GENEVA SCHOOL OF ECONOMICS AND MANAGEMENT
UNIVERSITÉ DE GENÈVE

Inference for multivariate time series models: a measure transportation approach

Davide La Vecchia
(joint work with M. Hallin & H. Liu)

Cyprus, Nov-2021

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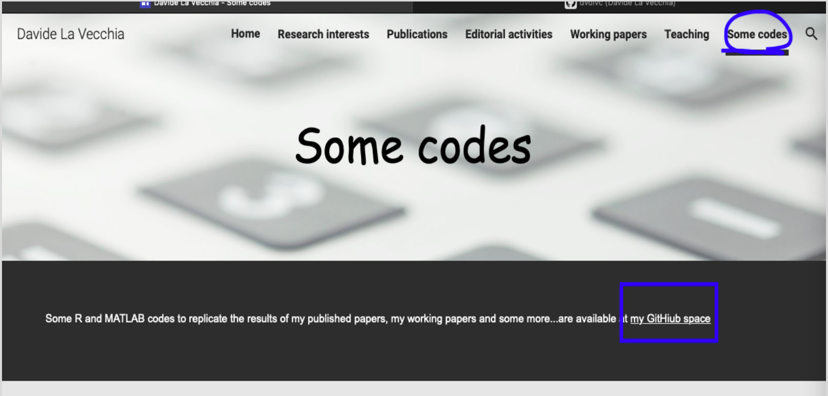
Talk based on

- *(HLLa) Hallin, La Vecchia & Liu, Center-outward R-estimation for semiparametric VARMA models, JASA, accepted in 2020.*
- *(HLLb) Hallin, La Vecchia & Liu, Rank-based testing for semiparametric VAR models: a measure transportation approach, (2021), working paper.*

For R-estimation, https://github.com/HangLiu10/RestVARMA. Other related codes available on my GitHub that you may join via my website:



Outline

- Monge-Kantorovich problem: measure transportation
- Motivation
  - Empirical and theoretical issues
  - Overview of our solution: center-outward ranks and signs estimators and tests
- The method at a glance:
  - Mapping the residuals to the unit ball: the measure transportation approach
  - Center-outward ranks and signs
  - Sketch of the construction for R-estimators and tests
- A real-data example
- Take home message

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Let α *and* β *denote two probability measures over (for simplicity)* $(\mathbb{R}^d, \mathcal{B}^d)$, *for* $d \geq 1$. *Let* $c : \mathbb{R}^{2d} \to \mathbb{R}$ *be a Borel-measurable cost function such that* $c(\mathbf{x}, \mathbf{y})$ *represents the cost of transporting* $\mathbf{x}$ *to* $\mathbf{y}$. *Then, find a measurable transport map* $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ *that achieves*

$$\inf_{\mathcal{T} \in M} \int_{\mathbb{R}^d} c[\mathbf{x}, \mathcal{T}(\mathbf{x})] \mathrm{d}\alpha \qquad (1)$$

where

$$M := \{\mathcal{T} : \mathbf{X} \to \mathbf{Y}\},$$

with $\mathbf{X} \sim \alpha$, $\mathbf{Y} \sim \beta$. *The map* $\mathcal{T}\#\alpha = \beta$ *does the "push forward of* α *to* β*".*

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$\Rightarrow$ The map solution to (1) is called the optimal transportation map.

Monge's problem remained open until the 1940s, when it was revisited by **Leonid Vitaliyevitch Kantorovich** (1912-1986; Nobel Prize in Economics in 1975) for the economic problem of optimal allocation of resources; see e.g. Villani (2008), Santambrogio (2015), Galichon (2016).

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In the Kantorovich primal problem, the objective is to find the optimal transportation plan γ, which solves

$$\inf_{\gamma\in\Gamma(\alpha,\beta)} \int_{\mathbb{R}^d\times\mathbb{R}^d} c(\mathbf{x},\mathbf{y})\mathrm{d}\gamma(\mathbf{x},\mathbf{y}), \tag{2}$$

where the infimum is over all coupling $(\mathbf{X},\mathbf{Y})$ of (α,β), belonging to $\Gamma(\alpha,\beta)$, the set of probability measures γ on $\mathbb{R}^d\times\mathbb{R}^d$, satisfying

$$\gamma(A\times\mathbb{R}^d)=\alpha(A) \text{ and } \gamma(\mathbb{R}^d\times B)=\beta(B),$$

for measurable sets $A,B\subset\mathbb{R}^d$.

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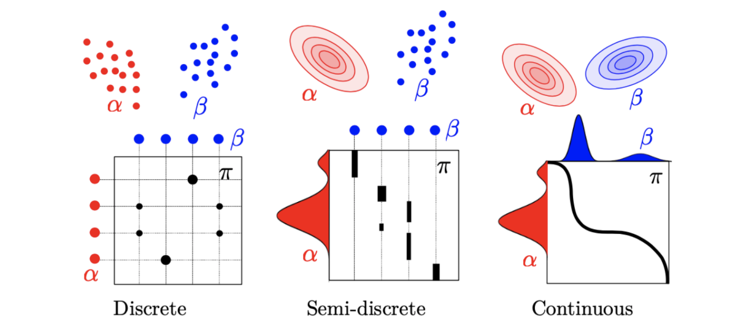
Remark

*Solving Kantorovich problem with the quadratic cost yields L_2-**optimal assignment**. The solution is an optimal transportation plan induced by the optimal map $\mathcal{T}$, which can be expressed as the gradient of a convex function.*

We make use of this theory to transport different types of measures, as depicted in Peyré & Cuturi (2019)



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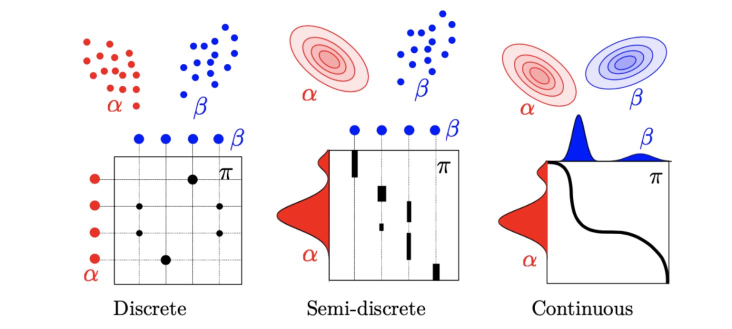
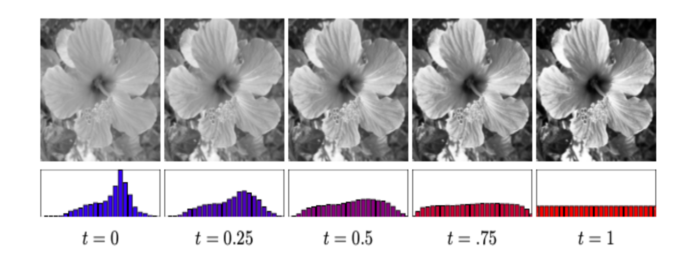


Some examples:

- PDEs: Jacoby equation, Monge-Ampére equation
- Differential geometry: geodesic, curvature, exponential mapping
- Machine learning: image processing, adversarial learning
- Statistics: Wasserstein distance based procedures

- Kids entertainment (actually, image processing in ML)



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Q: How is this related to semiparametric inference for multivariate time series?

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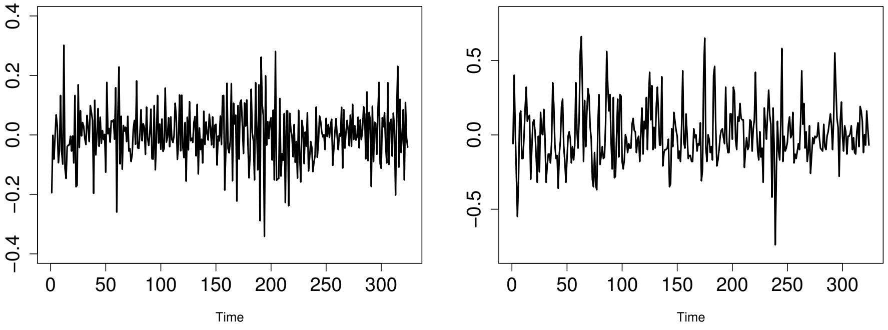
A: In short: measure transportation theory provides us with the tools needed to define ranks and signs in a multivariate setting. Those tools are pivotal for our novel theory.

Example (Ex 1: US Hstarts & Mortg)

We consider the seasonally adjusted series of **monthly** housing starts (`Hstarts`, left panel) and the 30-year conventional mortgage rate (`Mortg`, right panel—no need for seasonal adjustment) **from January 1989 to January 2016**, with a sample size $n = 325$.



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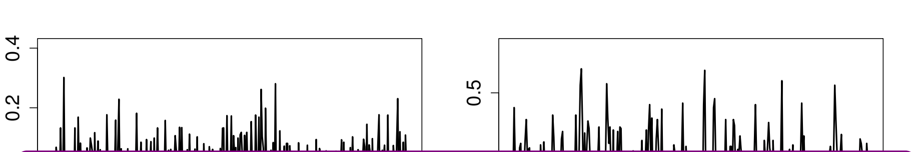

Comment

- Visual inspection suggests the presence of significant auto- and cross-correlations, as expected from macroeconomic theory
- `Hstarts` and `Mortg` series seems to be driven by skew innovations (with large positive values more likely than the negative ones) $\Rightarrow$ No Gaussian or elliptical innovation density!!

Issue (Empirical)

Despite the overwhelming empirical evidence of non-Gaussian and non-symmetric (non-elliptical) distributions, the statistical analysis of univariate and multivariate time series remains very deeply marked by explicit or implicit Gaussian assumptions:

- *Correlogram-based estimation methods*
- *Pseudo-Gaussian tests*
- *Gaussian quasi-likelihood and its possible robustification,*
- *Spectral methods (Whittle)*
- *...*

are ubiquitous in methodological developments as well as in daily practice.

Issue (Theoretical)

In support of the Gaussian procedures, there is a theoretical justification: Gaussian QMLE is asymptotically valid (viz., pseudo-Gaussian tests have correct asymptotic nominal size and Gaussian quasi-maximum-likelihood estimators (QMLEs) are root-n consistent) under a broad range $\mathcal{F}$ of non-Gaussian innovation densities f (typically, under finite fourth-order moments); see, e.g., Gourieroux, Monfort & Trognon (Econometrica, 1984) in the econometric literature...

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....but there exist theoretical aspects which are often overlooked...

Issue (Theoretical (cont'd))

- *Estimation (HLLa)*

  (i) *while achieving efficiency under Gaussian innovations, their asymptotic performance can be quite poor under non-Gaussian ones;*

  (ii) *due to technical reasons (the Fisher consistency requirement), the choice of a quasi-likelihood is always the most pessimistic one: quasi-likelihoods automatically are based on the least favourable innovation density;*

  (iii) *actual fourth-order moments may be infinite.*

- *Testing (HLLb)*

  (iv) *Asymptotic validity of Gaussian test holds pointwise, for any given $f \in \mathcal{F}$, but it fails to hold uniformly over $\mathcal{F}$.*

  (v) *Lack of power under alternative hypothesis*

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- In the univariate setting, rank-based procedures (estimation and testing) offer a solution to all these issues. R-estimation and testing allows for efficiency/power improvements in the context of location (Hodges and Lehmann 1956) and regression models with independent observations (Jurečková 1971, Koul 1971, van Eeden 1972, Jaeckel 1972). Later on, it was extended to autoregressive time series (Koul and Saleh 1993, Koul and Ossiander 1994, Terpstra et al. 2001, Hettmansperger and McKean 2008, Mukherjee and Bai 2002, Andrews 2008, 2012) and non-linear time series (Mukherjee 2007, Andreou and Werker 2015, Hallin and La Vecchia 2017, 2019).

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- In the multivariate setting, notions of multivariate ranks and signs have been proposed in the statistical literature. Among them, the componentwise ranks (Puri and Sen 1971), the spatial ranks (Oja 2010), the depth-based ranks (Liu 1992; Liu and Singh 1993). Those ranks and signs all have their own merits but also some drawbacks, which make them unsuitable for our needs. A notable exception are the Mahalanobis ranks and signs in Hallin and Paindaveine (2002, 2004a), under the restrictive assumption of elliptical innovation density.

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Comment

The problem with other notions of ranks is that they are not distribution-free (spatial ranks) and/or they do not feature essentially maximal ancillarity (depth-based ranks): this implies that semiparametric efficiency cannot be obtained—do not exploit completely spatial info; see Hallin, del Barrio, Cuesta Albertos, and Matran (AoS, 2020) and Hallin, Hlubinka and Hudecova (AoS, 2020).

General problem: R-estimation involves the ranking (an ordering) of some quantities. Unlike the real line, the space $\mathbb{R}^d$, $d \geq 2$ is not canonically ordered. As a consequence basic univariate concepts as

- (in population) quantile and distribution functions,
- (in samples) empirical quantiles (quantile contours), ranks, signs, etc.,

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Contribution

We tackle the general problem. We derive a novel class of semiparametric inference procedures, a family of estimators and tests (select the rank-based estimating function/test statistic 'you like') working for multivariate time series: the procedures are based on center-outward ranks and signs (related to measure transportation).

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Q: *Are our R-estimators and tests really better than Gaussian procedures?*

Example (Ex. 3 : semiparametric 2D time series (HLLa))

Let's consider the bivariate VAR(1) model

$$(\boldsymbol{I}_2 - \boldsymbol{A}L)\,\mathbf{X}_t = \boldsymbol{\epsilon}_t \quad t \in \mathbb{Z}, \qquad \text{where } \boldsymbol{A} = \begin{bmatrix} 0.2 & 0.3 \\ 0.6 & 1.1 \end{bmatrix}$$

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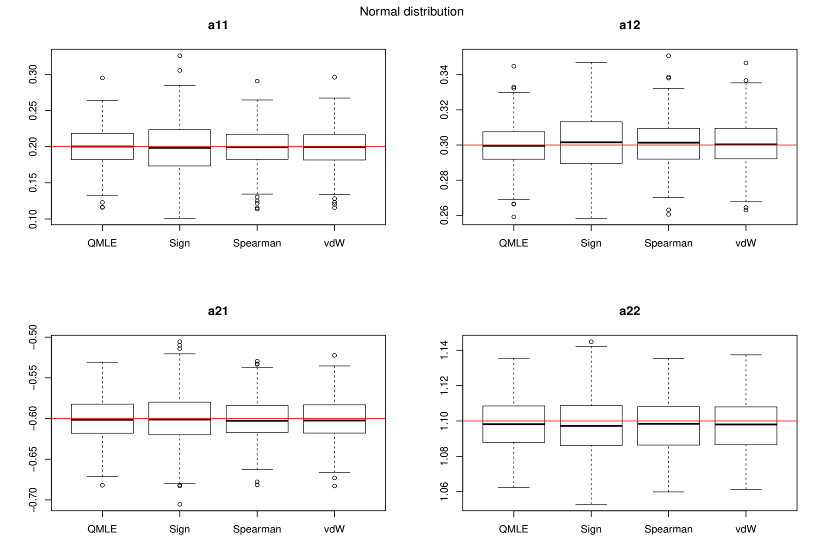
with $\mathbf{X}_t \in \mathbb{R}^2$ and innovation densities:

- a bivariate $\mathcal{N}(\mathbf{0}, \mathbf{I}_2)$ density
- a mixture of Gaussians
- a contaminated model, with additive outliers (AO)

Those densities were used to generate independent innovations $\boldsymbol{\epsilon}_t$ in $N = 300$ replications of length $n = 1000$ of the stationary solution of this VAR(1).

Example (cont'd, Gaussian innovations)

For each replication, **A** was estimated via Gaussian QMLE and **R-estimators** (defined using MK results and selecting a reference density)



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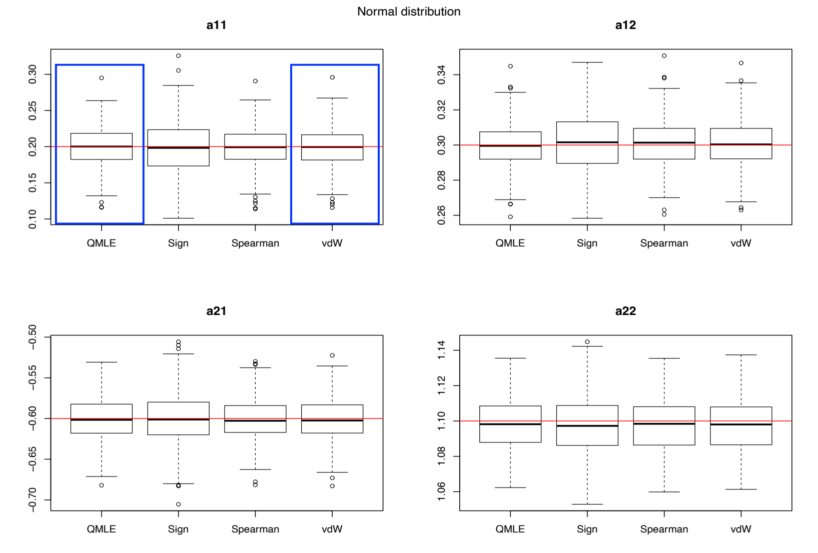
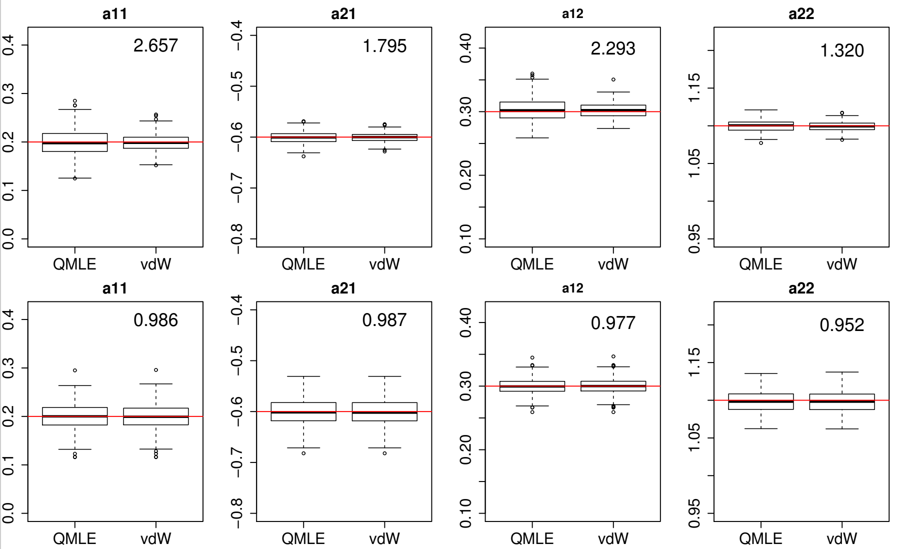
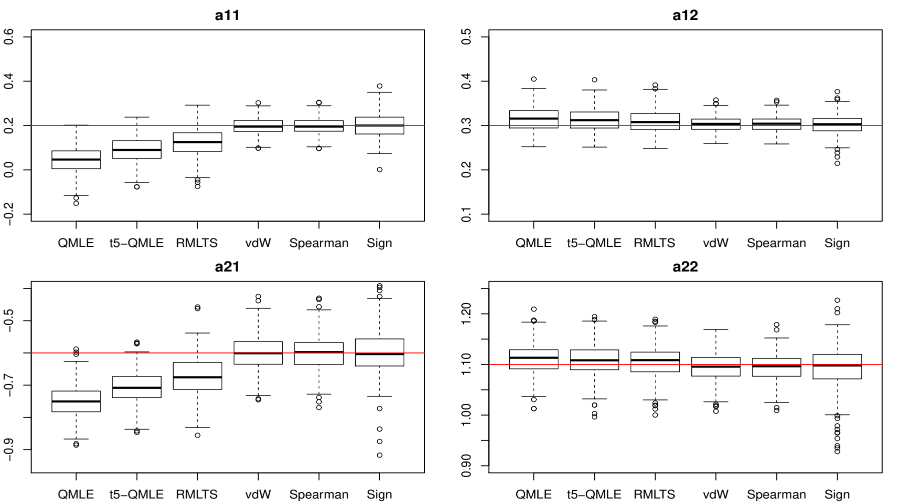


Example (MSE ratio: Mixture innovations (top) & Gaussian (bottom))



Example (cont'd, Outliers)



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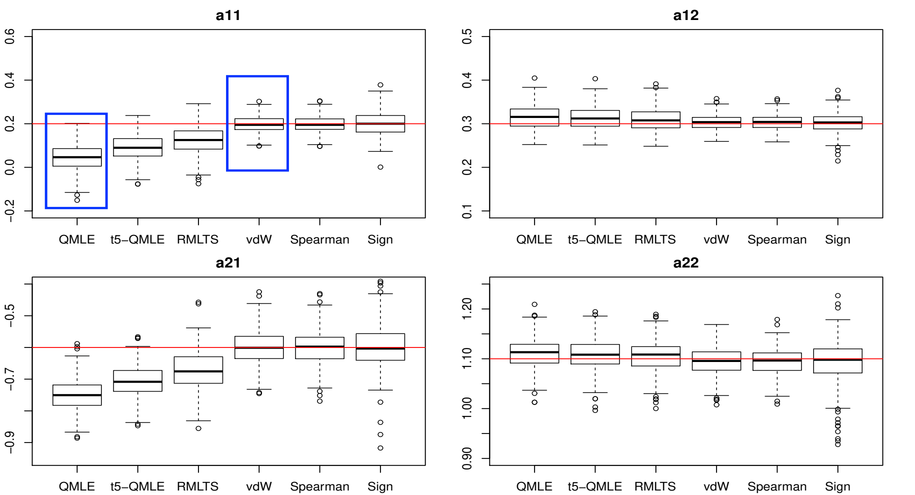

Example (cont'd, graphical intuition)

In these examples the innovation density is not spherical and/or elliptical.

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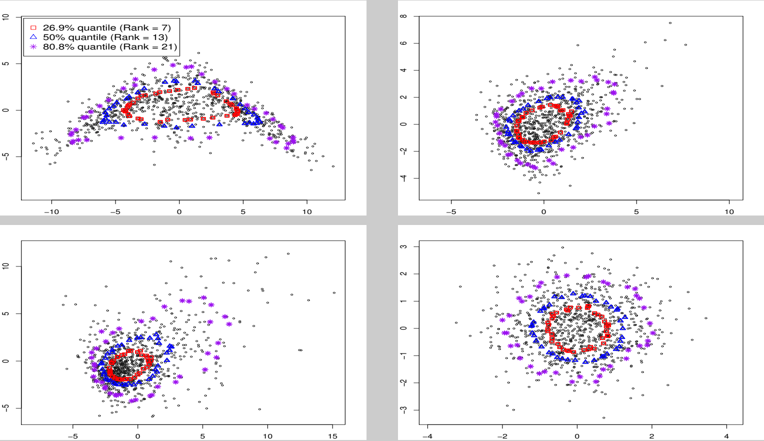
In these examples the innovation density is not spherical and/or elliptical. Empirical center-outward quantile contours computed from $n = 1000$ points drawn from (clockwise) the Gaussian mixture, the skew-normal and skew-t_3 and from a standard multivariate normal.



Example (White noise vs VAR(1) in a 3D model (HLLb))

Rejection frequencies (out of $N = 1000$ replications), under values $\ell\boldsymbol{B}$, $\ell = 0, 1, 1.5$ of the VAR(1) autoregression matrix, for a trivariate model ($d = 3$), and various innovation densities of the Gaussian and rank- and sign-tests of white noise against VAR(1); the sample size is $n = 1400$; the nominal level is $\alpha = 5\%$.

f Test	**0**	$\boldsymbol{B}$	$1.5\boldsymbol{B}$	f Test	**0**	$\boldsymbol{B}$	$1.5\boldsymbol{B}$
Normal				Mixture			
Gaussian	0.044	0.354	0.751	Gaussian	0.053	0.490	0.885
BC vdW	0.047	0.325	0.729	BCvdW	0.056	0.736	0.987
BC Spearman	0.048	0.285	0.696	BC Spearman	0.061	0.735	0.987
Sign	0.043	0.238	0.568	Sign	0.043	0.492	0.898
t_3				Skew-t_3			
Gaussian	0.035	0.376	0.763	Gaussian	0.038	0.412	0.819
BC vdW	0.060	0.464	0.888	BC vdW	0.056	0.759	0.992
BC Spearman	0.046	0.411	0.834	BC Spearman	0.058	0.728	0.990
Sign	0.044	0.422	0.832	Sign	0.052	0.501	0.924
AOs ($\boldsymbol{s} = (6, 6, 6)'$)				AOs ($\boldsymbol{s} = (9, 9, 9)'$)			
Gaussian	0.337	0.378	0.525	Gaussian	0.745	0.768	0.854
BC vdW	0.059	0.215	0.497	BC vdW	0.059	0.223	0.499
BC Spearman	0.053	0.186	0.448	BC Spearman	0.061	0.190	0.432
Sign	0.052	0.162	0.389	Sign	0.063	0.165	0.397

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Denote by

$$\mathcal{E}^{(n)} = \left\{ \mathrm{P}^{(n)}_{f,\boldsymbol{\theta}} \middle| \, f \in \mathcal{F}, \; \boldsymbol{\theta} \in \boldsymbol{\Theta} \right\}$$

a semiparametric model with Euclidean parameter of interest $\boldsymbol{\theta}$ and nuisance f (the density of the iid noise driving the data-generating process, whose CDF is F).

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a semiparametric model with Euclidean parameter of interest $\boldsymbol{\theta}$ and nuisance f (the density of the iid noise driving the data-generating process, whose CDF is F). For $\boldsymbol{X}_t^{(n)} \in \mathbb{R}^d$, $d \geq 1$, let $\boldsymbol{X}^{(n)} := \{\boldsymbol{X}_1^{(n)}, \ldots, \boldsymbol{X}_n^{(n)}\}$ be a finite realization of a *VARMA process* with distribution $\mathrm{P}^{(n)}_{f,\boldsymbol{\theta}}$, so the residual function

$$(\mathbf{X}^{(n)}, \boldsymbol{\theta}) \mapsto \mathbf{Z}^{(n)}(\boldsymbol{\theta})$$

is defined, where $\mathbf{Z}^{(n)}(\boldsymbol{\theta}) := \{\boldsymbol{Z}_1^{(n)}(\boldsymbol{\theta}), \ldots, \boldsymbol{Z}_n^{(n)}(\boldsymbol{\theta})\}$.

Univariate case, $d = 1$. For a fix $\boldsymbol{\theta}$, the residuals are $Z_1^{(n)}(\boldsymbol{\theta}), \ldots, Z_n^{(n)}(\boldsymbol{\theta})$ and the univariate ranks are a mapping of $Z_1^{(n)}, \ldots, Z_n^{(n)}$ to a regular grid on $(0, 1)$

$$\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}.$$

This mapping is related to the definition of the empirical version $F^{(n)}$ of the residuals distribution function F, for $t = 1, \ldots, n$,

$$\boxed{R_t^{(n)} := (n+1)F^{(n)}(Z_t^{(n)}).}$$

The corresponding (in population) distribution function is

$$F \;:\; z \mapsto F(z) := \mathrm{Pr}[Z \leq z]$$

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Remark

F strongly exploits the left-to-right nature of the order relation "$\leq$", a feature that does not make any sense in higher dimension (no left, no right, no up, no down!).

Let's consider the *center-outward* distribution function

$$\mathbf{F}_{\pm} := 2F - 1,$$

which is strictly increasing, with values in $(-1, 1)$, the unit ball $\mathbb{S}_1$ in $\mathbb{R}$.
We consider an empirical version $\mathbf{F}_{\pm}^{(n)}$ of $\mathbf{F}_{\pm}$ defined as the monotone increasing mapping of $Z_1^{(n)}, \ldots, Z_n^{(n)}$ to the regular grid (let $\lfloor n/2 \rfloor =: n_R$)

$$\left\{0, \pm\frac{1}{n_R + 1}, \ldots, \pm\frac{n_R}{n_R + 1})\right\} \quad \text{when } n \text{ is odd } (n = 2n_R + 1),$$

$$\left\{\pm\frac{1}{n_R + 1}, \ldots, \pm\frac{n_R}{n_R + 1}\right\} \quad \text{when } n \text{ is even } (n = 2n_R)$$

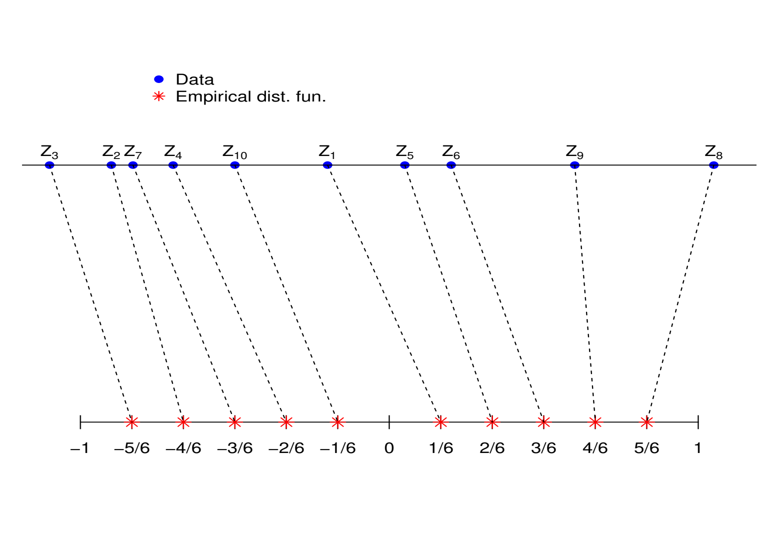
of the interval $(-1, 1)$ (the unit ball in $\mathbb{R}$). So, for $i = 1, \ldots, n$

$$\boxed{\mathbf{S}_i^{(n)} R_{\pm;i}^{(n)} := (n_R + 1)\mathbf{F}_{\pm}^{(n)}(Z_i^{(n)}).}$$

Clearly, F and $\mathbf{F}_{\pm}$ convey the same information and the "center" is

$$\mathrm{Med}(P_{f,\boldsymbol{\theta}}) = F^{-1}(1/2) = \mathbf{F}_{\pm}^{-1}(0).$$

Mapping induced by $\mathbf{F}_{\pm}^{(n)}$:



Remark

One can prove that $\mathbf{F}_{\pm}^{(n)}$ is the solution to on optimal transportation problem. In measure transportation words, $\mathbf{F}_{\pm}^{(n)}$ achieves the **optimal assignment with quadratic cost function** *from the $Z_t^{(n)}$'s to the n points of the grid over $(-1, 1)$. Hallin (2019): $\mathbf{F}_{\pm}$ is the unique gradient of a convex function (Kantorovich potential) pushing F forward to $\mathrm{U}_{(-1,1)}$.*

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$\Rightarrow$ This characterization can be adopted as the definition of $\mathbf{F}_{\pm}$. The huge advantage of this measure transportation-based definition is that it does not involve the canonical ordering of $\mathbb{R}$, and therefore readily extends to $\mathbb{R}^d$, $d \geq 2$.

Multivariate case, $d \geq 2$. Let $\mathbb{S}_d$ denote the open unit ball in $\mathbb{R}^d$. The *center-outward distribution function* $\boldsymbol{F}_{\pm}$ is the unique gradient of a convex function mapping $\mathbb{R}^d$ to $\mathbb{S}_d$ and pushing F forward to the spherical uniform distribution U_d over $\mathbb{S}_d$. This is an optimal mapping in the sense of **quadratic cost function**.

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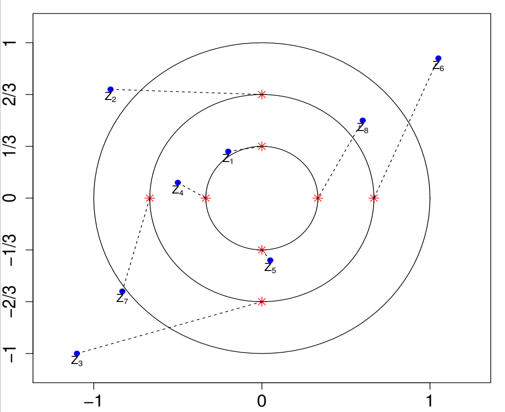

Based on this empirical center-outward distribution function, the *center-outward ranks* are

$$\boxed{R_{\pm,t}^{(n)} := (n_R + 1)\|\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_t^{(n)})\|,} \tag{3}$$

and the *center-outward signs* are

$$\mathbf{S}_{\pm,t}^{(n)} := \mathbf{S}_{\pm,t}^{(n)}(\theta) := \mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_t^{(n)}) I[\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_t^{(n)}) \neq \mathbf{0}] / \|\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_t^{(n)})\|. \tag{4}$$

In the above definitions, we let n factorize into $n = n_R n_S + n_0$, for $n_R, n_S, n_0 \in \mathbb{N}$ and $0 \leq n_0 < \min\{n_R, n_S\}$, where $n_R \to \infty$ and $n_S \to \infty$ as $n \to \infty$.

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Remark

The center-outward distribution function $\boldsymbol{F}_{\pm}$ is a homeomorphism between $\mathbb{R}^d \setminus \boldsymbol{F}_{\pm}^{-1}(\{\boldsymbol{0}\})$ and $\mathbb{S}_d \setminus \{\boldsymbol{0}\}$; see Figalli 2018. This ensures the existence of closed and nested quantile contours, obtained as the images under $\boldsymbol{F}_{\pm}^{-1}$ of nested hyperspheres.

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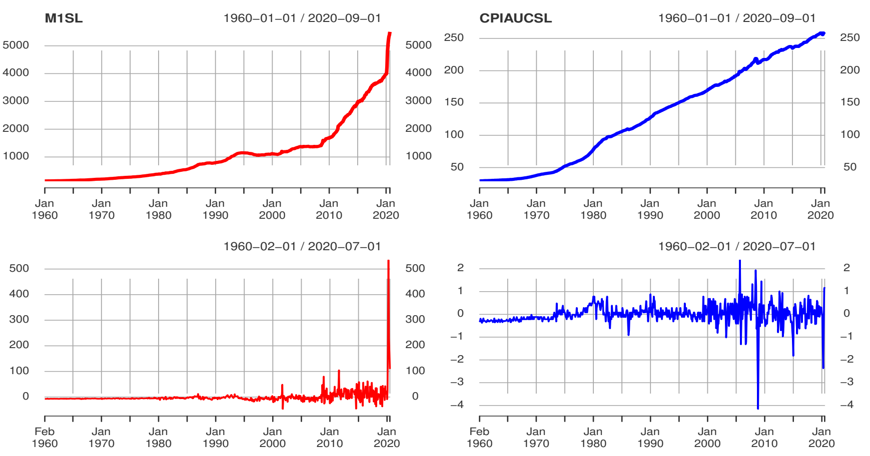
    - *Testing (HLLb): we use the new central sequence to define a class of tests and derive their* **asymptotics** *under the null (level) and under a sequence of local alternatives (power).*

Jump to math detail

Example (Ex 2: US M1SL & CPIAUCSL)

M1SL series in levels (left panel, in Billions of Dollars) and CPIAUCSL series in levels (right panel). **Monthly** records.



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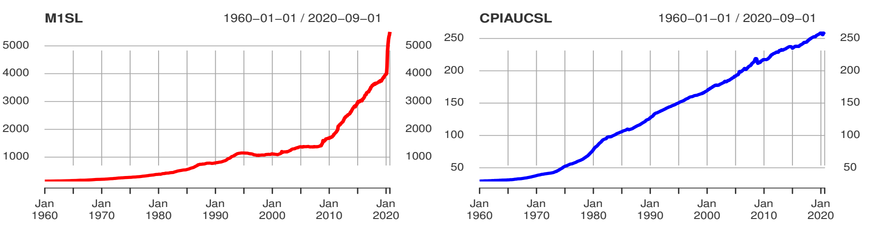

Comment

- The differentiated series seem to be correlated and display common movements (mainly in opposite directions)
- Starting from 2005, the trajectories of the differentiated series look increasingly asymmetric and spiky: March-September-2020 reveals a cluster of outlying values due to COVID-19-related policy decisions. $\Rightarrow$ No Gaussian or elliptical innovation density!!

Example (Ex 2 (cont'd): VAR order selection for M1SL and CPIAUCSL)

Values of test statistics and p-values (in parenthesis) of the Gaussian, vdW, Spearman, sign tests, along with their p-values (in brackets) under the null hypotheses of a VAR(p_0) model ($p_0 = 0, \dots, 7$).

p_0	Gaussian	vdW	Spearman	Sign
0	25.05 (4.91×10^{-5})	224.67 (0)	238.71 (0)	356.90 (0)
1	25.83 (3.42×10^{-5})	16.77 (0.002)	15.42 (0.004)	26.16 (2.94×10^{-5})
2	2.30 (0.681)	62.35 (9.30×10^{-13})	71.57 (1.07×10^{-14})	16.56 (0.002)
3		48.59 (7.18×10^{-10})	56.90 (1.30×10^{-11})	47.99 (9.49×10^{-10})
4		10.74 (0.030)	13.14 (0.011)	26.58 (2.41×10^{-5})
5		19.02 (7.81×10^{-5})	21.37 (2.67×10^{-5})	16.57 (0.002)
6		7.58 (0.108)	24.14 (7.49×10^{-5})	32.58 (1.45×10^{-6})
7			4.27 (0.371)	8.60 (0.072)

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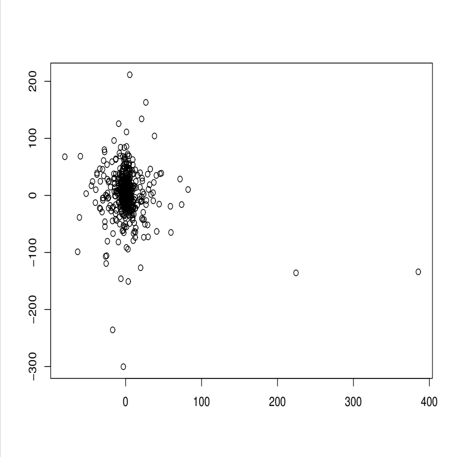
Comment

- sharp contrast between the conclusions of the Gaussian and rank-based methods; 2 lags vs 6-7 lags.
- this is in line with the simulation exercises: our educated guess is that the combination of skewness, kurtosis, and outliers (all due to the COVID-19 periods) are blurring the conclusions of Gaussian tests.

Example (Ex 2 (cont'd): VAR order selection for M1SL and CPIAUCSL)

To see this, we consider the scatter plot of the fitted residuals (VAR(7) model)



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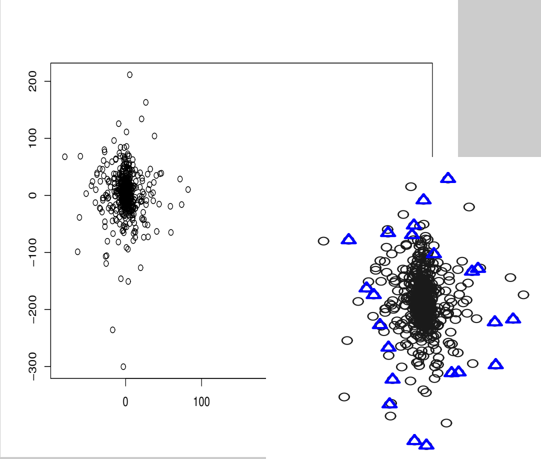

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Thank you

For questions: `davide.lavecchia@unige.ch`

Write for the log-likelihood ratio math detail

$$L^{(n)}_{\boldsymbol{\theta}_0+n^{-1/2}\boldsymbol{\tau}^{(n)}/\boldsymbol{\theta}_0;f} := \log \mathrm{dP}^{(n)}_{\boldsymbol{\theta}_0+n^{-1/2}\boldsymbol{\tau}^{(n)};f}/\mathrm{dP}^{(n)}_{\boldsymbol{\theta}_0;f},$$

where $\boldsymbol{\tau}^{(n)}$ is a bounded sequence of $\mathbb{R}^{p_1 d^2}$. Then,

Proposition

Under some regularity assumptions on the VAR process and on f, for any bounded sequence $\boldsymbol{\tau}^{(n)}$ in $\mathbb{R}^{p_1 d^2}$, under $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_0;f}$, as $n \to \infty$,

$$L^{(n)}_{\boldsymbol{\theta}_0+n^{-1/2}\boldsymbol{\tau}^{(n)}/\boldsymbol{\theta}_0;f} = \boldsymbol{\tau}^{(n)\prime}\boldsymbol{\Delta}^{(n)}_f(\boldsymbol{\theta}_0) - \frac{1}{2}\boldsymbol{\tau}^{(n)\prime}\boldsymbol{\Lambda}_f(\boldsymbol{\theta}_0)\boldsymbol{\tau}^{(n)} + o_{\mathrm{P}}(1), \qquad (5)$$

and $\boldsymbol{\Delta}^{(n)}_f(\boldsymbol{\theta}_0) \to \mathcal{N}(\mathbf{0}, \boldsymbol{\Lambda}_f(\boldsymbol{\theta}_0))$.

Parametrically efficient (in the Hájek-Le Cam asymptotic sense) and distribution-free inference procedures in LAN families (with given f) are possible when the LAN central sequence $\boldsymbol{\Delta}_f^{(n)}(\boldsymbol{\theta}_0)$ can be expressed in terms of signs and ranks.

More precisely when there exists some $\underset{\sim}{\boldsymbol{\Delta}}_f^{(n)}(\boldsymbol{\theta}_0)$ measurable with respects to the ranks and signs of the residuals $\boldsymbol{Z}_t^{(n)}(\boldsymbol{\theta}_0)$ such that

$$\underset{\sim}{\boldsymbol{\Delta}}_f^{(n)}(\boldsymbol{\theta}_0) - \boldsymbol{\Delta}_f^{(n)}(\boldsymbol{\theta}_0) = o_{\mathrm{P}}(1)$$

under $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_0;f}$ as $n \to \infty$.

In mathematical detail:

$$\boldsymbol{\Delta}_f^{(n)}(\boldsymbol{\theta}) := \boldsymbol{M}'_{\boldsymbol{\theta}} \boldsymbol{P}'_{\boldsymbol{\theta}} \boldsymbol{Q}_{\boldsymbol{\theta}}^{(n)\prime} \boldsymbol{\Gamma}_f^{(n)}(\boldsymbol{\theta}), \tag{6}$$

where $\boldsymbol{M}_{\boldsymbol{\theta}}$, $\boldsymbol{P}_{\boldsymbol{\theta}}$, and $\boldsymbol{Q}_{\boldsymbol{\theta}}^{(n)}$ (see the JASA paper for an explicit form) do not depend on f and

$$\boldsymbol{\Gamma}_f^{(n)}(\boldsymbol{\theta}) := \left((n-1)^{1/2}(\mathrm{vec}\boldsymbol{\Gamma}_{1,f}^{(n)}(\boldsymbol{\theta}))', \ldots, (n-i)^{1/2}(\mathrm{vec}\boldsymbol{\Gamma}_{i,f}^{(n)}(\boldsymbol{\theta}))', \ldots, (\mathrm{vec}\boldsymbol{\Gamma}_{n-1,f}^{(n)}(\boldsymbol{\theta}))'\right)'$$

with the so-called *f-cross-covariance matrices*

$$\boxed{\boldsymbol{\Gamma}_{i,f}^{(n)}(\boldsymbol{\theta}) := (n-i)^{-1} \sum_{t=i+1}^{n} \varphi_f(\boldsymbol{Z}_t^{(n)}(\boldsymbol{\theta})) \boldsymbol{Z}_{t-i}^{(n)\prime}(\boldsymbol{\theta}).}$$

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A rank-based, hence distribution-free, f-cross-covariance matrices of the form ($i = 1, \ldots, n-1$) is (for an adequate multivariate reference density f)

$$\boxed{\underset{\sim}{\boldsymbol{\Gamma}}_{i,f}^{(n)}(\boldsymbol{\theta}) := (n-i)^{-1} \sum_{t=i+1}^{n} \boldsymbol{\varphi}_f\left(\boldsymbol{F}_{\pm}^{-1}\left(\frac{R_{\pm,t}^{(n)}}{n_R+1}\boldsymbol{S}_{\pm,t}^{(n)}\right)\right)\boldsymbol{F}_{\pm}^{-1\prime}\left(\frac{R_{\pm,t-i}^{(n)}}{n_R+1}\boldsymbol{S}_{\pm,t-i}^{(n)}\right).}$$

In the JASA paper, we concentrate on rank-based cross-covariance matrices of the form $(i = 1, \ldots, n-1)$ of type:

$$\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i,J_1,J_2}(\boldsymbol{\theta}) := (n-i)^{-1} \sum_{t=i+1}^{n} J_1\left(\frac{R^{(n)}_{\pm,t}}{n_R+1}\right) J_2\left(\frac{R^{(n)}_{\pm,t-i}}{n_R+1}\right) \boldsymbol{S}^{(n)}_{\pm,t}\boldsymbol{S}^{(n)\prime}_{\pm,t-i}$$

to which $\underset{\sim}{\boldsymbol{\Gamma}}^{(n)}_{i,f}(\boldsymbol{\theta})$ reduces, with

$$J_1(u) = \varphi_{\mathfrak{f}}(F^{\star-1}_{d;\mathfrak{f}}(u))$$

and

$$J_2(u) = F^{\star-1}_{d;\mathfrak{f}}(u),$$

in the case of a **spherical reference** f with radial density $\mathfrak{f}$, yielding a rank-based version $\underset{\sim}{\boldsymbol{\Delta}}^{(n)}_{f}$ of $\boldsymbol{\Delta}^{(n)}_{f}$.

Jump Back