BERLIN,

Exercise 1

a) & b)

- Common PDE: Kaup-Kupershmidt equation

$$\frac{\partial u}{\partial t} = \frac{\partial^5 u}{\partial x^5} + 10\frac{\partial^3 u}{\partial x^3}u + 25\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial x} + 20u^2\frac{\partial u}{\partial x} \tag{1}$$

is a PDE fifth order.

- Member 1 PDE: Hunter-Saxton equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{2}$$

is a PDE second order.

- Member 2 PDE: Liouville equation

$$\nabla^2 u + e^{\lambda u} = 0 \tag{3}$$

is a PDE second order.

- Member 3 PDE: φ^4 - Equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \varphi + \varphi^3 = 0 \tag{4}$$

is a PDE second order.

c)

The Navier-Stokes-Equation describes the motion of the viscous fluid substances and is expressed for compressible fluid as

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p + \mu\nabla^2 u + f \tag{5}$$

with ρ the density, u velocity vector, p pressure, and μ kinematic viscosity of the fluid. Equation (5) is expressed in homogenous form by setting $f = 0$ as follows

$$\rho(\partial_t u + u \cdot \nabla u) + \nabla p - \mu\nabla^2 u = 0 \tag{6}$$

For $u(t,x) = (u_0 x_2(H - x_2), 0)^T$ with $u_0 \in \mathbb{R}$, $x = (x_1, x_2) \in \Omega = \mathbb{R} \times (0, H)$, and $t \in (0, \infty)$, the partial differentiations result

$$\frac{\partial u}{\partial t} = (0,0)^T \tag{7}$$
$$\nabla u = (0,0)^T \tag{8}$$
$$\nabla^2 u = (0,0)^T \tag{9}$$

since u is not t-dependent and u_1 and u_2 are not effected by x_1 and x_2, respectively. Equations 7, 8 and 9 show that u is a twice differentiable function, satisfying the homogenous Navier-Stokes PDE with a boundary condition in a domain $\Omega \in \mathbb{R}^2$, which is reffered to as the classical solution for second order PDEs.

For the given conditions, Equation (6) can be expressed as

$$\nabla p = 0 \tag{10}$$

This can be referred to a 2D-flow model of a fluid in a tube with a width of H at any certain height, which is observed along the gravity axis. Therefore, the pressure in the domain Ω is described as $p = const. \in [0, \infty)$.

Exercise 2

Given is the Toeplitz-Matrix

$$\underline{\underline{K_4}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{11}$$

and $f(x) = \frac{1}{2}\underline{x}^T \underline{\underline{K_4}}\underline{x} : \mathbb{R}^4 \to R$.

a)
$f(x)$ can be expressed by executing the matrix multiplication as

$$f(x) = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + x_4^2 \tag{12}$$

and the gradient of $f(x)$ is

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \\ \frac{\partial f}{\partial x_4} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4x_1 - x_2 \\ -x_1 + 4x_2 - x_3 \\ -x_2 + 4x_3 - x_4 \\ -x_3 + 4x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 - \frac{x_2}{2} \\ \frac{-x_1}{2} + 2x_2 - \frac{x_3}{2} \\ \frac{-x_2}{2} + 2x_3 - \frac{x_4}{2} \\ \frac{-x_3}{2} + 2x_4 \end{bmatrix} \tag{13}$$

and $\underline{\underline{K_4}}\underline{x}$ gives

$$\underline{\underline{K_4}}\underline{x} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 - \frac{x_2}{2} \\ \frac{-x_1}{2} + 2x_2 - \frac{x_3}{2} \\ \frac{-x_2}{2} + 2x_3 - \frac{x_4}{2} \\ \frac{-x_3}{2} + 2x_4 \end{bmatrix} \tag{14}$$

Hence, the statement $\nabla f(x) = \underline{\underline{K_4}}\underline{x}$ is verified, since Equation (13) and Equation (14) portray equal functionals.

b)
A real symmetric matrix $\underline{\underline{K_n}} \in \mathbb{R}^{n\times n}$ is considered as positive definite, if $\underline{x}^T \underline{\underline{K_n}}\underline{x} > 0, \forall x \in \mathbb{R}^n \backslash \{0\}$. $f(x)$ is a good example for fulfilment of this condition, since the $\underline{x}^T \underline{\underline{K_4}}\underline{x}$ is already expanded in Equation (12) that consists of sum of square terms, and therefore non-negative for all $x \in \mathbb{R}^n \backslash \{0\}$.

c)
In the first step of the induction $\det(K_1)$ for $n = 1$ is investigated. With

$$\det(\underline{\underline{K_1}}) = \det(2) = 2 \tag{15}$$

the expression $det(\underline{\underline{K_n}}) = n + 1$ is fulfilled.

In the second step, we prove a statement for a general n, assuming that the relation holds for every value up to n-1. The determinant $\det(\underline{\underline{K_n}})$ for $n \in \mathbb{N}$ is written with the Laplace expansion as follows

$$\begin{vmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & & & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{vmatrix}_n = 2 \cdot (-1)^{n+n} \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & & & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{vmatrix}_{n-1} + (-1)(-1)^{n+n-1} \begin{vmatrix} 2 & - \\ -1 & \\ & \ddots \end{vmatrix} \tag{16}$$

$$= 2\det(K_{n-1}) \tag{17}$$