Group 5 NumMat II Ex1

25. Oktober 2019

1 Discussion and Analyze of PDEs

1.1 Examples to PDEs

- Common PDE: Kaup-Kupershmidt equation

$$\frac{\partial u}{\partial t} = \frac{\partial^5 u}{\partial x^5} + 10\frac{\partial^3 u}{\partial x^3}u + 25\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x} + 20u^2\frac{\partial u}{\partial x} \tag{1}$$

  is a PDE fifth order.

- Member 1 PDE: Hunter-Saxton equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{2}$$

  is a PDE second order.

- Member 2 PDE: Liouville equation

$$\nabla^2 u + e^{\lambda u} = 0 \tag{3}$$

  is a PDE second order.

- Member 3 PDE: φ^4 - Equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \varphi + \varphi^3 = 0 \tag{4}$$

  is a PDE second order.

1.2 Navier-Stokes

The Navier-Stokes-Equation describes the motion of the viscous fluid substances and is expressed for compressible fluid as

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p + \mu\nabla^2 u + f \tag{5}$$

with ρ the density, u velocity vector, p pressure, and μ kinematic viscosity of the fluid. Equation (5) is expressed in homogenous form by setting $f = 0$ as follows

$$\rho(\partial_t u + u \cdot \nabla u) + \nabla p - \mu\nabla^2 u = 0 \tag{6}$$

For $u(t, x) = (u_0 x_2(H - x_2), 0)^T$ with $u_0 \in \mathbb{R}$, $x = (x_1, x_2) \in \Omega = \mathbb{R} \times (0, H)$, and $t \in (0, \infty)$, the partial differentiations result

$$\frac{\partial u}{\partial t} = (0, 0)^T \tag{7}$$

$$\nabla u = (0, 0)^T \tag{8}$$

$$\nabla^2 u = (0, 0)^T \tag{9}$$

since u is not t-dependent and u_1 and u_2 are not effected by x_1 and x_2, respectively. Equations 7, 8 and 9 show that u is a twice differentiable function, satisfying the homogenous Navier-Stokes PDE with a boundary condition in a domain $\Omega \in \mathbb{R}^2$, which is reffered to as the classical solution for second order PDEs.

For the given conditions, Equation (6) can be expressed as

$$\nabla p = 0 \tag{10}$$

This can be referred to a 2D-flow model of a fluid in a tube with a width of H at any certain height, which is observed along the gravity axis. Therefore, the pressure in the domain Ω is described as $p = \mathit{const.} \in [0, \infty)$.

2 Linear Algebra

Given is the Toeplitz-Matrix

$$K_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{11}$$

The function $f(x) = \frac{1}{2}x^T K_4 x : \mathbb{R}^4 \to R$ can be expressed as

$$f(x) = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + x_4^2 \tag{12}$$

and the gradient of $f(x)$ is

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \\ \frac{\partial f}{\partial x_4} \end{bmatrix} = \tag{13}$$