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Numerical Mathematics II for Engineers

Homework Assignment 3

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by **Group 5**

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Exercise 1

Given is $m \in \mathbb{N}_0$, assuming $\bar{\Omega} = [0,1]$ and $\bar{\Omega}_h = \{0, h, \ldots, (N+1)h\}$ with $h = \frac{1}{N+1}$ the corresponding grid. Considering that $x_n = nh \in \bar{\Omega}_h$ with sufficiently many neighbors to either side, for any j with $n - m \leq j \leq n + m$, requiring $x_j = jh \in \bar{\Omega}_h$, and $u_n = u(x_n)$ with $u \colon \bar{\Omega} \to \mathbb{R}$ is r times continuously differentiable.

a) u values on neighbor points with above mentioned conditions are expressed in Taylor expansion form about the x_n

$$u_h(x_n + jh) = u_h(x_n) + u'(x_n)\frac{(jh)^1}{1!} + u_h''\frac{(jh)^2}{2!} + \cdots + u_h(x_n)\frac{(jh)^k}{k!} + R \quad (1)$$

with k the derivative order and R the remainder. Applying this for each neighbor point, a linear equation system between the u and their derivatives is built up

$$\begin{bmatrix} u_h(x_{n-m}) \\ \vdots \\ u_h(x_{n+m}) \end{bmatrix} = \underline{\underline{S}} \begin{bmatrix} u(x_n) \\ \vdots \\ h^k u^{(k)}(x_n) \end{bmatrix} \text{ where } \underline{\underline{S}} \in \mathbb{R}^{k \times k} \text{ with } s_{j,k} = \frac{j^k}{k!}. \quad (2)$$

In this case, the symmetric stencils for spatial differentiations $\Delta_{h,r}$ up to k-th order are obtained by inverting $\underline{\underline{S}}$, where the remainders are neglected.

$$\underline{\underline{S}}^{-}1 \begin{bmatrix} u_h(x_{n-m}) \\ \vdots \\ u_h(x_{n+m}) \end{bmatrix} = \begin{bmatrix} u(x_n) \\ \vdots \\ h^k u^{(k)}(x_n) \end{bmatrix} \quad (3)$$

Since the $\underline{\underline{S}}^{-1}$ must be a symmetric matrix, applying the Taylor expansion procedure to m neighbor points on either sides limits the derivative order at $k = 2m + 1$ for u being $r = 2m + 2$ differentiable. Hence, the remainder can also be written in the matrix form as

$$\underline{\underline{R}} \in \mathbb{R}^{k \times k} \text{ with } r_{j,k} = \frac{(jh)^r}{r!} u^{(r)}(\xi) \quad \text{for} \quad \xi \in [x_n - kh, x_n + kh]. \tag{4}$$

b) Please refer to online submitted `a03ex01getstencil.m` file.

c) Please refer to online submitted `a03ex01getLaplace.m` file.

d) For $u(x) = \sin(2\pi x)$, Figure 1 illustrates the maximum error magnitude between the finite difference method with four different level of stencils and the analytical method applied on u''.

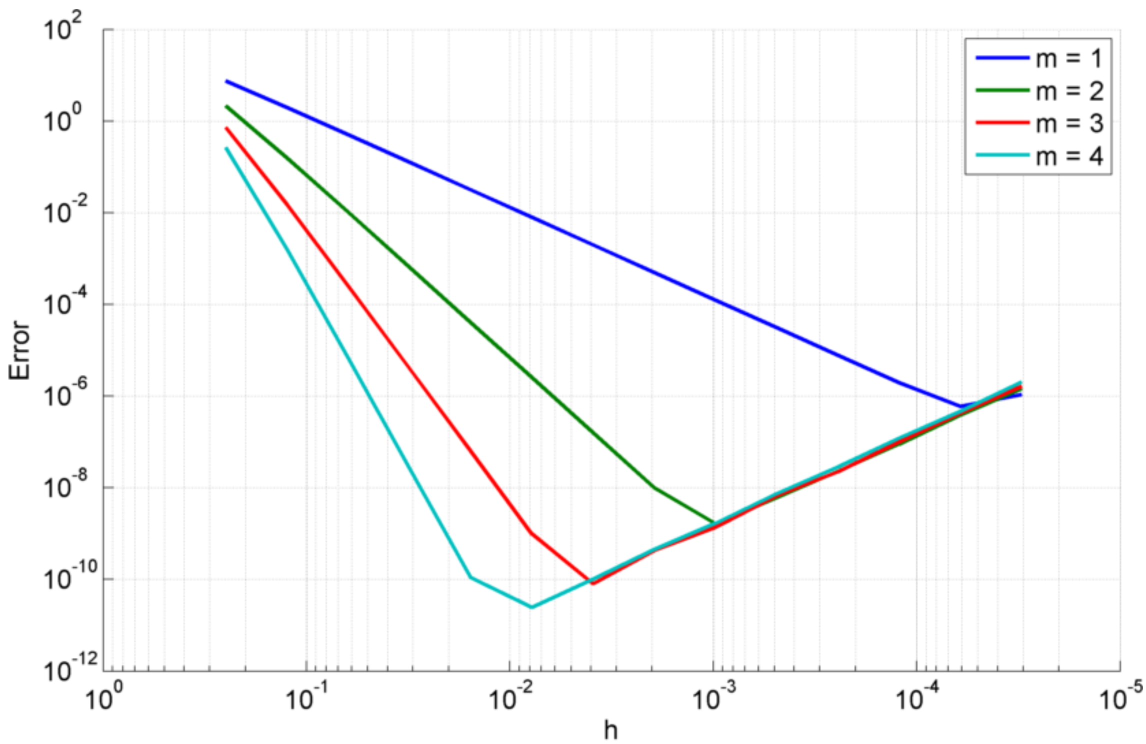

Figure 1 *Error of the finite difference stencils as a function of h with respect to m*

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