

Numerical Mathematics II for Engineers

Homework Assignment 2

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by **Group 5**

Kagan Atci	338131	Physical Engineering, M.Sc.
Navneet Singh	380443	Scientific Computing, M.Sc.
Daniel V. Herrmannsdoerfer	412543	Scientific Computing, M.Sc.

Exercise 1

Given is the following boundary problem of an annulus

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega = \{(x, y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2 + y^2} < 2\} \subset \mathbb{R}^2, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

with the boundary condition

$$g(x, y) = \begin{cases} x & \text{for } x^2 + y^2 = 2^2 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

a) $(x, y) \in \Omega$ is transformed to polar coordinates $(r, \varphi) \in \Omega_r$ using

$$x = r \cos(\varphi), \quad (3)$$

$$y = r \sin(\varphi) \quad (4)$$

with $r \in (1, 2]$ and $\varphi \in (0, 2\pi]$. Let $v : \Omega_r \rightarrow \mathbb{R}$ defined by $v(r, \varphi) = v(x, y)$. In this case, the partial derivatives v_x and v_y are expressed using chain rule as follows

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial v}{\partial \varphi} \frac{\partial \varphi}{\partial x}, \text{ also denoted as } u_x = u_r r_x + v_\varphi \varphi_x \quad (5)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial v}{\partial \varphi} \frac{\partial \varphi}{\partial y}, \text{ also denoted as } u_y = u_r r_y + v_\varphi \varphi_y \quad (6)$$

The second partial derivative of v with respect to x is obtained using product rule

$$v_{xx} = u_r r_{xx} + (u_r)_x r_x + v_\varphi \varphi_{xx} + (v_\varphi)_x \varphi_x \quad (7)$$

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By applying the chain rule from Equation (5) and Equation (6) into Equation (7), v_{xx} is written in the following form

$$v_{xx} = u_r r_{xx} + v_{rr} r_x^2 + 2v_{r\varphi} r_x \varphi_x + v_{\varphi} \varphi_{xx} + v_{\varphi\varphi} \varphi_x^2 \quad (8)$$

where a similar expression is obtained for y

$$v_{yy} = u_r r_{yy} + v_{rr} r_y^2 + 2v_{r\varphi} r_y \varphi_y + v_{\varphi} \varphi_{yy} + v_{\varphi\varphi} \varphi_y^2 \quad (9)$$

The laplace equation $\Delta v = v_{xx} + v_{yy}$ can be written in a proper semi-polar coordinate form by adding Equation (8) and Equation (9) and collecting the like terms

$$\begin{aligned} \Delta v &= v_{xx} + v_{yy} \\ &= u_r (r_{xx} + r_{yy}) + v_{rr} (r_x^2 + r_y^2) + 2v_{r\varphi} (r_x \varphi_x + r_y \varphi_y) + v_{\varphi} (\varphi_{xx} + \varphi_{yy}) + v_{\varphi\varphi} (\varphi_x^2 + \varphi_y^2) \end{aligned} \quad (10)$$

Now, expressions in parentheses are to be elaborated in the partial derivations with respect to polar coordinates. For this purpose, the relationship $x^2 + y^2 = r^2$ is differentiated with respect to x and y. Accordingly, the partial differentiation of r terms with respect Cartesian terms up to second order are obtained as

$$r_x = \frac{x}{r}, \quad (11)$$

$$r_{xx} = \frac{y^2}{r^3}, \quad (12)$$

$$r_y = \frac{y}{r}, \quad (13)$$

$$r_{yy} = \frac{x^2}{r^3}. \quad (14)$$

Similarly, the partial differentiation of φ terms with respect to x and y are obtained as

$$\varphi_x = -\frac{y}{r^2}, \quad (15)$$

$$\varphi_{xx} = \frac{2xy}{r^4}, \quad (16)$$

$$\varphi_y = \frac{x}{r^2}, \quad (17)$$

$$\varphi_{yy} = -\frac{2xy}{r^4}. \quad (18)$$

Employing the Equations 11 to 18 in Equation (10) gives

$$\begin{aligned} \Delta v &= u_r \left(\frac{y^2}{r^3} + \frac{x^2}{r^3} \right) + v_{rr} \left(\left(\frac{x}{r} \right)^2 + \left(\frac{y}{r} \right)^2 \right) + 2v_{r\varphi} \left(\frac{-xy}{r^3} + \frac{yx}{r^3} \right) + \dots \\ &\quad v_{\varphi} \left(\frac{2xy}{r^4} - \frac{2xy}{r^4} \right) + v_{\varphi\varphi} \left(\left(-\frac{x^2}{r^3} \right)^2 + \left(\frac{x^2}{r^3} \right)^2 \right) \\ &= \frac{1}{r} u_r + v_{rr} + 0 + 0 + \frac{1}{r^2} v_{\varphi\varphi}. \end{aligned}$$

Thus, for any v satisfying the Laplace equation $-\Delta v = 0$, v satisfies in polar coordinates the equation

$$-\left(v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi}\right) = 0 \quad (19)$$

b) For any v defined in a), the domain Ω_v is defined as

$$\Omega_v = \{(r, \varphi) \in \mathbb{R}^2: r \in (1, 2), \varphi \in (0, 2\pi]\} \quad (20)$$

with the boundary condition

$$h(r, \varphi) = \begin{cases} 2 \cos(\varphi) & \text{for } r = 2, \varphi \in (0, 2\pi] \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

c) Assuming that $v(r, \varphi) = R(r)\Phi(\varphi)$, $\forall r \in (0, 1]$, $\forall \varphi \in (0, 2\pi]$. If $v(r, \varphi)$ satisfies Equation (19), then this equation can be expressed in the following form

$$-\left(R_{rr}\Phi + \frac{1}{r}R_r\Phi + \frac{R}{r^2}\Phi_{\varphi\varphi}\right) = 0. \quad (22)$$

Placing the functions depended of r and φ in separate terms, Equation (22) is written as

$$\frac{r^2 R_{rr} + r R_r}{R} = -\frac{\Phi_{\varphi\varphi}}{\Phi} = -\lambda \quad (23)$$

where λ represent a constant real factor. Equation (23) enables $v(r, \varphi)$ to be considered as two different ODE's, as both R and Φ terms are equated with λ separately

$$\frac{\Phi_{\varphi\varphi}}{\Phi} = \lambda \iff \Phi_{\varphi\varphi} = \Phi\lambda \quad (24)$$

$$\frac{r^2 R_{rr} + r R_r}{R} = \lambda \iff (r^2 R_{rr} + r R_r) = -R\lambda. \quad (25)$$

d) The solution of $v(r, \varphi)$ consists of the superposition of Equation (24) and Equation (25) solutions, which are studied in three different cases for λ :

- **$\lambda = 0$:**

In this case, possible solutions are given by

$$\Phi(\varphi) = a\varphi + b \quad (26)$$

$$R(r) = c \ln r + d. \quad (27)$$

Since Φ has to be a periodic equation, $a = 0$, and $c = 0$, as the solution has to remain finite, as r goes to 0. With the applied conditions, the ansatz gives

$$v(r, \varphi) = bd = g = \text{const.} \quad (28)$$

- $\lambda < 0$:

Let $\lambda = -k^2$. Possible solutions for R and Φ are given by

$$\Phi(\varphi) = a_k \cosh(k\varphi) + b_k \sinh(k\varphi) \quad (29)$$

$$R(r) = c_k r^{-k} + d_k r^k \quad (30)$$

$v(r, \varphi)$ holds no solutions for this case, because Equation (29) implies a periodic function in neither terms, thus remains zero for Φ .

- $\lambda > 0$:

Let $\lambda = k^2$. Possible solutions are given by

$$\Phi(\varphi) = a_k \cos(k\varphi) + b_k \sin(k\varphi) \quad (31)$$

$$R(r) = c_k r^k + d_k r^{-k} \quad (32)$$

In this case, Φ shows a well periodic function with period 2π . Furthermore d_k must be $-c_k$, in order to apply the boundary condition on the inner edge of the disc

$$v(r, \varphi) = (r^k - r^{-k}) (A_k \cos(k\varphi) + B_k \sin(k\varphi)) \quad (33)$$

where $A_k = a_k c_k$ and $B_k = b_k c_k$.

The solution for $v(r, \varphi)$ is expressed as the superposition of Equation (28) and Equation (33)

$$v(r, \varphi) = g + \sum_{k=1} (r^k - r^{-k}) (A_k \cos(k\varphi) + B_k \sin(k\varphi)) \quad (34)$$

for A_k and $B_k \in \mathbb{R}$ are obtained by the Fourier expansion.

The solution of $v(r, \varphi)$ is to be investigated on the outer boundary with $r = 2$. So gives

$$v(2, \varphi) = g + \sum_{k=1} (2^k - 2^{-k}) (A_k \cos(k\varphi) + B_k \sin(k\varphi)) = 2 \cos(\varphi) \quad (35)$$

As Equation (35) implies, the boundary condition is already given in the Fourier-series form, so that the only solution exists for $k = 1$, $A_1 = \frac{4}{3}$, $B_1 = 0$ and $g = 0$. Hence the solution, that satisfies Equation (19) and Equation (21) in Ω_v is

$$v(r, \varphi) = \left(r - \frac{1}{r}\right) \frac{4}{3} \cos(\varphi). \quad (36)$$

e & f) The solution code can be found in `a02e01solution.m` and the plot code in `a02e01plot.m`. The surface plot of the Equation (36) solution is illustrated in Figure 1.

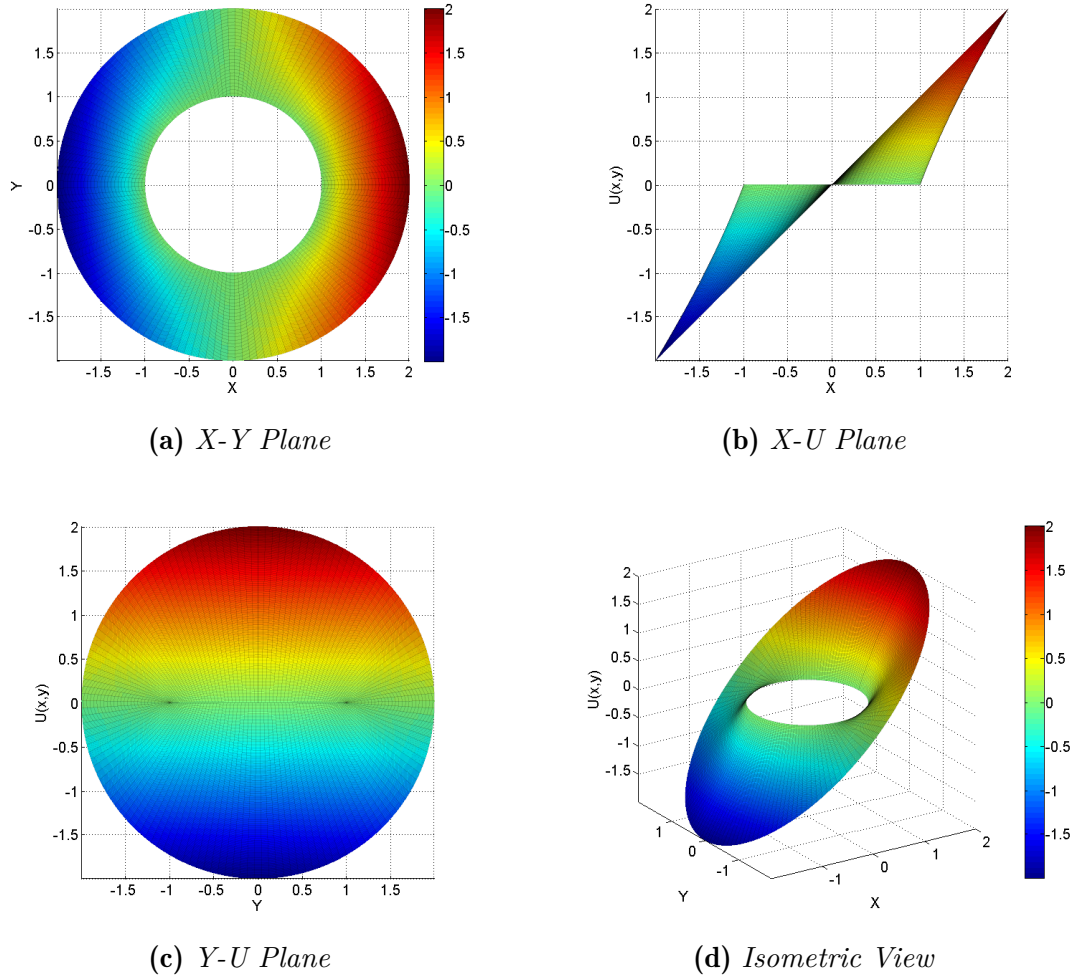


Figure 1 *Surface Plot of the Solution*

Exercise 2

a & b) See the handwritten solutions.

c & d) The solution and plot codes can be found in `NumMat_Ex2_2_Program.py`. An example of the solution results with respect to input data is illustrated in Figure 2.

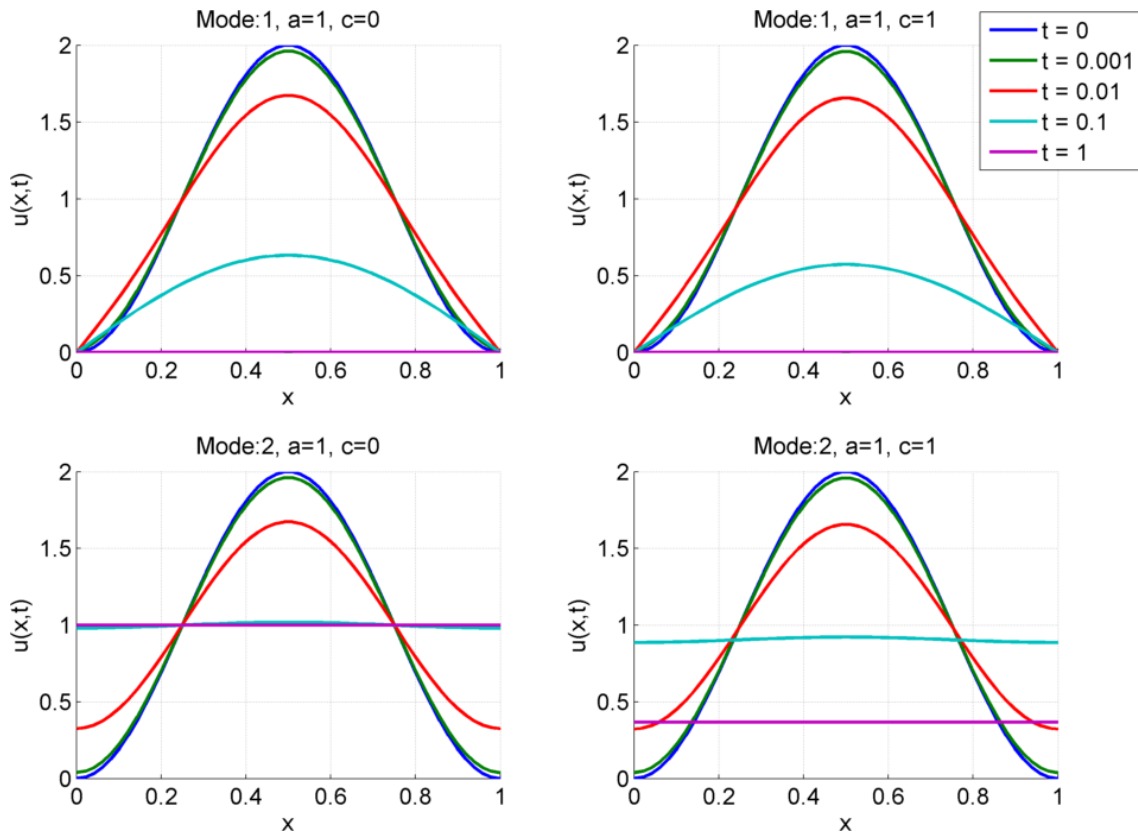


Figure 2 Results of the heat-equation with respect to boundary conditions, a and c

Exercise 3

Given is $u: [0, 1] \rightarrow \mathbb{R}$ as a sufficiently smooth function.

a) For $x \in [0, 1]$, following difference quotients are considered:

$$D^0 u(x) = \frac{u(x+h) - u(x-h)}{2h} \quad (37)$$

$$D^+ u(x) = \frac{u(x+h) - u(x)}{h} \quad (38)$$

$$D^- u(x) = \frac{u(x) - u(x-h)}{h} \quad (39)$$

$$D^+ D^- u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \quad (40)$$

with $h \in (0, \frac{1}{2})$ the given step size. For the given x interval holds $[x-h < 0, x+h > 1]$. Hence the x interval of the above mentioned quotients is

- $x \in [h, 1 - h]$ for D^0 ,
- $x \in [0, 1 - h]$ for D^+ ,
- $x \in [h, 1]$ for D^- ,
- $x \in [h, 1 - h]$ for D^+D^- .

b) The equality of $D^0(x) = \frac{1}{2}(D^+(x) + D^-(x))$ is investigated by employing Equations (38) and (39), and the result must give Equation (37).

$$\begin{aligned} \frac{1}{2}(D^+u(x) + D^-u(x)) &= \frac{1}{2} \left(\frac{u(x+h) - u(x)}{h} + \frac{u(x) - u(x-h)}{h} \right) \\ &= \frac{1}{2} \frac{u(x+h) - u(x-h)}{h} \end{aligned} \quad (41)$$

Equation (41) confirms the equality for $x \in [0, 1]$.

c) Similar to **b)**, both sides of $D^+D^-u(x) = D^-D^+u(x)$ are investigated separately.

$$\begin{aligned} D^+D^-u(x) &= D^+ \left[\frac{u(x) - u(x-h)}{h} \right] \\ &= \frac{1}{2} [D^+u(x) - D^+u(x-h)] \\ &= \frac{1}{h} \left[\frac{x(u+h) - u(x)}{h} - \frac{x(u-h+h) - u(x-h)}{h} \right] \\ &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \end{aligned} \quad (42)$$

$$\begin{aligned} D^-D^+u(x) &= D^- \left[\frac{u(x+h) - u(x)}{h} \right] \\ &= \frac{1}{2} [D^-u(x+h) - D^-u(x)] \\ &= \frac{1}{h} \left[\frac{x(u+h) - u(x+h-h)}{h} - \frac{x(u) - u(x-h)}{h} \right] \\ &= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \end{aligned} \quad (43)$$

Equations (42) and (43) confirm the equality for $x \in [0, 1]$.

d) Applying Taylor's formula on Equation (37) gives

$$\begin{aligned}
 D^0 u(x) &= \frac{1}{2} [\cancel{u(x)} + u'(x)h + \cancel{\frac{1}{2}u''(x)h^2} + \cancel{\frac{1}{6}u'''(x)h^3} + \dots \\
 &\quad \cancel{-u(x)} + u'(x)h - \cancel{\frac{1}{2}u''(x)h^2} + \frac{1}{6}u'''(x)h^3 + \dots] \\
 &= u'(x) + h^2 \underbrace{\left[\frac{1}{6}u'''(x) + R \right]}_{R_0}.
 \end{aligned} \tag{44}$$

Considering that $R_0 = \frac{1}{6} (u'''(\xi_1) + u'''(\xi_2))$, it holds

$$|R_0| \leq \frac{1}{6} \max |u'''(\xi)|, \text{ for } \xi, \xi_1, \xi_2 \in [x-h, x+h]. \tag{45}$$

Applying the same procedure on Equation (40) gives

$$\begin{aligned}
 D^{+-} u(x) &= \frac{1}{h^2} [\cancel{u(x)} + \cancel{u'(x)h} + \frac{1}{2}u''(x)h^2 + \cancel{\frac{1}{6}u'''(x)h^3} + \frac{1}{24}u^{(4)}h^4 + \dots \\
 &\quad \cancel{u(x)} - \cancel{u'(x)h} - \frac{1}{2}u''(x)h^2 - \cancel{\frac{1}{6}u'''(x)h^3} + \frac{1}{24}u^{(4)}h^4 + \dots \\
 &\quad \cancel{-2u(x)}] \\
 &= u'(x) + h^2 \underbrace{\left[\frac{1}{12}u^{(4)}(x) + R \right]}_{R_1}.
 \end{aligned} \tag{46}$$

Considering that $R_1 = \frac{1}{12} (u^{(4)}(\xi_1) + u^{(4)}(\xi_2))$, it holds

$$|R_1| \leq \frac{1}{12} \max |u^{(4)}(\xi)|, \text{ for } \xi, \xi_1, \xi_2 \in [x-h, x+h]. \tag{47}$$

$D^- D^0$ is obtained by applying Equation (39) on Equation (37)

$$\begin{aligned}
 D^- D^0 &= \frac{D^-}{2h} (u(x+h) - u(x-h)) \\
 &= \frac{1}{2h} \left(\frac{u(x+h) - u(x)}{h} - \frac{u(x-h) - u(x-2h)}{h} \right) \\
 &= \frac{1}{2h} (-u(x) + u(x+h) - u(x-h) + u(x-2h))
 \end{aligned} \tag{48}$$

Hence, the Taylor approximation of D^-D^0 is

$$\begin{aligned}
 D^-D^0 &= \frac{1}{2h^2} [\cancel{u(x)} + \cancel{u'(x)h} + \cancel{\frac{1}{2}u''(x)h^2} + R_1 + \dots \\
 &\quad \cancel{-u(x)} \\
 &\quad \cancel{-u(x)} + \cancel{u'(x)h} - \cancel{\frac{1}{2}u''(x)h^2} + R_2 + \dots \\
 &\quad \cancel{u(x)} - \cancel{u'(x)2h} + \frac{1}{2}u''(x)4h^2 - R_3] \\
 &= \frac{1}{2h^2} (2u''(x)h^2 + R_1 + R_2 - R_3) = u'' + \mathcal{O}(h)
 \end{aligned} \tag{49}$$

As Equation (49) implies, this differentiation quotient has a lower order, compared to above mentioned ones. This shows a lower rate in error decrease and thus less suitable for approximating the second derivative.