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Numerical Mathematics II for Engineers

Homework Assignment 7

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by **Group 5**

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Exercise 1

Considered is the parabolic problem $\partial_t u(t,x) + Lu(t,x) = f(x)$, with $(t,x) \in Q_T = (T_0, T) \times \Omega$, and $\Omega = (0,1) \subset \mathbb{R}$. With the elliptic operator $Lu = -a\partial_x^2 u + b\partial_x u$ acting on the spatial part of u, following Dirichlet boundary conditions and initial conditions are utilized:

$$\begin{aligned} u(t,0) &= g0 \in \mathbb{R}, \\ u(t,1) &= g1 \in \mathbb{R}, \\ u(T_0,x) &= u_0(x). \end{aligned}$$

Furthermore, it is supposed that the initial conditions satisfy the boundary conditions $u_0(0) = g_0$ and $u_0(1) = g_1$, where $a, b \in \mathbb{R}$ and $a > 0$.

a) Assuming $u \in C^{2,4}(\bar{Q}_T)$, for $a = 1$, $b = f = 0$, the considered parabolic problem is adjusted

$$u_t - u_{xx} = 0. \tag{1}$$

Accordingly, the Θ-Scheme is adapted to $f = 0$ for at any time point $k+1$

$$w = \underbrace{\frac{1}{\tau}\left(u_n^{k-1} - u_n^k\right)}_{D_t^+ u_n^k} + \underbrace{\left[\Theta L_h u_n^{k+1} + (1-\Theta) L_h u_n^k\right]}_{D^+ D^- u_n^k}. \tag{2}$$

Here, every u^{k+1} term is expanded using Taylor-Series, and thus the dependence of the problem is fixed to u^k

$$\begin{aligned} D_t^+ u_n^k &= \frac{1}{\tau}\left(u_n^k + u_{n,t}^k \cdot \tau + u_{n,tt}^k \,\frac{\tau^2}{2} + \mathcal{O}(\tau^3)\right) - \frac{1}{\tau} u_n^k \\ &= u_{n,t}^k + u_{n,tt}^k \,\frac{\tau}{2} + \mathcal{O}(\tau^2) && (3) \\ D^+D^- u_n^k &= \Theta L_h \left[u_n^k + u_{n,t}^k \,\tau + \mathcal{O}(\tau^2)\right] + (1-\Theta) L_h \left[u_n^k\right] \\ &= L_h u_n^k + \Theta L_h \left[u_{n,t}^k \,\tau + \mathcal{O}(\tau^2)\right]. && (4) \end{aligned}$$

In order to shorten the summation, Equation (3) and Equation (4) is written in a form that the first terms from both equations imply Equation (1)

$$\begin{aligned} w &= \underbrace{u_t^k + L_h u_n^k}_{u_t - u_{xx} + \mathcal{O}(h^2)} + \tau\left(\frac{u_{n,tt}^k}{2} + \Theta L_h u_{n,t}^k\right) + \mathcal{O}(\tau^2) \\ &= 0 + \mathcal{O}(h^2) + \tau\left(\frac{u_{n,t}^k}{2} + \Theta L_h u_n^k\right)_t + \mathcal{O}(\tau^2) \\ &= \tau\left(\frac{u_{n,t}^k}{2} + \Theta L_h u_n^k\right)_t \mathcal{O}(\tau^2 + h^2) \Rightarrow \left(\frac{u_{n,t}^k}{2} + \Theta L_h u_n^k\right)_t + \mathcal{O}(\tau + h^2). && (5) \end{aligned}$$

Hence, the maximum norm $||w||_{\tau,h,\infty}$ for w from Equation (5), gives the maximum value of w with an error of $\mathcal{O}(\tau + h^2)$.

Plugging $\Theta = \frac{1}{2}$ in Equation (5) yields

$$w = \frac{\tau}{2}\left(\underbrace{u_{n,t}^k + L_h u_n^k + \mathcal{O}(h^2)}_{=0}\right)_t + \mathcal{O}(\tau^2 + h^2) \tag{6}$$

The bracket term in Equation (6) vanishes, as it implies exactly Equation (1). Hence, only the consistency error of order $\mathcal{O}(\tau^2 + h^2)$ remains in $||w_{\Theta=1/2}||_{h,\tau,\infty}$.

b) Let $a = 1$, $b = 0$, $f = 0$, $g_0 = 0$, $g_1 = 1$, and, $u_0(x) = H(x - 1/2)$, with H being the Heaviside function. The solution of the problem is built up through the eigenvalue problem. Here, the solution of the inhomogeneous boundary condition is

$$u(t,x) = u_{hom}(x) + \underbrace{\sum_{k=1}^{n} \bar{a}_k v_k(x)\, e^{-\lambda_k t}}_{u_0 - u_{hom}} \quad \text{, with } \lambda_k = (k \cdot 2\pi)^2. \tag{7}$$

Replacing u_{hom} with x satisfies $Lu_{hom} = 0$, and therefore enables the variable separation to be directly applied on the sum term in Equation (7). Hence, the solution can be written in the following form by employing Fourier series for the first 300 terms

$$u(t,x) = x - \frac{1}{\pi}\left(\sum_{k=1}^{300} \eta_k \cdot \frac{1}{k}\sin\left(k\cdot 2\pi x\right)\right) e^{-(k\cdot 2\pi)^2 t} \tag{8}$$

where $\eta_k = \begin{cases} -1 & \text{for even } k \\ +1 & \text{for odd } k \end{cases}$.

For the implementation of the solution, please refer to the online submitted file `a07ex01getsol.m`.

c) Please refer to the online submitted `a07ex01getPDE.m` file.

d) In this exercise, the PDE from b) was solved iteratively for differentΘ, M, and N starting from $T_0 = 1E-5$ and to $T = 0.01$. The solutions are plotted along with the analytical solutions in Figure 1, Figure 2 and Figure 3 for $\Theta = [0, 1/2, 1]$ respectively. In following these result are discussed. For the implementation, please refer to online submitted `a07ex01d.m` and `a07ex01solve_d.m` files.

For the stability of the Θ-Schema, the diagonal dominance of M_h requires

$$1 - 2(1-\Theta)\frac{\tau}{h^2} \geq 0. \tag{9}$$

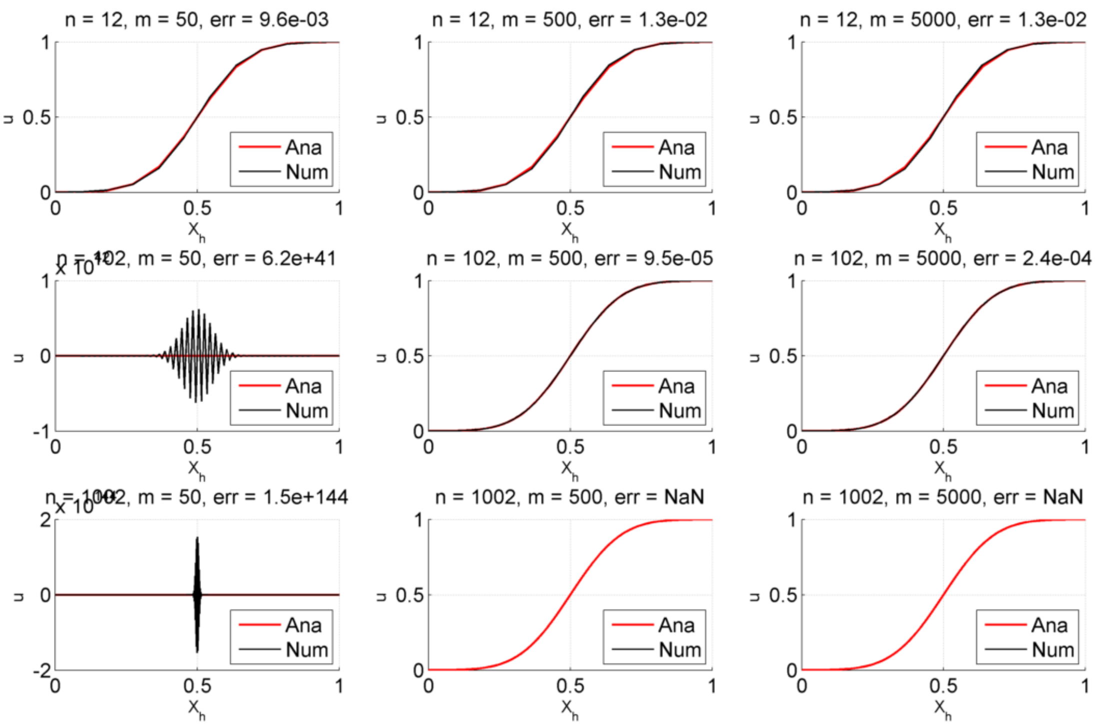

Figure 1 | Solutions of the PDE for Euler Explicit Integration, $\Theta = 0$

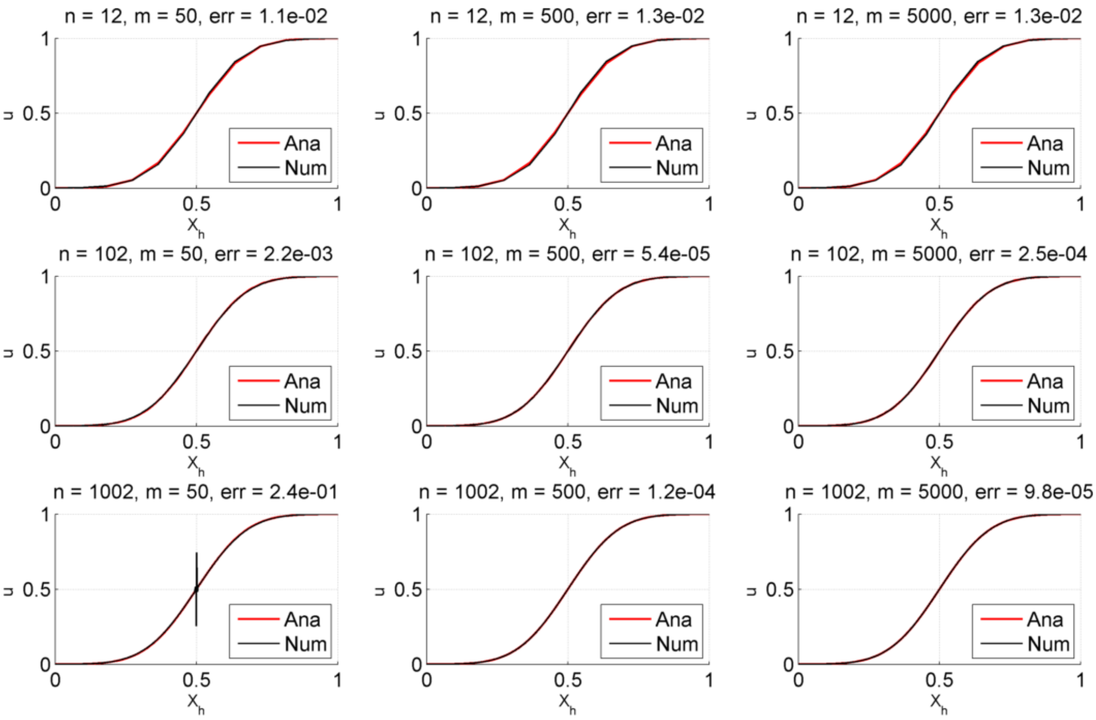

Figure 2 | Solutions of the PDE be for Euler Implicit Integration, $\Theta = 1/2$

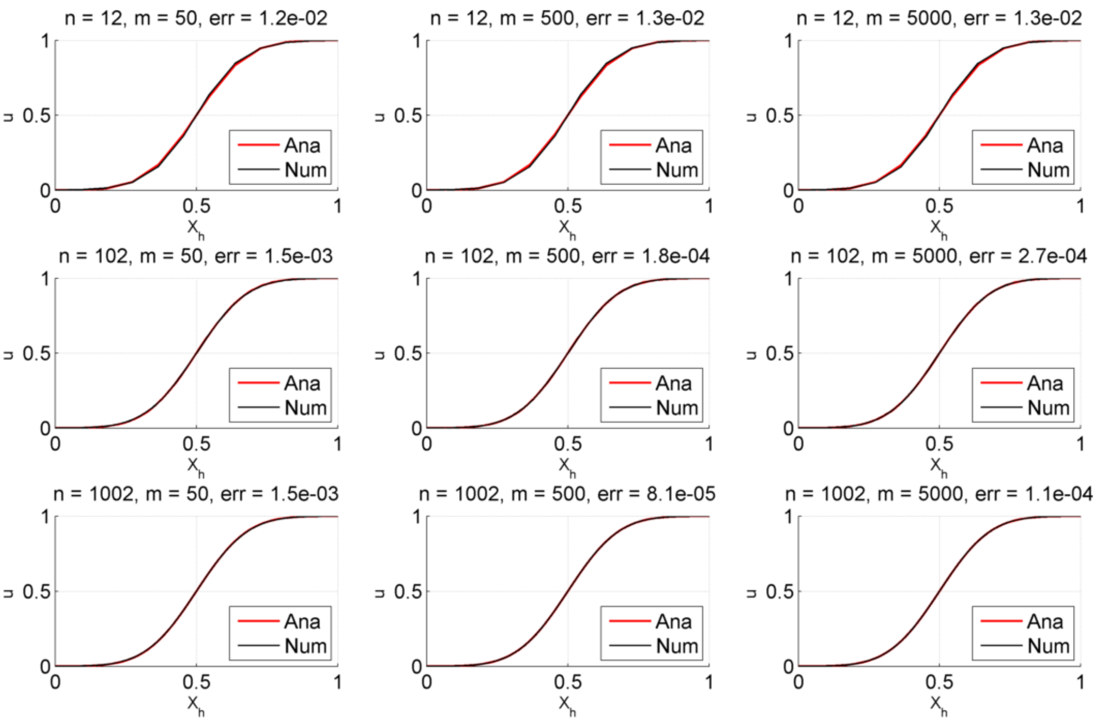

Figure 3 | Solutions of the PDE be for Crank-Nicholson Integration, $\Theta = 1$

In the explicit Euler method (Figure 1), this requirement is violated for the $n = 1002$ row, as well as for the solution from $n = 102, m = 50$. In contrast, the implicit Euler method (Figure 3) doesn't conflict with Equation (9), as $\Theta = 0$ yields 1 on the left hand side for every M and N.

e) With $\tau = \frac{T}{M}$, M is also considerable in violating the stability in terms of Equation (9) requirement. Results from d) show that Euler Explicit method may become rapidly hazardous, when M is variated. For Euler Implicit Method, the impact of vanishes. Therefore, an arbitrary $M > 0$ would satisfy the condition.

f) In this exercise, the PDE with $u_0(x) = 0$, $f(x) = 1$, $g_0 = g_1 = 0$, $a = 1/100$, $b = 1$ and $T = 100$ was solved iteratively for three different Θ and for common difference stencils with $N = 50$ and $M = 100$. The solutions are plotted along with the analytical solutions in Figure 4 respectively. In following these result are discussed. For the implementation, please refer to online submitted `a07ex01f.m` files.

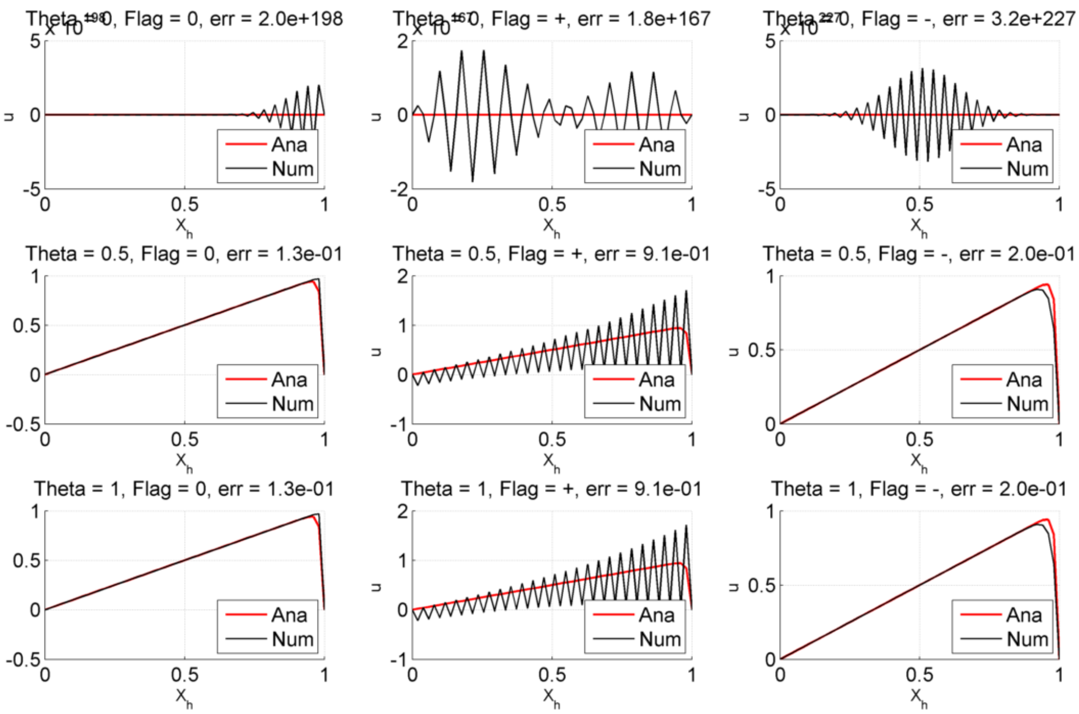

Figure 4 | Results of the Parabolic PDE for the considered integration methods, as well as difference stencils with $N = 50$, $M = 100$.

For $\Theta = 0$, all three solutions diverged due to not fulfilling the stability criteria in Equation (9). Furthermore, the forward difference stencil shows for $\Theta = 0.5$ and $\Theta = 1.0$ consistent results, if not converged.

Running the simulation with $N = 200$ improves the convergence of the solutions in Euler Implicit and Crank-Nicolson methods for all three stencils, whereas the results from Euler Explicit method remains diverged.

Exercise 2

a) Since the proposed function u does not depend on t we have $\partial_t u = 0$ and so we have

$$\begin{aligned} -\Delta u &= -(\partial_{xx} + \partial_{yy} + \partial_{zz})u \\ &= (k^2\pi^2 + l^2\pi^2 + m^2\pi^2)u \\ &= \lambda_{klm} u \end{aligned}$$

which is an eigenfunction with eigenvalue

$$\lambda_{klm} = \pi^2(k^2 + l^2 + m^2)$$

the normalization factor is given by

$$1 = \int_0^1 u^2 dV = a_{klm}^2 \int_0^1 sin(k\pi x)^2 dx \int_0^1 sin(l\pi y)^2 dy \int_0^1 sin(k\pi x)^2 dz$$

which for $k, l, m \in \mathbb{N}$ is

$$a_{klm} = \sqrt{8} \tag{10}$$

We can rewrite the inner product between u_{klm} and u_{nop} as

$$a_{klm} a_{nop} \int_0^1 sin(k\pi x) sin(n\pi x) dx \int_0^1 sin(l\pi y) sin(o\pi y) dy \int_0^1 sin(k\pi x) sin(p\pi x) dz \tag{11}$$

The integral of the first term for k with the corresponding normalization terms of a_{klm} and a_{nop} is of the form

$$sin(\pi k) cos(\pi n) - cos(\pi k) sin(\pi n) \tag{12}$$

Which for $k, n \in \mathbb{N}$ is equal to δ_k^n. We have a product of three of these kronecker deltas and obtain δ_{klm}^{nop} which is the definition of the function we wanted to show. To show that the weights w_{klm} are given by the equation

$$\frac{1}{\lambda_{klm}} \int_\Omega f(x) u_{klm}(x) dx dy dz \tag{13}$$

we just substitute into $Lu = f$ and note that since u_{klm} is a basis, we can also write $f(x) = \sum_{nop} w_{nop} u_{nop}$ as

$$
\begin{aligned}
f = Lu = -\Delta u = -\Delta \sum_{klm} w_{klm} u_{klm} &= \sum_{klm} w_{klm}(-\Delta u_{klm}) \\
&= \sum_{klm} \frac{1}{\lambda_{klm}} \int_\Omega f u_{klm} dxdydz (-\Delta u_{klm}) \\
&= \sum_{klm} \int_\Omega \left(\sum_{nop} w_{nop} u_{nop} \right) u_{klm} dxdydz \cdot u_{klm} \\
&= \sum_{klm} \left(\sum_{nop} \int_\Omega w_{nop} u_{nop} u_{klm} dxdydz \right) \cdot u_{klm} \\
&= \sum_{klm} \left(\sum_{nop} w_{nop} \int_\Omega u_{nop} u_{klm} dxdydz \right) \cdot u_{klm}
\end{aligned}
$$

.

Given that $\int_\Omega u_{nop} u_{klm} dxdydz = \delta_{klm}^{nop}$ this cancels out the inner sum, so we have:

$$
f = Lu = \sum_{klm} w_{klm} \cdot u_{klm}
$$

For $f = 1$ we have

$$
\begin{aligned}
w_{klm} &= \frac{1}{\lambda_{klm}} \int_\Omega u_{klm} dxdydz \\
&= \frac{a_{klm}}{\lambda_{klm}} \int_0^1 sin(k\pi x) dx \int_0^1 sin(l\pi y) dy \int_0^1 sin(k\pi x) dz \\
&= \frac{a_{klm}}{\lambda_{klm}} \frac{8}{\pi^3 klm}
\end{aligned}
$$

which since $k, l, m \in \mathbb{N}$ reduces to:

$$
w_{klm} = \begin{cases} \frac{a_{klm}}{\lambda_{klm}} \frac{8}{\pi^3 klm}, & \text{if } k, l, m \text{ all divisible by 2} \\ 0, & \text{otherwise} \end{cases} \tag{14}
$$

b) In this exercise, the sample code from assignment 4 was extended to $\Omega = (0, 1)^3$ and the problem $-\Delta u = 1$ was solved with homogeneous Dirichlet boundary conditions. Figure 5 illustrates the solution, whereas the error is plotted in Figure 6. For the implementation, please refer to the online submitted `a07ex02b.py` file.

c) Please refer to the online submitted `a07ex02c.py` file.

d) The error is plotted in Figure 7. Please refer to the online submitted `a07ex02d.py` and `a07ex02d_error.py` files.

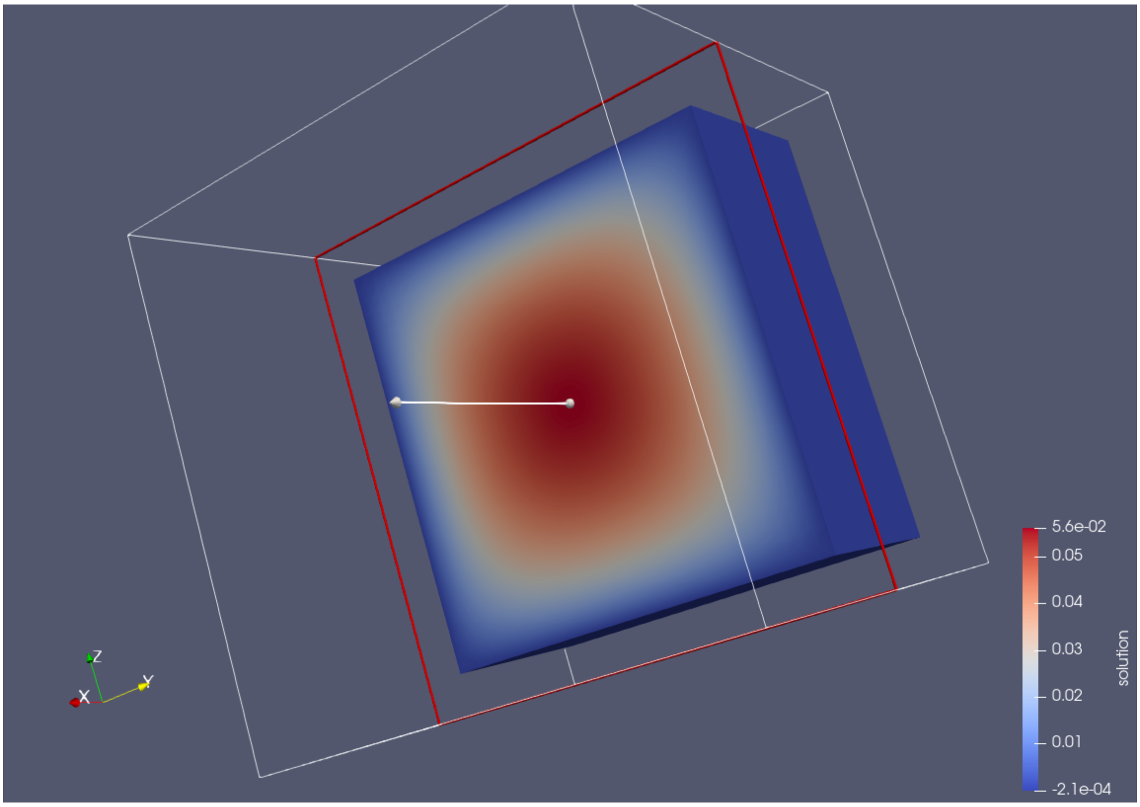

Figure 5 | Solution from 2a, very similar as 2b

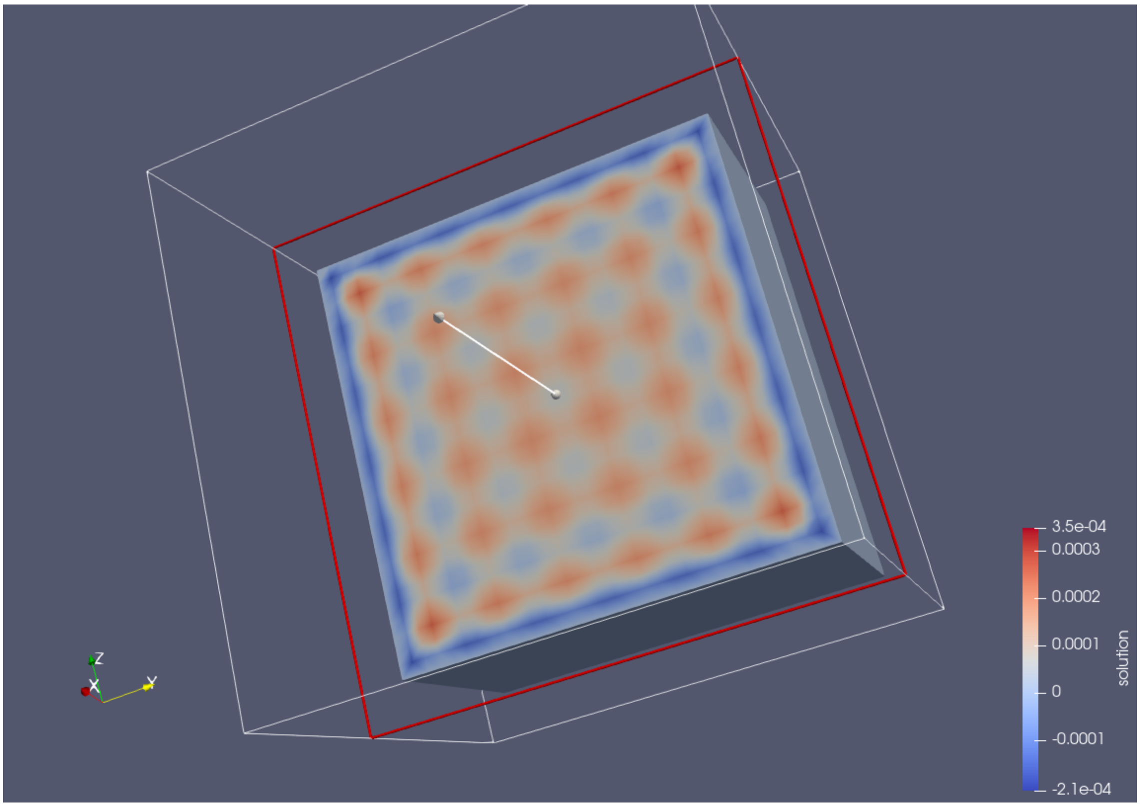

Figure 6 | Error between 2a an 2b. Note that the relative error is of about two orders of magnitude.

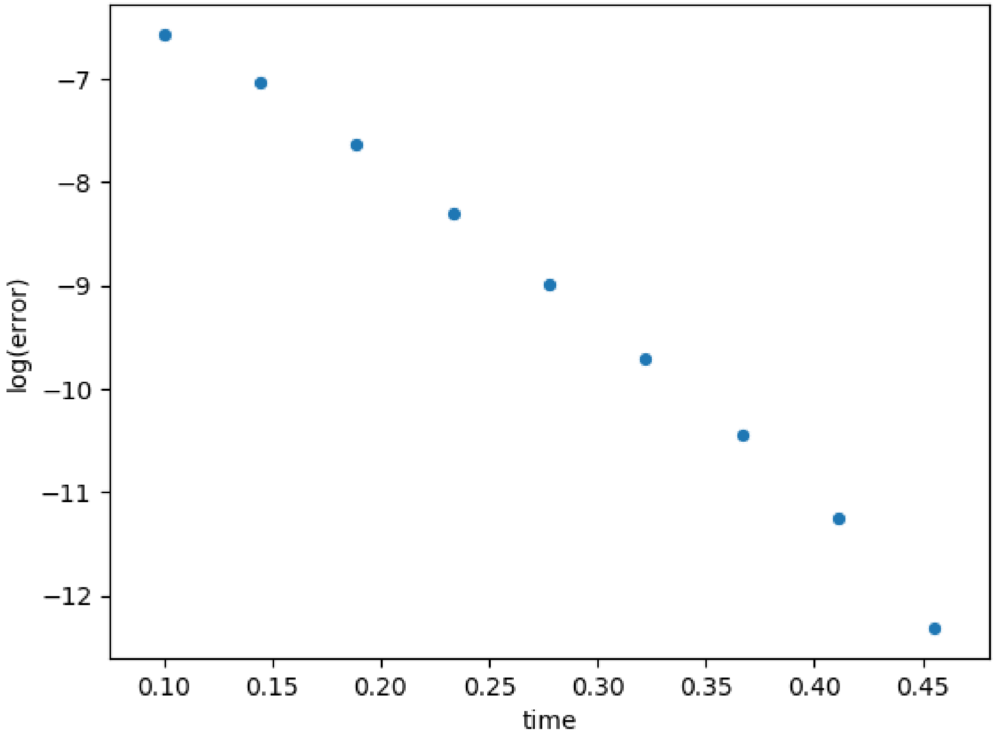

Figure 7 | Error between the exact solution and the one computed numerically using the implicit Euler method. Note that even though the error decreases, it stabilizes around a value of 0.000709, which was subtracted in order to better distinguish a possibly stable decreasing error rate