TECHNICAL UNIVERSITY OF BERLIN

Faculty II - Mathematics and Natural Sciences

Institute of Mathematics

Dr. D. Peschka, A. Selahi

Numerical Mathematics II for Engineers

Homework Assignment 3

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by **Group 5**

Kagan Atci	338131	Physical Engineering, M.Sc.
Navneet Singh	380443	Scientific Computing, M.Sc.
Daniel V. Herrmannsdoerfer	412543	Scientific Computing, M.Sc.

Exercise 1

Given is $m \in \mathbb{N}_0$, assuming $\bar{\Omega} = [0, 1]$ and $\bar{\Omega}_h = \{0, h, \ldots, (N+1)h\}$ with $h = \frac{1}{N+1}$ the corresponding grid. Considering that $x_n = nh \in \bar{\Omega}_h$ with sufficiently many neighbors to either side, for any j with $n - m \leq j \leq n + m$, requiring $x_j = jh \in \bar{\Omega}_h$, and $u_n = u(x_n)$ with $u \colon \bar{\Omega} \to \mathbb{R}$ is r times continuously differentiable.

a) u values on neighbor points with above mentioned conditions are expressed in Taylor expansion form about the x_n

$$u_h(x_n + jh) = u_h(x_n) + u'(x_n)\frac{(jh)^1}{1!} + u_h''\frac{(jh)^2}{2!} + \cdots + u_h(x_n)\frac{(jh)^k}{k!} + R \quad (1)$$

with k the derivative order and R the remainder. Applying this for each neighbor point, a linear equation system between the u and their derivatives is built up

$$\begin{bmatrix} u_h(x_{n-m}) \\ \vdots \\ u_h(x_{n+m}) \end{bmatrix} = \underline{\underline{S}} \begin{bmatrix} u(x_n) \\ \vdots \\ h^k u^{(k)}(x_n) \end{bmatrix} \quad \text{where } \underline{\underline{S}} \in \mathbb{R}^{k \times k} \text{ with } s_{j,k} = \frac{j^k}{k!}. \quad (2)$$

In this case, the symmetric stencils for spatial differentiations $\Delta_{h,r}$ up to k-th order are obtained by inverting $\underline{\underline{S}}$, where the remainders are neglected.

$$\underline{\underline{S}}^{-1} \begin{bmatrix} u_h(x_{n-m}) \\ \vdots \\ u_h(x_{n+m}) \end{bmatrix} = \begin{bmatrix} u(x_n) \\ \vdots \\ h^k u^{(k)}(x_n) \end{bmatrix} \quad (3)$$

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Since the $\underline{\underline{S}}^{-1}$ must be a symmetric matrix, applying the Taylor expansion procedure to m neighbor points on either sides limits the derivative order at $k = 2m + 1$ for u being $r = 2m + 2$ differentiable. Hence, the remainder can also be written in the matrix form as

$$\underline{\underline{R}} \in \mathbb{R}^{k \times k} \text{ with } r_{j,k} = \frac{(jh)^r}{r!} u^{(r)}(\xi) \text{ for } \xi \in [x_n - kh, x_n + kh]. \tag{4}$$

b) Please refer to online submitted `a03ex01getstencil.m` file.

c) Please refer to online submitted `a03ex01getLaplace.m` file.

d) For $u(x) = \sin(2\pi x)$, Figure 1 illustrates the maximum error magnitude between the finite difference method with four different level of stencils and the analytical method applied on u''.
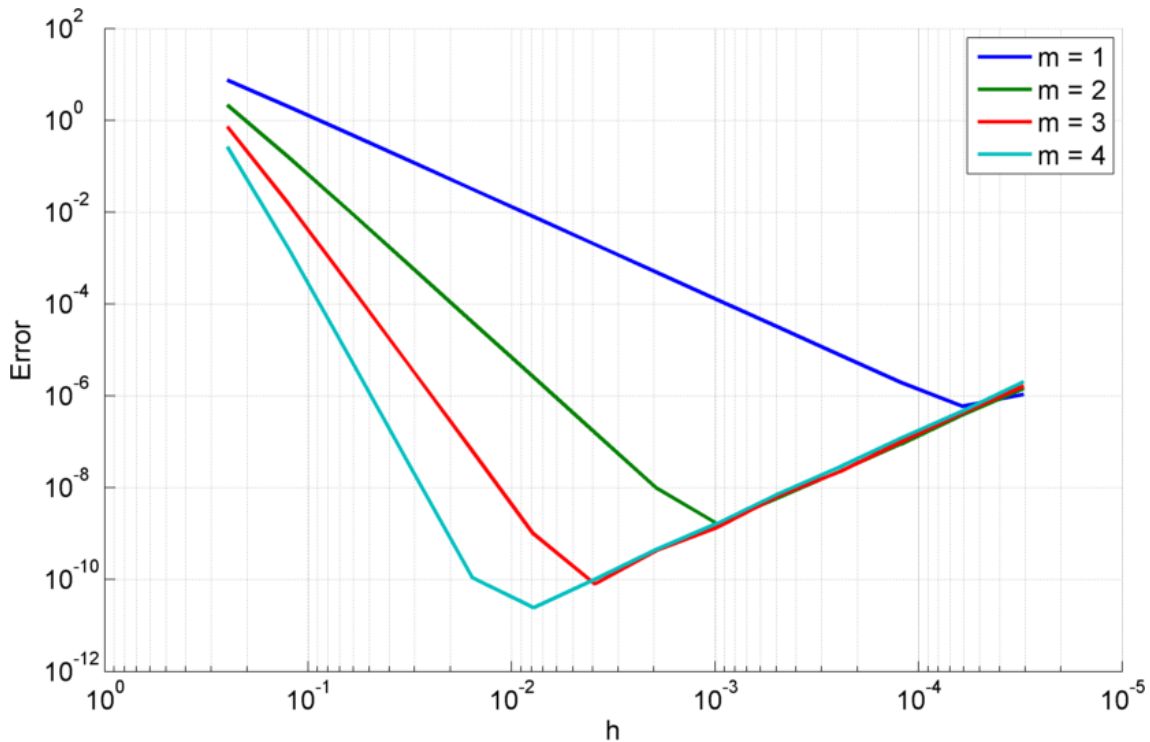

Figure 1 *Maximum error $\max_j |L_h(x_j)u_h(x_j) - u''(x_j)|$ of the finite difference stencils as a function of h with respect to m*

Accordingly, a straight conclusion regarding to an exponential error reduction by increasing the mesh resolution (deceasing h) can be made in the first place. This is also considered as an improvement in convergence. Furthermore, increasing the number of neighbor points in constructing the finite difference stencil leads to the improvement of convergence at even higher rates. However, the error, which in theory is supposed to go to 0 continuously for $\lim_{h \to 0}$, stops this behavior at a

certain h for every m. This is issue is associated with the rounding error of the computer. The more neighbor points are used for the stencil, the sooner can this limit be reached. Therefore, this limiting fact must be taken into account during geometry meshing.