Technische Universität Berlin
Institut für Mathematik
Dr. D. Peschka, A. Selahi

WS 2019/20

Numerical Mathematics II for Engineers

Homework assignment 6 : *Eigenvalues and boundaries.*

Programming assignments can be solved in MATLAB/Python/Julia.
Use sparse matrices where appropriate !

Deadline : submit until Friday December 6, 2019.

1. Excercise : Elliptic eigenvalue problems **9 points**

Consider the elliptic eigenvalue problem of finding an eigenfunction/eigenvector pair (u, λ) such that $Lu = \lambda u$ in Ω supplemented with suitable boundary conditions on $\partial\Omega$.

a) Let $\Omega = (0, 1) \in \mathbb{R}^1$ and $Lu = -u'' + u$ with a.1) homogeneous Dirichlet boundary conditions and a.2) homogenous Neumann boundary conditions (with noncentral differences) and a.3) periodic boundary conditions. Create the matrix L_h and solve the eigenvalue problem numerically. For $h = 1/(N + 1)$ and $N = 200$ compare the $2nd$ and $20st$ eigenvalue/vector (ordered by magnitude) with the exact eigenpair. For a.3 : What is the impact of using a noncompact 5-point stencil on the eigenpairs ? Create a MATLAB/Python/Julia file solving this problem in a single file script named `a06ex01_solveEVPa`. Use `eigs` (MATLAB/Python scipy).*

b) Let $\Omega = \{(x, y) \in \mathbb{R}^2 : (x - \frac{L}{2})^2 + (y - \frac{L}{2})^2 < R\}$ for given $L, R \in \mathbb{R}$ and $R > 0$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$. Reproduce the computation of the lecture (notes) and state the first 6 distinct eigenvalues (ordered by magnitude) for radii $R = 1$ and $R = 2$.

c) Discretize Ω from b) as in the assignment 5, exercise 1 and build the corresponding discrete Laplace operator $L_h = -\Delta_h$ for the disc with $L = 2.2, R = 1$. Use `eigs` (MATLAB/Python scipy) to solve the (generalized) eigenvalue problem using one of the two approaches mentioned in the hint below and compare with the 6 exact eigenvalues from b) for $h = 1/(N + 1)$ for $N = 2^p - 1$ and $p = 3, 6, 9$. Plot the first few eigenfunctions. Create a MATLAB/Python/Julia file solving this problem in a single file script named `a06ex01_solveEVPc`.*

Hint : Two ways to solve such a problem numerically :

i) Use the full L_h and enforce homogeneous Dirichlet/Neumann boundary conditions by solving the generalized eigenvalue problem $L_h u_h = \lambda M_h u_h$, where M_h is diagonal and $(M_h)_{ii} = 1$ for $x_i \in \Omega_h$ and $(M_h)_{ii} = 0$ for $x_i \in \Gamma_h$.

ii) Use the reduced matrix L_h and solve the usual eigenvalue problem $L_h u_h = \lambda u_h$.

$*$ You can add further MATLAB/Python/Julia functions if convenient.

Please turn the page !

2. Exercise : Elliptic problem with space-dependent coefficients **9 points**

Let $\Omega = \{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$ and find $\hat{u} : \Omega \to \mathbb{R}$ solving the usual Poisson problem

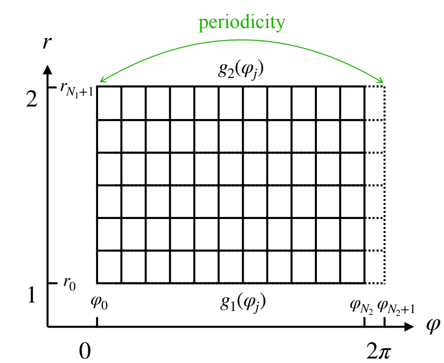
$$-\Delta \hat{u} = \hat{f} \quad \text{in} \quad \Omega, \qquad\qquad \hat{u} = \hat{g} \quad \text{on} \quad \Gamma = \partial\Omega,$$

for given functions $\hat{f} : \Omega \to \mathbb{R}$ and $\hat{g} : \Gamma \to \mathbb{R}$. Transforming to polar coordinates $\Omega_r = (1,2)\times(0,2\pi)$ and $\hat{v}(r,\varphi) = v(r\cos\varphi, r\sin\varphi)$ for $(\hat{u},\hat{f},\hat{g}) \to (u, f, g_1 = g_{r=1}, g_2 = g_{r=2})$ gives the elliptic BVP with space-dependent coefficients

$$-\left(\frac{\partial^2}{\partial r^2}u + \frac{1}{r}\frac{\partial}{\partial r}u + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}u\right) = f, \qquad \text{in } \Omega_r \tag{BVP}$$

with Dirichlet boundary conditions $u(r = 1, \varphi) = g_1(\varphi)$ and $u(r = 2, \varphi) = g_2(\varphi)$ and periodic boundary conditions $u(r, \varphi = 0) = u(r, \varphi = 2\pi)$ and $u^{(k)}(r, \varphi = 0) = u^{(k)}(r, \varphi = 2\pi)$ for a k-times continuously differentiable function $u(r, \varphi)$.

a) Discretize the elliptic PDE problem `[Lh,xx,yy,gamma1,gamma2]=a06ex02getPDE(N1,N2)` where `xx(k,l)=1+(k-1)*h1` and `yy(k,l)=(l-1)*h2` are polar coordinates for $k = 1...N+2$ and $l = 1...N+1$. The matrix is $L_h \in \mathbb{R}^{(N_1+2)(N_2+1)\times(N_1+2)(N_2+1)}$ and `gamma1,gamma2` denote the index slice for the Dirichlet boundary $r = 1$, $r = 2$.



Step 1 : Discretize with $(r_i, \varphi_j) \in \bar{\Omega}_{r,h}$ with $r_i = 1 + ih_1$ and $\varphi_j = jh_2$ with grid sizes $h_1 = 1/(N_1 + 1)$ and $h_2 = 2\pi/(N_2 + 1)$ and introduce a lexicographical ordering for grid functions $u_{ij} = u_h(r_i, \varphi_j)$ for $i = 0...N_1 + 1$ and $j = 0...N_2 + 1$. Don't count $j = 0, N_2 + 1$ separately (periodic boundary conditions)!

Step 2 : Introduce Dirichlet boundary conditions $u_{0j} = g_1(\varphi_j)$ and $u_{(N_1+1)j} = g_2(\varphi_j)$ for $j = 0...N_2$ and periodic boundary conditions $u_{i(N_2+1)} = u_{i0}$ for $i = 0...N_1 + 1$. The image above might help for Step 1 & 2.

Step 3 : Implement the operator L_h based on the finite difference stencils below.

b) Show that consistent finite difference stencils for the operator in (BVP) are

$$\frac{u_{(i+1)j} - 2u_{ij} + u_{(i-1)j}}{h_1^2}, \qquad \frac{1}{r_i}\frac{u_{(i+1)j} - u_{(i-1)j}}{2h_1}, \qquad \frac{1}{r_i^2}\frac{u_{i(j+1)} - 2u_{ij} + u_{i(j-1)}}{h_2^2}.$$

c) Use the code to solve the problem from assignment 2, exercise 1 numerically and determine the error for $N_1 = 10, N_2 = 80$. Plot the solution.

d) Solve this problem with the programming solution of assignment 5, exercise 1 for $N = 2^{10} - 1$. Plot the solution and determine the error. Compare with c).

See next page!

3. Exercise (optional) : Non-constant coefficients in 1D **4 points**

Consider $-(k(x)u'(x))' = f(x)$ on $\Omega = (0,1)$ with $u(0) = u(1) = 0$.

a) Implement the discrete operator L_h as indicated in the lecture (advanced stencils) or by applying the derivative and conversion into a problem in standard form with $a = k$, $b = k'$ and $c = 0$ (adjust function arguments accordingly). Write a program

```
[Lh,xh,fh]=a06ex03getPDE(k,f,g,N)
```

where `k,f,g` are function handles and $h = 1/(N+1)$ the grid size. Build the full or the reduced system `Lh` with the respective right-side `fh` and mesh `xh`.

b) Solve for $k(x) = 2 + \tanh(\epsilon^{-1}(x - 1/2))$ and $f = 1$ and discuss the solutions behavior for $\epsilon = 1/10, 1/100, 1/1000$. Choose $h = 1/(N+1)$ appropriately.

c) Can you guess/determine the solution as $\epsilon \to 0$?

Hint : How would you discretize the standard form with $a(x), b(x)$ looking at the space-dependent stencils in 2b)? You can also implement the advanced stencil discussed in the lecture (notes).

total sum : 18(+4) points