

## Numerical Mathematics II for Engineers

### Homework Assignment 2

Submitted on November 4th, 2019

by **Group 5**

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### Exercise 1

Given is the following boundary problem of an annulus

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega = \{(x, y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2 + y^2} < 2\} \subset \mathbb{R}^2, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

with the boundary condition

$$g(x, y) = \begin{cases} x & \text{for } x^2 + y^2 = 2^2 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

a)  $(x, y) \in \Omega$  is transformed to polar coordinates  $(r, \varphi) \in \Omega_r$  using

$$x = r \cos(\varphi), \quad (3)$$

$$y = r \sin(\varphi) \quad (4)$$

with  $r \in (1, 2]$  and  $\varphi \in (0, 2\pi]$ . Let  $v : \Omega_r \rightarrow \mathbb{R}$  defined by  $v(r, \varphi) = v(x, y)$ . In this case, the partial derivatives  $v_x$  and  $v_y$  are expressed using chain rule as follows

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \varphi} \frac{\partial \varphi}{\partial x}, \text{ also denoted as } u_x = u_r r_x + v_\varphi \varphi_x \quad (5)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \varphi} \frac{\partial \varphi}{\partial y}, \text{ also denoted as } u_y = u_r r_y + v_\varphi \varphi_y \quad (6)$$

The second partial derivative of  $v$  with respect to  $x$  is obtained using product rule

$$v_{xx} = u_r r_{xx} + (u_r)_x r_x + v_\varphi \varphi_{xx} + (v_\varphi)_x \varphi_x \quad (7)$$

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By applying the chain rule from Equation (5) and Equation (6) into Equation (7),  $v_{xx}$  is written in the following form

$$v_{xx} = u_r r_{xx} + v_{rr} r_x^2 + 2v_{r\varphi} r_x \varphi_x + v_{\varphi} \varphi_{xx} + v_{\varphi\varphi} \varphi_x^2 \quad (8)$$

where a similar expression is obtained for y

$$v_{yy} = u_r r_{yy} + v_{rr} r_y^2 + 2v_{r\varphi} r_y \varphi_y + v_{\varphi} \varphi_{yy} + v_{\varphi\varphi} \varphi_y^2 \quad (9)$$

The laplace equation  $\Delta v = v_{xx} + v_{yy}$  can be written in a proper semi-polar coordinate form by adding Equation (8) and Equation (9) and collecting the like terms

$$\begin{aligned} \Delta v &= v_{xx} + v_{yy} \\ &= u_r (r_{xx} + r_{yy}) + v_{rr} (r_x^2 + r_y^2) + 2v_{r\varphi} (r_x \varphi_x + r_y \varphi_y) + v_{\varphi} (\varphi_{xx} + \varphi_{yy}) + v_{\varphi\varphi} (\varphi_x^2 + \varphi_y^2) \end{aligned} \quad (10)$$

Now, expressions in parentheses are to be elaborated in the partial derivations with respect to polar coordinates. For this purpose, the relationship  $x^2 + y^2 = r^2$  is differentiated with respect to x and y. Accordingly, the partial differentiation of r terms with respect Cartesian terms up to second order are obtained as

$$r_x = \frac{x}{r}, \quad (11)$$

$$r_{xx} = \frac{y^2}{r^3}, \quad (12)$$

$$r_y = \frac{y}{r}, \quad (13)$$

$$r_{yy} = \frac{x^2}{r^3}. \quad (14)$$

Similarly, the partial differentiation of  $\varphi$  terms with respect to x and y are obtained as

$$\varphi_x = -\frac{y}{r^2}, \quad (15)$$

$$\varphi_{xx} = \frac{2xy}{r^4}, \quad (16)$$

$$\varphi_y = \frac{x}{r^2}, \quad (17)$$

$$\varphi_{yy} = -\frac{2xy}{r^4}. \quad (18)$$

Employing the Equations 11 to 18 in Equation (10) gives

$$\begin{aligned} \Delta v &= u_r \left( \frac{y^2}{r^3} + \frac{x^2}{r^3} \right) + v_{rr} \left( \left( \frac{x}{r} \right)^2 + \left( \frac{y}{r} \right)^2 \right) + 2v_{r\varphi} \left( \frac{-xy}{r^3} + \frac{yx}{r^3} \right) + \dots \\ &\quad v_{\varphi} \left( \frac{2xy}{r^4} - \frac{2xy}{r^4} \right) + v_{\varphi\varphi} \left( \left( -\frac{x^2}{r^3} \right)^2 + \left( \frac{x^2}{r^3} \right)^2 \right) \\ &= \frac{1}{r} u_r + v_{rr} + 0 + 0 + \frac{1}{r^2} v_{\varphi\varphi}. \end{aligned}$$

Thus, for any  $v$  satisfying the Laplace equation  $-\Delta v = 0$ ,  $v$  satisfies in polar coordinates the equation

$$-\left(v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi}\right) = 0 \quad (19)$$

b) For any  $v$  defined in a), the domain  $\Omega_v$  is defined as

$$\Omega_v = \{(r, \varphi) \in \mathbb{R}^2 : r \in (1, 2), \varphi \in (0, 2\pi]\} \quad (20)$$

with the boundary condition

$$\partial\Omega_v = \begin{cases} 2 \cos(\varphi) & \text{for } r = 2, \varphi \in (0, 2\pi] \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

c)

## Exercise 2

a)

b)

c)