2a) Given three points $\{(x_{i-1}, u(x_{i-1})), (x_i, u(x_i)), (x_{i+1}, u(x_{i+1}))\}$ there is only one parabola passing through them $p(x) = ax^2+bx+c$ so we can assume that the second derivative of $u(x_i)$ and this second degree approximation parabola will coincide. $p''(x_i) = 2a = u''(x_i)$

We set up a linear system of equations to find a:

$$\underbrace{\begin{pmatrix} 1 & x_{i-1} & x_{i-1}^2 \\ 1 & x_i & x_i^2 \\ 1 & x_{i+1} & x_{i+1}^2 \end{pmatrix}}_{V} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} u(x_{i-1}) \\ u(x_i) \\ u(x_{i+1}) \end{pmatrix}$$

V is the Vandermonde matrix, which has an inverse as long as $x_i \neq x_j$, which in our case is true, of the form:

$$V^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{x_{i-1}-x_i} & \frac{1}{x_i - x_{i-1}} & 0 \\ \frac{1}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} & \frac{1}{(x_i-x_{i-1})(x_i-x_{i+1})} & \frac{1}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} \end{pmatrix}$$

So we have, for $x_i - x_{i-1} = h_i$, $h_i + h_{i+1} = x_{i+1} - x_{i-1}$, $h_{i+1} = x_{i+1} - x_i$

$$u''(x_i) = 2 \cdot \left[\frac{1}{h_i(h_i+h_{i+1})} u(x_{i-1}) - \frac{1}{h_i h_{i+1}} u(x_i) + \frac{1}{(h_i+h_{i+1})h_{i+1}} u(x_{i+1}) \right]$$

2b) we rewrite $u(x_{i-1}) = u(x_i - h_i)$, $u(x_{i+1}) = u(x_i + h_{i+1})$ and expand the taylor series:

$$u''(x_i) = 2 \cdot \left[\begin{array}{l} + \frac{1}{h_i(h_i+h_{i+1})} \left[u(x_i) - u'(x_i)h_i + \frac{1}{2} u''(x_i) h_i^2 - \frac{1}{6} u'''(\xi_1) h_i^3 \right] \\ - \frac{1}{h_i h_{i+1}} \quad u(x_i) \\ + \frac{1}{(h_i+h_{i+1})h_{i+1}} \left[u(x_i) + u'(x_i) h_{i+1} + \frac{1}{2} u''(x_i) h_{i+1}^2 + \frac{1}{6} u'''(\xi_2) h_{i+1}^3 \right] \end{array} \right]$$

$= D^+ D^- u(x_i) + R_i$, $|R| \leq \frac{1}{3} u'''(\xi)(h_{i+1} - h_i) \Rightarrow R_i = O(h^1)$

Since the dominating R_i term depends only on u''', u does not need to be $\in C^4$