TECHNICAL UNIVERSITY OF BERLIN

Faculty II - Mathematics and Natural Sciences

Institute of Mathematics

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Numerical Mathematics II for Engineers

Homework Assignment 6

Submitted on

by **Group 5**

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Exercise 1

Considered is the elliptic eigenvalue problem of finding an eigenfunction / eigenvector pair (u, λ) such that $Lu = \lambda u$ in Ω supplemented with suitable boundary conditions on $\partial \Omega$.

a) Taking $Lu = -u'' + u$ in $\Omega = (0, 1)$, the discrete eigenvalues yielded through

$$u(x) = \tag{1}$$

$$\lambda_k = \frac{4}{h^2} \sin\left(\frac{h\pi k}{2}\right) + 1 \tag{2}$$

for the k-th eigenvalue. In following, the eigenvalues are derived from L_h using non-compact 3 point stencil with $N = 200$ and $h = 1/(N+1)$ respectively for

- Homogeneous Dirichlet boundary conditions,

- Homogeneous Neumann boundary conditions,

- Periodic boundary conditions.

The list of the first 20 Eigenvalues are stated in Table 1, while the eigenvectors of the second and the 20th eigenvalues are compared in Figure 1.

For the implementation, please refer to the online submitted `a06ex01_solveEVPa.m` file.

#	Analytical	Dirichlet	Neumann	Periodic
20	**3.916799e+03**	**3.916799e+03**	**3.916799e+03**	**3.916799e+03**
19	3.537820e+03	3.537820e+03	3.537820e+03	3.537820e+03
18	3.177715e+03	3.177715e+03	3.177715e+03	3.177715e+03
17	2.836574e+03	2.836574e+03	2.836574e+03	2.836574e+03
16	2.514479e+03	2.514479e+03	2.514479e+03	2.514479e+03
15	2.211508e+03	2.211508e+03	2.211508e+03	2.211508e+03
14	1.927736e+03	1.927736e+03	1.927736e+03	1.927736e+03
13	1.663233e+03	1.663233e+03	1.663233e+03	1.663233e+03
12	1.418062e+03	1.418062e+03	1.418062e+03	1.418062e+03
11	1.192283e+03	1.192283e+03	1.192283e+03	1.192283e+03
10	9.859529e+02	9.859529e+02	9.859529e+02	9.859529e+02
9	7.991206e+02	7.991206e+02	7.991206e+02	7.991206e+02
8	6.318321e+02	6.318321e+02	6.318321e+02	6.318321e+02
7	4.841284e+02	4.841284e+02	4.841284e+02	4.841284e+02
6	3.560454e+02	3.560454e+02	3.560454e+02	3.560454e+02
5	2.476146e+02	2.476146e+02	2.476146e+02	2.476146e+02
4	1.588622e+02	1.588622e+02	1.588622e+02	1.588622e+02
3	8.981017e+01	8.981017e+01	8.981017e+01	8.981017e+01
2	**4.047520e+01**	**4.047520e+01**	**4.047520e+01**	**4.047520e+01**
1	1.086940e+01	1.086940e+01	1.086940e+01	1.086940e+01

Table 1 | List of the first 20 Eigenvalues ordered by magnitude. 2nd and 20th Eigenvalues are written in bold to be distinguished easily.

(a) 2nd Eigenvalue
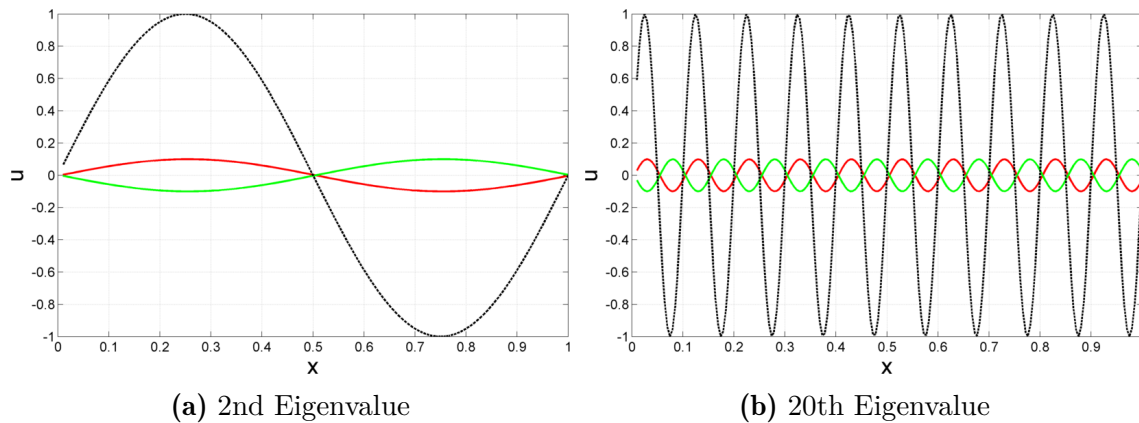
(b) 20th Eigenvalue

Figure 1 | Comparison of the Eigenvectors between discretized EV (black dashed), Dirichlet B.C. (red), Neumann B.C. (blue) and Periodic B.C. (green) for the 2nd and 20th Eigenvalue respectively.

b)   Considering

$$-\Delta v = \lambda v \ \text{ in } \Omega, \tag{3}$$
$$v = 0 \ \text{ on } \partial\Omega \tag{4}$$

with $\Omega = \{(x, y) \in \mathbb{R}^2 : (x - L/2)^2 + (y - L/2)^2 < R\}$ for given $L, R \in \mathbb{R}$ and $R > 0$. The Eigenvalues of such problem can be solved analytically utilizing the Bessel function of first kind $J_n(\xi)$ with $\xi = \sqrt{\lambda}R$ at its zero. Taking the corresponding k-th zeros form the lecture notes, the first six Eigenvalues for $R = 1$ and $R = 2$ are stated in Table 2.

|	1	2	3	4	5	6
R = 1	5.7832	14.6820	14.6820	26.3746	26.3746	30.4713
R = 2	1.4458	3.6705	3.6705	6.59360	6.5936	7.6178

Table 2 | First six Eigenvalues of the disc problem with respect to $R = 1$ and $R = 2$

The solution was implemented in the form of a function submitted in `a06ex01_solveEVPb.m`. The function takes an arbitrary radius $R > 0$ as input and gives the first six Eigenvalues and their n values.

c)   In this exercise, the domain Ω from assignment 5, exercise 1 has been discretized with respect to the disc problem in lecture notes with $L = 2.2$ and $R = 1$ and the the Laplace Operator L_h has been built accordingly. The numerical Eigenvalues have been numerically computed via reduced matrix L_h for $h = 1/(N + 1)$ applied on three mesh resolutions with $N = 7$, $N = 63$, and $N = 511$. A comparison between the numerical and the exact Eigenvalues from **b)** is listed in Table 3. The first four distinct eigenfunctions are demonstrated in Figure 2. For the implementation, please refer to online submitted `a06ex01_solveEVPc.m` file.

#	Discretisized	N = 7	N = 63	N = 511
6	3.047128e+01	2.975207e+01	3.151469e+01	3.061249e+01
5	2.637459e+01	2.975207e+01	2.733002e+01	2.650670e+01
4	2.637459e+01	2.975207e+01	2.724406e+01	2.648730e+01
3	1.468200e+01	1.923312e+01	1.520505e+01	1.475030e+01
2	1.468200e+01	1.923312e+01	1.520505e+01	1.475030e+01
1	5.783207e+00	8.714178e+00	5.993756e+00	5.810162e+00

Table 3 | List of the first 6 Eigenvalues ordered by magnitude for different mesh resolution and the discretized analytical solution.

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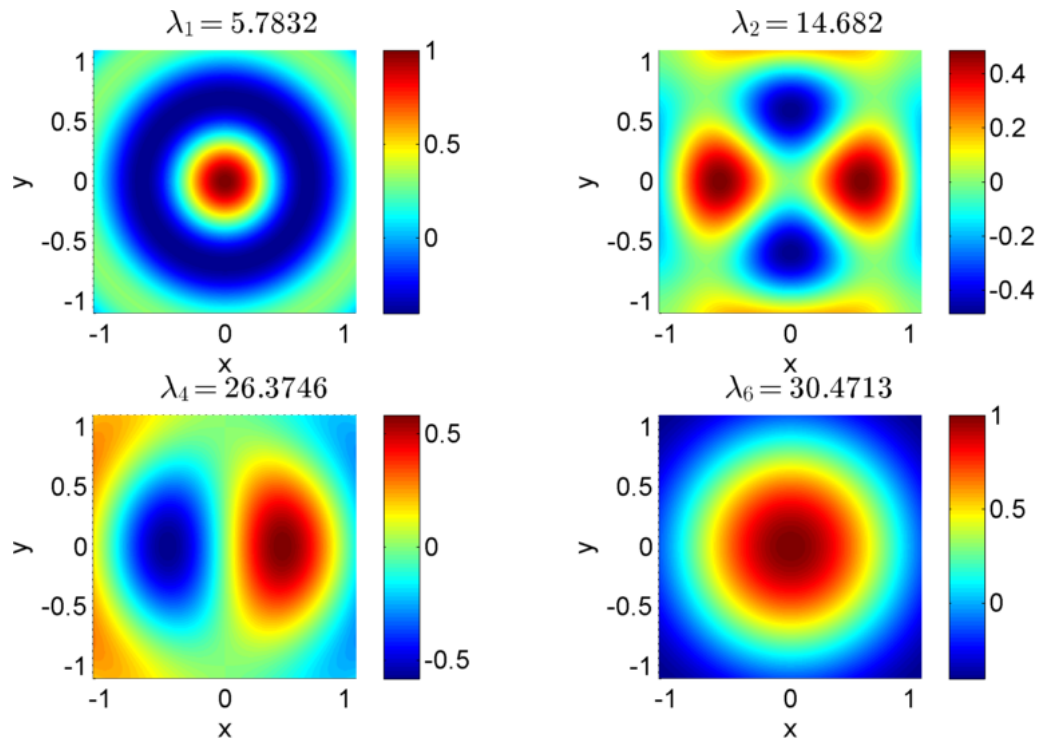
Figure 2 | Eigenfunctions of the first 4 distinct eigenvalues. The value of the data has been respectively labeled with the color bar next to the plot.

Exercise 2

a) The discrete algorithm is implemented in a similar fashion to the one for section $II.5$ in the script, changing the lexicographic indexing to have the angle φ vary over i and r over j. The operators are constructed in blocks by using Kronecker products of matrices containing the inverse radius coefficients for $\frac{1}{r_{ij}}\partial_r u_{ij}$ and $\frac{1}{r_{ij}^2}\partial_{\varphi\varphi}u_{ij}$. Multiplied with the corresponding adjacency matrices. The part a) and c) are thus jointly implemented in the program a06ex02_getPDE.py

b) We say the finite difference method is consistent if

$$f_h - L_h R_h u_h \xrightarrow{h \to 0} 0$$

Wlog let us take the maximum norm, that way we can write, by writing out the action of the restriction operator and the L_h matrix:

$$max(f_h - L_h R_h u) \xrightarrow{h \to 0} 0 =$$

So lets consider the particular index (ij) for which the maximal difference is reached, knowing that the difference stencil jumping considering neighbouring indices j is the difference operator with respect to r and the one considering neighbouring i the one for φ

$$|(\partial_{rr}u_{ij} - D_j^+ D_j^-) + \frac{1}{r_{ij}}(\partial_r u_{ij} - D_j^0) + \frac{1}{r_{ij}^2}(\partial_{\varphi\varphi}u_{ij} - D_i^+ D_i^-)|$$

By the triangle inequality we can bound the expression by the sum of the maximums and take out the constant factors $\frac{1}{r_{ij}}$ and $\frac{1}{r_{ij}^2}$ since they are always positive.

$$|\partial_{rr}u_{ij} - D_j^+ D_j^-| + \frac{1}{r_{ij}}|\partial_r u_{ij} - D_j^0| + \frac{1}{r_{ij}^2}|\partial_{\varphi\varphi}u_{ij} - D_i^+ D_i^-|$$

Since we have the $r_{ij} \geq 1$ we can again bound by dropping the inverse factors in front of the last two summands. It is now clear that the total sum will tend to zero for small h, since all the differences are of order $O(h^2)$, so their sum will be too.

"Documentation/Figures/""a06ex02c_uh".png

Figure 3

c) Figure **??** shows the plot of the anulus represented as a square in polar co-ordinates. Notice how the boundary values are satisfied at the upper and lower horizontal edges and how the periodicity condition holds across the left and right vertical ones.

The max error for the given N_1 and N_2 is 0.111

"Documentation/Figures/""a06ex02c_error".png

Figure 4 | Error between the exact solution and the one obtained by the method
in a). Notice the change in scale of the heat map

d) The error of this method is around 0.15 and does not seem to decrease with
further refinement of the mesh. Another difference with respect to the method used
in c is that here the discrete method consistently underestimates the solution, while
in c) it is bot under and overestimated

"Documentation/Figures/""a06ex02d_uh".png

Figure 5 | Plot of the function values obtained using a modified version of the program from a05ex01
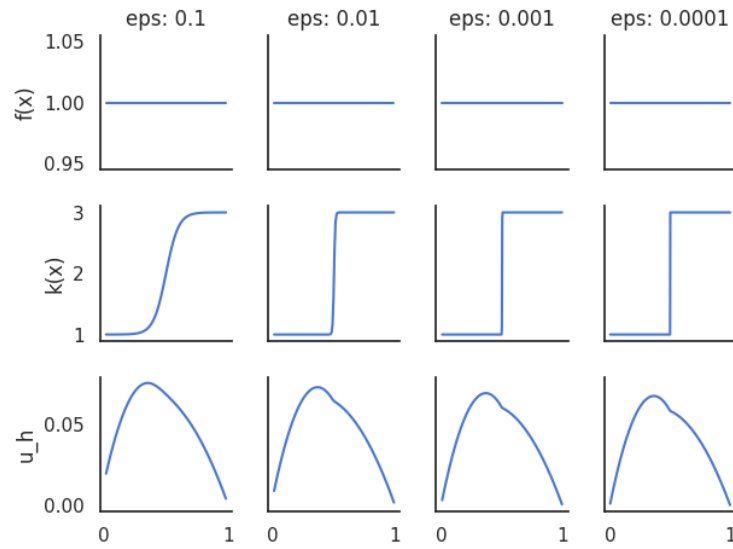
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"Documentation/Figures/""a06ex02d_error".png
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Figure 6 | Error between the exact solution and the one obtained using a modified version of the program from a05ex01. Notice the change in scale of the vertical axis

Exercise 3

a) Please refer to the online submitted `a06e03getPDE.py` file.

b and c) The grid size was empirically defined as $N = 20 + 10\sqrt{\varepsilon^{-1}}$.

Figure 7 | Functions $f(x)$, $k(x)$ and u_h evaluated for $\varepsilon = 0.1, 0.01, 0.001, 0.0001$

The function $k(x)$ will behave as a Heaviside step function for $\varepsilon \to 0$.