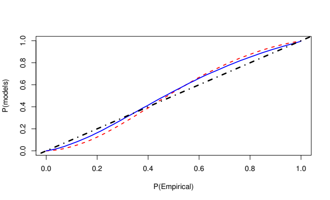
Rook



Queen

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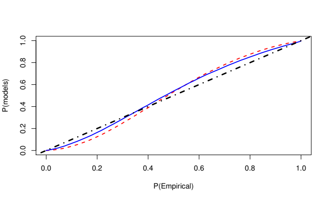
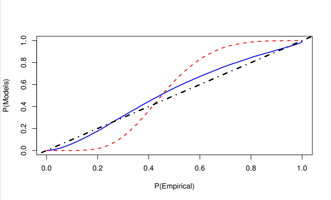


Queen



Comparison with asymptotic theory (cont'd)

Is the saddlepoint density approximation doing any better?

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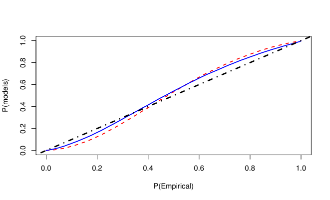

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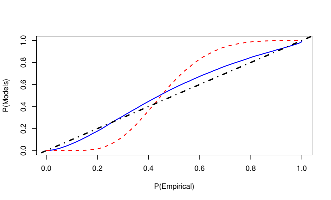
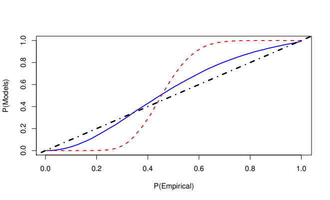


Queen torus



"All together now" ($n = 24$ and W_n is Rook)

We study the level of the test $\mathcal{H}_0 : \lambda_0 = 0$ vs $\mathcal{H}_1 : \lambda_0 > 0$.

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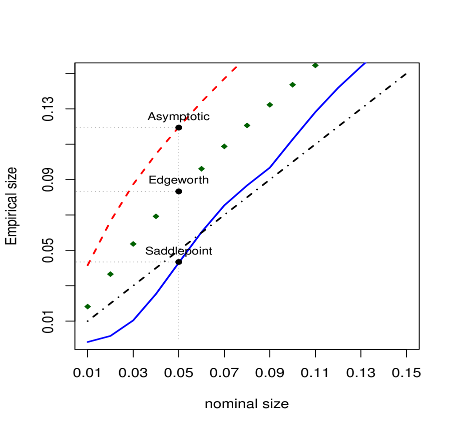

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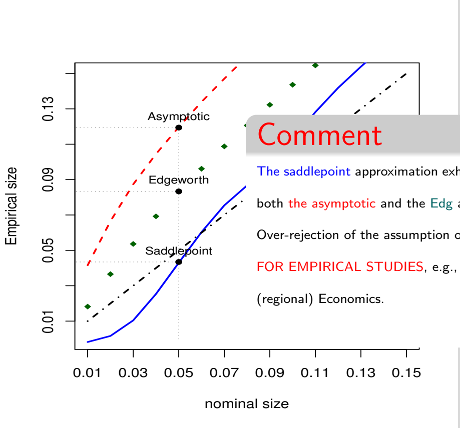

Comment

The saddlepoint approximation exhibits only mild size distortions, whilst both the asymptotic and the Edg approx display **severe size distortion**.

Over-rejection of the assumption of no spatial correlation $\Rightarrow$ ALARM FOR EMPIRICAL STUDIES, e.g., in (applied) Econometrics and/or (regional) Economics.

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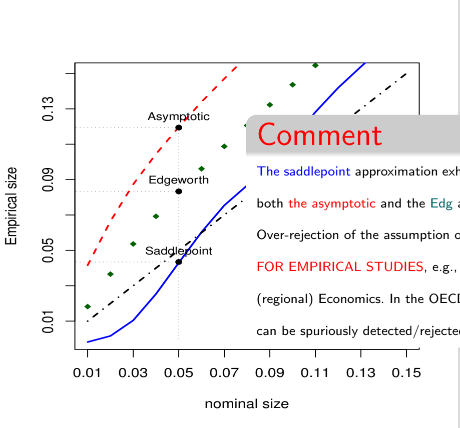

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Test results for the OCDE countries

Report the p-values (testing in presence of **nuisance parameters**, Robinson et al. (2003), AoS and see La Vecchia et al. (2022), Stat. Science) of Saddlepoint (SAD) and first-order asymptotic (ASY) approximation.

		Inverse distance			7 nearest neighbours		
		1960-1970	1971-1985	1986-2000	1960-1970	1971-1985	1986-2000
$\lambda = 0$	*SAD*	1.0000	0.0096	**0.0000**	0.9998	0.2248	**0.0000**
	ASY	1.0000	0.0116	0.5679	0.9987	0.0130	0.1123
$\rho = 0$	*SAD*	0.1134	0.0024	0.1217	0.3232	**0.0403**	0.9993
	ASY	0.5890	0.2261	0.9578	0.7101	0.3898	0.9998
$\lambda = \rho = 0$	*SAD*	0.1414	0.0000	0.0000	0.2603	0.0000	0.0000
	ASY	0.4615	0.0000	0.0000	0.5042	0.0000	0.0000

Test results for the OCDE countries

Remark

- *In the sub-period 86-00, there are large differences between p-values under the two approximations. We find that there is no spatial dependence of investing rates across countries for that period, and vice-versa for the asymptotic approximation. This is in line with the overrejection of the ASY that we find in the Monte Carlo experiments for* λ.

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- *In the sub-period 71-85 under 7NN W_n, the ASY test does not find evidence against $\rho = 0$, while the SAD test rejects this composite hypothesis. Thus, the SAD test indicates a spillover through the contemporary shocks between countries. This spillover goes through the innovations, i.e., through the unexpected part of the model dynamics, a finding not documentable when one relies on the first-order asymptotic theory.*

Take home message

- First-order asymptotics and Edgeworth expansions may deliver poor inference in the setting of dependent processes (like e.g. spatial autoregressive models with and without covariates) in small samples since they exhibit severe absolute and relative distortions in the tail areas.

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Jump to Parametric Bootstrap

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- (Interesting) Connections with the statistical analysis of network data and optimal transportation theory deserve further investigation

Thank you

For questions: davide.lavecchia@unige.ch

Gaussian MLE

The log-likelihood reads as:

$$
\begin{aligned}
\ell_{n,T}(\theta) &= -\frac{n(T-1)}{2}\ln(2\pi\sigma^2) + (T-1)[\ln|S_n(\lambda)| \\
&+ \ln|R_n(\rho)|] - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\tilde{V}'_{nt}(\zeta)\tilde{V}_{nt}(\zeta),
\end{aligned}
$$

where the **transformed errors, which are iid**

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\tilde{V}_{nt}(\zeta) = R_n(\rho)[S_n(\lambda)\tilde{Y}_{nt} - \tilde{X}_{nt}\beta],
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$$\tilde{V}_{nt}(\zeta)=R_n(\rho)[S_n(\lambda)\tilde{Y}_{nt}-\tilde{X}_{nt}\beta],$$

are obtained using the transformed variables and transformed covariates

$$\tilde{Y}_{nt}=Y_{nt}-\sum_{t=1}^{T}Y_{nt}/T \text{ and } \tilde{X}_{nt}=X_{nt}-\sum_{t=1}^{T}X_{nt}/T.$$

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The MLE $\hat{\theta}_{n,T}$ for θ is obtained solving $\hat{\theta}_{n,T} = \arg\max_{\theta\in\Theta}\ell_{n,T}(\theta)$. This is an **M-estimator** related to the log-likelihood score function, which can be written as a quadratic form in $\tilde{V}_{nt}(\zeta)$.

HOA for the Gaussian MLE

Set $d = \dim(\Theta)$, the MLE is the solution to

$$\frac{1}{n}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}(T-1)^{-1}\psi_{i,t,1}(\hat{\theta}_{n,T}), \ldots, \sum_{i=1}^{n}(T-1)^{-1}\psi_{i,t,d}(\hat{\theta}_{n,T})\right)' = 0,$$

with $\psi_{i,t,j}$ represents the j-th component of the likelihood score, at time t in location i. The M-functional ϑ related to the MLE is implicitly defined as the unique functional root of:

$$\mathbb{E}\left\{\sum_{t=1}^{T}(T-1)^{-1}\,\psi_{nt}\left[\vartheta(P_{\theta_0})\right]\right\} = 0, \tag{4}$$

or equivalently via the asymptotic maximization

$$\theta_0 = \arg\max_{\theta\in\Theta}\mathbb{E}[\ell_{n,T}(\theta_0)] = \vartheta(P_{\theta_0})$$

The finite sample version of the M-functional is the M-estimator $\hat{\theta}_{n,T} = \vartheta(P_{n,T})$.

HOA for the Gaussian MLE (cont'd)

Under additional assumptions, the following expansion holds:

$$\vartheta(P_{n,T}) - \vartheta(P_{\theta_0}) = \underbrace{\frac{1}{n}\sum_{i=1}^{n} IF_{i,T}(\psi, P_{\theta_0})}_{\text{1st order}} + \underbrace{\frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{i,j,T}(\psi, P_{\theta_0})}_{\text{2nd order}} + O_P(m^{-3/2}), \tag{5}$$

where, for $M_{i,T}(\psi, P_{\theta_0}) = \mathbb{E}\left[-(T-1)^{-1}\sum_{t=1}^{T} \partial\psi_{i,t}(\theta)/\partial\theta\Big|_{\theta=\theta_0}\right]$, we have

$$IF_{i,T}(\psi, P_{\theta_0}) = M_{i,T}^{-1}(\psi, P_{\theta_0})(T-1)^{-1}\sum_{t=1}^{T}\psi_{i,t}(\theta_0), \tag{6}$$

and (the expression of $\Gamma_{i,j,T}(\psi, P_{\theta_0})$ is provided in the paper)

$$\begin{aligned}
\varphi_{i,j,T}(\psi, P_{\theta_0}) \quad = \quad & IF_{i,T}(\psi, P_{\theta_0}) + IF_{j,T}(\psi, P_{\theta_0}) + M_{i,T}^{-1}(\psi, P_{\theta_0})\Gamma_{i,j,T}(\psi, P_{\theta_0}) \\
& + M_{i,T}^{-1}(\psi, P_{\theta_0})\left\{(T-1)^{-1}\sum_{t=1}^{T}\frac{\partial\psi_{j,t}(\theta)}{\partial\theta}\Big|_{\theta_0} IF_{i,T}(\psi, P_{\theta_0})\right. \\
& \left. + (T-1)^{-1}\sum_{t=1}^{T}\frac{\partial\psi_{i,t}(\theta)}{\partial\theta}\Big|_{\theta=\theta_0} IF_{j,T}(\psi, P_{\theta_0})\right\}
\end{aligned} \tag{7}$$

HOA for the Gaussian MLE (cont'd)

Let q be a function from $\mathbb{R}^d$ to $\mathbb{R}$, which has continuous and nonzero gradient at $\theta = \theta_0$ and continuous second derivative at $\theta = \theta_0$. Then, the following expansion holds:

$$q[\vartheta(P_{n,T})] - q[\vartheta(P_{\theta_0})] = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} h_{i,j,T}\left(\psi, P_{\theta_0}\right) + O_P(m^{-3/2}),$$

where

$$\begin{aligned} h_{i,j,T}\left(\psi, P_{\theta_0}\right) &= g_{i,T}\left(\psi, P_{\theta_0}\right) + g_{j,T}\left(\psi, P_{\theta_0}\right) + \gamma_{i,j,T}(\psi, P_{\theta_0}) \\ &= \frac{1}{2}\left\{IF'_{i,T}(\psi, P_{\theta_0}) + IF'_{j,T}(\psi, P_{\theta_0}) + \varphi'_{i,j,T}(\psi, P_{\theta_0})\right\} \frac{\partial q(\vartheta)}{\partial \vartheta}\Big|_{\theta=\theta_0} \\ &+ \frac{1}{2} IF'_{i,T}(\psi, P_{\theta_0}) \frac{\partial^2 q(\vartheta)}{\partial\vartheta\partial\vartheta'}\Big|_{\theta=\theta_0} IF_{j,T}(\psi, P_{\theta_0}), \end{aligned}$$

with

$$g_{i,T}(\psi, P_{\theta_0}) = \frac{1}{2}\left(IF'_{i,T}(\psi, P_{\theta_0}) \frac{\partial q(\vartheta)}{\partial\vartheta}\Big|_{\theta=\theta_0}\right),$$

$$\gamma_{i,j,T}(\psi, P_{\theta_0}) = \frac{1}{2}\left(\varphi'_{i,j,T}(\psi, P_{\theta_0}) \frac{\partial q(\vartheta)}{\partial\vartheta}\Big|_{\theta=\theta_0} + IF'_{i,T}(\psi, P_{\theta_0}) \frac{\partial^2 q(\vartheta)}{\partial\vartheta\partial\vartheta'}\Big|_{\theta=\theta_0} IF_{j,T}(\psi, P_{\theta_0})\right).$$

HOA for the Gaussian MLE (cont'd)

We derive the Edgeworth expansion for $q[\vartheta(P_{n,T})] - q[\vartheta(P_{\theta_0})]$, working along the same lines as in Bickel et al., 1986, AoS. Our derivation of the saddlepoint density approximation to $f_{n,T}(z)$ is based on the tilted-Edgeworth expansion for U-statistics of order two.

HOA for the Gaussian MLE (cont'd)

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The Edgeworth expansion $\Lambda_m(z)$ for the c.d.f. F_m of $\sigma_{n,T}^{-1}\{q[\vartheta(P_{n,T})] - q[\vartheta(P_{\theta_0})]\}$ is

$$\Lambda_m(z) \quad = \quad \Phi(z) - \phi(z)\left\{ n^{-1/2}\frac{\kappa_{n,T}^{(3)}}{3!}(z^2-1) + n^{-1}\frac{\kappa_{n,T}^{(4)}}{4!}(z^3-3z) + n^{-1}\frac{\kappa_{n,T}^{(3)}}{72}(z^5-10z^2+15z)\right\} \tag{8}$$

where $z \in \mathcal{A}$, $\sigma_{n,T}$ is the standard deviation of $q[\vartheta(P_{n,T})] - q[\vartheta(P_{\theta_0})]$, $\Phi(z)$ and $\phi(z)$ are the c.d.f. and p.d.f of a standard normal r.v. respectively, $\kappa_{n,T}^{(3)}n^{-1/2}$ and $\kappa_{n,T}^{(4)}n^{-1}$ are the approximate third and fourth cumulants of $\sigma_{n,T}^{-1}\{q[\vartheta(P_{n,T})] - q[\vartheta(P_{\theta_0})]\}$, as defined in the paper. Then

$$\sup_z |F_m(z) - \Lambda_m(z)| = o(m^{-1}).$$

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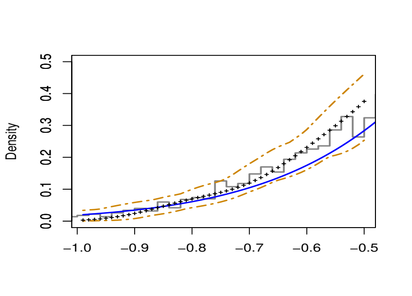
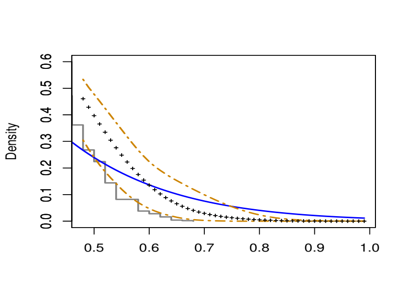
Comparison with Parametric Bootstrap

The parametric bootstrap represents a computer-based competitor: we compare our saddlepoint approximation to the one obtained by bootstrap, for $B = 499$ bootstrap repetitions. We display the functional boxplots (as obtained iterating the procedure 100 times) of the bootstrap approximated density, for sample size $n = 24$ and for W_n is Queen.





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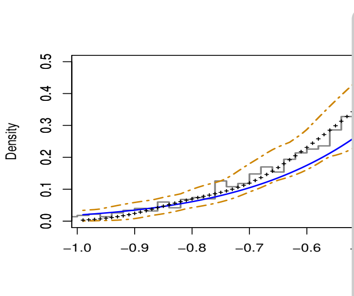

The bootstrap median functional curve is close to the actual density, the range between the quartile curves illustrates that the bootstrap approximation has a variability, which depends on B. However, larger values of B entail bigger computational costs: when $B = 499$, the bootstrap is almost as fast as the saddlepoint density approximation, but for $B = 999$, it is three times slower.

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