Technical University of Berlin
Faculty II - Mathematics and Natural Sciences
Institute of Mathematics
Dr. D. Peschka, A. Selahi

Numerical Mathematics II for Engineers

Homework Assignment 4

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by **Group 5**

Kagan Atci	338131	Physical Engineering, M.Sc.
Navneet Singh	380443	Scientific Computing, M.Sc.
Riccardo Parise	412524	Scientific Computing, M.Sc.
Daniel V. Herrmannsdoerfer	412543	Scientific Computing, M.Sc.

Exercise 1

Considered is the Laplace operator $Lu = -\Delta$ in $\Omega = (0, L)^2$ and $u \in C^4(\bar{\Omega})$.

a) Let $\underline{x} = [x_1, x_2]^T \in \Omega_h$. Applying the Laplace operator in the form of finite difference on $u(\underline{x})$ takes place by differentiating $u(\underline{x})$ with respect to x_1 and x_2 one at a time. The $D^- D^+$ stencil is employed as the 2nd order finite difference operator with one neighbor on either side at the distance h.

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{u(x_1 + h, x_2) - 2u(x_1, x_2) + u(x_1 - h, x_2)}{h^2} + \mathcal{O}(h^2) \tag{1}$$

$$\frac{\partial^2 u}{\partial x_2^2} = \frac{u(x_1, x_2 + h) - 2u(x_1, x_2) + u(x_1, x_2 - h)}{h^2} + \mathcal{O}(h^2) \tag{2}$$

With remainders neglected in Equations (1) and (2), the Laplace operator can be approximated by adding both equations

$$\Delta u = \frac{u(x_1 + h, x_2) + u(x_1, x_2 + h) - 4u(x_1, x_2) + u(x_1 - h, x_2) + u(x_1, x_2 - h)}{h^2} + \mathcal{O}(h^2). \tag{3}$$

Since u is a four times continuously differentiable function in $\bar{\Omega}$ and the remainder is $\mathcal{O}(h^2)$, it holds $||f_h - L_h R_h u||_h \in \mathcal{O}(h^2)$ as $h \to 0$, where f_h is the right hand side of the BVP, L_h the finite difference matrix and R_h the restriction operator within $\bar{\Omega}$. Hence, Equation (3), also called as the 5-point stencil, is considered as a 2nd order consistent approximation of Δu.

b) Unlike in the Exercise 1c of Assignment 3, a more practical way exists in building the $\underline{\underline{L_h}}$ in a sparse form using row & column indices and the element value. The command `sparse(ii, jj, aa)` in MATLAB serves exactly to this purpose, with `ii` and `jj` representing the vectors containing the row and column indices of the non-zero elements and `aa` the value of those elements respectively, or $L_h(ii(k), jj(k)) = aa(k)$ with k being the integer in the range between 1 and the number of non-zero elements. The key advantage of `sparse` is the handling of indices in an arbitrary sequence within the aforementioned vectors, such that the stencil coefficients can be written in lexicographically form one under the other at any arrangement.

c) Please refer to online submitted `a04ex01_Lh5.m` file.

d & e) The exercises d and e are combined in in the online submitted `a04ex01_solve.m` file. The function is called with a single input parameter `FLAG` that refers to the form of the numerical differentiation with

- `FULL`: Solves the problem with full matrix,
- `RED`: Solves the problem with reduced differentiation matrix and accordingly modified right hand side.

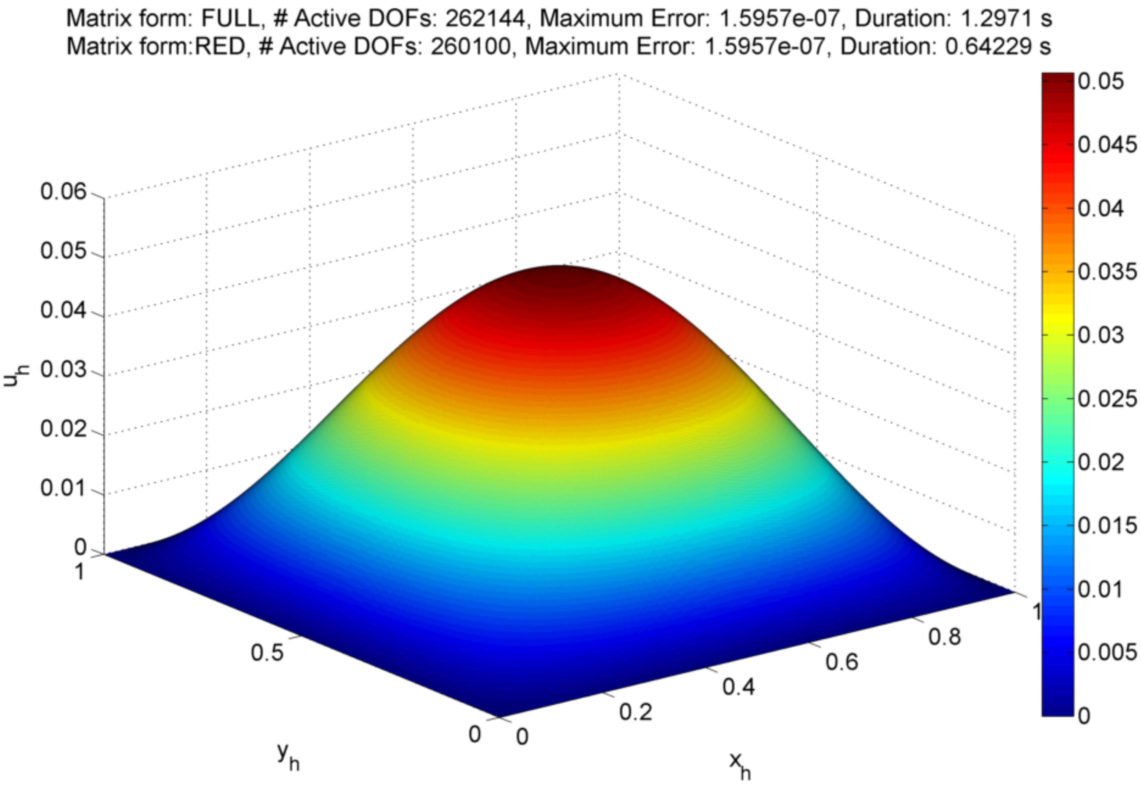

Figure 1 *Solution of* $L_h u_h = \sin(\pi x)\sin(\pi y)$ *with remarking the active dof number, maximum error* $\max_{x,y\in\Omega_h} |u_h(x,y) - u(x,y)|$ *and process duration for both flags. Since the error is same in both flags, the plot of the latter flag is omitted and the remark title is added instead.*

The solution of $L_h u_h = \sin(\pi x) \sin(\pi y)$ with homogeneous Dirichlet boundary conditions with either differentiation forms was simulated with 512×512 points. The result is illustrated in Figure 1. It reveals that the error in both methods remain same, whereas the number of DOFs dropped significantly in the reduced method, thus nearly halving the process duration.

Exercise 2

a)

b)

Exercise 3

a)

b)

c)

d)