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Numerical Mathematics II for Engineers

Homework assignment 8 : *Finite elements and software.*

Programming assignments can be solved in MATLAB/Python/Julia.

Deadline : submit until December, Friday 20, 2019.

1. Exercise : Weak formulation **5 points**

Consider the elliptic problem for the electric potential φ

$$\begin{aligned} -\big(\epsilon(x)\varphi'(x)\big)' &= \varrho(x) && \text{in } \Omega = (0,1), \\ \varphi &= 0 && \text{at } \{0,1\} = \partial\Omega, \end{aligned}$$

where the relative permittivity $\epsilon : \Omega \to \mathbb{R}$ is

$$\epsilon(x) = \begin{cases} 1 & \text{for } 0 < x < 1/2 \\ \bar{\epsilon} & \text{for } 1/2 \le x < 1 \end{cases}$$

and with given charge density $\varrho : \Omega \to \mathbb{R}$. The interpretation is that we have different materials in $0 < x < 1/2$ and in $1/2 < x < 1$, so that the permittivity might jump.

a) Show that the general requirement $\lim_{\delta\to 0} \int_{1/2-\delta}^{1/2+\delta} \varrho(x)\,\mathrm{d}x = 0$ i.e., the interface carries no extra charges, leads to the transmission condition

$$\lim_{\delta\to 0} \Big[\epsilon(1/2+\delta)\varphi'(1/2+\delta) - \epsilon(1/2-\delta)\varphi'(1/2-\delta)\Big] = 0.$$

b) Derive the weak form of the problem. **Hint :** Assume the test functions fulfill the boundary conditions and φ can be integrated by parts separately on $(0, 1/2)$ and $(1/2, 1)$. The solution and the test functions are continuous.

c) Find the weak solution for $\bar{\epsilon} = 2$, $\varrho(x) = -1$. **Hint :** Try to combine polynomials on $(0, 1/2)$ and on $(1/2, 1)$ with proper continuity of $\varphi, \epsilon\varphi'$.

d) Is the solution u a classical solution $u \in C^2(\Omega)$?

2. Exercise : Equivalence to a minimization problem **3 points**

Let $A \in \mathbb{R}^{N\times N}$ be a symmetric and positive definite matrix and $b \in \mathbb{R}^N$ be arbitrary. Show that the following statements are equivalent.
— The vector $x \in \mathbb{R}^N$ is the solution to the linear system $Ax = b$.
— The vector $x \in \mathbb{R}^N$ is the minimizer of the expression $J(x) = \frac{1}{2}x^{\mathsf{T}}Ax - b^{\mathsf{T}}x$.

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3. Exercise : **5 points**

Consider the functions $\hat{\varphi}_1(y) = 1 - y$ and $\hat{\varphi}_2(y) = y$ defined for $y \in [0, 1]$.

a) Explicitly compute the matrices $M \in \mathbb{R}^{2\times 2}$ and $S \in \mathbb{R}^{2\times 2}$ whose entries are defined

$$\hat{M}_{ij} = \int_0^1 \hat{\varphi}_i(y)\hat{\varphi}_j(y)\mathrm{d}y \quad \text{and} \quad \hat{S}_{ij} = \int_0^1 \hat{\varphi}_i'(y)\hat{\varphi}_j'(y)\mathrm{d}y,$$

respectively.

b) Define a transformation $x = T(y) = (b-a)y+a$ with $a, b \in \mathbb{R}$ and $a < b$. Moreover, define $\varphi_i(x)$ for $x \in [a, b]$ via $\varphi_i(T(y)) = \hat{\varphi}_i(y)$ for $i = 1, 2$. Compute the matrices $M \in \mathbb{R}^{2\times 2}$ and $S \in \mathbb{R}^{2\times 2}$

$$M_{ij} = \int_{T(0)}^{T(1)} \varphi_i(x)\varphi_j(x)\mathrm{d}x \quad \text{and} \quad \bar{S}_{ij} = \int_{T(0)}^{T(1)} \varphi_i'(x)\varphi_j'(x)\mathrm{d}x,$$

for general a, b. Use integration by substitution! Express the resulting matrices M, S using $\hat{M}, \hat{S}$. Note : The matrices defined above will play an important role for constructing finite element schemes later.

4. Exercise : Software (Triangle) **5 points**

Download and compile `Triangle` and make sure you understand the example `box.poly` from its documentation (also provided on the course website).

a) Refine the `box.poly` to obtain a mesh with > 100 vertices

b) Generate a disc domain `disc.poly` and refine to have > 200 vertices.

c) Generate an annulus domain with inner radius 1 and outer radius 2 `ann.poly` and refine to have > 500 vertices.

d) Load the mesh into MATLAB/Python/Julia using the provided function `readtria` and state the number of elements and of vertices/points for each mesh.

e) (Optional) Plot the meshes. (+2 points)

total sum : 18 (+2) points