Eddy Viscosity in Two and Three Dimensions

ROBERT H. KRAICHNAN

Dublin, New Hampshire 03444

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ABSTRACT

The test-field model for isotropic turbulence is used to examine the effective eddy viscosity acting on wavenumbers $<k_m$ due to interactions with subgrid-scale wavenumbers, defined as wavenumbers $>k_m$. In both two and three dimensions, the effective eddy viscosity for $k \ll k_m$ is independent of k and of local spectrum shape. In two dimensions, this asymptotic eddy viscosity is negative. The physical mechanism responsible for the negative eddy viscosity is the interaction of large-spatial-scale straining fields with the secondary flow associated with small-spatial-scale vorticity fluctuations. This process is examined without appeal to turbulence approximations. For $k_m - k \ll k_m$, the effective eddy viscosity rises sharply to a cusp at $k = k_m$ if k_m lies in a long energy-transferring inertial range in either two or three dimensions or in a long enstrophy-transferring inertial range in two dimensions. The cusp behavior is associated with a diffusion in wavenumber due to random straining, by large spatial scales, of structures with wavenumber close to k_m. This behavior makes the use of a k-independent eddy viscosity substantially inaccurate for the three-dimensional inertial range. In the two-dimensional enstrophy inertial range, the cusp region contributes most of the enstrophy transfer across k_m. The transfer function is squeezed into a region about k_m whose width is of order k_0, where k_0 is a characteristic wavenumber at the bottom of the enstrophy range. If $k_m \gg k_0$, the shape of the transfer function does not have a universal form but instead depends on the spectrum shape near k_0. Representation of this transfer by an eddy viscosity seems highly unjustified.

1. Introduction

This paper is intended to illustrate some limitations on the use of an eddy viscosity to represent the dynamical effects of small-scale turbulence, especially in two-dimensional flows. Our principal tool will be the approximation of turbulent energy transfer by closure formulas of the direct-interaction family.

The basic idea of eddy viscosity is that scales of motion of given size are acted on by smaller scales as if the latter were an augmentation of the equilibrium thermal agitation. Throughout the history of the study of turbulence, this concept has been highly useful in visualizing and parameterizing turbulent transport processes and the passage of turbulent energy between different scales. With the advent of large-scale computer simulations of flows, eddy-viscosity parameters have been used to represent the effects of subgrid-scale motions. Parameterization of subgrid turbulence is discussed, for example, by Smagorinsky (1963) and Leith (1972).

In three-dimensional flow, the analogy between subgrid scales of turbulence and thermal agitation is flawed because the motion displays a continuous distribution of scale sizes. In typical laboratory and geophysical flows, the time and space scales of almost all the thermal agitation are well separated from the smallest hydrodynamic scales which are appreciably excited. It is this which permits representation of molecular viscous and diffusive effects by local differential operators. In contrast, the division of the hydrodynamic motion into subgrid scales and explicit scales is arbitrary.[1] The largest subgrid scales are as large as the smallest explicit scales, and this means that their interaction cannot be correctly described by a simple operator of the form $\nu_{\text{eddy}}\nabla^2$ (Corrsin, 1974). In two-dimensional flow, the eddy-viscosity concept meets with additional and more severe difficulties, connected with the reverse flow of kinetic energy from small scales to large scales (Fjørtoft, 1953). Molecular viscosity does not exist in a strictly two-dimensional fluid (Dorfman and Cohen, 1970). There is no transformation of hydrodynamic energy into microscale thermal agitation, and the thermal equilibrium of such a fluid is itself anomalous (Kraichnan, 1975b). In practice, we are not interested in two-dimensional fluids but rather in the two-dimensional macroscopic hydrodynamics of a three-dimensional fluid. Nevertheless, the non-existence of molecular viscosity in two dimensions correctly suggests a severe breakdown of the eddy-viscosity representation of subgrid scales. In the present paper, we shall analyze in some detail the dynamics of interaction between

[1] We shall use the terms explicit scales and explicit wavenumbers to denote those scales and wavenumbers which are not included in the subgrid-scale motion.

explicit and subgrid scales in two dimensions. In order to clarify the phenomena, we shall also treat the subgrid-scale representation of a passive scalar field advected by turbulence. Some of the difficulties with eddy viscosity in two dimensions have been discussed previously by Leith (1971, 1972).

We have tried to include some background for the present work by giving, in Section 2, a fairly detailed review of the turbulence approximations which are used. The references cited therein give more information, including the results of numerical tests of the approximations. The reader who wants a minimum of involvement with the mathematical development can start with Section 7, which refers back to key equations in the preceding text. Section 5, which treats the physical mechanism of the negative eddy viscosity encountered in two-dimensional flow, is largely self-contained and can be read without the analysis which goes before it.

With the exception of Section 5, the analysis is confined to the effects of subgrid scales on energy and enstrophy transfer in isotropic turbulence. Thus the results are not directly applicable to the use of an eddy viscosity to represent subgrid scales in computer simulation of a single flow, which is locally anisotropic everywhere. However, the difficulties with eddy viscosity which we bring out reflect physical mechanisms which can be expected to operate in general.

2. Energy-transfer approximations

The direct-interaction approximation (Kraichnan, 1959) provides an economical and internally consistent analytical framework in which to investigate eddy-viscosity concepts. An advantage over simpler phenomenological formulations is that it automatically includes the nonlocalness effects described in Section 1 and applies to inhomogeneous, anisotropic flows as well as to homogeneous flows. The direct-interaction approximation (DIA) appears to be the simplest turbulence approximation with these properties that consistently handles turbulence statistics at the level of the space-time velocity covariance. The approximation is invariant to Fourier transformation of the velocity field; it can be derived and manipulated either in x space or wavevector space. The DIA leads also to a family of related approximations of similar structure. In this section, we shall present some basic equations of the DIA itself and of one relative, the test-field model (TFM). We shall deal with homogeneous turbulence, for which it is more efficient to carry through the analysis in wavevector space.

The DIA equations can be derived in several ways (Kraichnan, 1959; Herring and Kraichnan, 1972). What follows now is intended as a plausibility argument rather than a derivation. Consider incompressible Navier-Stokes flow in a box of side L with cyclic boundary conditions. If the velocity field $\tilde{\mathbf{u}}(\mathbf{x},t)$ is expanded in the Fourier modes allowed by these boundary conditions,

$$\tilde{\mathbf{u}}(\mathbf{x},t)=\sum_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{x})\mathbf{u}(\mathbf{k},t), \tag{2.1}$$

then the Navier-Stokes equation can be written in the form

$$(\partial/\partial t+\nu k^2)u_i(\mathbf{k},t) = -ik_m P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p},t)u_m(\mathbf{q},t). \tag{2.2}$$

Here ν is kinematic viscosity and

$$P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2 \tag{2.3}$$

incorporates the effects of the pressure term and serves to maintain the incompressibility property

$$k_iu_i(\mathbf{k},t)=0. \tag{2.4}$$

The total kinetic energy per unit mass is

$$\tfrac{1}{2}\sum_{\mathbf{k}} u_i(\mathbf{k},t)u_i^*(\mathbf{k},t).$$

It is conserved by the nonlinear right-hand side of (2.2). Moreover, there is a detailed conservation property for each triad of interacting wavevectors. Suppose that only the terms involving a particular pair $(\mathbf{p},\mathbf{q})$ were retained in (2.2) and that, correspondingly, only the terms involving the pair $(\mathbf{k}, -\mathbf{q})$ were retained in the equation of motion for $\mathbf{u}(\mathbf{p},t)$ and only the terms involving the pair $(\mathbf{k}, -\mathbf{p})$ were retained in the equation of motion for $\mathbf{u}(\mathbf{q},t)$. The resulting equations conserve the triad kinetic energy

$$\tfrac{1}{2}[|\mathbf{u}(\mathbf{k})|^2+|\mathbf{u}(\mathbf{p})|^2+|\mathbf{u}(\mathbf{q})|^2].$$

Note here that $\mathbf{u}(\mathbf{k})=\mathbf{u}^*(-\mathbf{k})$ because $\tilde{\mathbf{u}}(\mathbf{x},t)$ is real.

The conservation property implies that certain phase correlations develop among the Fourier amplitudes for different wavevectors; the amplitudes cannot remain statistically independent under the Navier-Stokes equation. To see this suppose that $u_j(\mathbf{p})$ and $u_m(\mathbf{q})$ on the right-hand side of (2.2) *were* independent random variables. Then the right-hand side would be a random forcing term for $u_i(\mathbf{k})$. On the average, this would pump energy into mode $\mathbf{k}$, for every $\mathbf{k}$, thereby contradicting the conservation of energy by the totality of nonlinear terms.

The DIA may be regarded as a device for neglecting the statistical correlations among the terms contributing to the right-hand side of (2.2), while systematically correcting the energy budget by adding an additional term which has the form of a generalized eddy damping. Thus the approximation is equivalent to replacing (2.2) by a modified, model dynamical equation. The model equation is most simply expressed in the case of isotropic turbulence, in which case it has the explicit form

(Kraichnan, 1970; Leith, 1971; Herring and Kraichnan, 1972)

$$(\partial/\partial t+\nu k^2)u_i(\mathbf{k},t)+\int_0^t \eta(k,t,s)u_i(\mathbf{k},s)ds = -ik_m P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \xi_j(\mathbf{p},t)\xi_m(\mathbf{q},t). \quad (2.5)$$

Here $\xi_i(\mathbf{p},t)$, which replaces $u_i(\mathbf{p},t)$ in the nonlinear terms, is a multivariate-normal velocity field which is related to the field $u_i(\mathbf{p},t)$ only in that they have the same covariance functions:

$$\langle \xi_i(\mathbf{k},t)\xi_j^*(\mathbf{k},t')\rangle = \langle u_i(\mathbf{k},t)u_j^*(\mathbf{k},t')\rangle, \quad (2.6)$$

where the angle braces denote averages over ensemble. The function $\eta(k,t,s)$ in the damping term which restores energy conservation is a functional of the covariance and Green's function of the u field. It is given by

$$\eta(k,t,s)=\pi k\int\int_\Delta b_3(k,p,q)G(p,t,s)U(q,t,s)pqdpdq, \quad (2.7)$$

where the scalar function $G(p,t,s)$ is the Green's function associated with (2.5) and the covariance scalar $U(q,t,s)$ is defined by

$$U(k,t,s)=(L/2\pi)^3\langle u_i(k,t)u_i^*(k,s)\rangle. \quad (2.8)$$

The coefficient $b_3(k,p,q)$, which arises from dot products of P_{ij} operators, is given by

$$b_3(k,p,q)=\tfrac{1}{2}k^{-4}\sin^2\alpha[(k^2-q^2)(p^2-q^2)+k^2p^2], \quad (2.9)$$

where α is the internal angle opposite k in a triangle whose sides are k, p and q. The integration in (2.7) extends over all wavenumbers p and q which can form such a triangle. Eqs. (2.7)–(2.9) are for three dimensions; the two-dimensional forms will be given later.

The time $t=0$ which forms the lower integration bound in (2.5) is an initial time at which the u field is assumed to be multivariate normal. The system of equations is completed by equations of motion for G and U, which follow from (2.5). They are

$$(\partial/\partial t+\nu k^2)G(k,t,t')+\int_0^t \eta(k,t,s)G(k,s,t')=0, \quad G(k,t',t')=1, \quad (2.10)$$

$$(\partial/\partial t+\nu k^2)U(k,t,t') = \pi k\int\int_\Delta b_3(k,p,q)\left[\int_0^t G(k,t',s)U(p,t,s)U(q,t,s)ds - \int_0^{t'} G(p,t,s)U(k,t',s)U(q,t,s)ds\right]pqdpdq. \quad (2.11)$$

In (2.11), the second term on the right-hand side arises from the η term in (2.5). Eqs. (2.10) and (2.11), together with the symmetry property

$$U(k,t,t')=U(k,t',t), \quad (2.12)$$

can be integrated forward together from initial values $U(k,0,0)$ to yield $G(k,t,t')(t>t')$ and $U(k,t,t')$ for all t and t' greater than zero.

Before discussing the properties and degree of uniqueness of (2.5) and the relations that follow from it, we shall present the equations describing the balance of energy. The total kinetic energy of hydrodynamic motion, per unit mass, may be written as

$$\int_0^\infty E(k,t)dk,$$

where the energy spectrum $E(k,t)$ is given by

$$E(k,t)=2\pi k^2U(k,t,t). \quad (2.13)$$

$E(k,t)$ obeys

$$(\partial/\partial t+2\nu k^2)E(k,t)=T(k,t), \quad (2.14)$$

where the energy transfer function $T(k,t)$ is found by use of (2.11) to be

$$T(k,t)=4\pi^2k^3\int\int_\Delta b_3(k,p,q)\int_0^t ds[G(k,t,s)U(p,t,s) - G(p,t,s)U(k,t,s)]U(q,t,s)pqdpdq. \quad (2.15)$$

The function $T(k,t)$, as given by (2.15), obeys identically both the overall and detailed conservation conditions on the nonlinear dynamics. The overall conservation relation is

$$\int_0^\infty T(k,t)dk=0. \quad (2.16)$$

To express the detailed conservation relations, we write

$$T(k,t)=\frac{1}{2}\int\int_\Delta T(k,p,q,t)dpdq, \quad (2.17)$$

where $T(k,p,q,t)=T(k,q,p,t)$ is the total contribution to $T(k,t)$ from interaction of wavenumber k with wavenumbers p and q. Thus

$$T(k,p,q,t)=\tilde{T}(k,p,q,t)+\tilde{T}(k,q,p,t),$$

where $\tilde{T}(k,p,q,t)$ is the integrand of the (p,q) integration in (2.15). The detailed conservation relation is then

$$T(k,p,q,t)+T(p,q,k,t)+T(q,k,p,t)=0. \quad (2.18)$$

If the analysis leading from (2.5) to (2.15) is traced in detail, it is found that the term in (2.15) involving $G(k,t,s)$ arises from the random driving term on the right-hand side of (2.5), while the term involving $G(p,t,s)$ arises from the $\eta(k,t,s)$ term on the left-hand side of (2.5). It can be shown that the $G(k,t,s)$ term in (2.15), and in the corresponding expression for

$T(k,p,q,t)$, is typically positive. It is always positive for small t and remains positive for all t unless the correlations $U(p,t,s)$ and $U(q,t,s)$ are very unlikely functions of t and s. We therefore call this term the *input* term in (2.15). The term in $G(p,t,s)$, arising from $\eta(k,t,s)$, is therefore clearly necessary to maintain energy conservation. We shall call it the *output* term. It is a dynamical damping term analogous to a radiation damping.

The DIA form (2.7) for $\eta(k,t,s)$ is obtained in the formal derivations of the DIA without explicit appeal to the condition that conservation be maintained. Now, if we grant a model equation of the form (2.5), with the random right-hand side as specified, how unique is (2.7)? The answer, which we shall not prove, is that the detailed conservation condition, together with the requirement that the model system exhibit an absolute inviscid statistical equilibrium identical to that of the real system, leads uniquely to (2.7). We are assuming here that the isotropic turbulence is non-helical (reflection-invariant). If it is not, the DIA damping is altered. It remains uniquely determined if we add helicity conservation to the list of conditions and include inviscid equilibria with helicity. We should also remark here that in two dimensions the uniqueness of the DIA form for $\eta(k,t,s)$ follows if enstrophy conservation is included in the list of conditions rather than helicity conservation (there is no helicity in two-dimensional flow).

In summary, then, if phase correlations in the nonlinear part of the Navier-Stokes equation (2.2) are neglected, energy conservation and other fundamental consistency requirements lead to the DIA model system (2.5)–(2.7), in which there appears the generalized eddy-damping function $\eta(k,t,s)$. We shall see in subsequent sections that under suitable limiting conditions the η term in (2.5) goes over into an eddy viscosity term of the type $\nu_{\text{eddy}}\nabla^2$, but in general we note that the η term is nonlocal in time and has a complicated k dependence, with the result that it is also nonlocal in space when back-Fourier-transformed. This nonlocalness is an essential consequence of the continuity of scale sizes, and it is mandated by the energy-conservation condition. The eddy-damping on wavenumber k takes the ordinary local form only when the contributing wavenumbers p and $q \gg k$. We also note from (2.5) that no pair (p,q) act on k purely as a dynamical damping; there is always in addition the random forcing term on the right-hand side.

Several relatives of the DIA can be obtained by altering the form of the random nonlinear terms in the model equation or by modifying the final statistical equations (2.10) and (2.11) (Leith, 1971; Herring and Kraichnan, 1972). We shall use one of these, the test-field model (TFM), for all explicit calculations in the present paper. In the TFM (Kraichnan, 1971a), the right-hand side of (2.5) is replaced by nonlinear terms which are a white noise in time. This is a bad approximation for time correlations but it has two advantages. First, the resulting equations are local in time and are therefore simpler to handle. Second, it gives a better representation of inertial-range dynamics than the DIA because the flexibility is provided to remove some spurious effects which arise in the DIA.

The TFM model equation, replacing (2.5), is

$$[\partial/\partial t+\nu k^2+\eta(k,t)]u_i(\mathbf{k},t)$$
$$=-ik_m P_{ij}(\mathbf{k})w(t)\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}}[\theta_{kpq}(t)]^{\frac{1}{2}}\xi_j(\mathbf{p},t)\xi_m(\mathbf{q},t). \quad (2.19)$$

Here $\xi_i(\mathbf{k},t)$ is again a random velocity field, related to $u_i(\mathbf{k},t)$ by (2.6). The new function $w(t)$ is a random, zero-mean, white noise process which satisfies

$$\langle w(t)w(t')\rangle=\delta(t-t'). \quad (2.20)$$

The function $\theta_{kpq}(t)$ is a characteristic interaction time for the wavenumber triad (k,p,q). It makes (2.19) dimensionally correct, and it replaces correlation times which arise in the DIA as integrals over time-displaced correlations. Thus, there is no time integration in the dynamical damping term in (2.19), and $\eta(k,t)$ is given by

$$\eta(k,t)=\pi k\int\int_{\Delta} b_3(k,p,q)\theta_{pqk}(t)U(q,t)pqdpdq, \quad (2.21)$$

where $U(q,t)\equiv U(q,t,t)$.

The characteristic times $\theta_{kpq}(t)$ for the TFM are determined by equations which measure the self-distortion of the velocity field by advection and pressure forces (Kraichnan, 1971a). In essence, $\theta_{kpq}(t)$ is the root-mean-square reciprocal of an effective shear which acts, by its distorting effect, to limit the build-up of phase correlations among the triad (k,p,q). Eqs. (2.19)–(2.21) lead to an energy equation of form (2.14), with $\tilde{T}(k,p,q,t)$ [defined by (2.17) *et seq.*] given by

$$\tilde{T}(k,p,q,t)=4\pi^2k^3b_3(k,p,q)pq$$
$$\times[\theta_{kpq}(t)U(p,t)-\theta_{pqk}(t)U(k,t)]U(q,t). \quad (2.22)$$

This relation, together with an equation for $\theta_{kpq}(t)$ which we shall not reproduce here, form a closed set which permit the determination of $U(k,t)$ and $\theta_{kpq}(t)$ from initial values $\theta_{kpq}(0)=0$ and $U(k,0)$. The TFM transfer function $T(k,t)$ obeys the conservation conditions (2.16) and (2.18). The TFM also satisfies other consistency requirements satisfied by the DIA for the single-time quantities $U(k,t)$. However, it does not give a satisfactory representation of time-displaced correlations, and this is the price paid for representing the nonlinear terms as a wihte-noise process. Further details of the TFM equations and their properties are given by Herring and Kraichnan (1972) and by Leith and Kraihcnan (1972).

In a steady state, where $U(k,t)$ has the time-independent value $U(k)$, the DIA transfer function also

takes the form (2.17), (2.22), but with

$$\theta_{kpq}=\int_0^t G(k,t,s)U(p,t,s)U(q,t,s)ds/[U(p)U(q)], \quad t\to\infty. \quad (2.23)$$

To reach such a steady state, some kind of external driving must be added to the Navier-Stokes equation, and to the DIA and TFM model equations, to replenish the energy removed by viscosity. We shall not introduce such driving forces explicitly because they do not change the form of the transfer functions in either DIA or TFM.

3. Three-dimensional inertial range

The total rate of energy transfer per unit mass to all wavenumbers above a given wavenumber k_m is

$$\Pi(k_m)=\int_{k_m}^{\infty} T(k)dk=-\int_0^{k_m} T(k)dk, \quad (3.1)$$

where the two integral expressions are equal because of (2.16). We have omitted the time arguments in (3.1). The total contribution to $T(k)$ from triads (k,p,q) such that $k<k_m$ and p and/or $q>k_m$ is

$$T(k|k_m)=\frac{1}{2}\int\int_{\Delta}' T(k,p,q)dpdq, \quad k<k_m, \quad (3.2)$$

where the integration $\int\int_{\Delta}'$ signifies p and/or $q>k_m$. It follows from the detailed conservation property (2.18) that

$$\Pi(k_m)=-\int_0^{k_m} T(k|k_m)dk. \quad (3.3)$$

Finally, an *effective eddy viscosity* acting on modes of wavenumber k due to dynamical interaction with wavenumbers $>k_m$ may be defined by

$$\nu(k|k_m)=-T(k|k_m)/[2k^2E(k)], \quad k<k_m. \quad (3.4)$$

If $k\ll k_m$, the condition that k, p and q form a triangle means that $|p-q|\leqslant k\ll q$ for all contributions to the integral in (3.2). In this case, all the quantities that enter $T(k,p,q)$ can be expanded in powers of $p-q$, with the result that the p integration can be performed. We shall omit the somewhat tedious algebra and state the final result, which is

$$\nu(k|k_m)=(2\pi/15)\int_{k_m}^{\infty}\theta_{qqk}[7U(q)+qdU(q)/dq]q^2dq, \quad k\ll k_m. \quad (3.5)$$

This is the leading term in an expansion in powers of k/k_m.

If $q\gg k$, then the interaction time θ_{qqk} becomes independent of k unless the energy spectrum rises very rapidly toward small wavenumbers. Thus (3.5) is independent of k for sufficiently well-behaved spectra and therefore conforms to the usual concept of what an eddy viscosity should be. The analysis leading to (3.5) shows that for $k\ll k_m$ the quantity $\nu(k|k_m)$ arises from the output term of the transfer function exclusively; that is, it comes from the η term in the model equation (2.5) or (2.19).

Now we wish to examine the explicit form taken by $T(k|k_m)$ and $\nu(k|k_m)$ in the case of the Kolmogorov inertial range, where

$$E(k)=C\epsilon^{2/3}k^{-5/3}. \quad (3.6)$$

Here C is a numerical constant of order 1 and ϵ is the rate of dissipation of kinetic energy by viscosity per unit mass. Energy conservation implies that for k in the inertial range, where the direct effect of viscosity is negligible,

$$\Pi(k)=\epsilon, \quad (3.7)$$

independent of k.

The TFM yields a Kolmogorov inertial range (Kraichnan, 1971b), wherein

$$\theta_{kpq}=[C^{\frac{1}{2}}\mu\epsilon^{\frac{1}{3}}(k^{\frac{2}{3}}+p^{\frac{2}{3}}+q^{\frac{2}{3}})]^{-1}, \quad (3.8)$$

with μ another numerical constant of order one. The energetic interaction among the wavenumbers in this range has been found to be local in the following sense. Let us write

$$\Pi(k)=\epsilon\int_0^1 Q(v)v^{-1}dv, \quad (3.9)$$

where $Q(v)$ is a measure of the contribution to $\Pi(k)$ from different shapes of wavenumber triangle. Here v is defined as the ratio of the smallest of the three wavenumber legs of the triangle to middle-sized leg. It has been found that

$$Q(v)\sim v^{\frac{4}{3}}\ln(1/v), \quad v\ll 1, \quad (3.10)$$

which shows that triangles with very large wavenumber ratios contribute negligibly to the inertial-range transfer. A plot of $Q(v)$ is given by Kraichnan (1971b).

If (3.6) and (3.8) are substituted into (3.5), the asymptotic eddy viscosity (3.5) becomes

$$\nu(k|k_m)=\tfrac{1}{12}C^{\frac{1}{2}}\mu^{-1}\epsilon^{\frac{1}{3}}k_m^{-\frac{4}{3}}, \quad k\ll k_m. \quad (3.11)$$

The full functions $T(k|k_m)$ and $\nu(k|k_m)$ for k and k_m both in the inertial range can be obtained by performing the integrations in (3.2) numerically. Fig. 1 shows $\nu(k|k_m)/\nu(0|k_m)$, where $\nu(0|k_m)$ denotes the right-hand side of (3.11), and the function

$$\Pi(k|k_m)=-\epsilon^{-1}\int_k^{k_m} T(k'|k_m)dk'. \quad (3.12)$$

The latter [note (3.3)] is the fraction of the total energy transfer across k_m which comes from wavenumbers $k'>k$. The third curve in Fig. 1 shows $\nu_i(k|k_m)$, which

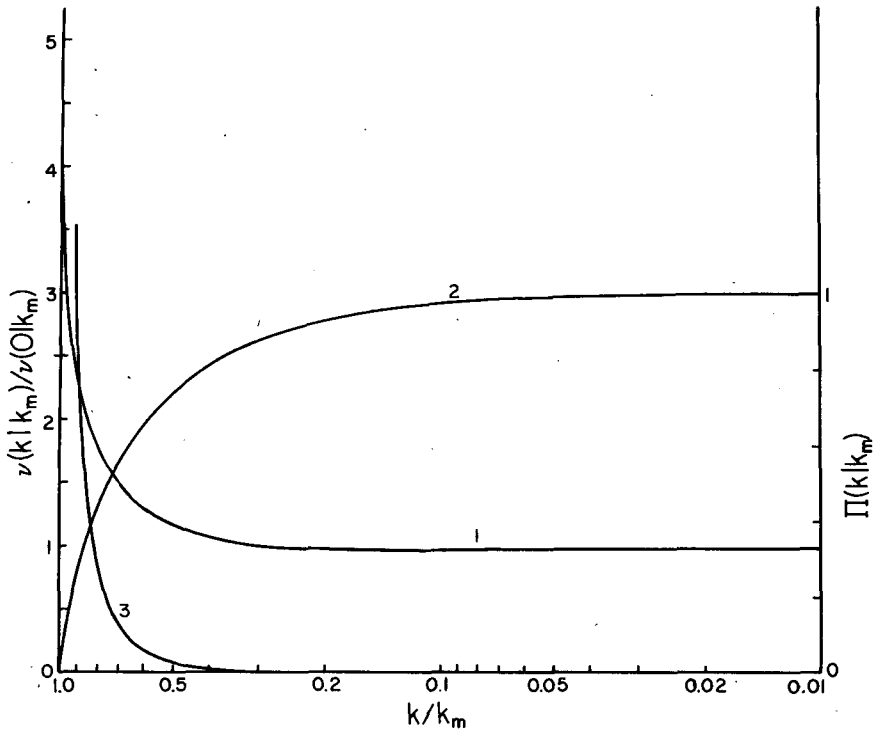

FIG. 1. Energy transfer across k_m in the three-dimensional inertial range. Curve 1, normalized eddy viscosity $\nu(k|k_m)/\nu(0|k_m)$; curve 2, integrated transfer $\Pi(k|k_m)$; curve 3, input contribution $-\nu_i(k|k_m)/\nu(0|k_m)$.

we define as the part of $\nu(k|k_m)$ which arises from the *input* term [involving $U(p)U(q)$] in (2.22).

Fig. 1 indicates that for $k/k_m<0.5$, $\nu(k|k_m)$ is within about 15% of its asymptotic value (3.11). These k contribute approximately 25% of the total transfer across k_m. About 50% of the total transfer comes from $k>0.75k_m$, and for these wavenumbers $\nu(k|k_m)/\nu(0|k_m)>1.6$. At $k=k_m$, $\nu(k|k_m)$, and consequently $T(k|k_m)$, has a cusp where it rises to $\sim 5.24\nu(0|k_m)$. Near the cusp, $\nu(k_m|k_m)-\nu(k|k_m)$ goes like $(k_m-k)^{\frac{1}{2}}$. The $\nu_i(k|k_m)$ contribution is a negligible fraction of $\nu(k|k_m)$ for $k<0.5k_m$, rises to 50% and 100% of $\nu(k|k_m)$ at $k\sim 0.8k_m$ and $k\sim 0.9k_m$, respectively, and diverges like $(k_m-k)^{-2/3}$ as $k\to k_m$. The nonlocalness of energy transfer in wavenumber displayed in Fig. 1 is similar to that inferred from a phenomenological model by Tennekes and Lumley (1972).

The transition from constant eddy viscosity at $k\ll k_m$ to the cusp-like behavior at $k_m-k\ll k_m$ expresses some basic physics of the Navier-Stokes equation. If $k\ll k_m$, the wavenumber triads contributing to $T(k|k_m)$ all have p and $q\gg k$. This gives a separation of scales sufficient to make the analogy to molecular viscosity a good one. The dynamical mechanism at work is then the random displacement, by the high-wavenumber p and q excitation, of momentum associated with the low-wavenumber k excitation. This implies, on the average, a decrease in the kinetic energy of the k excitation, and energy conservation requires that the lost energy appear in the p and q excitation. If $k_m-k\ll k_m$, this eddy-viscosity mechanism is still at work, giving contributions to $T(k|k_m)$ which involve p and $q\gg k_m$. However, there is now an additional mechanism involving triads with q (or p)$\ll k_m$. If q is the low wavenumber, this mechanism consists of coherent straining of the high-wavenumber (k and p) excitation by the random shear associated with q. The result is a diffusion process *in wavenumber* in which there is a two-way exchange, by stretching and unstretching, across the boundary k_m. The input term in (2.22) then describes the "unstretching" whereby excitation at wavenumbers slightly greater than k_m is transformed to excitation at k, and the output term describes the opposite process. The two terms nearly cancel for low q triads and the slight excess of output over input gives the net contribution to $T(k|k_m)$ which is responsible for the rise of $\nu(k|k_m)$ to a finite cusp at $k=k_m$.

We have so far considered, and shown in Fig. 1, the ideal asymptotic case in which the inertial range continues to $k=0$. If, instead, the spectrum is cut off at a bottom wavenumber k_0, then the very low q contributions to $T(k|k_m)$ are removed. The result is a rounding off of the cusp in $\nu(k|k_m)$ and the removal of the divergence of $\nu_i(k|k_m)$ at $k=k_m$. If $k_0\ll k_m$, it is clear from comparing the integrated transfer curve in Fig. 1 with the $\nu(k|k_m)$ curve that the order of 50% of the total inertial-range energy transfer comes from the region dominated by the coherent straining mechanism, with

$\nu(k|k_m)$ substantially greater than $\nu(0|k_m)$. Or, to put it differently, only about half of the energy transfer can validly be ascribed to an eddy-viscosity mechanism analogous to molecular viscosity.

What now are the implications for subgrid-scale representation? If a spectral representation of the velocity field is used (Orszag, 1971) and the Fourier modes are reasonably dense (spacing between allowed wavevectors in k space small compared to wavenumber), then $T(k|k_m)$ gives directly the effect on the energy of the explicit modes due to interaction with subgrid modes above the cutoff wavenumber k_m. As noted by Leith (1971), the inertial-range spectrum $E(k)$ will be preserved undistorted right up to the cutoff if molecular viscosity in the truncated TFM model equation is formally augmented by the function $\nu(k|k_m)$ plotted in Fig. 1. Truncation of the model equation here means removing from both the η term and the right-hand side of (2.19) every contribution that involves a wavenumber$>k_m$. Favorable tests of the TFM against computer simulations (Herring and Kraichnan, 1972) suggest that the same $\nu(k|k_m)$ should give a reasonably good representation of the energetic effects of subgrid scales if the Navier-Stokes equation itself is truncated.

However, there is already a serious difficulty even if consideration is restricted to the wavenumber spectrum and more refined details of the flow are ignored. This is that $\nu(k|k_m)$ for any $k<k_m$ is actually a functional of $E(k')$ for all $k'<k_m$. The normalized function $\nu(k|k_m)$ shown in Fig. 1 is correct only if the spectrum is that given by (3.6) and is quasi-stationary. Thus this eddy-viscosity function will not, in general, give correct results if applied to simulations of flows which are evolving and do not satisfy (3.6). Only for $k\ll k_m$, where (3.5) is valid, does the dependence on $E(k')$ disappear.

Despite these difficulties in principle with eddy-viscosity representation of subgrid scales, it may nevertheless be possible to use even very crude eddy viscosities in practical applications. Fig. 1 shows that the asymptotic constant eddy viscosity is a good approximation to within one octave in wavenumber from k_m. Correspondingly, some experience with simulation of moderate Reynolds number turbulence (Orszag and Patterson, 1972; Herring *et al.*, 1974) and with treatment of such turbulence by closure approximations (Kraichnan, 1964; Herring and Kraichnan, 1972) indicates surprising insensitivity of the spectrum to the nature of the cutoff at k_m. Even a cutoff with zero eddy viscosity (conservative reflection of energy back to lower wavenumbers) is compensated principally by a spurious rise of the spectrum within an octave of k_m. The spectrum at lower wavenumbers is almost unaffected. The need for a faithful representation of subgrid-scale effects on explicit scales may be strong only when computing economy does not permit the luxury of a throw-away inaccurate octave of explicit scales. And we must note here that the highest octave in fact contains most of the modes.

4. Two-dimensional energy inertial range

It has been suggested (Kraichnan, 1967; Leith, 1968; Batchelor, 1969) that two-dimensional turbulence at high Reynolds number can exhibit two distinct inertial ranges, which can exist simultaneously: an energy-transferring range in which energy is passed down to smaller wavenumbers and an enstrophy-transferring range in which enstrophy is passed up to larger wavenumbers. In a steady state, energy and enstrophy are fed to the flow at wavenumbers lying between the two ranges. The TFM yields both ranges. Computer simulations are consistent with the existence of these ranges but do not prove their existence (Lilly, 1971, 1972; Herring *et al.*, 1974).

In the present section, we shall carry out an analysis of subgrid-scale, effective-eddy-viscosity phenomena in the two-dimensional energy inertial range so as to parallel as closely as possible the three-dimensional analysis of Section 3. The practical relevance is minimal, but this analysis is valuable because it contrasts two- and three-dimensional dynamics under as symmetric conditions as possible and serves as a basis for the more relevant case of the enstrophy inertial range.

We shall start by indicating how the basic equations of Section 2 are altered in two dimensions. Eqs. (2.1)–(2.6) are unchanged. Eq. (2.7) for the DIA dynamic damping function is changed to

$$\eta(k,t,s)=k^2\int\int_\Delta b_2(k,p,q)G(p,t,s)U(q,t,s)dpdq, \quad (4.1)$$

where

$$b_2(k,p,q)=2k^{-4}(k^2-q^2)(p^2-q^2)\sin\alpha. \quad (4.2)$$

In (2.8), $(L/2\pi)^3$ is changed to $(L/2\pi)^2$, while (2.13) becomes

$$E(k,t)=\pi kU(k,t,t). \quad (4.3)$$

Eq. (2.10) is unchanged, as is (2.12), and (2.11) changes by the replacement

$$\pi kpqb_3(k,p,q)\rightarrow k^2b_2(k,p,q).$$

There is no change in (2.14). The DIA energy-transfer formula (2.15) changes by the replacement

$$4\pi^2k^3b_3(k,p,q)pq\rightarrow 2\pi k^3b_2(k,p,q).$$

Eqs. (2.16)–(2.20) are unchanged, as is (2.23). The TFM expressions for the damping function [(2.21)] and the transfer function [(2.22)] become, respectively,

$$\eta(k,t)=k^2\int\int_\Delta b_2(k,p,q)\theta_{pqk}(t)U(q,t)dpdq, \quad (4.4)$$

$$\tilde{T}(k,p,q,t)=2\pi k^3b_2(k,p,q)$$
$$\times[\theta_{kpq}(t)U(p,t)-\theta_{pqk}(t)U(k,t)]U(q,t). \quad (4.5)$$

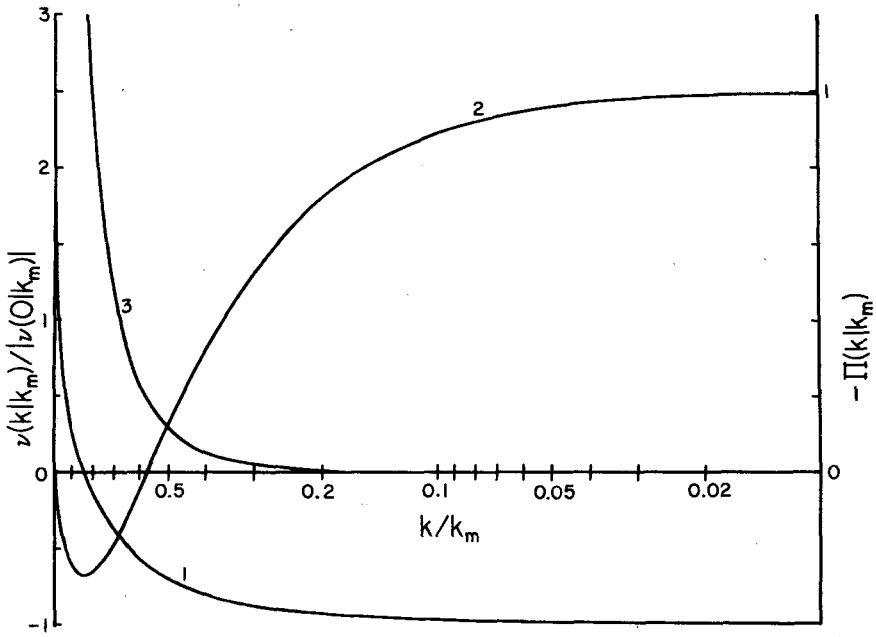

FIG. 2. Energy transfer across k_m in the two-dimensional energy inertial range. Curve 1, normalized eddy viscosity $\nu(k|k_m)/|\nu(0|k_m)|$; curve 2, integrated transfer $-\Pi(k|k_m)$; curve 3, input contribution $-\nu_i(k|k_m)/|\nu(0|k_m)|$.

Now we are equipped to consider eddy viscosity formulas in two dimensions. The defining relations (3.1)–(3.4) are unchanged from three dimensions, but the expansion procedure that gave (3.5) now gives

$$\nu(k|k_m)=(\pi/4)\int_{k_m}^{\infty}\theta_{qqk}\frac{d}{dq}[q^2U(q)]dq. \qquad (4.6)$$

This formula differs from (3.5) in several interesting ways. First, there is no term in $U(q)$ alone; $U(q)$ appears in the integrand only as the derivative of the enstrophy mode intensity $q^2U(q)$. Second, as a result, $\nu(k|k_m)$ is negative if the enstrophy mode intensity is a decreasing function of q for $q>k_m$. Third, the integrand is a total derivative except for the q dependence of θ_{qqk}. This means that any addition to the spectrum $U(q)$ for $q>k_m$ which vanishes at $q=k_m$ would add nothing to $\nu(k|k_m)$ were it not for the q dependence of θ_{qqk}. In common with (3.5), Eq. (4.6) gives $\nu(k|k_m)$ independent of k for $k\ll k_m$ because then θ_{qqk} becomes independent of k; it therefore remains reasonable to continue to call $\nu(k|k_m)$ an eddy viscosity. The differences we have listed imply that the physical basis of eddy viscosity in two dimensions differs radically from the three-dimensional case. In Section 5 we shall explore the explicit dynamical mechanisms which are responsible.

The TFM yields (3.6) and (3.8) in the two-dimensional energy inertial range, but ϵ, as defined by (3.7), is negative corresponding to the fact that energy is flowing toward smaller wavenumbers (Kraichnan, 1971b). The asymptotic relation (3.10) continues to hold. But the numerical decrease of $Q(v)$ with v is slower in two dimensions than in three indicating that the energy-transferring interactions are less local in wavenumber.

Fig. 2 shows the results of a numerical evaluation of $\nu(k|k_m)$ for the two-dimensional energy inertial range, together with the functions $\nu_i(k|k_m)$ and $\Pi(k|k_m)$ as defined in Section 3. The principal qualitative differences from three dimensions are the negative asymptotic value $\nu(0|k_m)$, the negative asymptotic value $\Pi(0|k_m)=\Pi(k_m)$, and the fact that $\nu(k|k_m)$ and $\Pi(k|k_m)$ change sign near $k=k_m$. The value obtained by substituting (3.6) and (3.8) into (4.6) is

$$\nu(k|k_m)=-\tfrac{1}{16}C^{\frac{1}{2}}\mu^{-1}|\epsilon|^{\frac{1}{3}}k_m^{-\frac{4}{3}}, \quad k\ll k_m, \qquad (4.7)$$

which is to be compared with (3.11). The constants C and μ have different numerical values in two and three dimensions (Kraichnan, 1971b).

Fig. 2 indicates that the overall energy transfer is dominated by relatively small values of k/k_m (smaller than in these dimensions) for which the asymptotic negative eddy viscosity $\nu(0|k_m)$ is a fair approximation. Thus 50% of the total transfer comes from the approximate domain $k<0.3k_m$ for which $\nu(k|k_m)$ is within 15% of the asymptotic value and about 85% comes from $k<0.5k_m$, for which $\nu(k|k_m)$ is within 30% of the asymptotic value. For $k=0.3k_m$, the input-term contribution is about 5% of $\nu(0|k_m)$ and rises to about 30% of $\nu(0|k_m)$ at $k=0.5k_m$.

The cusp behavior for $k_m-k\ll k_m$ is essentially the same as in Fig. 1 and is due to the same physical process: diffusion in wavenumber across k_m due to coherent stretching and unstretching by very small wavenumber excitation. The cusp in $\nu(k|k_m)$ has $\nu(k_m|k_m)\sim 2.1|\nu(0|k_m)|$. The divergence in $\nu_i(k|k_m)$

at $k=k_m$ is also similar to the three-dimensional case. The positive regions of $\nu(k|k_m)$ and $\Pi(k|k_m)$ for small k_m-k represent a flow of enstrophy from below to above the boundary k_m and serve to make the total enstrophy transfer zero, in accord with the underlying theory of the two-dimensional energy transferring range (Kraichnan, 1967).

5. Mechanism of the negative eddy viscosity

The inviscid vorticity equation in two dimensions can be written

$$(\partial/\partial t+\mathbf{u}\cdot\nabla)\omega=0, \tag{5.1}$$

where we drop the tilde and represent the velocity field and vorticity in x space by $\mathbf{u}(\mathbf{x},t)$ and $\omega(\mathbf{x},t)$. This is identical with the advection equation of a passive scalar field in the absence of molecular diffusion. The difference is that ω and $\mathbf{u}$ are related by

$$\omega=\partial u_2/\partial x_1-\partial u_1/\partial x_2, \tag{5.2}$$

while there is no such functional relation in the case of the passive scalar.

We consider scales of motion small enough that most of the shear which acts on these scales arises from much larger spatial scales, with, consequently, much higher kinetic energy. The sheared small scales should then negligibly react on the shearing scales and should be strained as if the vorticity field associated with the small scales were a passive scalar (Kraichnan, 1975a). We wish now to examine this asymptotic case in order to expose the mechanisms responsible for the peculiar negative eddy viscosity found in Section 4. We shall show that the passage of kinetic energy from small scales to the large-scale straining field can be regarded as an interference, or interaction, between the large-scale field and a secondary flow associated with the small scales.

The discussion is clearest if the velocity potential associated with the small-scale motion has the form

$$\left.\begin{aligned}\psi(\mathbf{x})&=k^{-2}f(\mathbf{x})\cos(kx_2)\\ f(\mathbf{x})&=\exp[-\tfrac{1}{2}(x_1^2+x_2^2)/D^2]\end{aligned}\right\}, \tag{5.3}$$

i.e., a co-sinusoidal variation localized in a domain of size D. The associated velocity field is

$$\left.\begin{aligned}u_1&=-\partial\psi/\partial x_2\\ &=k^{-1}f[\sin(kx_2)+(x_2/kD^2)\cos(kx_2)]\\ u_2&=\partial\psi/\partial x_1=-k^{-1}(x_1/kD^2)f\cos(kx_2)\end{aligned}\right\}. \tag{5.4}$$

We take $kD\gg1$ so that the spectrum of ω is concentrated about wavenumber k and let this small-scale motion exist in the presence of a uniform straining field whose velocity potential is

$$V=-axy, \tag{5.5}$$

where a is a parameter.

It follows directly from (5.1) that the effect of this potential straining flow on the small-scale motion is to squeeze the vorticity field along the x_2 axis and stretch it along the x_1 axis so that areas remain constant and the value of the vorticity at each point moving with the straining remains constant. The result is that to leading order in an expansion in powers of $(kD)^{-1}$ the effect on the vorticity field is a distortion of the shaping function f into elliptical form together with a change of the central wavenumber k according to

$$dk/dt=ak. \tag{5.6}$$

Then (5.4) shows that the kinetic energy of the small-scale motion changes according to

$$dF/dt=-2aE, \tag{5.7}$$

to leading order. The initial kinetic energy according to (5.4) is $\pi D^2/4$ to leading order so that the initial rate of decrease of kinetic energy, associated with the transfer of vorticity to higher wavenumber, is

$$(dE/dt)_{t=0}=-\pi aD^2/2. \tag{5.8}$$

We shall now show that the kinetic energy thus lost shows up as kinetic energy of interaction between the straining field and the self-generated secondary flow associated with the small-scale motion. If $\mathbf{v}$ is the velocity field due to the potential (5.5), then this interaction energy is

$$\int\mathbf{v}\cdot\mathbf{u}d^2x=\int[(\partial V/\partial x_2)(\partial\psi/\partial x_2)+(\partial V/\partial x_1)(\partial\psi/\partial x_1)]d^2x. \tag{5.9}$$

Partial integration transforms this to

$$\int\mathbf{v}\cdot\mathbf{u}d^2x=-\int V\nabla^2\psi d^2x=-\int V\omega d^2x. \tag{5.10}$$

The rate of change of the interaction energy is then

$$\frac{d}{dt}\int\mathbf{v}\cdot\mathbf{u}d^2x=-\int V(d\omega/dt)d^2x. \tag{5.11}$$

Note that the total velocity field is $\mathbf{v}+\mathbf{u}$.

Since V varies slowly, only secular contributions to $d\omega/dt$ (those which survive averaging over distances large compared to k^{-1}) can affect the integral in (5.11) to leading order in $1/kD$. The secular contributions must be due to self-interaction of the small-scale motion (secondary flow) since the straining motion $\mathbf{v}$ leaves unchanged the vorticity of each fluid element. To find the development of the secondary flow, we again use (5.1) but now with $\mathbf{u}$ as the velocity field (5.4). The initial value $(d\omega/dt)_{t=0}$ follows by straightforward substitution. The full expression has a number of terms, but the secular part, obtained by averaging the sine and cosine functions over distances large compared

to k^{-1} but small compared to D, is

$$[(d\omega/dt)_{t=0}]_{\text{secular}}=(2x_1x_2/D^4)f^2 \qquad (5.12)$$

to leading order in $1/kD$. If (5.12) and (5.5) are substituted into (5.11), the result is

$$\frac{d}{dt}\int \mathbf{v}\cdot\mathbf{u}d^2x=\pi aD^2/2, \quad t=0, \qquad (5.13)$$

which precisely balances the loss (5.8) of kinetic self-energy.

The secondary flow whose initial buildup is described by (5.12) consists of four vortices of size $\sim D$, one in each quadrant of the (x_1,x_2) plane. Each vortex rotates in the opposite sense from those in the adjacent quadrants, and the combined flow brings fluid toward the origin along the x_2 axis and removes it along the x_1 axis. Thus this vortex flow reinforces the velocity field near the origin associated with (5.5), and the interaction energy given by (5.13) is positive. If, however, a is negative, then the central wavenumber of the small-scale motion decreases and the energy E increases, according to (5.6) and (5.7), so that correspondingly the interaction energy (5.13) is negative.

If a small-scale motion has the form of a compact blob of vorticity, or an assembly of uncorrelated blobs, a steady straining will eventually draw a typical blob out into an elongated shape, with corresponding thinning and increase of typical wavenumber. The typical result will be a decrease of the kinetic energy of the small-scale motion and a corresponding reinforcement of the straining field, essentially as described above. Now it is important to note that this energy transfer between small-scale and large-scale motion differs in an important respect from the negative-eddy-viscosity effects discussed in Section 4. There, we found an asymptotically constant eddy coefficient for small wavenumbers, so that the rate of energy transfer was proportional to the *energy* in the large scales (small wavenumbers). But according to (5.6) or (5.13), the rate of energy transfer with constant straining is proportional to the *amplitude* of the large-scale motion.

This difference can be traced to the implicit assumption in the TFM formulas of Section 4 that the effective straining is not constant but instead has an effective lifetime fixed by the interaction-time parameters [e.g., θ_{qqk} in (4.6)]. In the actual fluid, the effective lifetime of straining by large scales is limited in two ways: by change in the straining field itself and by rotation of the small scales relative to the straining field. Both effects show up in the TFM formulas. If the effective shear acting on the small scales represented by $U(q)$ in (4.6) is dominated by wavenumbers $\ll q$, then θ_{kqq} is found from the TFM equations to be approximately the eddy-circulation time or the correlation time of the large-scale straining motion, whichever is shorter. On the other hand, if the shear and rotation acting on motions of wavenumber q are due primarily to interactions that are local in wavenumber, or scale size, then θ_{kqq} is the order of the eddy-circulation time or correlation time of the motions of scale $1/q$.

Consider now what happens if we repeat the analysis of the interaction of the small-scale field (5.3) with the large-scale field (5.5) but simulate variation in the effective straining by orienting the initial wavevector $\mathbf{k}$ of the small-scale field at a random angle ϕ to the x_2 axis instead of having it parallel to the axis as before. Further, we can make a in (5.5) a randomly varying function of time $a(t)$, with zero mean. The central wavenumber k of the small-scale field is then subjected to randomly alternating stretchings and squeezings which constitute a diffusion process in wavenumber rather than a monotonic increase or decrease as before.

The evolution of the central wavevector $\mathbf{k}(t)$ then obeys

$$dk_1/dt=-a(t)k_1, \quad dk_2/dt=a(t)k_2, \qquad (5.14)$$

so that, taking $k_1(0)=k(0)\sin\phi$, $k_2(0)=k(0)\cos\phi$, we have

$$[k(t)]^2=[e^{-2\beta}\sin^2\phi+e^{2\beta}\cos^2\phi][k(0)]^2, \qquad (5.15)$$

where

$$\beta(t)=\int_0^t a(s)ds. \qquad (5.16)$$

Minor manipulation of (5.15) gives

$$[k(t)]^2/[k(0)]^2 =\cosh[2\beta(t)]+\sinh[2\beta(t)]\cos(2\phi). \qquad (5.17)$$

Now, if we average (5.17) over a uniform distribution of the random angle ϕ in the interval $(0, 2\pi)$, we have

$$\langle[k(t)]^2\rangle/[k(0)]^2=\cosh[2\beta(t)], \qquad (5.18)$$

where there is no averaging yet over the distribution of $a(t)$. Unless the correlation time of $a(t)$ vanishes, $\beta(t)$ will typically grow with t, so that in most realizations the average over random orientations gives a growth of the wavenumber $k(t)$ in mean square. Noting (5.4), and the fact that the kinetic energy of the small-scale motion is quadratic in velocity, we see that the kinetic energy is proportional to $[k(t)]^{-2}$. Averaging this over the initial angle ϕ, we have

$$\langle[k(t)]^{-2}\rangle[k(0)]^2=(1/2\pi)\int_0^{2\pi}[\cosh(2\beta) +\sinh(2\beta)\cos(2\phi)]^{-1}d\phi=1. \qquad (5.19)$$

The definite integral is a standard form.

Eq. (5.19) says that there is no change in kinetic energy averaged over initial orientation, despite the mean-square increase in $k(t)$. This is possible because there are always some values of ϕ such that the right-hand side of (5.17) is less than 1. As $\beta(t)$ increases in magnitude with t, the range of such ϕ becomes exponentially small. Thus the constancy of $\langle[k(t)]^{-2}\rangle$ at large t

depends on the contributions of a very few exceptional realizations in which $k(t)$ decreases strongly and, consequently, the kinetic energy increases strongly.

Although (5.19) was obtained under special assumptions [the field (5.5) is irrotational and the orientation of its principal axes is fixed in time], it is of general validity for the straining of ω by a large-scale velocity field, according to (5.1). Cocke (1969) has shown that the evolution of the vector $\mathbf{k}$ due to any quasi-uniform straining motion takes the form

$$[k(t)]^2 = k_i(0) W_{ij}(t) k_j(0), \tag{5.20}$$

where $W_{ij}(t)$ is a symmetric matrix.[2] Moreover, if $W_{ij}(t)$ is transformed to principal axes so that it is diagonal, the product of the diagonal elements is 1, in consequence of incompressibility. If now we have an isotropic distribution of $\mathbf{k}(0)$, or of $W_{ij}(t)$, and ϕ again measures the angle between $\mathbf{k}(0)$ and the principal axes, we again obtain (5.15), where $e^{-2\beta}$ and $e^{2\beta}$ represent the two eigenvalues of $W_{ij}(t)$. The rest of the analysis goes as before.

We are now in a position to understand two features of (4.6). First, if θ_{qqk} is dominated by low-wavenumber straining, in correspondence to our present discussion, it is independent of q and the integrand of (4.6) is a total derivative. Thus any excitation, described by $U(q)$, which is totally confined to $q<k_m$, gives zero contribution to the effective eddy viscosity exerted on $k \ll q$. This is a direct consequence of (5.19) which says that low-wavenumber straining of the small scales gives a diffusion process in wavenumber with *no* average loss of kinetic energy. By conservation, there is then no net gain of kinetic energy by the straining scales. On the other hand, if k_m falls within the small-scale excitation, the diffusion of the excitation to smaller k occurs at wavenumbers $<k_m$ and is not counted in (4.6) which then includes only the outward diffusion. The latter *does* involve a net loss of kinetic energy by the small scales and thus gives rise to a negative contribution to the eddy viscosity. [Note that the role of q in (4.6) is played by k in the analysis of the present Section.]

Second, it is clear from symmetry that if the orientation of $\mathbf{k}(t)$ relative to the straining field is random, or changes randomly with time, then the amplitude of the strain, which is signed, cannot contribute to the averaged diffusion process in wavenumber. This shows up in (5.18), where the right-hand side is even in the time-integrated strain $\beta(t)$. Thus the primary controlling quantity is essentially the square of the strain, expressed in suitably isotropically-averaged form. This is consistent with (4.6), wherein a k-independent eddy viscosity $\nu(k|k_m)$ implies energy transfer proportional to the mean-squared strain of the straining wavenumbers (here k).

[2] See also Kraichnan (1974), Section 2.

6. Two-dimensional enstrophy inertial range

The small scales of a passive scalar field subjected to random straining by motions of much larger spatial scale exhibit an equilibrium range of wavenumbers in which the wavenumber spectrum of the scalar variance goes as k^{-1} (Batchelor, 1959; Kraichnan, 1974). Since (5.1) is the same as the scalar advection equation, a similar range, with enstrophy taking the place of the scalar variance, should exist for two-dimensional turbulence, provided that the excitation in the range is weak enough that it makes negligible contribution to the effective straining field and thus does not upset the assumption that the straining motions are at much smaller wavenumbers than those in the equilibrium range (Kraichnan, 1967; Leith, 1968; Batchelor, 1969). The weakness assumption is rather artificial and the more physically reasonable case is one in which the effective strain acting on wavenumbers in the enstrophy inertial range comes from much smaller wavenumbers still in the range itself. This implies logarithmic corrections to the k^{-1} enstrophy spectrum (Kraichnan, 1971b; Leith and Kraichnan, 1972). In either case, with or without the logarithmic corrections, the enstrophy inertial range exhibits asymptotically k-independent enstrophy transfer rate, as the length of the range becomes infinite, and asymptotically zero energy transfer rate.

We wish now to examine the transfer $T(k|k_m)$ and effective eddy viscosity $\nu(k|k_m)$ as given by (3.2), (3.4) and (4.5) when the grid-scale cutoff wavenumber k_m falls within the enstrophy inertial range. We shall do this first for the uncorrected k^{-1} range and then include the logarithmic corrections. An enstrophy transfer function can be defined by

$$Z(k) = 2k^2 T(k), \quad Z(k|k_m) = 2k^2 T(k|k_m). \tag{6.1}$$

For k in the uncorrected k^{-1} range, $Z(k)$ and $Z(k|k_m)$ behave precisely like the corresponding transfer functions of a converted passive scalar.

We let the straining field be dominated by excitation at wavenumbers $q<k_0$, where $k_0 \ll k_m$. Then the triangle condition of wavenumber interactions implies that $Z(k|k_m)$ in the k^{-1} range is nonzero only for $k_m - k \leqslant k_0$, where by (3.2)

$$Z(k|k_m) = 2k^2 \int_{k_m-k}^{k_0} dq \int_{k_m}^{k+q} T(k,p,q)\,dp. \tag{6.2}$$

The limits in (6.2) express the triangle condition and the factor $\frac{1}{2}$ in (3.2) has disappeared because in (6.2) only p is permitted above k_m while in (3.2) either p or q can be the high wavenumber. Under our assumption about the straining field, the TFM equations for the interaction times give θ_{kpq} and θ_{pkq} in (4.5) independent of their arguments for k in the k^{-1} range. Their approxi-

mate value is

$$\theta^{-2}\approx 2\pi\int_0^{k_0} U(q)q^3dq. \tag{6.3}$$

That is, θ is approximately the reciprocal shear associated with the straining field or, alternately, the eddy-circulation time of that field.

If the coefficient $b_2(k,p,q)$ in (4.5) is expanded in powers of k_0/k_m, (6.2) can be reduced to a simple form. The terms in $U(k)U(p)$ are negligible because of our assumption of weak excitation near k_m. The remaining terms give, to leading order in k_0/k_m,

$$Z(k\,|\,k_m)=8\pi k_m{}^3\theta\int_{k_m-k}^{k_0} dqU(q)\int_{k_m}^{k+q} dp\,\sin(k,q)\times[\Omega(p)-\Omega(k)], \tag{6.4}$$

where $\sin(k,q)$ is the sine of the interior angle opposite p in the triangle with sides (k,p,q) and $\Omega(k)=k^2U(k)$ measures the mean enstrophy per mode. So far we have not used the fact that $\Omega(k)$ has the specific dependence k^{-2} [the enstrophy spectrum function is $2\pi k\Omega(k)$]. Eq. (6.4) is identical with the corresponding transfer formula for the variance of a passive scalar (cf. Kraichnan, 1968).

A further reduction of (6.4) is obtained by expanding $\sin(k,q)$ and $\Omega(p)$ in powers of $p-k$ and thereby carrying out explicitly the integration over p. The result to leading order in k_0/k_m is

$$Z(k\,|\,k_m)=\theta(8\pi/3)k_m{}^3\Omega'(k_m)\times\int_{k_m-k}^{k_0}[q^2-(k_m-k)^2]^{\frac{3}{2}}U(q)q^{-1}dq, \tag{6.5}$$

where $\Omega'(k)=d\Omega(k)/dk$. The total rate of enstrophy transfer across the wavenumber k_m is

$$\chi(k_m)=-\int_{k_m-k_0}^{k_m} Z(k\,|\,k_m)dk. \tag{6.6}$$

We now insert (6.5) in (6.6) and let $x=k_m-k$. Then the order of integration can be reversed according to

$$\int_0^{k_0}dx\int_x^{k_0}dq=\int_0^{k_0}dq\int_0^{q}dx,$$

and the x integration can be carried out explicitly. The result of this is

$$\chi(k_m)=-(\pi^2/2)[k_m{}^3\Omega'(k_m)]\theta\int_0^{k_0}U(q)q^3dq. \tag{6.7}$$

The total energy transfer from the region $(k_m-k_0,\,k_m)$ to leading order in k_0/k_m follows immediately from (3.3), (6.1) and (6.7); it is

$$\Pi_1(k_m)=-(\pi^2/4)[k_m\Omega'(k_m)]\theta\int_0^{k_0}U(q)q^3dq. \tag{6.8}$$

We use the subscript 1 because there is also an energy transfer across k_m associated with the negative eddy viscosity (4.6) exerted on wavenumbers in the range $(0,k_0)$. Exchanging k and q in (4.6) to conform with present notation and carrying out the integration, we find for this transfer

$$\Pi_2(k_m)=2\pi\int\nu(q\,|\,k_m)U(q)q^3dq=-(\pi^2/2)\Omega(k_m)\theta\int_0^{k_0}U(q)q^3dq, \tag{6.9}$$

where we have used $E(q)=\pi qU(q)$ and have assumed $\Omega(\infty)=0$ in carrying out the integration in (4.6).

Now consider specifically the case $\Omega(k)\propto k^{-2}$. The right-hand side of (6.7) is then independent of k_m so that there is constant enstropy transfer. Also, $\Pi_1(k_m)$ and $\Pi_2(k_m)$ are equal and opposite in sign, so that the net energy transfer across k_m vanishes. Both facts are well-known properties of the enstropy inertial range. We neglect the enstrophy transfer associated with (6.9) since it is down from (6.7) by a factor of order $k_0{}^2/k_m{}^2$. In consequence of (6.3), the right-hand sides of (6.7)-(6.9) are actually proportional to θ^{-1}; that is, to the characteristic shear rate associated with the large-scale straining motion.

To what extent is it reasonable to think of subgrid-scale interactions in terms of an effective eddy viscosity in the present case? According to (4.6), the effect on the large-scale straining motions is well-representable, in the sense that the eddy viscosity is independent of wavenumber [k in (4.6), q in the notation of the present section]. However, this eddy viscosity is negative and, moreover, it has a negligible effect on the dynamics of the large scales because these scales have energy very large compared to that of the k^{-1} range wavenumbers. The question of practical importance for possible applications to simulations is how well (6.5) can be modelled by an effective eddy viscosity. It would appear that the answer is very poorly. First, the integral in (6.5) would have to be independent of k for the eddy viscosity $\nu(k\,|\,k_m)$ to be k-independent [we note $k\approx k_m$ in (6.5)]. Instead the integral varies strongly as k moves in the range $(k_m-k_0,\,k_m)$ and vanishes for k outside this range. In particular, $\nu(k\,|\,k_m)$ does not scale with k_m as it does for the three- and two-dimensional energy inertial ranges. Instead it scales with the nonlocal parameter k_0. Thus as $k_0/k_m\to 0$, the significant contributions to $Z(k\,|\,k_m)$ or $\nu(k\,|\,k_m)$ are squeezed into a vanishingly small fractional deviation $(k_m-k)/k_m$.

To illustrate the k-independence implied by (6.5), we can construct an easily integrated example by taking

$$U(q)\propto q^2\exp(-q^2/k_0{}^2), \tag{6.10}$$

which falls off rapidly for $q>k_0$ instead of dropping abruptly to zero, and correspondingly replacing the upper integration limit in (6.5) by infinity. Then we

find

$$Z(k|k_m)/Z(k_m|k_m)=\exp[-(k_m-k)^2/k_0^2]. \quad (6.11)$$

There is not only a scaling with k_0 instead of with k_m alone, but also the shape of $Z(k|k_m)$ for k near k_m depends on the shape of $U(q)$ for the low q which dominate the effective straining.

If the enstrophy inertial range is produced by cascade from lower wavenumbers where energy and enstrophy are injected into the fluid, most of the effective strain acting on wavenumbers high in the range is due to lower wavenumbers in the inertial range itself, in distinction to the artificial case treated above wehre the straining excitation is separate and at wavenumbers below the inertial range. The k^{-1} dependence for the enstrophy spectrum is then not self-consistent, because each octave of a k^{-1} range would make an equal contribution to the effective strain acting on higher wavenumbers, with the result that the effective strain would increase with wavenumber. Self-consistency is restored if the spectrum decreases slightly faster than k^{-1} (Kraichnan, 1971b). Thus,

$$\left.\begin{aligned} E(k)&=C'\chi^{\frac{2}{3}}k^{-3}[\ln(k/k_0)]^{-\frac{1}{3}}\\ U(k)&=E(k)/\pi k\\ \Omega(k)&=k^2U(k)\end{aligned}\right\}, \quad (6.12)$$

where C' is a numerical constant of order 1 and χ is the constant rate of enstrophy transfer to higher wavenumbers. The TFM equations for θ_{kpq} in the log-corrected range give

$$\theta_{kpq}=[\mu(k)+\mu(p)+\mu(q)]^{-1}, \quad (6.13)$$

where $\mu(k)$ is a modal dynamic damping rate given by

$$\left.\begin{aligned}\mu(k)&=\tfrac{3}{4}(2\chi)^{\frac{1}{3}}I^{\frac{1}{3}}[\ln(k/k_0)]^{\frac{1}{3}}\\ I&=\int_0^1(1+\tfrac{1}{2}z^{\frac{4}{3}})^{-1}dz=0.75624\end{aligned}\right\} \quad (6.14)$$

(Leith and Kraichnan, 1972). The wavenumber k_0 in (6.12) marks the bottom of the enstrophy inertial range, and it is assumed that wavenumbers below k_0 make a negligible contribution to the effective strain acting on the range.

The logarithmic corrections make only minor changes in the analysis that leads to (6.4) and (6.5). If the enstrophy range is long, the strain acting on wavenumbers near a typical wavenumber k_m in the range still comes from $q\ll k_m$. This implies, on examination, that the $U(p)U(k)$ term in (4.5) is still negligible and that θ_{kpq} is well approximated by $[2\mu(k)]^{-1}$. To leading order in k_0/k_m and $[\ln(k/k_0)]^{-\frac{1}{3}}$, the equation which replaces (6.5) is found to be

$$Z(k|k_m)=(4\pi/3)[k_m^3\Omega'(k_m)][\mu(k_m)]^{-1}\times\int_{k_m-k}^{\infty}[q^2-(k_m-k)^2]^{\frac{1}{2}}U(q)q^{-1}dq, \quad (6.15)$$

where $\Omega(k_m)$ and $U(q)$ are given by (6.12). The upper integration limit in (6.15) should properly be some wavenumber $q_1\ll k$ below which most of the effective strain lies, but since $U(q)$ falls off like q^{-4} (times log correction), there is negligible error in replacing the limit by ∞.

With (6.12) inserted, (6.15) gives a fall-off of $Z(k|k_m)$ away from k_m which goes like $(\Delta k)^{-1}[\ln(\Delta k/k_0)]^{-\frac{1}{3}}$ for $\Delta k\gg k_0(\Delta k=k_m-k)$. For $\Delta k\sim k_0$, the behavior of $Z(k|k_m)$ again depends on the specific form of $U(q)$ for wavenumbers $\sim k_0$, where (6.12) must be modified. In order to illustrate the behavior, we can make the simple assumption that (6.12) holds down to $k=ek_0$, where the log factor is 1, and then falls abruptly to zero. The resultant form for $Z(k|k_m)$ is

$$Z(k|k_m)=(4\pi/3)[k_m^3\Omega'(k_m)]\times[\mu(k_m)]^{-1}k_0^{-1}J(\Delta k/k_0), \quad (6.16)$$

where

$$J(x)=\int_{x,e}^{\infty}(z^2-x^2)^{\frac{1}{2}}z^{-5}(\ln z)^{-\frac{1}{3}}dz, \quad (6.17)$$

and the lower limit in (6.17) is the lesser of x and e. As in the case of the uncorrected k^{-1} range, the scaling with k_0 is evident. A plot of $J(\Delta k/k_0)$ is given in Fig. 3.

For large x, $J(x)$ asymptotically approaches

$$J_\infty(x)=\tfrac{1}{4}(1+3\pi/8)x^{-1}(\ln x)^{-\frac{1}{3}}, \quad (6.18)$$

and $J_\infty(\Delta k/k_0)$ is also plotted in Fig. 3.

The approximations which lead to (6.15) and (6.16) are valid only for $k_m-k\ll k_m$. This region contains all the significant contributions to $\chi(k_m)$ when the enstrophy range is long and extends far below k_m. However, the energy transfer function $T(k|k_m)$ is significant also in the region $k\ll k_m$ where the negative eddy-viscosity formula (4.6) is applicable. If (6.12)–(6.14) are substituted into (4.6) and an integration by parts is performed, the terms involving the derivative of θ are down by a logarithmic factor from the integrated part so that the asymptotic result is

$$\begin{aligned}\nu(q|k_m)&=-(\pi/8)[\mu(k_m)]^{-1}\Omega(k_m)\\&=-\tfrac{1}{6}(2I)^{-\frac{1}{3}}C'\chi^{\frac{1}{3}}k_m^{-2}[\ln(k_m/k_0)]^{-\frac{1}{3}}.\end{aligned} \quad (6.19)$$

The energy transfer associated with the negative viscosity is then

$$\begin{aligned}T_2(q|k_m)&=2\nu(q|k_m)q^2E(q)\\&=\tfrac{1}{3}(2I)^{-\frac{1}{3}}(C')^2\chi k_m^{-2}[\ln(k_m/k_0)]^{-\frac{1}{3}}\times q^{-1}[\ln(q/k_0)]^{-\frac{1}{3}}.\end{aligned} \quad (6.20)$$

What happens now in the intermediate region where neither asymptotic expression for energy transfer is valid? The total energy transfer across k_m due to wavenumbers for which $k_m-k\ll k_m$ in the asymptotically long inertial range is $\sim\chi k_m^{-2}$. If (6.20) is integrated over q from $q=ek_0$ to a wavenumber q_1, the result is $\sim\chi k_m^{-2}$ for $q_1\sim k_m$. [Note that $\int x^{-1}(\ln x)^{-\frac{1}{3}}dx$

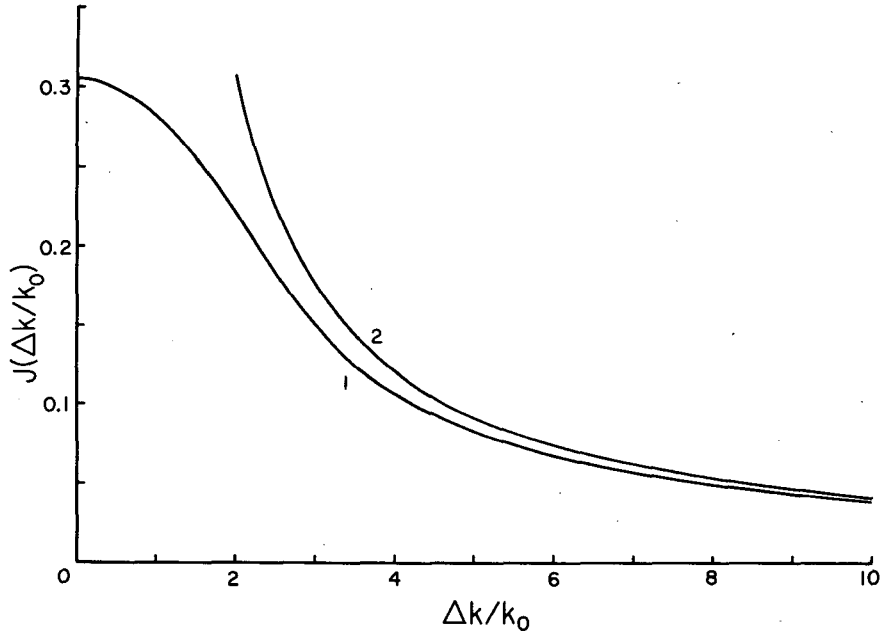

FIG. 3. The functions $J(\Delta k/k_0)$, curve 1, and $J_\infty(\Delta k/k_0)$, curve 2, for the log-corrected enstrophy inertial range, as given by (6.17) and (6.18).

$=(3/2)(\ln x)^{\frac{1}{2}}$.] This implies that (6.20) becomes inapplicable for q the order of, say, $k_m/2$ and that $T(q|k_m)$ changes sign at a wavenumber of this order. This is corroborated by the fact that $T_2(q|k_m)$ given by (6.20) and the transfer $Z(k|k_m)/2k^2$ obtained from (6.16) become equal in magnitude at a wavenumber of this order.

7. Conclusions and discussion

In this paper we have illustrated some theoretical limitations on the representation of subgrid scales by an eddy viscosity. Our approach has been to examine, by means of closure approximations and especially the test-field model (TFM) closure, the contributions of subgrid scales to the energy and enstrophy balance of the explicit scales in the asymptotic energy and enstrophy inertial ranges. Thus a cutoff wavenumber k_m was chosen which divides the subgrid scales ($>k_m$) from the explicit scales ($<k_m$). Expressions were then found for the contribution $T(k|k_m)$ to the net energy transfer rate into wavenumber $k<k_m$ due to all nonlinear processes (advection and pressure forces) which involve any wavenumber $>k_m$. A corresponding effective eddy viscosity was defined by (3.4), in which $E(k)$ is the energy spectrum function of the isotropic turbulence, and a corresponding enstrophy transfer rate is defined by (6.1). The precise definition of $T(k|k_m)$ itself is given by (3.2) and the associated equations.

If the analogy with molecular viscosity were perfect, the effective eddy viscosity $\nu(k|k_m)$ would be independent of both k and $E(k)$. We have noted that in general this cannot be expected because there is a continuity of scale size between explicit and subgrid scales. However, for $k\ll k_m$, there is a clean separation of scales, analogous to the separation between hydrodynamic and thermal agitation scales, and we indeed found that $\nu(k|k_m)$ is so independent. The asymptotic expressions for $\nu(k|k_m)$ at $k\ll k_m$ are given by (3.5) in three dimensions and by (4.6) in two dimensions, according to the TFM. In the case of the energy-transferring inertial range [$E(k)\propto k^{-5/3}$], these equations reduce to the respective forms (3.11) and (4.7), wherein ϵ is the overall rate of energy transfer through the inertial range and μ is a parameter of order 1 associated with the TFM.

The nature and magnitude of the departure of $\nu(k|k_m)$ from its asymptotic constant value when k is comparable to k_m is shown by Figs. 1 and 2 for the three- and two-dimensional energy-transferring ranges, respectively. In three dimensions approximately 25% of the total transfer across k_m comes from $k/k_m<0.5$, where $\nu(k|k_m)$ is within about 15% of its asymptotic value. About 50% of the total transfer comes from $k>0.75k_m$, where $\nu(k|k_m)$ is rising sharply above its asymptotic value. We found that the cusp in $T(k|k_m)$ at $k=k_m$ is associated with the coherent stretching and unstretching of structures of wavenumber $\sim k_m$ due to straining by low wavenumbers (large spatial scales). This is a dynamical mechanism very different in nature from molecular viscosity. Here $T(k|k_m)$ is not proportional to $E(k)$ but instead involves close balance between an *output* term in the TFM transfer expression which is so proportional and an *input* term which is not (see Section 2). The shape of the cusp very close to k_m is not universal in form when the inertial range is of finite extent but instead depends on the spectrum shape in the energy-containing range of wavenumbers.

The curve for $\nu(k|k_m)$ in the two-dimensional energy-transferring inertial range shown in Fig. 2 resembles that for three dimensions. The cusp behavior at $k=k_m$ is associated with the same physical mechanism of coherent straining. The striking difference is that the asymptotic constant value of $\nu(k|k_m)$ for $k\ll k_m$ is negative. This is a necessary consequence of the fact that energy flow in the two-dimensional range is reversed, going from larger to smaller wavenumbers. The mechanism of the negative eddy viscosity is discussed in some detail in Section 5, where it is found to be due to interaction of the straining field of the large-scale motion (here the small wavenumber k) with a secondary flow associated with the small-scale motion (here wavenumbers $>k_m$). A corollary of that discussion is that the ordinary eddy-viscosity mechanism in three dimensions, whereby small scales are parasitic on the energy of the large scales, is intrinsically three-dimensional and cannot take place if the small-scale motion is coplanar with the local large-scale motion. Fig. 2 illustrates that the interaction of wavenumbers in the two-dimensional energy inertial range is rather less local in wavenumber than for the three-dimensional range. About 50% of the total transfer takes place at wavenumbers $k<0.3k_m$, where the function $\nu(k|k_m)$ is within 15% of its asymptotic value.

The general character of the effective eddy viscosity $\nu(k|k_m)$ in Fig. 2, a constant negative asymptotic region together with a positive cusp behavior near $k=k_m$, carries over to the enstrophy-transferring inertial range in two dimensions. However, there are also crucially important differences. Also, the cusp region dominates the enstrophy transfer and thereby is of greater interest in the enstrophy-transferring range.

If the effective rate of strain acting on motions in the enstrophy inertial range is dominated by strong excitations at wavenumbers below this range (an artificial assumption), then the enstrophy spectrum in the range goes like k^{-1} so that $E(k)$ is like k^{-3}. The straining of vorticity at wavenumbers in the range is then precisely analogous to the straining of a passive scalar field, with the dynamically unimportant exception that, in the case of vorticity, there is an energy-balancing reaction on the straining scales expressed by the asymptotic constant negative eddy viscosity. The effect of the straining on enstrophy transfer from wavenumbers k below and near k_m to wavenumbers above k_m is described by (6.4) and (6.5) which give the cusplike part of the transfer function and are identical with corresponding transfer expressions for a passive scalar field.

Two important features of (6.4) and (6.5) are that (i) the enstrophy transfer is concentrated in a region of width k_0 at k_m, where k_0 is a characteristic wavenumber of the straining velocity field; and (ii) the transfer is not proportional to the enstrophy intensity $\Omega(k)$ as one would expect by analogy to ordinary viscosity but to the derivative $d\Omega(k)/dk$. The first feature means that the transfer curve does not scale with k_m alone but with k_0, a parameter from outside the range. If k_m is increased, the transfer remains concentrated in the width k_0 and thus is squeezed into an ever-decreasing fraction of k_m. A specific example is given by (6.10) and (6.11). The second feature means that in x space the transfer cannot be well approximated by an operator of the form $\nu_{\text{eddy}}\nabla^2$. Instead, the enstrophy transfer is better described as a diffusion process in wavenumber space.

In an enstrophy inertial range arising naturally from input of energy and enstrophy at the bottom of the range, the effective straining field is dominated by excitations in the range itself. However, it is still true that, if the range is long, the effective strain acting on wavenumbers well within the range comes from much smaller wavenumbers, although these wavenumbers now are themselves in the enstrophy range. The result is a logarithmic correction to the k^{-1} spectrum. These corrections leave the conclusions of the preceding paragraph intact. The enstrophy transfer is still concentrated in a region of width $\ll k_m$ immediately below k_m. The shape of the transfer curve in this region depends on the spectrum shape in the vicinity of k_0, where now k_0 is characteristic of the bottom of the enstrophy-transferring range. The log-corrected transfer function is given by (6.15), and an explicit illustration for a simple choice of spectrum function shape is given in (6.16) and Fig. 3.

The enstrophy inertial range asymptotically has constant enstrophy transfer, independent of k, and zero energy transfer when it is infinitely long. The net zero energy transfer comes about because the upward energy transfer associated with enstrophy transfer in the vicinity of k_m is exactly compensated by the energy transfer to the principal straining wavenumbers associated with the negative region of the function $\nu(k|k_m)$. The physical mechanism for this is the interaction of straining field with secondary flow discussed in Section 5. An interesting result obtained there is that an average over randomly oriented quasi-uniform straining of small scales gives zero net energy transfer to the straining field, i.e., zero net eddy viscosity. This is because, in addition to the overall tendency to strain the small scales into higher wavenumber (and therefore, in view of enstrophy conservation, lower kinetic energy) structures, there is also some probability of reverse straining, into *lower* wavenumber structures. The distribution of the latter is highly intermittent and represents a gain of energy of the small-scale structures which just compensates, on the average, the loss from the typical straining to higher wavenumbers. We get a nonzero, negative eddy viscosity $\nu(k|k_m)$ for $k\ll k_m$ because we include in $\nu(k|k_m)$ only contributions involving the subgrid wavenumbers $>k_m$, while the reverse straining in effect involves only wavenumbers $<k_m$. If there is no artificial division made into explicit and subgrid wavenumbers, and the *total* transfer function is considered, only the lowest wavenumbers contributing

substantially to the overall straining field actually experience a net negative eddy viscosity.

Our analysis has been confined to the effects of subgrid scales on energy and enstrophy transfer is isotropic turbulence. The results are not directly applicable to the use of eddy viscosities to represent subgrid scales in the computer simulation of a single flow, which is locally nonisotropic everywhere. However, the theoretical difficulties with the eddy-viscosity concept we have brought out are based on physical mechanisms which can be expected to operate also in computer simulations. We feel, therefore, that the theoretical basis for the use of simple eddy viscosities to represent subgrid scales is substantially insecure. Why then have they worked so well in practice? Apparently this is largely because the flow has built-in compensatory mechanisms. The effect of a crude and inaccurate term to represent the passage of energy or enstrophy through the boundary at k_m has the principal effect of distorting the flow in a relatively restricted wavenumber range below k_m. It remains to be seen whether the use of more accurate, and thereby more complicated, representations of subgrid scales pays off. For it to do so, the increased accuracy near k_m, and thereby the implied possibility of lowering k_m in a given calculation, must overbalance the added computational load of carrying the more complex subgrid-scale representation. We intend in a later paper to discuss some modifications of conventional eddy-viscosity formulas which are suggested by the present work, and to take up also the problem of representing the random excitation of explicit scales by subgrid scales. In our present discussion of isotropic turbulence, this phenomenon is lumped into what we have termed the input term in the energy transfer expression. But in the computation of a single flow, the random excitation cannot be handled at all by an eddy-viscosity-like treatment.

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