Statistical Theory of Turbulence

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Introduction and Summary of Parts I–IV

Since the time of Osborne Reynolds it has been known that turbulence produces virtual mean stresses which are proportional to the coefficient of correlation between the components of turbulent velocity at a fixed point in two perpendicular directions. The significance of correlation between the velocity of a particle at one time and that of the same particle at a later time, or between simultaneous velocities at two fixed points was discussed in 1921 by the present writer in a theory of "Diffusion by Continuous Movements." The recent improvements in the technique of measuring turbulence have made it possible actually to measure some of the quantities envisaged in the theory and thus to verify some of the relationships then put forward.

The theory has also been developed in several directions which were not originally contemplated. The theory, as originally put forward, provided a method for defining the scale of turbulence when the motion is defined in the Lagrangian manner, and showed how this scale is related to diffusion. It is now shown that it can be applied either to the Lagrangian or to the Eulerian conceptions of fluid flow.

Where turbulence is produced in an air stream with a definite scale by means of a honeycomb or regular screen, either conception can be used to define a length which is related to certain measurable properties of flow and is a definite fraction of the mesh-length, M, of the turbulence-producing screen.

The Lagrangian conception leads to a length l_1, which is analogous to the "Mischungsweg" of Prandtl. Experiments on diffusion behind screens, Part IV, show that $l_1 = 0 \cdot 1$ M. The Eulerian conception leads to a definite length l_2 which might be regarded as the average size of an eddy. Correlation measurements with a hot wire, Part II, show that l_2 is about equal to $0 \cdot 2$ M.

The theory applied in the Eulerian manner to these correlation measurements also contains implicitly a definition of λ, "the average size of the smallest eddies," which are responsible for the dissipation of energy by viscosity.

It is proved that

$$\overline{W} = 15\mu\,(\overline{u^2}/\lambda^2,),$$

where $\overline{u^2}$ is the mean square variation in one component of velocity and $\overline{W}$ is the rate of dissipation of energy. This relationship is verified experimentally (Part II).

The relationship between λ and M is discussed and it is predicted that turbulence in an air stream moving with velocity U will die down so that

$$\frac{U}{\sqrt{\overline{u^2}}} = A\frac{x}{M} + B,$$

provided that the scale of turbulence is determined by the mesh-length M where A is a universal constant and B depends on the choice of the origin taken for x (the down-stream co-ordinate); u is a component of turbulent velocity. This theoretical relationship is compared with results of experiments carried out in wind tunnels in England and in America.

The theory is applied in Part III, to determine the distribution of dissipation across the section of a parallel wall channel (two-dimensional pipe) and it is shown that in the region near the walls turbulent energy is produced more rapidly than it is dissipated. In the central region the reverse is the case.

In Part IV the results of diffusion experiments made in America and at the National Physical Laboratory are discussed and it is shown that a complete set of such measurements can give $\sqrt{\overline{v^2}}$, l_1, and a length λ_η which may be regarded as a measure of the "smallest size of eddy" in the Lagrangian system. λ_η is connected, through the Lagrangian equations of motion, with the average spatial rate of change in pressure, namely

$$\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}}$$

by the formula

$$\sqrt{\overline{\left(\frac{\partial p}{\partial y}\right)^2}} = \sqrt{2}\,\rho\,\frac{\overline{v^2}}{\lambda_\eta}.$$

Finally it is shown that the theory leads to the prediction that λ_η is a constant multiple of λ. The only set of experiments which exists at present gives $\lambda_\eta = 2\lambda$ approximately.

All the above results are subject to the restriction that the "Reynolds Number of Turbulence," namely $l\sqrt{\overline{u^2}}/\nu$, is greater than some number which must be determined by experiment.

Part I

At an early stage in the development of the theory of turbulence the idea arose that turbulent motion consists of eddies of more or less definite range of sizes. This conception combined with the already existing ideas of the Kinetic Theory of Gases led Prandtl and me independently to introduce the length l which is often called a " Mischungsweg " and is analogous to the " mean free path " of the Kinetic Theory. The length l could only be defined in relation to the definite but quite erroneous conception that lumps of air behave like molecules of a gas, preserving their identity till some definite point in their path, when they mix with their surroundings and attain the same velocity and other properties as the mean value of the corresponding property in the neighbourhood. Such a conception must evidently be regarded as a very rough representation of the true state of affairs. If we consider a number of particles or small volumes of fluid starting from some definite level and carrying, say, heat in a direction transverse to the mean stream lines, their average distance from the level at which they started will go on increasing indefinitely so that we can only consider a " Mischungsweg " in relation to some arbitrary time of flight during which we must consider that the particles preserve their individual properties distinct from those of their surroundings. Clearly this is an arbitrary conception and if pursued logically probably leads to a definitely wrong result. The only way in which a small volume can lose its heat is by conductivity to its surroundings. A decrease in molecular conductivity would therefore lead to an increasing time during which the small volume would retain its heat distinct from its surroundings and consequently a decrease in conductivity would necessarily lead to an increase in the " Mischungsweg." In all theories which make use of l it is assumed that l depends only on the *dynamical* conditions of the fluid and is nearly independent of such physical constants as thermal conductivity.

In all applications of " Mischungsweg " theories the length l is considered only in relation to further, more or less arbitrary, assumptions concerning the effect of turbulence on the mean motion or of the mean motion on turbulence. It appears as a fictitious length, the existence of which is detected only by observations of the distribution of mean velocity, temperature, etc.

The difficulty of defining a " Mischungsweg," or scale of turbulence, without recourse to some definite hypothetical physical process which bears no relation to reality does not arise in such applications. The difficulty, however, still exists and it led me, some years ago, to introduce the idea* that the scale of turbulence and its statistical properties in general can be given an exact interpretation by considering the correlation between the velocities at various points of the field at one instant of time or between the velocity of a particle at one instant of time and that of the same particle at some definite time, ξ, later. Some general relations applicable to either of these two aspects of the turbulent field were discussed, and the application of the definitions used in the second of them to diffusion in one dimension was worked out in detail. In this application of the theory the particles are conceived to move irregularly but with continuous velocity, v and $\overline{v^2}$ is supposed to be independent of time. The diffusion of particles starting from a point ($y = 0$) is shown to depend on the correlation R_ξ between the velocity of a particle at any instant and that of the same particle after an interval of time ξ. In continuous turbulent movements R_ξ must be a function of ξ such that $R_\xi = 1$ when $\xi = 0$ and $R_\xi \to 0$ when ξ is large.

If $\overline{Y^2}$ is the mean square of the distance through which the particles have diffused in time t it was proved that

$$\tfrac{1}{2}\frac{d}{dt}\overline{(Y^2)} = \overline{Yv} = \overline{v^2}\int_0^t R_\xi \, d\xi. \tag{1}$$

If the time of diffusion is small so that R_ξ has not departed appreciably from its initial value $1 \cdot 0$, (1) becomes

$$\tfrac{1}{2}\frac{d}{dt}\overline{(Y^2)} = \overline{v^2}t$$

so that

$$\sqrt{\overline{Y^2}} = v't, \tag{2}$$

where $v' = \sqrt{\overline{v^2}}$.

If the diffusion is taking place in a stream of air moving with velocity U and if the spread is observed at a small distance x down-stream from the source $t = x/U$ so that

$$\frac{\sqrt{\overline{Y^2}}}{x} = \frac{v'}{U}. \tag{3}$$

If the irregular motion is of such a character that it is possible to define a time T such that $R_\xi = 0$ for all values of ξ greater than T, so that there is no correlation between the velocities of a particle at the beginning and end of the time interval T, then

$$\overline{Yv} = \overline{v^2}\int_0^T R_\xi d_\xi, \tag{4}$$

* 'Proc. Lond. Math. Soc.,' vol. 20, p. 196 (1921).

$\overline{Yv}$ is therefore constant for all values of $t > T$ in spite of the fact that the value of $\overline{Y^2}$ is continually increasing and $\overline{v^2}$ is constant.

Under these circumstances it is possible to define a length l_1, such that

$$l_1 \sqrt{\overline{v^2}} = \overline{v^2} \int_0^T R_\xi \, d_\xi = \tfrac{1}{2} \frac{d}{dt} (\overline{Y^2}). \tag{5}$$

It will be seen from (5) that the length l_1, defined as

$$l_1 = \sqrt{\overline{v^2}} \int_0^T R_\xi \, d_\xi, \tag{6}$$

bears the same relationship to diffusion by turbulent motion that the mean free path does to molecular diffusion. In this sense it is very similar to the "Mischungsweg," l, but with this important difference that the question of mixture does not arise in defining it.

As is pointed out above, theories which depend essentially on the idea of mixture by subdivision and ultimate molecular diffusion lead to the expectation that the "Mischungsweg" will depend very greatly on the molecular diffusive power of the fluid. In the theory of diffusion by continuous movements the length l_1 bears no relation to any process of mixture, indeed it is equally valid if mixture never takes place. The effect of molecular diffusion would be to prevent the fluid from becoming ever increasingly "spotty," *i.e.*, it would tend to prevent a continual increase in the deviations of the measurable properties of the fluid from their mean value in the neighbourhood. Mixture has no effect in this theory on the diffusive power of turbulent motion.

Correlation in the Turbulent Field when described in the Eulerian Manner

In a loose way it has been thought that the "Mischungsweg" length l is related to, and even may be taken as a measure of the average size of the larger eddies in turbulent flow. It will be noticed that in the original "Mischungsweg" theories, and also in the theory of diffusion by continuous movements, everything is defined in a Lagrangian manner, *i.e.*, by following the paths of particles. When a field of eddying flow is considered as an entity in itself, apart from its effect as a diffusive agent, it is more usual to think in terms of the Eulerian conception of fluid flow, *i.e.*, a field of stream lines conceived to exist in space at one instant of time. Any ideas we may have about "the size of an eddy" are likely to be formulated in the Eulerian system. For this reason it would not be possible to connect directly the size of an eddy, even if it could be accurately defined, with the value of l or of l_1 as defined by (6) in the Lagrangian system. At the same time it seems to be a matter of considerable theoretical interest to investigate the statistical properties of a field of turbulent flow when described in the Eulerian manner, with a view to defining a length which may represent in some definite way the "size of an eddy."

The correlation theory developed in my paper, "Diffusion by Continuous Movements," is equally applicable in this case and may be used to formulate another definition of the scale of turbulence. It is clear that whatever we may mean by the diameter of an eddy a high degree of correlation must exist between the velocities at two points which are close together when compared with this diameter. On the other hand, the correlation is likely to be small between the velocity at two points situated many eddy diameters apart. If, therefore, we imagine that the correlation R_y between the values of u at two points distant y apart in the direction of the y co-ordinate has been determined for various values of y we may plot a curve of R_y against y, and this curve will represent, from the statistical point of view, the distribution of u along the y axis. If R_y falls to zero at, say, $y = Y$, then a length l_2 can be defined such that

$$l_2 = \int_0^\infty R_y\, dy = \int_0^Y R_y\, dy. \tag{7}$$

This length l_2 may be regarded as the analogue in the Eulerian system of l_1, which is defined in the Lagrangian system. It may be taken as a possible definition of the "average size of the eddies."

Experimental Methods for Measuring l_1 and l_2

The compensated hot wire is capable of being used to measure several of the quantities which are necessarily considered in any statistical theory of turbulence.

(1) $\overline{u^2}$ can be measured by means of a hot wire anemometer. If the amplified disturbances are passed through a wire the heat produced can give rise to a current in a thermojunction, which will cause a deflection in a galvanometer proportional to $\overline{u^2}$.

(2) If two hot wires are set up at a distance y apart transverse to a stream of air and the currents produced by variations in u at the two points are sent through the two coils of an electric dynamometer, the resulting deflection will be proportional to $\overline{u_0 u_y}$ where u_0 and u_y are the

velocities at the two points. In this way $R_y = \overline{u_0 u_y}/\overline{u^2}$ can be measured. By repeating these measurements for a number of different distances of separation y between the two hot wires, R_y can be determined for all values of y and hence by integration l_2 can be found. The (R_y, y) curve has already been obtained in certain cases by Messrs. Simmons and Salter at the National Physical Laboratory by this method (*see* fig. 1 of Part II).

Another method is to arrange two equal hot wires on two arms of a Wheatstone bridge thus measuring $\overline{(u_1 - u_0)^2}$. If $\overline{u_0^2}$ and $\overline{u_1^2}$ are measured independently at the two stations, $\overline{u_0 u_1}$ can be found from the relationship

$$\overline{u_0^2} + \overline{u_1^2} - \overline{(u_0 - u_1)^2} = 2\overline{u_0 u_1}. \tag{8}$$

Yet another method due to Prandtl* is to pass the currents from the two hot wires through coils which cause deflections of a spot of light in two directions at right angles to one another. If the two hot wires are identical and so close together that the correlation is nearly 1·0, the spot of light moves over a very elongated elliptic area, the long axis of which is at 45°, to the deflections caused by either of the wires in the absence of disturbances from the other. By measuring the ratio of the principal axes of the elliptical blackened areas produced on a photographic plate by the moving spot of light during a prolonged exposure, it is possible to calculate R_y. This method is specially suitable for measurements when the correlation is very high, *i.e.*, $1 - R_y$ is small. It is not so suitable for small correlations as the electric dynamometer method. Correlation measurements made in this way are shown in fig. 1 of Part III of this paper.

(3) By introducing heat at a concentrated source or a line source in an air stream and measuring the spreading of the heat to leeward of the source it should be possible to measure the quantity $\frac{d}{dt}\overline{Y^2}$ which occurs in (1) and hence to find $\int_0^t R_\xi \, d_\xi$ for various values of t. If this reaches a constant value at some distance down-stream then l_1 can be found. This method was suggested in my paper on "Diffusion by Continuous Movements." Up to the present, however, the theory has only been applied to cases like that of diffusion in the atmosphere† where there is no *a priori* reason to suppose that any definite scale of turbulence can be

* Prandtl and Reichardt, "Einfluss von Wärmeschichtung auf die Eigenschaften einer Turbulenter Strömung." Deutsche Forschung, p. 110 (1934).

† Sutton, 'Proc. Roy. Soc.,' A, vol. 135, p. 143 (1932).

defined. Indeed, Mr. O. G. Sutton has shown that the best representation of diffusion in the air near the ground is obtained by assuming $R_\xi \propto \xi^{-n}$ so that R_ξ does not vanish however great ξ may be. In fact $\int_0^t R_\xi \, d_\xi$ increased continuously with increase in t so that l_1, defined as in equation (6), would have no definite value.

The turbulence which occurs in wind tunnels is produced or controlled by a honeycomb with cells of a definite size. In a wind tunnel, therefore, there is an *a priori* reason why the turbulence might be expected to be of some definite scale. In fact, it might be expected that both l_1 and l_2 would be some definite fraction of the mesh of the cells. Under these circumstances the diffusion equations (1) and (6) reduce to

$$\tfrac{1}{2}\frac{d}{dt}\overline{Y^2} = l_1 v'. \tag{9}$$

This expression is valid when the distance x of the points at which measurements of $\overline{Y^2}$ are made from the point or line source of diffusion is so great that $R_\xi = 0$ where $\xi = x/U$ and U is the mean speed of the air stream.

Application of Diffusion Equation when Turbulence is Decaying

In the air stream behind a grid or honeycomb the turbulence is not constant. It decreases as the distance down-stream increases. The preceding theory cannot then be applied without further investigation.

If $\overline{v^2}$ is considered as a function of t the diffusion equation is

$$\tfrac{1}{2}\frac{d}{dt}\overline{Y^2} = \overline{v_t \int_0^t v_{t-\xi}\, d\xi} \tag{10}$$

for $Y = \int_0^t v_{t-\xi}\, d\xi$ and $\frac{d}{dt}(\overline{Y^2})$ is the rate of increase in $\overline{Y^2}$ at time t after the beginning of the diffusion from a concentrated source.

If ${}_tR_{t-\xi}$ is the coefficient of correlation between the velocity at time t and that at time $t - \xi$, (10) may be written

$$\tfrac{1}{2}\frac{d}{dt}\overline{Y^2} = v'_t \int_0^t v'_{t-\xi}\,({}_tR_{t-\xi})\, d\xi, \tag{11}$$

where v'_t, $v'_{t-\xi}$ are written for $\sqrt{\overline{v^2}_t}$, $\sqrt{\overline{v^2}_{t-\xi}}$.

When the average condition of the turbulent motion is constant with respect to time ${}_tR_{t-\xi}$ is the same as ${}_tR_{t+\xi}$ or R_ξ and is a function of ξ

only, so that (11) is identical with (1). When v' is not constant, it is not possible to proceed beyond (11), but the existing experimental evidence seems to show that turbulent diffusion is proportional to the speed, so that if matter from a concentrated source is diffused over an area down-stream from the source, an increase in the speed of the whole system (*i.e.*, proportional increases in turbulent and mean speed) leaves the distribution of matter in space unchanged (though the absolute concentration is reduced). The condition that this may be so is that ${}_t\mathrm{R}_{t-\xi}$ is a function of η only where

$$d\eta = v'\,d\xi = (v'/\mathrm{U})\,dx \tag{12}$$

and $x = \mathrm{U}t$ is the distance down-stream from the source.

The equation which represents the lateral spread of matter or heat from a concentrated source is therefore

$$\tfrac{1}{2}\frac{\mathrm{U}}{v'}\frac{d}{dx}(\overline{\mathrm{Y}^2}) = \int_0^{\eta_x} \mathrm{R}_\eta\,d\eta, \tag{13}$$

where

$$\eta_x = \int_0^x \frac{v'}{\mathrm{U}}\,dx, \tag{14}$$

and R_η is the correlation between the velocities of a particle at times t_1 and t_2 when $\eta = \int_{t_1}^{t_2} v'\,dt$. If R_η falls to zero at a finite value of η, say $\eta = \eta_1$, and remains zero for all greater values of η, $\int_0^\eta \mathrm{R}_\eta\,d\eta$ is finite. If l_1 be written for $\int_0^{\eta_1} \mathrm{R}_\eta d\eta$ then (11) becomes

$$\tfrac{1}{2}\frac{\mathrm{U}}{v'}\frac{d}{dx}(\overline{\mathrm{Y}^2}) = l_1. \tag{15}$$

This is the same expression as that found for turbulence which is not decaying.*

It is worth noticing that (13) may be expressed in the form

$$\tfrac{1}{2}\frac{d}{d\eta}(\overline{\mathrm{Y}^2}) = \int_0^{\eta_x} \mathrm{R}_\eta\,d\eta. \tag{16}$$

When η_x is small so that $\mathrm{R}_\eta = 1$ over the range from 0 to x, (16) becomes

$$\tfrac{1}{2}\frac{d}{d\eta}(\overline{\mathrm{Y}^2}) = \eta. \tag{17}$$

The integral of (17) is

$$\overline{\mathrm{Y}^2} = \eta^2 \quad \text{or} \quad \sqrt{\overline{\mathrm{Y}^2}} = \eta. \tag{18}$$

* *See* equations (1) and (6).

When the turbulence is constant $\eta = xv'/U$ so that (18) reduces to the previous expression (3) for the spread of matter near a concentrated source. If the turbulence is not constant and if $\overline{Y^2}$ and v'/U are measured at a number of values of x, then both η and $\frac{1}{2}\frac{U}{v'}\frac{d}{dx}(\overline{Y^2})$ can be found. Thus $\int_0^{\eta} R_\eta\, d\eta$ can be plotted against η and R_η can be found graphically from this experimental curve.

Micro-Turbulence and Dissipation of Energy

Besides the motions which are chiefly responsible for the diffusive power of turbulence the whole field may be in a state of micro-turbulence, *i.e.*, there may exist very small-scale eddies which, though they play a very small part in diffusion, yet may be the principal agents in the dissipation of energy. They may also be the principal causes of the effects of turbulence on the boundary layer in wind tunnel work because the absolute magnitude of the space rates of change in pressure may depend on them.

Dissipation of Energy

The rate of dissipation of energy in a fluid at any instant depends only on the viscosity, μ, and on the instantaneous distribution of velocity. If, therefore, the representation of the essential statistical properties of the velocity field can be expressed by the R_y curve and similar correlation curves it must be possible to deduce from them the rate of dissipation of energy. This would in general involve a complicated analysis, but the problem can be much simplified if the field of turbulent flow is assumed to be isotropic.

Isotropic Turbulence

In isotropic turbulence the average value of any function of the velocity components, defined in relation to a given set of axes, is unaltered if the axes of reference are rotated in any manner. That there is a strong tendency to isotropy in turbulent motion has long been known. It has been shown by Fage and Townend,* for instance, that the average values of the three components of velocity in the central region of a pipe of square section are nearly equal to one another. In the atmosphere the same phenomenon has been observed; though, as might be expected, the

* Townend, 'Proc. Roy. Soc.,' A, vol. 145 (1934) (*see* fig. 15, p. 203).

vertical components are smaller near the ground than the horizontal ones, this inequality decreases with height above the ground.*

The assumption of isotropy immediately introduces many simplifications both into the statistical representation of turbulence and into the expression for the mean rate of dissipation of energy.

The general expression for the rate of dissipation is

$$\mathrm{W} = \mu \left\{ 2\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial v}{\partial y}\right)^2} + 2\overline{\left(\frac{\partial w}{\partial z}\right)^2} + \overline{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2} + \overline{\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2} + \overline{\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2} \right\}. \tag{19}$$

Making the assumption that the turbulence is statistically isotropic, the relations

$$\left.\begin{aligned} &\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial y}\right)^2} = \overline{\left(\frac{\partial w}{\partial z}\right)^2} \\ \text{and}\quad &\overline{\left(\frac{\partial u}{\partial y}\right)^2} = \overline{\left(\frac{\partial u}{\partial z}\right)^2} = \overline{\left(\frac{\partial v}{\partial x}\right)^2} = \overline{\left(\frac{\partial v}{\partial z}\right)^2} = \overline{\left(\frac{\partial w}{\partial x}\right)^2} = \overline{\left(\frac{\partial w}{\partial y}\right)^2} \\ \text{and}\quad &\overline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}} = \overline{\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}} = \overline{\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}} \end{aligned}\right\} \tag{20}$$

are immediately obtained so that

$$\frac{\mathrm{W}}{\mu} = 6\overline{\left(\frac{\partial u}{\partial x}\right)^2} + 6\overline{\left(\frac{\partial u}{\partial y}\right)^2} + 6\overline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}. \tag{21}$$

Equation (21) contains three types of term. It will now be shown that these are all related to one another so that if the value of one is known the other two are known.

That relationships can be found between the mean values of squares and products of $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, .., etc., is obvious. The simplest relationship is obtained as follows. The condition of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

so that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = -2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial x}\right). \tag{22}$$

The conditions of statistical isotropy therefore lead to the relationship

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = -2\,\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}} \tag{23}$$

* Taylor, 'Q. J. R. Met. Soc.,' vol. 53, p. 210 (1927).

or

$$\frac{\overline{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}}}{\sqrt{\overline{\left(\dfrac{\partial u}{\partial x}\right)^2}}\sqrt{\overline{\left(\dfrac{\partial v}{\partial y}\right)^2}}} = -\tfrac{1}{2}. \tag{24}$$

In other words there is a definite correlation coefficient between $\partial u/\partial x$ and $\partial v/\partial y$ equal to $-\frac{1}{2}$.

Mean Value of General Quadratic Function of $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$, ..., etc.

Consider the most general possible expression for the mean value of any quadratic function of the nine quantities

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial z}.$$

In general there are 36 possible combinations of 9 things taken 2 at a time. Thus the most general quadratic expression contains 45 terms, namely the 9 squares of the quantities concerned and the 36 combinations of 2.

When the motion is statistically istotropic the 45 terms fall into 10 groups, each of which contains 3 or 6 means which are equal to one another; for example, one group containing 3 equal terms consists of

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2}, \overline{\left(\frac{\partial v}{\partial y}\right)^2}, \text{ and } \overline{\left(\frac{\partial w}{\partial z}\right)^2}.$$

Another containing 6 equal terms consists of

$$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}, \overline{\frac{\partial u}{\partial x}\frac{\partial w}{\partial y}}, \overline{\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}}, \overline{\frac{\partial v}{\partial y}\frac{\partial w}{\partial x}}, \overline{\frac{\partial w}{\partial z}\frac{\partial u}{\partial y}}, \overline{\frac{\partial w}{\partial z}\frac{\partial v}{\partial x}}.$$

The 10 possible independent mean values will be denoted by a_1, a_2, ..., a_{10} according to the scheme laid out in Table I where the top row of the table gives the type term and all other terms of the same type can be obtained by permuting symmetrically the elements of the type term.

The symbol which represents the mean value of any term of a type is given in the second row and the number of independent terms in each group is given in the last row.

In terms of these symbols (21) becomes

$$W/\mu = 6a_1 + 6a_3 + 6a_8. \tag{25}$$

TABLE I

Type	$\overline{\left(\frac{\partial u}{\partial x}\right)^2}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}$	$\overline{\left(\frac{\partial u}{\partial y}\right)^2}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}$	$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$	$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}}$	$\overline{\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}}$	Total
Symbol	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
Number of terms in group	3	6	6	3	6	3	6	3	6	3	45

I now propose to prove that the 10 values, $a_1, a_2, \ldots a_{10}$, are interconnected, so that if the value of any one of them, which is not zero, is known all the rest are known. For this purpose it is necessary to prove 9 linear relationships. One such relationship has already been proved (*see* equation (23)). Expressed in the symbols of Table I (23) may be written

$$a_1 = -2a_6. \tag{26}$$

Further relationships may be obtained as follows. Take any one of the 45 possible terms in the most general quadratic expression involving the 9 partial differentials of (25). Transform u, v, w, x, y, z, by rotation of the axes to u', v', w', x', y', z'. The transformed expression will still be quadratic but will contain terms of other types than the original one. When the mean values of the terms in the transformed expression are considered it is a necessary consequence of the definition of isotropy that the value of each is equal to that of the type term in the group in which they are classed. A simple transformation is obtained by rotating the axes through 45° about the axis of z so that

$$\left.\begin{aligned} \sqrt{2}\,x' &= x+y \\ \sqrt{2}\,y' &= -x+y \\ z' &= z \end{aligned}\right\} \quad \left.\begin{aligned} \sqrt{2}\,u' &= u+v \\ \sqrt{2}\,v' &= -u+v \\ w' &= w \end{aligned}\right\}. \tag{27}$$

Hence

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \tfrac{1}{2}\left(\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial y'}\right) \\ \frac{\partial v}{\partial x} &= \tfrac{1}{2}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial y'}\right) \\ \frac{\partial w}{\partial x} &= \frac{1}{\sqrt{2}}\left(\frac{\partial w'}{\partial x'} - \frac{\partial w'}{\partial y'}\right) \end{aligned}\right| \quad \begin{aligned} \frac{\partial u}{\partial y} &= \tfrac{1}{2}\left(\frac{\partial u'}{\partial x'} - \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial y'}\right) \\ \frac{\partial v}{\partial y} &= \tfrac{1}{2}\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial y'}\right) \\ \frac{\partial w}{\partial y} &= \frac{1}{\sqrt{2}}\left(\frac{\partial w'}{\partial x'} + \frac{\partial w'}{\partial y'}\right) \end{aligned}\ \Bigg|$$

$$\left.\begin{aligned} \frac{\partial u}{\partial z} &= \frac{1}{\sqrt{2}}\left(\frac{\partial u'}{\partial z'} - \frac{\partial v'}{\partial z'}\right) \\ \frac{\partial v}{\partial z} &= \frac{1}{\sqrt{2}}\left(\frac{\partial u'}{\partial z'} + \frac{\partial v'}{\partial z'}\right) \\ \frac{\partial w}{\partial z} &= \frac{\partial w'}{\partial z'} \end{aligned}\right|. \tag{28}$$

Take, for example, $\overline{\left(\frac{\partial u}{\partial x}\right)^2} = a_1$. By squaring the transformed expression for $\left(\frac{\partial u}{\partial x}\right)$, taking the mean value and substituting the symbol for the corresponding type term from Table I it will be found that

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = a_1 = \tfrac{1}{4}\left\{\left(\frac{\partial u'}{\partial x'}\right)^2 + \left(\frac{\partial v'}{\partial x'}\right)^2 + \ldots \frac{\partial u'}{\partial x'}\frac{\partial v'}{\partial x'} \ldots\right\}$$

$$= \tfrac{1}{2}(a_1 + a_3 - a_5 - a_2 + a_6 + a_8 - a_2 - a_5). \quad (29)$$

Similarly

$$\overline{\left(\frac{\partial v}{\partial y}\right)^2} = a_1 = \tfrac{1}{2}(a_1 + a_3 + a_5 + a_2 + a_6 + a_8 + a_2 + a_5), \quad (30)$$

$$\overline{\left(\frac{\partial u}{\partial y}\right)^2} = a_3 = \tfrac{1}{2}(a_1 + a_3 - a_5 + a_2 - a_6 - a_8 + a_2 - a_5), \quad (31)$$

$$\overline{\left(\frac{\partial v}{\partial x}\right)^2} = a_3 = \tfrac{1}{2}(a_1 + a_3 + a_5 - a_2 - a_6 - a_8 - a_2 + a_5). \quad (32)$$

From these equations it will be found that

$$a_2 = a_5 = 0 \quad (33)$$

and

$$a_1 - a_3 - a_6 - a_8 = 0. \quad (34)$$

No further relations can be derived by transforming the type terms corresponding with a_2, a_5, a_6, or a_8. Proceeding to terms involving w or z

$$\overline{\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}} = a_{10} = \tfrac{1}{2}\left\{\left(\frac{\partial u'}{\partial z'}\right)^2 - \left(\frac{\partial v'}{\partial z'}\right)^2\right\} = \tfrac{1}{2}(a_3 - a_3) = 0, \quad (35)$$

$$\overline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}} = a_9 = \frac{1}{2\sqrt{2}}(a_2 - a_9 + a_4 - a_7 + a_7 - a_4 + a_9 - a_2) = 0, \quad (36)$$

$$\overline{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = a_7 = \frac{1}{2\sqrt{2}}(a_2 - a_9 - a_4 + a_7 + a_7 - a_4 - a_9 + a_2). \quad (37)$$

hence since

$$a_2 = a_9 = 0 \qquad a_7(\sqrt{2} - 1) + a_4 = 0 \quad (38)$$

$$\overline{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}} = a_4 = \frac{1}{2\sqrt{2}}(a_2 - a_9 + a_4 - a_7 - a_7 + a_4 - a_9 + a_2),$$

and hence

$$a_4(\sqrt{2} - 1) + a_7 = 0 \quad (39)$$

combining (38) with (39)

$$a_4 = a_7 = 0. \quad (40)$$

Summing up the results so far obtained 6 of the 10 independent types of mean are zero, namely, a_2, a_4, a_5, a_7, a_9, a_{10} and there are two independent relationships between the remaining 4 means, namely,

$$a_1 = -2a_6 \quad \text{and} \quad a_1 - a_3 - a_6 - a_8 = 0.$$

Of these the first depends on incompressibility and isotropy. The second depends only on isotropy.

One further relationship can be obtained by volume integration of the general dissipation expression (19). This integration is well known*: it is

$$\iiint \frac{W}{\mu}\,\partial x\,\partial y\,\partial z = \iiint \left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 \right.$$
$$\left. + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right\} dx\,dy\,dz$$

$$= \iiint (\xi^2 + \eta^2 + \zeta^2)\,dx\,dy\,dz - \iint \frac{\partial}{\partial n}(q^2)\,dS + 2\iint \begin{vmatrix} l & m & n \\ u & v & w \\ \xi & \eta & \zeta \end{vmatrix} dS, \tag{41}$$

where

$$\xi = (\partial w/\partial y) - (\partial v/\partial z), \text{ etc.}$$

and the integrals are taken over the cloud surface S and through its volume. If the closed surface is large compared with the scale of the turbulence the surface integrals are small compared with the volume integrals which may therefore be neglected. Taking the mean value of all the quantities in (41) and expressing the result for isotropic turbulence in terms of the symbols of Table I, (41) becomes

$$\overline{W}/\mu = 6a_1 + 6a_3 + 6a_8 = 6a_3 - 6a_8. \tag{42}$$

Hence

$$a_1 + 2a_8 = 0, \tag{43}$$

solving (26), (34), and (43) it will be seen that

$$a_1 = \tfrac{1}{2}a_3 = -2a_6 = -2a_8. \tag{44}$$

Three obvious corollaries to this result may be noticed:

(1) The correlation coefficient between $\partial u/\partial x$ and $\partial v/\partial y$ is $-\frac{1}{2}$.

(2) The correlation coefficient between $\partial u/\partial y$ and $\partial v/\partial x$ is $-\frac{1}{4}$.

(3) When the mean value of any one of the four possible types of quadratic terms which are not zero is known all the rest are known, so that the mean value of any quadratic function of the space rates of change

* *See*, for instance, the chapter on viscosity in Lamb's "Hydrodynamics."

of velocity is also known. In particular the dissipation may be expressed in terms of $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$. The correct expression is

$$\overline{W}/\mu = 6a_1 + 6a_3 + 6a_8 = 3a_3 + 6a_3 - 1\cdot 5a_3 = 7\cdot 5\,\overline{\left(\frac{\partial u}{\partial y}\right)^2}. \tag{45}$$

Statistical Representation of Microturbulence

The value of $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$ is clearly related to the way in which the value of R_y falls off from its initial value $1\cdot 0$ as y increases from zero. I have proved,* in fact, that

$$R_y = 1 - \frac{1}{\lfloor 2}\frac{y^2}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial y}\right)^2} + \frac{1}{\lfloor 4}\frac{y^4}{\overline{u^2}}\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2} - \ldots \tag{46}$$

The curvature of the R_y curve at $y = 0$ is therefore a measure of $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$ so that

$$\overline{\left(\frac{\partial u}{\partial y}\right)^2} = \overline{2u^2}\,\underset{y\to 0}{\mathrm{Lt}}\left(\frac{1-R_y}{y^2}\right). \tag{47}$$

The significance of the expression (47) can best be appreciated by defining a length λ such that

$$\frac{1}{\lambda^2} = \underset{y\to 0}{\mathrm{Lt}}\left(\frac{1-R_y}{y^2}\right), \tag{48}$$

λ^2 is then a measure of the radius of curvature of the R_y curve at $y = 0$. If the curve is drawn on such a scale that its height is H (corresponding with $R_y = 1$ at $y = 0$) the radius of curvature at $y = 0$ is $\lambda^2/2H$.

Another interpretation of λ may be found by describing the parabola which touches the R_y curve at the origin. This parabola will cut the axis $R_y = 0$ at the point $y = \lambda$. λ may roughly be regarded as a measure of the diameters of the *smallest* eddies which are responsible for the dissipation of energy.

Connection between Dissipation of Energy and Correlation Function R_y

Combining (45) with (47) and (48), the dissipation is related to the correlation function R_y by the equation

$$\overline{W} = 15\,\mu\overline{u^2}\,\underset{y\to}{\mathrm{Lt}}\,\frac{1-R_y}{y^2} \tag{49}$$

* 'Proc. Lond. Math. Soc.,' vol. 20, p. 205, equation (14), (1921).

or

$$\overline{W} = 15\mu\overline{u^2}/\lambda^2. \tag{50}$$

Since $\overline{u^2}$ and R_y can be measured directly by means of the hot wire technique referred to earlier, the relationship (49) can be verified if $\overline{W}$ can be measured by other means. The way in which this can be done and the comparison between this statistical theory and the results of observation will be discussed later. In the meantime it may be noticed that if the Reynolds's stresses in geometrically similar fields of flow are proportional to $\overline{u^2}$ or u'^2, $\overline{W}$ is proportional to u'^3, so that λ is proportional to $(u')^{-\frac{1}{2}}$, and since λ is proportional to the curvature of the R_y curve at $y = 0$ we are led to the prediction that the curvature of the R_y curve at its summit, $y = 0$, will be proportional to $1/u'$. In the limit for very high values of u' the R_y curve may be expected to have a *pointed* top.

Suggestion for Experimental Test in Wind Tunnel of Predicted Correlation Relations

It has been shown how measurements of correlation between the readings of two hot wires at points close together in a transverse section of a pipe or wind tunnel can give the value of $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$. If similar measurements could be made in a line parallel to the main stream, values of $\overline{\left(\frac{\partial u}{\partial x}\right)^2}$ could be obtained in the same way. Equation (44) shows that

$$a_1 = \overline{\left(\frac{\partial u}{\partial x}\right)^2} = \tfrac{1}{2}\,a_3 = \tfrac{1}{2}\overline{\left(\frac{\partial u}{\partial y}\right)^2}$$

and referring to equation (47) which is equally true when x is substituted for y, it will be seen that for the correlation to fall a given amount from its coincidence value $1\cdot0$ the separation of the two hot wires must be $\sqrt{2}$ times as great when one lies up- or down-stream from the other as it is when they lie across the stream.

This is a definite new theoretical prediction which could be tested. If difficulty is found in working with one hot wire down-stream from the other, measurements might be made with the two wires mounted at a fixed distance r apart on a rotating holder, and the variation in the correlation R as the holder is rotated might be found.

The correlation between the values of u observed at two points situated at a short distance, r, apart in a line making an angle θ to the wind direction is*

$$R = 1 - \frac{r^2}{2\overline{u^2}}\overline{\left(\frac{\partial u}{\partial x'}\right)^2}, \tag{51}$$

* Compare equation (47).

where $x' = x\cos\theta + y\sin\theta$. Since

$$\frac{\partial u}{\partial x'} = \cos\theta\,\frac{\partial u}{\partial x} + \sin\theta\,\frac{\partial u}{\partial y}$$

(51) becomes, in the notation of Table I,

$$\mathrm{R} = 1 - \frac{r^2}{2\overline{u^2}}\,(a_1\cos^2\theta + a_3\sin^2\theta + 2a_2\cos\theta\sin\theta). \tag{52}$$

When the turbulence is isotropic this is

$$1 - \mathrm{R} = \frac{r^2}{2\overline{u^2}}\,\overline{\left(\frac{\partial u}{\partial y}\right)^2}\,(\cos^2\theta + \tfrac{1}{2}\sin^2\theta).$$

hence from the definition† of λ

$$1 - \mathrm{R} = \frac{r^2}{\lambda^2}\,(\cos^2\theta + \tfrac{1}{2}\sin^2\theta). \tag{53}$$

It appears, therefore, that $1 - \mathrm{R}$ should vary in the ratio 2 : 1 as the holder is rotated for the maximum to the position to maximum to minimum correlation.

Dimensional Relationship between λ and Scale of Turbulence

It has been shown by v. Karman that if the surface stress in a pipe is expressed in the form $\tau = \rho v_{\times}^2$ then

$$\frac{\mathrm{U}_c - \mathrm{U}}{v_{\times}} = f\left(\frac{r}{a}\right), \tag{54}$$

where U_c is the maximum velocity in the middle of the pipe and U is the velocity at radius r. This relationship is associated with the conception that the Reynolds's stresses are proportional to the squares of the turbulent components of velocity. It seems that the rate of dissipation of energy in such a system must be proportional, so far as changes in linear dimensions, velocity, and density are concerned, to $\rho u'^3/l$, where l is some linear dimension defining the scale of the system. For turbulence produced by geometrically similar boundaries therefore

$$\mathrm{W} = \text{constant}\left(\frac{\rho u'^3}{l}\right) = 15\,\frac{\mu u'^2}{\lambda^2}.$$

For such systems therefore

$$\frac{\lambda^2}{l^2} = \mathrm{C}\,\frac{\nu}{lu'}, \tag{55}$$

† *See* equation (48).

where C depends on the position relative to the solid boundaries of the point at which observations are made and on the element used for defining l.

Application to Air Stream Behind Regular Grids or Honeycombs

Formula (55) is specially well adapted for discussing the decay of turbulence in an air stream behind a grid or honeycomb, because it has been found that at a certain distance down-stream the stream becomes statistically uniform, *i.e.*, the " wind shadow " of the grid disappears and the mean velocity becomes uniform. Under these circumstances it seems that the C of formula (55) must be a constant for any definite form of grid. The researches of Schlichting* have shown that at a short distance behind a cylindrical obstacle the wake assumes a definite form. The width of the wake and the velocity of the air in the middle of the wake depend on the drag coefficient of the obstacle so that obstacles of very varied cross-sections produce identical wakes provided their drag coefficients are identical. For this reason it may be expected that if a regular grid or honeycomb is constructed the scale of the turbulent motion produced by it at any distance down-stream beyond the point where the " wind-shadow " has disappeared will depend only on the form and mesh size of the grid, and not on the cross-section of the bars or sheets from which it is constructed. On the other hand, the velocities of the turbulent components will certainly depend on the drag coefficient of the bars themselves as well as on the distance down-stream from the grid at which measurements are made.

These considerations lead to the prediction that if only one form of mesh is considered, say a square mesh, and if the length l in (55) is taken as M, the mesh length, *i.e.*, the side of each square of the mesh, then the constant C in (55) will be an absolute constant independent of the form of the bars of the grid. We are thus led to a definite expression for λ/M namely,

$$\frac{\lambda}{M} = A\sqrt{\frac{\nu}{Mu'}}, \tag{56}$$

where A is an absolute constant for all grids of a definite type, *e.g.*, for all square-mesh grids or honeycombs.

* 'Ingen. Arch.,' vol. 1, p. 533 (1930).

Prediction of Law of Decay of Turbulence Behind Grids and Honeycombs

We are now in a position to predict the way in which turbulence may be expected to decay when a definite scale has been given to it as the air stream passes through a regular grid or honeycomb.

The rate of loss of kinetic energy of the turbulence per unit volume is

$$-\tfrac{1}{2}\rho U \frac{d}{dx}(\overline{u^2}+\overline{v^2}+\overline{w^2}),$$

which in an isotropic field of turbulence is

$$-\tfrac{3}{2}\rho U \frac{d}{dx}(\overline{u^2}).$$

This must be equal to the rate of dissipation W, so that

$$-\tfrac{3}{2}\rho U \frac{d}{dx}(\overline{u^2}) = 15\mu\frac{\overline{u^2}}{\lambda^2}. \tag{57}$$

This equation is capable of experimental verifications independently of the relationship (56) between λ and M because, as has been shown, λ is connected with R_y through (48) and R_y can be measured instrumentally.

On the other hand, if the relationship (56) between λ and M is assumed to hold it is possible to calculate the law of decay of turbulence. Substituting for λ from (56), (57) becomes

$$-\frac{U}{u'^3}\frac{d}{dx}(u'^2) = \frac{10}{MA^2}, \tag{58}$$

and integrating (58) the following very simple law of decay is predicted,

$$\frac{U}{u'} = \frac{5x}{A^2M} + \text{constant}. \tag{59}$$

This expression should be applicable to all cases where the turbulence is of a definite scale. The linear law of increase in U/u' should therefore apply to all wind tunnels where the scale of turbulence is controlled by a honeycomb or grid, and the value of the constant A determined experimentally, using (59), should be universal for all square grids. Thus, the turbulence behind a square-section honeycomb with long cells should obey the same law of decay as that produced by a square-mesh grid of flat slats or a square-mesh grid of round bars, and the values of A should be identical in all these cases.

For other types of grid or honeycomb, *e.g.*, with hexagonal or triangular cells or a grid of parallel slats or plates, the constant A determined experimentally by applying (59) to observed values of u' at different distances down the air stream might be expected to assume other values.

Expected Limitations to Predicted Linear Law of Decay of Turbulence

This is a very comprehensive prediction, but it is subject to certain limitations. In the first place it cannot be expected to apply when $\mathrm{M}u'/\nu$ is small, for equations of the type (56) are not true when $\mathrm{M}u'/\nu$ is small. In fact if (56) were supposed to hold when $\mathrm{M}u'/v$ is small λ would be greater than M, a condition which is clearly impossible at any rate near the grid.

A second restriction is that the formula cannot be expected to apply in the region immediately behind the grid where the mean velocity is variable, *i.e.*, where the " shadow " of the grid is still distinct. It is found experimentally that when the diameter of the bars of the grid is small compared with M the shadow may extend to as much as 20 M or 30 M behind the grid, but when the bars are as broad as $\frac{1}{4}$ M the shadow disappears a few mesh lengths down-stream from the grid.

A third limitation may be expected to operate when the turbulence is not entirely due to the grid through which the stream passes. If, for instance, a very turbulent stream passes through a grid consisting of thin wires arranged in a large-scale mesh the scale of the turbulence in the stream might hardly be affected by its passage through the grid.

Summary of Results and Theoretical Predictions

(1) When the turbulence of a definite scale is produced or controlled in a stream of air by a honeycomb or grid of regularly spaced bars the scale of turbulence can be investigated in two ways. If the Lagrangian conception of fluid motion is adopted the scale of turbulence can be defined in reference to the correlation R_ξ between the velocity of a particle and that of the same particle at time ξ later. This conception is suited for discussing experiments on diffusion of heat from a concentrated source.

(2) If the diffusive spread of heat or matter from a line source is measured near the source it is proportional to the distance from the source and measures the transverse component of turbulent velocity independently of the scale.

(3) If the diffusive spread is measured at a number of positions extending far down-stream from the source a length l_1 analogous to the mean free path in kinetic theory of gases can be determined. It is anticipated that this will be some definite fraction of the mesh size M of the honeycomb or grid.

(4) Measurements of correlation between simultaneous values of the velocity at points distributed along a line can determine a length l_2 which measures the scale of turbulence from the standpoint of the Eulerian representation of fields of flow.

Both these lengths may be expected to be some definite fraction of the mesh length M, at any rate when the turbulence is not very small.

(5) A third length λ can be defined in relation to the dissipation of energy by the equation $\frac{\overline{W}}{\mu} = 15 \frac{\overline{u^2}}{\lambda^2}$. This length may be taken to represent roughly the diameters of the smallest eddies into which the eddies defined by the scales l_1 or l_2 will break up.

(6) If the rate of dissipation is proportional to the cube of the velocity, as it is where the Reynolds's stresses are proportional to the squares of the turbulent components of velocity, λ is proportional to $\sqrt{\frac{l\nu}{u'}}$.

In turbulence due to a square mesh honeycomb of mesh length M, $\frac{\lambda}{M} = A\sqrt{\frac{\nu}{Mu'}}$, where A is a constant. This formula is inapplicable when Mu'/ν is small.

(7) Using this value for λ it is shown that the law of decay of turbulence is such that U/u' increases linearly with x in accordance with equation (59).

(8) λ is also directly connected with the correlation between simultaneous measurements of velocity at fixed points separated by a small distance. This correlation can be measured by suitable apparatus so that the theory can be verified experimentally.

(9) In isotropic turbulence the mean value of any quadratic expression of the space rates of change in the velocity is known when the mean value of any one of the terms in it which is not zero is known. This leads to the prediction, which might be verified experimentally, that if the correlation between a component of velocity at a fixed point O and that at a neighbouring variable point P is measured, the surfaces of equal correlation are prolate spheroids with P as centre, the long axis is $\sqrt{2}$ times the equatorial axis and is directed in the direction in which the velocity component is measured. This statement is identical in substance though not in form with that given on p. 439.

Concluding Remarks

Of these results and predictions (1), (2) and (3) are substantially identical with the conclusions put forward in 1921 in my paper, "Diffusion by Continuous Movements," where the suggestion that the diffusive power of turbulence should be used for the purpose of measuring the scale of turbulence and the turbulent components was first made. Recently experiments of this nature have been made by C. B. Schubauer* and (2) has been verified, as will be shown in Part IV of the present paper. Mr. Schubauer, however, worked quite independently of my previous work and indeed gives an empirical explanation of his experimental results. Conclusions (4) to (9) are, I believe, new. It will be shown in Part II that all these results, except (9), have now been verified experimentally and shown to be true. Experimental work is now in hand to test the truth of (9).

Statistical Theory of Turbulence—II

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Measurements of Correlation in the Eulerian Representation of Turbulent Flow

The methods described in Part I have been used by Mr. L. F. G. Simmons, of the National Physical Laboratory, to find experimentally the correlation between the turbulent components of velocity u_0 and u_y at two points distant y apart in a direction transverse to the stream. The measurements were made at mean speed $U = 25$ feet per second in a wind tunnel behind a honeycomb with 0·9-inch square mesh. The results are shown in fig. 1 where the ordinates are $R_y = \dfrac{\overline{u_0 u_y}}{\overline{u^2}}$ and the abscissae are the corresponding values of y. It will be seen that the R_y curve is apparently rounded at the top and that R_y falls to 0·08 at $y = 0·38$ inches. No measurements were made beyond this point, but extrapolation seems to show that $R_y = 0$ when y is about 0·5 inches, *i.e.*, when y is slightly greater than $\frac{1}{2}$M.

* 'Rep. Nat. Adv. Ctee. Aero., Wash.,' No. 524 (1935).