Appendix

Full expansion for obstacle control barrier function

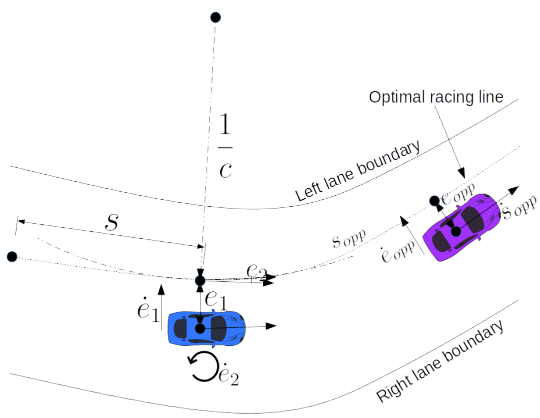

Fig. 18: Illustration for the derivation of obstacle control barrier function.

We define the safety function $h(\cdot)$ as follows:

$$h(\mathbf{x}, \mathbf{x}_{opp}) = \left(\frac{s - s_{opp}}{d_s}\right)^2 + \left(\frac{e_1 - e_{opp}}{d_e}\right)^2 - 1 \tag{13}$$

where s, s_{opp}, e_1, e_{opp} defined before are illustrated in Fig. 18. d_s and d_e are the longitudinal and lateral safe distances from the target opponent vehicle, $\mathbf{x}$ and $\mathbf{x}_{opp}$ are the states of the ego vehicle and the opponent vehicle. We reformulate the vehicle dynamic model (see Eq. 2) in a control offline form as $\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}$.

The first-order CBF requires $\dot{h}(\mathbf{x}, \mathbf{x}_{opp}) + \lambda h(\mathbf{x}, \mathbf{x}_{opp}) \geq 0$. By applying the chain rule, we have $\nabla_{\mathbf{x}} h(\cdot)\dot{\mathbf{x}} + \nabla_{\mathbf{x}_{opp}} h(\cdot)\dot{\mathbf{x}}_{opp} + \lambda h(\cdot) \geq 0$, i.e.,

$$\begin{aligned}
&\nabla_{\boldsymbol{x}} h(\cdot)[f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}] + \nabla_{\mathbf{x}_{opp}} h(\cdot)\dot{\mathbf{x}}_{opp} + \lambda h(\cdot) \geq 0 \\
&\text{thus, } \left[\frac{2(e_1 - e_{opp})}{d_e}\ 0\ 0\ 0\ 0\ \frac{2(s - s_{opp})}{d_s}\ 0\right](f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) \\
&- \frac{2(e_1 - e_{opp})}{d_e}\dot{e}_{opp} - \frac{2(s - s_{opp})}{d_s}\dot{s}_{opp} \\
&+ \lambda\left(\left(\frac{s - s_{opp}}{d_s}\right)^2 + \left(\frac{e_1 - e_{opp}}{d_e}\right)^2 - 1\right) \geq 0
\end{aligned} \tag{14}$$

where the Lie derivitive $L_g h(\boldsymbol{x}) := \nabla_{\boldsymbol{x}} h((\boldsymbol{x}))g(\mathbf{x}) = 0$. This implies that using the first-order CBF doesn't directly contain any constraint on the control commands $\mathbf{u}$. We continue taking derivatives until the constraints of the control appear. We use the second-order CBF by defining $\phi(\mathbf{x}, \mathbf{x}_{opp}) = \dot{h}(\mathbf{x}, \mathbf{x}_{opp}) + \lambda h(\mathbf{x}, \mathbf{x}_{opp})$. For the new safety constraint $\dot{\phi}(\cdot) + \lambda\phi(\cdot) \geq 0$, we have

$$\begin{aligned}
&\ddot{h}(\cdot) + \lambda\dot{h}(\cdot) + \lambda(\dot{h}(\cdot) + \lambda h(\cdot)) \geq 0 \\
&\text{i.e., } \ddot{h}(\cdot) + 2\lambda\dot{h}(\cdot) + \lambda^2 h(\cdot) \geq 0 \\
&\text{By expanding, we get} \\
&2\frac{(v_x - \dot{e}_1 e_2 - \dot{s}_{opp})^2}{d_s} + 2\frac{(\dot{e}_1 - \dot{e}_{opp})^2}{d_e} \\
&+ 2\frac{(s - s_{opp})}{d_s}(K_p p - K_f v_x - \Bigg(-\frac{2C_f + 2C_r}{mv_x}\dot{e}_1 \\
&+ \frac{2C_f + 2C_r}{m}e_2 - v_x\left(v_x + \frac{2C_f l_f - 2C_r l_r}{mv_x}\right)c + \frac{4C_f}{m}\delta\Bigg)e_2 \\
&- \dot{e}_1\dot{e}_2) - 2\frac{(e_1 - e_{opp})}{d_s}\Bigg(\frac{2C_f + 2C_r}{mV_x}\dot{e}_1 + \frac{2C_f + 2C_r}{m}e_2 \\
&- \frac{-2C_f l_f + 2C_r l_r}{mV_x}\dot{e}_2 v_x\left(v_x + \frac{2C_f l_f - 2C_r l_r}{mv_x}\right)c + \frac{4C_f}{m}\delta\Bigg) \\
&+ 2\lambda\Bigg(2\frac{(s - s_{opp})}{d_s}(v_x - \dot{e}_1 e_2 - \dot{s}_{opp}\Bigg) \\
&+ 2\frac{(e_1 - e_{opp})}{d_e}(\dot{e}_1 - \dot{e}_{opp})) \\
&+ \lambda^2\left(\left(\frac{s - s_{opp}}{d_s}\right)^2 + \left(\frac{e_1 - e_{opp}}{d_e}\right)^2 - 1\right) \geq 0
\end{aligned} \tag{15}$$