Ex 2a) Through an argument of seperation of variables we define:

$u(x,t) = f(t)\,g(x)$, so we have the PDE

$$f'(t)\,g(x) - a\,f(t)\,g''(x) + c\,f(t)\,g(x) = 0$$

$$\Leftrightarrow \frac{f'(t)}{f(t)} + c = \frac{a\,g''(x)}{g(x)} = k \quad \text{a System of ODEs}$$

① $f'(t) = (k-c)\,f(t)$

$\Rightarrow f(t) = d\cdot e^{(k-c)t}$, where d is an integration constant

② $g''(x) = \frac{k}{a}\,g(x)$

for $k = 0$ we have

$g(x) = c_1 x + c_2$

which is either the trivial solution or a strictly (de-)increasing function, which cannot fulfill the initial conditions $u(0,0) = u(0,1) = 0$.

for $k = \mu^2 > 0$ we have, since $a > 0$

$g(x) = c_1 e^{\frac{\mu}{\sqrt{a}}x} + c_2$

which also cannot fulfill the initial boundary conditions for the same reason as before

for $k = -\mu^2 < 0$ we have

$g(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x)$

from the ~~initial~~ boundary condition we have:

$u(t,0) = f(t)\,g(0) = 0 \xrightarrow{f(t)\neq 0\,\forall t} g(0) = 0$

$u(t,1) = f(t)\,g(1) = 0 \rightarrow g(1) = 0$

$\Rightarrow g(0) = c_1 \sin(\mu 0) + c_2 \cos(\mu 0) = 0 \Rightarrow c_2 = 0$

$g(1) = c_1 \sin(\mu 1) = 0 \Rightarrow \mu = \pi \cdot n,\ n \in \mathbb{N}_0$

There is no restriction on c_1, so the solution will be a linear combination of functions such that the initial condition is fullfilled, the corresponding coeficcents are the ones given in the hint, so we have:

$$u(t,x) = \sum_n^\infty \left[e^{-(\pi^2 n^2 c)t} \cdot \left(\underbrace{\frac{16}{\pi(4n-n^3)}}_{= d \cdot c} \sin(\pi n x) \right) \right] \text{ for odd } n, \ 0 \text{ for even } n.$$

~~2b) The stationary solutions are given by the exponential decaying factor $e^{-(\pi^2 n^2 + c)t}$~~

ii) von Neumann b.c.:

The derivation in the case of von Neumann b.c. is the same up to the constraints, which now are of the form:

~~$u(t,0)$~~

$g'(0) = c_1 \mu \cos(\mu 0) - c_2 \mu \sin(\mu 0) = 0 \Rightarrow c_1 = 0$

$g'(1) = c_2 \sin(\mu 1) = 0 \Rightarrow \mu = \pi \cdot n,\ n \in \mathbb{N}_0$

In this case the linear combination that satisfies the b.c. and initial conditions is easily constructed:

$$u(t,x) = e^{-(0^2 \cdot \pi^2 + c)t} - e^{-(2^2 \pi^2 + c)t} \cos(2\pi x)$$

2b) The stationary Solution is given by the exponentially decaying term. $e^{-(\pi^2 n^2 + c) t}$

The exponent is always strictly negative and tends to $-\infty$ for $t \to \infty$, except when $n = 0$ and $c = 0$, which means that for $t \to \infty$ we obtain ~~[illegible]~~ only non zero terms for the second condition (von Neumann).

i) Because all terms contain an $n \geq 1$ all terms vanish as $t \to \infty$

$u_{t \to \infty}(x) = 0$

ii) Here the first term survives if $c = 0$, giving

$u_{t \to \infty}(x) = 1$