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# Numerical Mathematics II for Engineers

Homework Assignment 2

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by **Group 5**

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## Exercise 1

Given is the following boundary problem of an annulus

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega = \{(x,y) \in \mathbb{R}^2 \colon 1 \leq \sqrt{x^2+y^2} < 2\} \subset \mathbb{R}^2, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with the boundary condition

$$g(x,y) = \begin{cases} x & \text{for } x^2+y^2 = 2^2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

**a)** $(x,y) \in \Omega$ is transformed to polar coordinates $(r,\varphi) \in \Omega_r$ using

$$x = r\cos(\varphi), \tag{3}$$
$$y = r\sin(\varphi) \tag{4}$$

with $r \in (1,2]$ and $\varphi \in (0,2\pi]$. Let $v : \Omega_r \to \mathbb{R}$ defined by $v(r,\varphi) = v(x,y)$. In this case, the partial derivatives $v_x$ and $v_y$ are expressed using chain rule as follows

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \varphi}\frac{\partial \varphi}{\partial x}, \text{also denoted as } u_x = u_r r_x + v_\varphi \varphi_x \tag{5}$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \varphi}\frac{\partial \varphi}{\partial x}, \text{also denoted as } u_y = u_r r_y + v_\varphi \varphi_y \tag{6}$$

The second partial derivative of $v$ with respect to $x$ is obtained using product rule

$$v_{xx} = u_r r_{xx} + (u_r)_x r_x + v_\varphi \varphi_{xx} + (v_\varphi)_x \varphi_x \tag{7}$$

By applying the chain rule from Equation (5) and Equation (6) into Equation (7), $v_{xx}$ is written in the following form

$$v_{xx} = u_r r_{xx} + v_{rr} r_x^2 + 2v_{r\varphi} r_x \varphi_x + v_\varphi \varphi_{xx} + v_{\varphi\varphi} \varphi_x^2 \tag{8}$$

where a similar expression is obtained for y

$$v_{yy} = u_r r_{yy} + v_{rr} r_y^2 + 2v_{r\varphi} r_y \varphi_y + v_\varphi \varphi_{yy} + v_{\varphi\varphi} \varphi_y^2 \tag{9}$$

The laplace equation $\Delta v = v_{xx}+v_{yy}$ can be written in a proper semi-polar coordinate form by adding Equation (8) and Equation (9) and collecting the like terms

$$\begin{aligned}\Delta v &= v_{xx} + v_{yy} \\ &= u_r(r_{xx} + r_{yy}) + v_{rr}(r_x^2 + r_y^2) + 2v_{r\varphi}(r_x\varphi_x + r_y\varphi_y) + v_\varphi(\varphi_{xx} + \varphi_{yy}) + v_{\varphi\varphi}(\varphi_x^2 + \varphi_y^2)\end{aligned} \tag{10}$$

Now, expressions in parentheses are to be elaborated in the partial derivations with respect to polar coordinates. For this purpose, the relationship $x^2 + y^2 = r^2$ is differentiated with respect to x and y. Accordingly, the partial differentiation of $r$ terms with respect Cartesian terms up to second order are obtained as

$$r_x = \frac{x}{r}, \tag{11}$$

$$r_{xx} = \frac{y^2}{r^3}, \tag{12}$$

$$r_y = \frac{y}{r}, \tag{13}$$

$$r_{yy} = \frac{x^2}{r^3}. \tag{14}$$

Similarly, the partial differentiation of $\varphi$ terms with respect to x and y are obtained as

$$\varphi_x = -\frac{y}{r^2}, \tag{15}$$

$$\varphi_{xx} = \frac{2xy}{r^4}, \tag{16}$$

$$\varphi_y = \frac{x}{r^2}, \tag{17}$$

$$\varphi_{yy} = -\frac{2xy}{r^4}. \tag{18}$$

Employing the Equations 11 to 18 in Equation (10) gives

$$\begin{aligned}\Delta v &= u_r\left(\frac{y^2}{r^3} + \frac{x^2}{r^3}\right) + v_{rr}\left(\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right) + 2v_{r\varphi}\left(\frac{-xy}{r^3} + \frac{yx}{r^3}\right) + \dots \\ &\quad v_\varphi\left(\frac{2xy}{r^4} - \frac{2xy}{r^4}\right) + v_{\varphi\varphi}\left(\left(-\frac{x^2}{r^3}\right)^2 + \left(\frac{x^2}{r^3}\right)^2\right) \\ &= \frac{1}{r}u_r + v_{rr} + 0 + 0 + \frac{1}{r^2}v_{\varphi\varphi}.\end{aligned}$$

Thus, for any $v$ satisfying the Laplace equation $-\Delta v = 0$, $v$ satisfies in polar coordinates the equation

$$-\left(v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi}\right) = 0 \tag{19}$$

**b)** For any $v$ defined in a), the domain $\Omega_v$ is defined as

$$\Omega_v = \{(r,\varphi) \in \mathbb{R}^2 \colon r \in (1,2), \varphi \in (0,2\pi]\} \tag{20}$$

with the boundary condition

$$h(r,\varphi) = \begin{cases} 2\cos(\varphi) & \text{for } r = 2, \varphi \in (0,2\pi] \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

**c)** Assuming that $v(r,\varphi) = R(r)\Phi(\varphi)$, $\forall r \in (0,1]$, $\forall \varphi \in (0,2\pi]$. If $v(r,\varphi)$ satisfies Equation (19), then this equation can be expressed in the following form

$$\left(R_{rr}\Phi + \frac{1}{r}R_r\Phi + \frac{R}{r^2}\Phi_{\varphi\varphi}\right) = 0. \tag{22}$$

Placing the functions depended of $r$ and $\varphi$ in separate terms, Equation (22) is written as

$$-\frac{r^2R_{rr} + rR_r}{R} = \frac{\Phi_{\varphi\varphi}}{\Phi} = \lambda \tag{23}$$

where $\lambda$ represent a constant real factor. Equation (23) enables $v(r,\varphi)$ to be considered as two different ODE's, as both $R$ and $\Phi$ terms are equated with $\lambda$ separately

$$\frac{\Phi_{\varphi\varphi}}{\Phi} = \lambda \iff \Phi_{\varphi\varphi} = \Phi\lambda \tag{24}$$

$$-\frac{r^2R_{rr} + rR_r}{R} = \lambda \iff -\left(r^2R_{rr} + rR_r\right) = R\lambda. \tag{25}$$

**d)** The solution of $v(r,\varphi)$ consists of the superposition of Equation (24) and Equation (25) solutions, which are studied in three different cases for $\lambda$:

- $\boldsymbol{\lambda = 0}$:
  In this case, possible solutions are given by

$$\Phi(\varphi) = a\varphi + b \tag{26}$$

$$R(r) = c\ln r + d. \tag{27}$$

Since $\Phi$ has to be a periodic equation, $a = 0$, and $c = 0$, as the solution has to remain finite, as $r$ goes to 0. With the applied conditions, the ansatz gives

$$v(r, \varphi) = bd = g = const. \tag{28}$$

- **$\boldsymbol{\lambda < 0}$:**
  Let $\lambda = -k^2$. Possible solutions for $R$ and $\Phi$ are given by

  $$\Phi(\varphi) = a_k \cosh(k\varphi) + b_k \sinh(k\varphi) \tag{29}$$
  $$R(r) = c_k r^{-k} + d_k r^k \tag{30}$$

  $v(r, \varphi)$ holds no solutions for this case, because Equation (29) implies a periodic function in neither terms, thus remains zero for $\Phi$.

- **$\boldsymbol{\lambda > 0}$:**
  Let $\lambda = k^2$. Possible solutions are given by

  $$\Phi(\varphi) = a_k \cos(k\varphi) + b_k \sinh(k\varphi) \tag{31}$$
  $$R(r) = c_k r^k + d_k r^{-k} \tag{32}$$

  In this case, $\Phi$ shows a well periodic function with period $2\pi$. Furthermore $d_k$ must be 0, in order to keep $R$ finite, as $r$ goes to zero

  $$v(r, \varphi) = r^k \left(A_k \cos(k\varphi) + B_k \sin(k\varphi)\right) \tag{33}$$

  where $A_k = a_k c_k$ and $B_k = b_k c_k$.

The solution for $v(r, \varphi)$ is expressed as the superposition of Equation (28) and Equation (33)

$$v(r, \varphi) = g + \sum_{k=1} r^k \left(A_k \cos(k\varphi) + B_k \sin(k\varphi)\right) \tag{34}$$

for $A_k$ and $B_k \in \mathbb{R}$ are obtained by the Fourier expansion.

The solution of $v(r, \varphi)$ is to be investigated on the boundary with $r = 2$. So gives

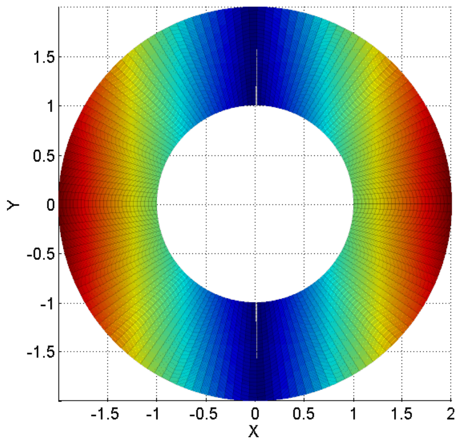
$$v(2, \varphi) = g + \sum_{k=1} 2^k \left(A_k \cos(k\varphi) + B_k \sin(k\varphi)\right) \tag{35}$$

As Equation (35) implies, the boundary condition is already given in the Fourier-series form, so that the only solution exists for $k = 1$, $A_1 = 0$, $B_1 = 0$ and $g = 0$. Hence the solution, that satisfies Equation (19) and Equation (21) in $\Omega_v$ is

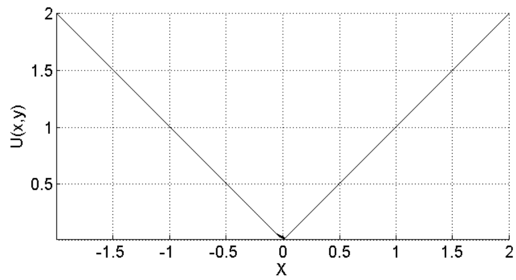
$$v(r, \varphi) = r \cos(\varphi). \tag{36}$$

**e & f)** The solution code can be found in `a02e01solution.m` and the plot code in `a02e01plot.m`. The surface plot of the Equation (36) solution is illustrated in

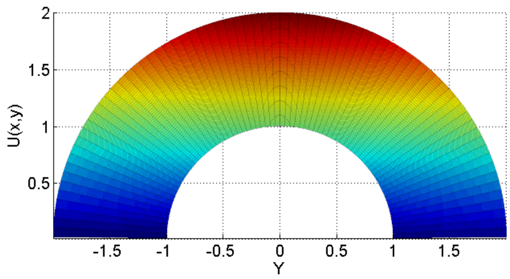
Figure 1.



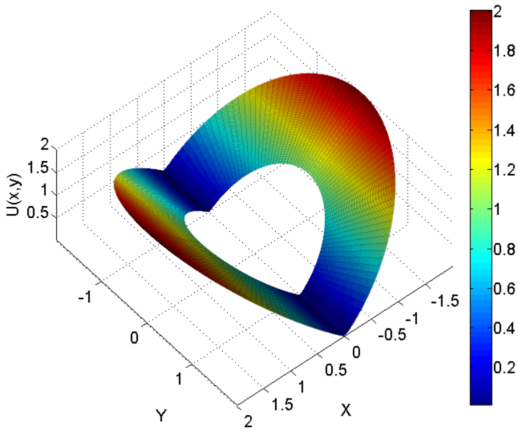

**(a)** *X-Y Plane*




**(b)** *X-U Plane*




**(c)** *Y-U Plane*




**(d)** *Isometric View*

**Figure 1** *Surface Plot of the Solution*

# Exercise 2