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Numerical Mathematics II for Engineers

Homework Assignment 2

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by **Group 5**

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Exercise 1

Given is the following boundary problem of an annulus

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega = \{(x,y) \in \mathbb{R}^2 \colon 1 \leq \sqrt{x^2+y^2} < 2\} \subset \mathbb{R}^2, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with the boundary condition

$$g(x,y) = \begin{cases} x & \text{for } x^2+y^2 = 2^2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

a) $(x,y) \in \Omega$ is transformed to polar coordinates $(r,\varphi) \in \Omega_r$ using

$$x = r\cos(\varphi), \tag{3}$$
$$y = r\sin(\varphi) \tag{4}$$

with $r \in (1,2]$ and $\varphi \in (0,2\pi]$. Let $v : \Omega_r \to \mathbb{R}$ defined by $v(r,\varphi) = v(x,y)$. In this case, the partial derivatives v_x and v_y are expressed using chain rule as follows

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \varphi}\frac{\partial \varphi}{\partial x}, \text{also denoted as } u_x = u_r r_x + v_\varphi \varphi_x \tag{5}$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \varphi}\frac{\partial \varphi}{\partial x}, \text{also denoted as } u_y = u_r r_y + v_\varphi \varphi_y \tag{6}$$

The second partial derivative of v with respect to x is obtained using product rule

$$v_{xx} = u_r r_{xx} + (u_r)_x r_x + v_\varphi \varphi_{xx} + (v_\varphi)_x \varphi_x \tag{7}$$

By applying the chain rule from Equation (5) and Equation (6) into Equation (7), v_{xx} is written in the following form

$$v_{xx} = u_r r_{xx} + v_{rr} r_x^2 + 2v_{r\varphi} r_x \varphi_x + v_\varphi \varphi_{xx} + v_{\varphi\varphi} \varphi_x^2 \tag{8}$$

where a similar expression is obtained for y

$$v_{yy} = u_r r_{yy} + v_{rr} r_y^2 + 2v_{r\varphi} r_y \varphi_y + v_\varphi \varphi_{yy} + v_{\varphi\varphi} \varphi_y^2 \tag{9}$$

The laplace equation $\Delta v = v_{xx}+v_{yy}$ can be written in a proper semi-polar coordinate form by adding Equation (8) and Equation (9) and collecting the like terms

$$\begin{aligned}\Delta v &= v_{xx} + v_{yy} \\ &= u_r(r_{xx} + r_{yy}) + v_{rr}(r_x^2 + r_y^2) + 2v_{r\varphi}(r_x\varphi_x + r_y\varphi_y) + v_\varphi(\varphi_{xx} + \varphi_{yy}) + v_{\varphi\varphi}(\varphi_x^2 + \varphi_y^2)\end{aligned} \tag{10}$$

Now, expressions in parentheses are to be elaborated in the partial derivations with respect to polar coordinates. For this purpose, the relationship $x^2 + y^2 = r^2$ is differentiated with respect to x and y. Accordingly, the partial differentiation of r terms with respect Cartesian terms up to second order are obtained as

$$r_x = \frac{x}{r}, \tag{11}$$

$$r_{xx} = \frac{y^2}{r^3}, \tag{12}$$

$$r_y = \frac{y}{r}, \tag{13}$$

$$r_{yy} = \frac{x^2}{r^3}. \tag{14}$$

Similarly, the partial differentiation of φ terms with respect to x and y are obtained as

$$\varphi_x = -\frac{y}{r^2}, \tag{15}$$

$$\varphi_{xx} = \frac{2xy}{r^4}, \tag{16}$$

$$\varphi_y = \frac{x}{r^2}, \tag{17}$$

$$\varphi_{yy} = -\frac{2xy}{r^4}. \tag{18}$$

Employing the Equations 11 to 18 in Equation (10) gives

$$\begin{aligned}\Delta v &= u_r\left(\frac{y^2}{r^3} + \frac{x^2}{r^3}\right) + v_{rr}\left(\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right) + 2v_{r\varphi}\left(\frac{-xy}{r^3} + \frac{yx}{r^3}\right) + \ldots \\ &\quad v_\varphi\left(\frac{2xy}{r^4} - \frac{2xy}{r^4}\right) + v_{\varphi\varphi}\left(\left(-\frac{x^2}{r^3}\right)^2 + \left(\frac{x^2}{r^3}\right)^2\right) \\ &= \frac{1}{r}u_r + v_{rr} + 0 + 0 + \frac{1}{r^2}v_{\varphi\varphi}.\end{aligned}$$

Thus, for any v satisfying the Laplace equation $-\Delta v = 0$, v satisfies in polar coordinates the equation

$$-\left(v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi}\right) = 0 \tag{19}$$

b) For any v defined in a), the domain Ω_v is defined as

$$\Omega_v = \{(r,\varphi) \in \mathbb{R}^2 \colon r \in (1,2), \varphi \in (0, 2\pi]\} \tag{20}$$

with the boundary condition

$$h(r,\varphi) = \begin{cases} 2\cos(\varphi) & \text{for } r = 2, \varphi \in (0,2\pi] \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

c) Assuming that $v(r,\varphi) = R(r)\Phi(\varphi)$, $\forall r \in (0,1]$, $\forall \varphi \in (0,2\pi]$. If $v(r,\varphi)$ satisfies Equation (19), then this equation can be expressed in the following form

$$-\left(R_{rr}\Phi + \frac{1}{r}R_r\Phi + \frac{R}{r^2}\Phi_{\varphi\varphi}\right) = 0. \tag{22}$$

Placing the functions depended of r and φ in separate terms, Equation (22) is written as

$$\frac{r^2R_{rr} + rR_r}{R} = -\frac{\Phi_{\varphi\varphi}}{\Phi} = -\lambda \tag{23}$$

where λ represent a constant real factor. Equation (23) enables $v(r,\varphi)$ to be considered as two different ODE's, as both R and Φ terms are equated with λ separately

$$\frac{\Phi_{\varphi\varphi}}{\Phi} = \lambda \iff \Phi_{\varphi\varphi} = \Phi\lambda \tag{24}$$

$$\frac{r^2R_{rr} + rR_r}{R} = \lambda \iff \left(r^2R_{rr} + rR_r\right) = -R\lambda. \tag{25}$$

d) The solution of $v(r,\varphi)$ consists of the superposition of Equation (24) and Equation (25) solutions, which are studied in three different cases for λ:

- $\boldsymbol{\lambda = 0}$:
  In this case, possible solutions are given by

$$\Phi(\varphi) = a\varphi + b \tag{26}$$

$$R(r) = c\ln r + d. \tag{27}$$

Since Φ has to be a periodic equation, $a = 0$, and $c = 0$, as the solution has to remain finite, as r goes to 0. With the applied conditions, the ansatz gives

$$v(r, \varphi) = bd = g = const. \tag{28}$$

- $\boldsymbol{\lambda < 0}$:
  Let $\lambda = -k^2$. Possible solutions for R and Φ are given by

$$\Phi(\varphi) = a_k \cosh(k\varphi) + b_k \sinh(k\varphi) \tag{29}$$
$$R(r) = c_k r^{-k} + d_k r^k \tag{30}$$

  $v(r, \varphi)$ holds no solutions for this case, because Equation (29) implies a periodic function in neither terms, thus remains zero for Φ.

- $\boldsymbol{\lambda > 0}$:
  Let $\lambda = k^2$. Possible solutions are given by

$$\Phi(\varphi) = a_k \cos(k\varphi) + b_k \sin(k\varphi) \tag{31}$$
$$R(r) = c_k r^k + d_k r^{-k} \tag{32}$$

  In this case, Φ shows a well periodic function with period 2π. Furthermore d_k must be $-c_k$, in order to apply the boundary condition on the inner edge of the disc

$$v(r, \varphi) = (r^k - r^{-k})\,(A_k \cos(k\varphi) + B_k \sin(k\varphi)) \tag{33}$$

  where $A_k = a_k c_k$ and $B_k = b_k c_k$.

The solution for $v(r, \varphi)$ is expressed as the superposition of Equation (28) and Equation (33)

$$v(r, \varphi) = g + \sum_{k=1} (r^k - r^{-k})\,(A_k \cos(k\varphi) + B_k \sin(k\varphi)) \tag{34}$$

for A_k and $B_k \in \mathbb{R}$ are obtained by the Fourier expansion.

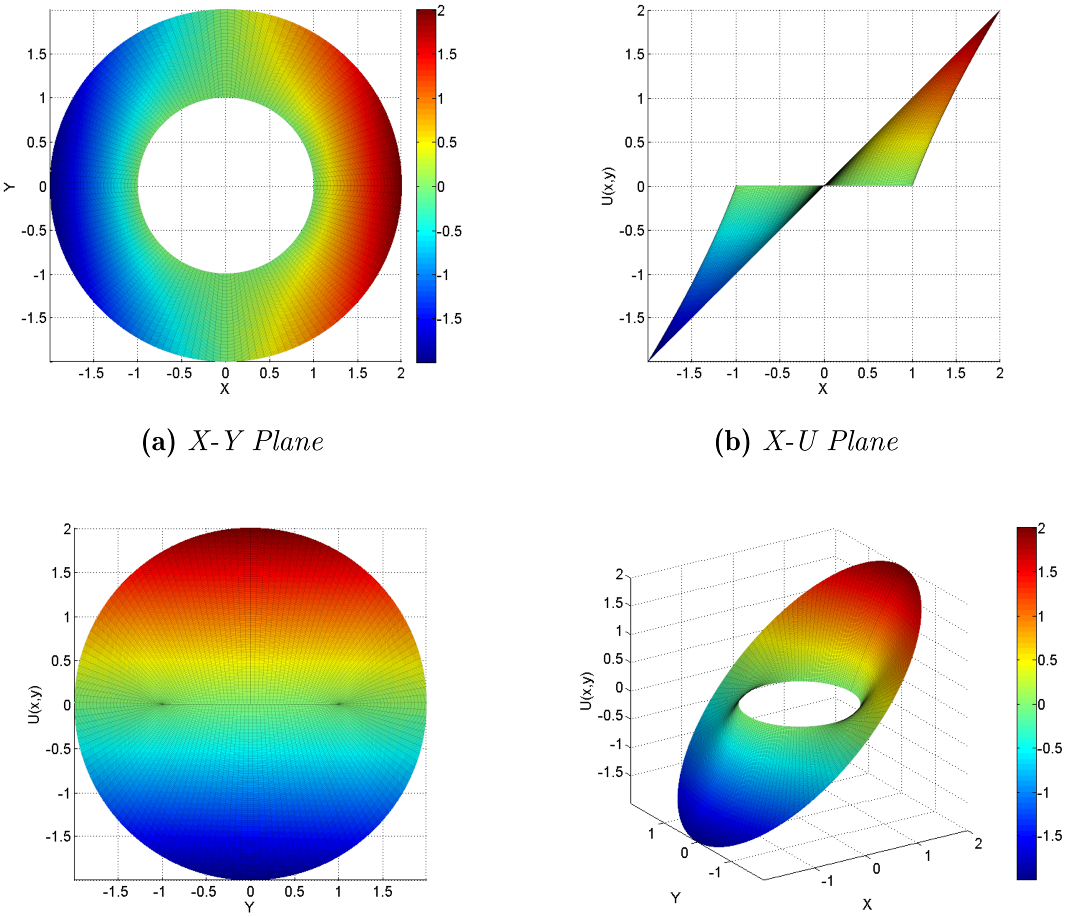
The solution of $v(r, \varphi)$ is to be investigated on the outer boundary with $r = 2$. So gives

$$v(2, \varphi) = g + \sum_{k=1} (2^k - 2^{-k})\,(A_k \cos(k\varphi) + B_k \sin(k\varphi)) = 2\cos(\varphi) \tag{35}$$

As Equation (35) implies, the boundary condition is already given in the Fourier-series form, so that the only solution exists for $k = 1$, $A_1 = \frac{4}{3}$, $B_1 = 0$ and $g = 0$. Hence the solution, that satisfies Equation (19) and Equation (21) in Ω_v is

$$v(r, \varphi) = \left(r - \frac{1}{r}\right)\frac{4}{3}\cos(\varphi). \tag{36}$$

e & f) The solution code can be found in `a02e01solution.m` and the plot code in `a02e01plot.m`. The surface plot of the Equation (36) solution is illustrated in Figure 1.



(a) *X-Y Plane* **(b)** *X-U Plane*

(c) *Y-U Plane* **(d)** *Isometric View*

Figure 1 *Surface Plot of the Solution*

Exercise 2

a & b) See the handwritten solutions.

c & d) The solution and plot codes can be found in `NumMat_Ex2_2_Program.py`. An example of the solution results with respect to input data is illustrated in Figure 2.

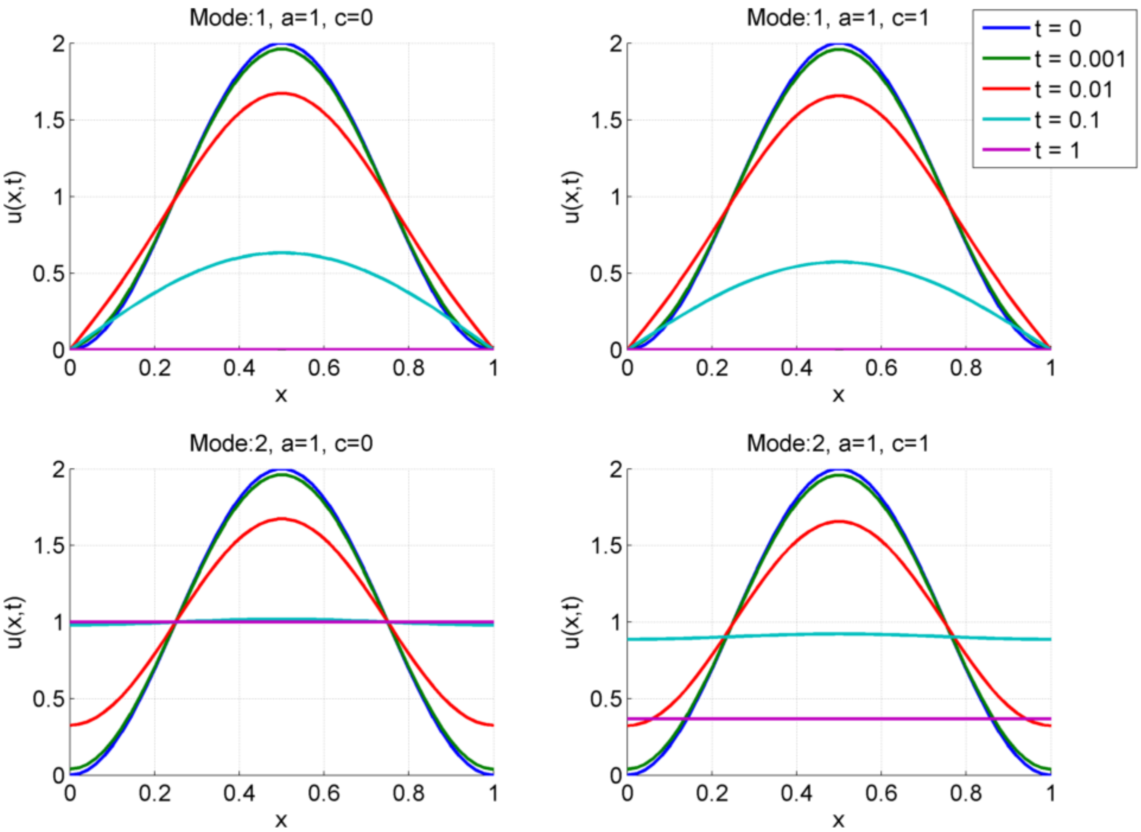


Figure 2 *Results of the heat-equation with respect to boundary conditions, a and c*

Exercise 3

Given is $u\colon [0,1] \to \mathbb{R}$ as a sufficiently smooth function.

a) For $x \in [0,1]$, following difference quotients are considered:

$$D^0u(x) = \frac{u(x+h) - u(x-h)}{2h} \tag{37}$$

$$D^+u(x) = \frac{u(x+h) - u(x)}{h} \tag{38}$$

$$D^-u(x) = \frac{u(x) - u(x-h)}{h} \tag{39}$$

$$D^+D^-u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \tag{40}$$

with $h \in \left(0, \frac{1}{2}\right)$ the given step size. For the given x interval holds $[x-h < 0, x+h > 1]$. Hence the x interval of the above mentioned quotients is

- $x \in [h, 1-h]$ for D^0,
- $x \in [0, 1-h]$ for D^+,
- $x \in [h, 1]$ for D^-,
- $x \in [h, 1-h]$ for D^+D^-.

b) The equality of $D^0(x) = \frac{1}{2}\left(D^+(x) + D^-(x)\right)$ is investigated by employing Equations (38) and (39), and the result must give Equation (37).

$$\begin{aligned}
\frac{1}{2}(D^+u(x) + D^-u(x)) &= \frac{1}{2}\left(\frac{u(x+h)-u(x)}{h} + \frac{u(x)-u(x-h)}{h}\right) \\
&= \frac{1}{2}\frac{u(x+h)-u(x-h)}{h}
\end{aligned} \tag{41}$$

Equation (41) confirms the equality for $x \in [0, 1]$.

c) Similar to **b)**, both sides of $D^+D^-u(x) = D^-D^+u(x)$ are investigated separately.

$$\begin{aligned}
D^+D^-u(x) &= D^+\left[\frac{u(x)-u(x-h)}{h}\right] \\
&= \frac{1}{2}\left[D^+u(x) - D^+u(x-h)\right] \\
&= \frac{1}{h}\left[\frac{x(u+h)-u(x)}{h} - \frac{x(u-h+h)-u(x-h)}{h}\right] \\
&= \frac{u(x+h)-2u(x)+u(x-h)}{h^2}
\end{aligned} \tag{42}$$

$$\begin{aligned}
D^-D^+u(x) &= D^-\left[\frac{u(x+h)-u(x)}{h}\right] \\
&= \frac{1}{2}\left[D^-u(x+h) - D^-u(x)\right] \\
&= \frac{1}{h}\left[\frac{x(u+h)-u(x+h-h)}{h} - \frac{x(u)-u(x-h)}{h}\right] \\
&= \frac{u(x+h)-2u(x)+u(x-h)}{h^2}
\end{aligned} \tag{43}$$

Equations (42) and (43) confirm the equality for $x \in [0, 1]$.

d) Applying Taylor's formula on Equation (37) gives

$$\begin{aligned}
D^0u(x) &= \frac{1}{2}[\cancel{u(x)} + u'(x)h + \cancel{\frac{1}{2}u''(x)h^2} + \frac{1}{6}u'''(x)h^3 + \cdots \\
&\qquad \cancel{-u(x)} + u'(x)h \cancel{-\frac{1}{2}u''(x)h^2} + \frac{1}{6}u'''(x)h^3 + \cdots] \\
&= u'(x) + h^2 \underbrace{\left[\frac{1}{6}u'''(x) + R\right]}_{R_0}.
\end{aligned} \tag{44}$$

Considering that $R_0 = \frac{1}{6}\left(u'''(\xi_1) + u'''(\xi_2)\right)$, it holds

$$|R_0| \leq \frac{1}{6}\max|u'''(\xi)|, \text{ for } \xi, \xi_1, \xi_2 \in [x-h, x+h]. \tag{45}$$

Applying the same procedure on Equation (40) gives

$$\begin{aligned}
D^{+-}u(x) &= \frac{1}{h^2}[\cancel{u(x)} \cancel{+u'(x)h} + \frac{1}{2}u''(x)h^2 \cancel{+\frac{1}{6}u'''(x)h^3} + \frac{1}{24}u^{(4)}h^4 + \cdots \\
&\qquad \cancel{u(x)} \cancel{-u'(x)h} - \frac{1}{2}u''(x)h^2 \cancel{-\frac{1}{6}u'''(x)h^3} + \frac{1}{24}u^{(4)}h^4 + \cdots \\
&\qquad \cancel{-2u(x)}] \\
&= u'(x) + h^2 \underbrace{\left[\frac{1}{12}u^{(4)}(x) + R\right]}_{R_1}.
\end{aligned} \tag{46}$$

Considering that $R_1 = \frac{1}{12}\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right)$, it holds

$$|R_1| \leq \frac{1}{12}\max\left|u^{(4)}(\xi)\right|, \text{ for } \xi, \xi_1, \xi_2 \in [x-h, x+h]. \tag{47}$$

D^-D^0 is obtained by applying Equation (39) on Equation (37)

$$\begin{aligned}
D^-D^0 &= \frac{D^-}{2h}\left(u(x+h) - u(x-h)\right) \\
&= \frac{1}{2h}\left(\frac{u(x+h) - u(x)}{h} - \frac{u(x-h) - u(x-2h)}{h}\right) \\
&= \frac{1}{2h}\left(-u(x) + u(x+h) - u(x-h) + u(x-2h)\right)
\end{aligned} \tag{48}$$

Hence, the Taylor approximation of D^-D^0 is

$$\begin{aligned} D^-D^0 = \frac{1}{2h^2}[&\cancel{u(x)} + \cancel{u'(x)h} + \cancel{\frac{1}{2}u''(x)}h^2 + R_1 + \cdots \\ &\cancel{-u(x)} \\ &\cancel{-u(x)} \cancel{+u'(x)h} - \cancel{\frac{1}{2}u''(x)}h^2 + R_2 + \cdots \\ &\cancel{u(x)} \cancel{-u'(x)2h} + \frac{1}{2}u''(x)4h^2 - R_3] \\ &= \frac{1}{2h^2}\left(2u''(x)h^2 + R_1 + R_2 - R_3\right) = u'' + \mathcal{O}(h) \end{aligned} \tag{49}$$

As Equation (49) implies, this differentiation quotient has a lower order, compared to above mentioned ones. This shows a lower rate in error decrease and thus less suitable for approximating the second derivative.

Ex 2a) Through an argument of separation of variables we define:

$u(x,t) = f(t)\,g(x)$, so we have the PDE

$$f'(t)\,g(x) - a\,f(t)\,g''(x) + c\,f(t)\,g(x) = 0$$

$$\Leftrightarrow \frac{f'(t)}{f(t)} + c = \frac{a\,g''(x)}{g(x)} = k \quad \text{a System of ODEs}$$

① $f'(t) = (k-c)\,f(t)$

$\Rightarrow f(t) = d \cdot e^{(k-c)t}$, where d is an integration constant

② $g''(x) = \frac{k}{a}\,g(x)$

<u>for $k = 0$</u> we have

$g(x) = c_1 x + c_2$

which is either the trivial solution or a strictly (de-)increasing function, which cannot fulfill the initial conditions $u(0,0) = u(0,1) = 0$.

<u>for $k = \mu^2 > 0$</u> we have, since $a > 0$

$g(x) = c_1 e^{\frac{\mu}{\sqrt{a}}x} + c_2$

which also cannot fulfill the initial boundary conditions for the same reason as before

<u>for $k = -\mu^2 < 0$</u> we have

$g(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x)$

from the boundary condition we have:

$$u(t,0) = f(t)\,g(0) = 0 \xrightarrow{f(t)\neq 0\ \forall t} g(0) = 0$$

$$u(t,1) = f(t)\,g(1) = 0 \rightarrow g(1) = 0$$

$$\Rightarrow g(0) = c_1 \sin(\mu 0) + c_2 \cos(\mu 0) = 0 \Rightarrow c_2 = 0$$

$$g(1) = c_1 \sin(\mu 1) = 0 \Rightarrow \mu = \pi \cdot n,\ n \in \mathbb{N}_0$$

There is no restriction on c_1, so the solution will be a linear combination of functions such that the initial condition is fullfilled, the corresponding coeficcents are the ones given in the hint, so we have:

$$u(t,x) = \sum_n^{\infty} \left[e^{-(\pi^2 n^2 + c)t} \cdot \left(\underbrace{\frac{16}{\pi(4n - n^3)}}_{= d \cdot c} \sin(\pi n x) \right) \right] \text{ for odd } n, \ 0 \text{ for even } n.$$

~~2b) The stationary solutions are given by the exponential decaying factor~~ $\cancel{e^{-(\pi^2 n^2 + c)t}}$

ii) von Neumann b.c.:

The derivation in the case of von Neumann b.c. is the same up to the constraints, which now are of the form:

$$g'(0) = c_1 \mu \cos(\mu 0) - c_2 \mu \sin(\mu 0) = 0 \Rightarrow c_1 = 0$$

$$g'(1) = c_2 \sin(\mu 1) = 0 \Rightarrow \mu = \pi \cdot n,\ n \in \mathbb{N}_0$$

In this case the linear combination that satisfies the b.c. and initial conditions is easily constructed:

$$u(t,x) = e^{-(0^2 \cdot \pi^2 + c)t} - e^{-(2^2 \pi^2 + c)t} \cos(2\pi x)$$

2b) The stationary Solution is given by the exponentially decaying term. $e^{-(\pi^2 n^2 + c) t}$

The exponent is always strictly negative and tends to $-\infty$ for $t \to \infty$, except when $n = 0$ and $c = 0$, which means that for $t \to \infty$ we obtain only non zero terms for the second condition (von Neumann).

i) Because all terms contain an $n \geq 1$ all terms vanish as $t \to \infty$

$$u_{t \to \infty}(x) = 0$$

ii) Here the first term survives if $c = 0$, giving

$$u_{t \to \infty}(x) = 1$$