CS 615

D

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$|A| = 1$

$$x' = \begin{vmatrix} x & -\sin\theta \\ y & \cos\theta \end{vmatrix}$$

$$x' = \cos\theta\, x - \sin\theta\, y$$

$$y' = \begin{vmatrix} \cos\theta & x \\ \sin\theta & y \end{vmatrix}$$

$$y' = -\sin\theta\, x + \cos\theta\, y$$

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = \cos\theta \frac{\partial}{\partial x'} - \sin\theta \frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = -\sin\theta \frac{\partial}{\partial x'} + \cos\theta \frac{\partial}{\partial y'}$$

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta \frac{\partial}{\partial x'} - \sin\theta \frac{\partial}{\partial y'}\right)\left(\cos\theta \frac{\partial}{\partial x'} - \sin\theta \frac{\partial}{\partial y'}\right)$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\theta \frac{\partial^2}{\partial x'^2} - 2\cos\theta\sin\theta \frac{\partial^2}{\partial x' \partial y'} + \sin^2\theta \frac{\partial^2}{\partial x' \partial y'}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta \frac{\partial^2}{\partial x'^2} + 2\sin\theta\cos\theta \frac{\partial^2}{\partial x' \partial y'} + \cos^2\theta \frac{\partial^2}{\partial x' \partial y'}$$

$$\nabla^2 f = f_{xx} + f_{yy}$$

$$= \cos^2\theta\, f_{x'x'} - 2\cos\theta\sin\theta\, f_{x'y'} + \sin^2\theta\, f_{y'y'} + \sin^2\theta\, f_{x'x'} + 2\cos\theta\sin\theta\, f_{x'y'} + \cos^2\theta\, f_{y'y'}$$

$$\nabla^2 f = f_{x'x'} + f_{y'y'} = f_{xx} + f_{yy}$$