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WS 2019/20

Numerical Mathematics II for Engineers

Homework assignment 7 : *Problems with time.*

Programming assignments can be solved in MATLAB/Python/Julia.

Deadline : submit until December, Friday 13, 2019.

1. Excercise : The Θ-scheme in $Q_T = (T_0, T) \times \Omega$ and $\Omega = (0,1) \subset \mathbb{R}^1$ **12 points**

Consider the parabolic problem

$$\partial_t u(t,x) + Lu(t,x) = f(x) \qquad (t,x) \in Q_T,$$

with the elliptic operator $Lu = -a\partial_x^2 u + b\partial_x u$ acting on the spatial part of u. We use Dirichlet boundary conditions $u(t,0) = g_0 \in \mathbb{R}$ and $u(t,1) = g_1 \in \mathbb{R}$ and initial conditions $u(T_0, x) = u_0(x)$. Suppose the initial conditions satisfy the boundary conditions $u_0(0) = g_0$ and $u_0(1) = g_1$ and $a, b \in \mathbb{R}$ with $a > 0$. As in the lecture $u_i^k = u_h(t^k, x_i)$ for $t^k = T_0 + k\tau$ for $k = 0 \dots M$ and $\tau M = (T - T_0)$.

a) Let $a = 1, b = f = 0$ and assume $u \in C^{2,4}(\bar{Q}_T)$. Show the Θ scheme is consistent in $\|w\|_{\tau,h,\infty} = \max_{n,k} |w_n^k|$ with $\mathcal{O}(\tau + h^2)$. If $u \in C^{3,4}(\bar{Q}_T)$, $\Theta = 1/2$, then $\mathcal{O}(\tau^2 + h^2)$.

b) Let $a = 1$, $b = f = 0$ and $u_0(x) = H(x - 1/2)$ (Heaviside function H at $T_0 = 0$) and $g_0 = 0, g_1 = 1$. Find the exact solution of the problem using the Fourier series

$$u_0(x) - x = -\tfrac{1}{\pi}\left(\tfrac{1}{1}\sin(w) - \tfrac{1}{2}\sin(2w) + \tfrac{1}{3}\sin(3w) + \ldots\right),$$

for $w = 2\pi x$ using a separation ansatz. Write a function `u=a07ex01getsol(xh,t)` returning the exact solution at time `t` and `xh` (use 300 terms in the series).

c) Write a function `[Ah,Mh,fh] = a07ex01getPDE(xh,f,consts,theta,tau,flag)`, so that the discrete solution $u_h^k \in \mathbb{R}^{N+2}$ at time t^{k+1} is computed

$$A_h u_h^{k+1} = M_h u_h^k + \tau f_h,$$

using the Θ-scheme, where `f` is a function handle, `consts=[a b]`, `flag='+','-','0'` indicates the discretization of u' in L. As usual `xh`$= (0, h, 2h, ..., 1) \in \mathbb{R}^{N+2}$. Use $u_i^{k+1} = u_i^k$ for $i = 0$ and $i = N+2$ to satisfy the discrete boundary conditions.

d) Write a function `[uM,err]=a07ex01d(N,M,T,Ah,Mh,fh)` solving the problem for $N = \{10, 100, 1000\}$, $T = 0.01$, $M = \{50, 500, 5000\}$ for the explicit/implicit Euler and Crank-Nicolson scheme. Plot the solutions $u_h^M \in \mathbb{R}^{N+2}$ (overlayed 1D plots), compute the error `err` $= \|u_n^k - u(t^k, x_n)\|_{\tau,h,\infty}$ and discuss your observations.

Please turn the page!

e) For which M we have stability ? Discuss advantages of the different methods.

f) Now let $u_0(x) = 0$ at $T_0 = 0$, $f(x) = 1$, $g_0 = g_1 = 0$, $a = 1/100$, $b = 1$, $T = 100$. Solve the problem for $N = 50$ numerically, plot the solution, and discuss the effect of different choices `flag='+','-','0'` and $\Theta = \{0, 1/2, 1\}$. Discuss what changes for $N = 200$? Compare with the solution u_ϵ from Assignment 4, Exercise 3. Use the file/script/function `a07ex01f`.

Remark : Reuse functions whenever possible ! E.g, reuse `a07ex01getsol` from b) to compute `err` in d). For the discretization of L reuse solutions of previous assignments.

Hint : Since H in b)-e) is nonsmooth, start the solution at $T_0 = 10^{-5}$ with $u_0(x)$ given by `u0=a07ex01getsol(xh,T0)`. Use $\tau = (T - T_0)/M$ and $t^k = T_0 + k\tau$ defined above.

2. Exercise : Implicit Euler method in 3D (or 2D) **10 points**

Let $\mathbf{x} \in \Omega = (0,1)^3 \subset \mathbb{R}^3$ and $t \in (0, \infty)$ and consider the heat equation

$$\partial_t u(t, \mathbf{x}) + Lu(t, \mathbf{x}) = f(\mathbf{x}) \tag{1}$$

for $Lu = -\Delta u$ with homogeneous Dirichlet boundary conditions $u(t, \mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$.

a) Show $u_{klm}(\mathbf{x}) = a_{klm} \sin(k\pi x) \sin(l\pi y) \sin(m\pi z)$ is an eigenfunction $(k, l, m \in \mathbb{N})$. What is the eigenvalue λ_{klm} ? Compute a_{klm} so that $\int_\Omega |u_{klm}(\mathbf{x})|^2 \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = 1$ and then show

$$\int_\Omega u_{klm}(\mathbf{x})\, u_{nop}(\mathbf{x}) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \begin{cases} 1 & k = n, l = o, m = p \\ 0 & \text{otherwise} \end{cases}.$$

First seek solutions of $Lu = f$ with homogeneous Dirichlet boundary conditions. Assume $u_{klm}(\mathbf{x})$ form a basis, so that we can expand $u(\mathbf{x}) = \sum_{k,l,m \in \mathbb{N}} w_{klm} u_{klm}(\mathbf{x})$. Show the coefficients $w_{klm} \in \mathbb{R}$ are given by

$$w_{klm} = \tfrac{1}{\lambda_{klm}} \int_\Omega f(\mathbf{x}) u_{klm}(\mathbf{x}) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z.$$

Compute $u(\mathbf{x})$ for $f(\mathbf{x}) = 1$ and write `u=a07ex02getsola(xx,yy,zz,R)` which returns the truncated Fourier series as $u_h \in \mathbb{R}^{N+2 \times N+2 \times N+2}$ for $k, l, m = 1, ..., R$.

b) Extend the sample code from assignment 4 to $\Omega = (0,1)^3$ and solve $-\Delta u = 1$ with homogeneous Dirichlet boundary conditions with $h = 1/(N + 1)$ as large as possible. Compare with the solution from a) and state the error in the maximum norm. Use the file/script/function `a07ex02b(N)`.

c) Extend the numerical solution from b) to solve the heat equation using the implicit Euler method with initial data $u(t = 0, \mathbf{x}) = 0$. Solve the problem numerically in the file/function `uh=a07ex02c(N,M,T)`.

See next page !

d) Compare the solution from c) with the exact solution at $T = 0.1$. Assume the solution can be expanded $u(t,\mathbf{x}) = \sum_{k,l,m} w_{klm}(t) u_{klm}(\mathbf{x})$. Derive ODEs for the coefficients $w_{klm}(t)$ and solve. Return a truncated expansion for $k, l, m = 1, ..., R$ of the exact solution u via the file/function `u=a07ex02getsold(T,xx,yy,zz,R)`. Measure the error using $\max_{x\in\Omega_h} |u(t^M, x) - u_h(t^M, x)|$ for $t^M = M\tau = T$.

Hint 1 : $x'(t) = ax(t) + b$ with $x(0) = 0$ is solved by $x(t) = \frac{b}{a}(e^{at} - 1)$.

Hint 2 : If you like, you can also solve this exercise in 2D. Then you can solve larger domains and plotting is easier. Then introduce $u_{kl}(\mathbf{x}) = a_{kl} \sin(k\pi x) \sin(l\pi y)$, λ_{kl}, v_{kl} instead of the quantities with three indices. With a direct solver $N = 30 - 50$ should be possible in 3D, otherwise you might want to try a (built-in) iterative solver such as `pcg` depending on your platform.

Hint 3 : In order to help with visualization, we provide `outvtk_structured3d` to output the data to VTK file format (ParaView) for rectilinear 3D meshes. Download ParaView at `https://www.paraview.org/download/` for free for Windows, Linux and macOS.

Hint 4 : If there are problems with a,b) or you like another test case, you can also use $f(\mathbf{x}) = 0$ and the eigenfunction $u_0(\mathbf{x}) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ of L as alternative/additional initial data for c,d).

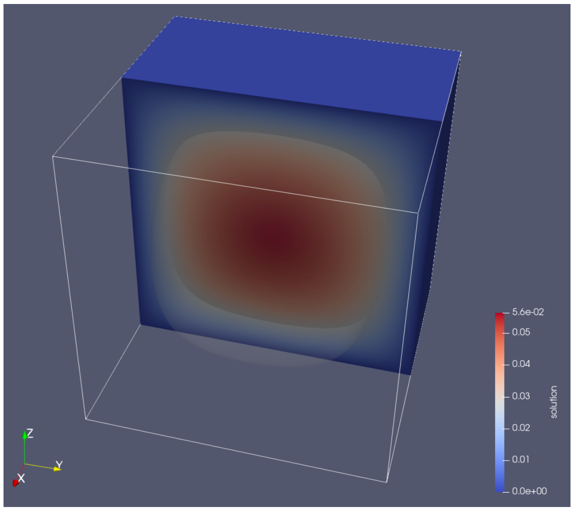

FIGURE 1 – Paraview visualization of solution $u(\mathbf{x})$ for b) with $N = 200$ (clip + contour).

total sum : 22 points