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Numerical Mathematics II for Engineers

Homework Assignment 5

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by **Group 5**

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Exercise 1

Given is the Poisson problem on general domains $\bar{\Omega} \subset [0, L]^2 \subset \mathbb{R}^2$ for given functions $f, g \colon \mathbb{R}^2 \to \mathbb{R}^2$, considering

$$-\Delta u = f \text{ in } \Omega, \; u = g \text{ on } \partial\Omega. \tag{1}$$

a) The considered domain is

$$\bar{\Omega} = \{(x, y) \in (0, L)^2 \colon \left(x - \frac{L}{2}\right)^2 + \left(y - \frac{L}{2}\right)^2 \leq 1\} \tag{2}$$

with $L = 2.2$ and built-up in a $\mathbb{R}^N \times \mathbb{R}^N$ environment, where $N = 256$. As illustrated in Figure 1, the points contained in Ω_h have all direct neighbors in $\bar{\Omega}_h$, where points on Γ_h having neighbors outside of Ω. For the algorithm, please refer to *Boundary Allocation* section in online submitted `a05ex01_get_laplace.m` file.

b) Please refer to '*Boundary Allocation*' section in online submitted `a05ex01_get_laplace.m` file.

c) Please refer to '*Differentiation Matrix*' section in online submitted `a05ex01_get_laplace.m` file.

d) The problem from Equation (1) and Equation (2) was solved numerically using 256×256 points with $f = 1$ and $g = 0$. The solution is plotted in Figure 2.

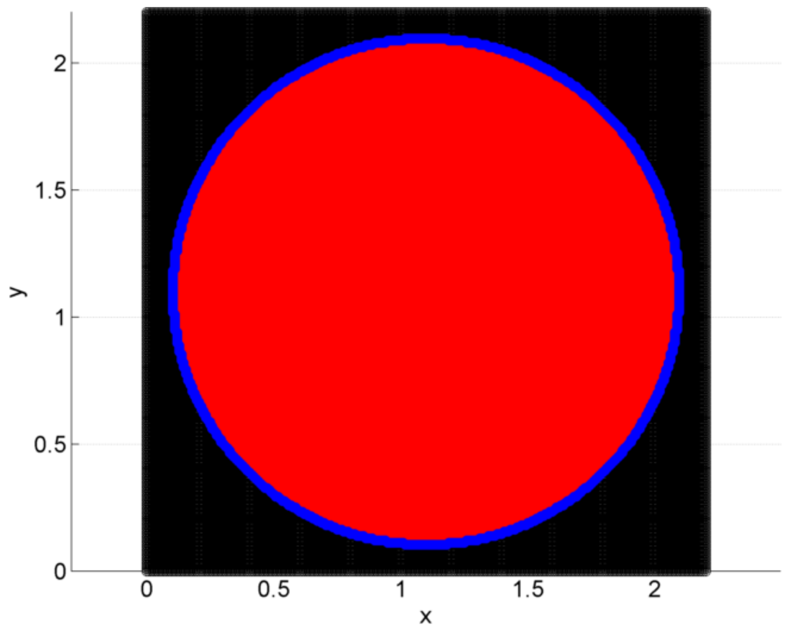


Figure 1 | The domain and sub-domain in 2D. The red area represents Ω_h and the blue circle Γ_h, which in combination makes the sub-domain $\bar{\Omega}_h$. The black area is the part of the domain that is not included into further calculations. The dot-wise plot is due to the high amount of grid points not recognizable.

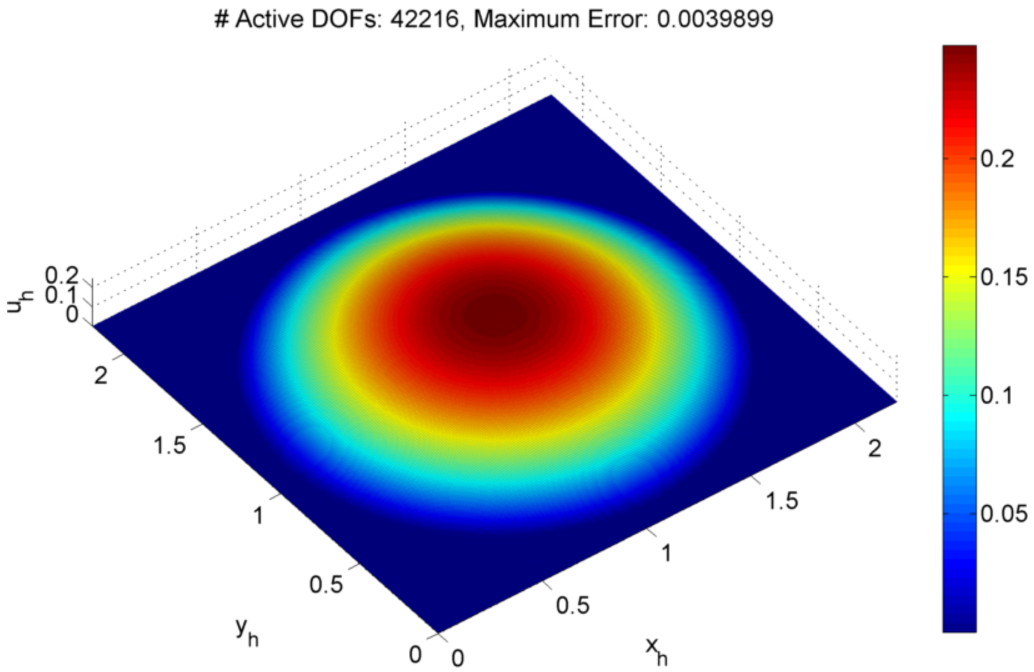


Figure 2 | Solution of the Equation (2) with $N = 256$. The magnitude of the U values are labeled with the color bar. Number of active DOFs (also the length of the L_h matrix) and the maximum error between the analytical and the numerical solution is given in the figure title.

e) In this sub-exercise, Equation (1) and Equation (2) with the previously mentioned boundary conditions were successively solved for 2^N points, with $N \in [2, 9] \subset \mathbb{N}$. The error between the analytical and numerical solution, calculated in the form of maximum maximum norm $||R_h u - u_h||_{\infty,h}$, is displayed in Figure 3.

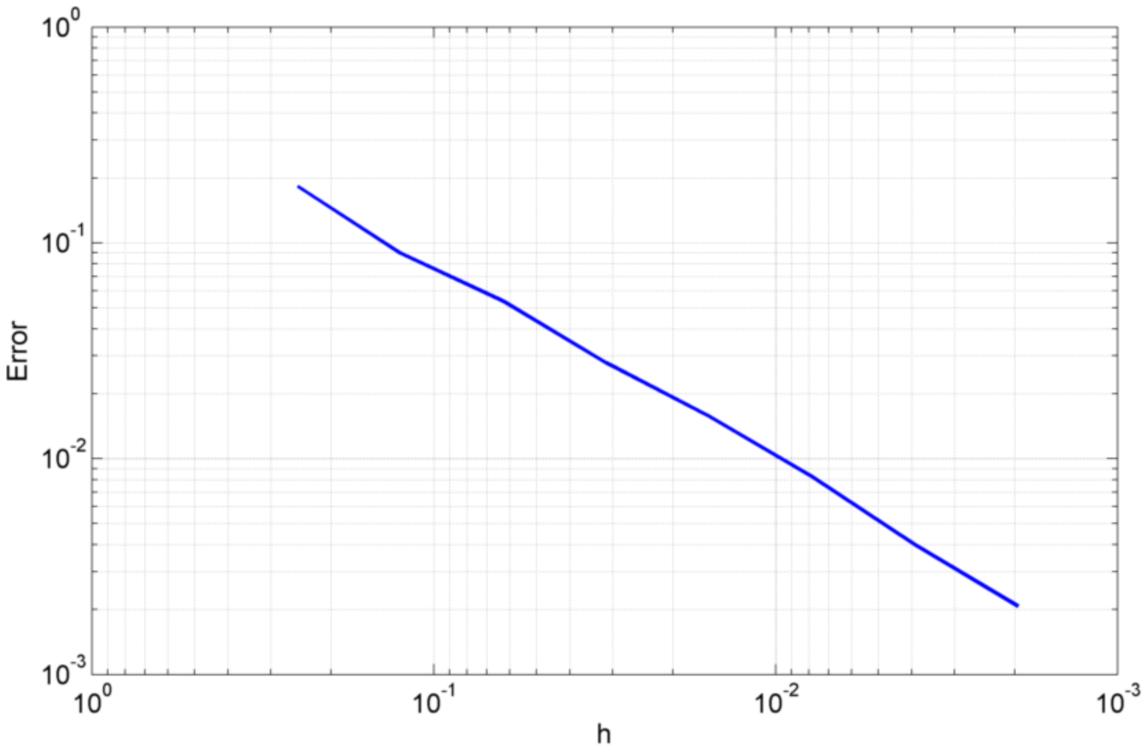


Figure 3 | Logarithmic diagram of $||R_h u - u_h||_{\infty,h}$. The direction of the x-axis was flipped, so that h is decreasing towards right the right hand side. Hence, a more comfortable overview can be attained.

The order of the convergence is obtained writing the convergence in the form of monomial equation $Error = a \cdot h^k$, with a the proportion factor, and k the order of the convergence. Taking the logarithmic line linear, the slope of the line k yields

$$k = \frac{\log \left(Error(h_9)/Error(h_2) \right)}{\log \left(h_9/h_2 \right)} \approx 0.92282\ . \tag{3}$$

f) Considering

$$\begin{aligned} \bar{\Omega} = \{(x, y) \in (0, L)^2 \colon & (x - L/12) & - & \quad 0.05 \cos(30 \cdot y/L) \geq 0 & \wedge \\ & (x - L/1.1) & - & \quad 0.1 \cos(30 \cdot y/L) \leq 0 & \wedge \\ & (y - L/1.09) & - & \quad 0.1 \sin(10 \cdot \pi \cdot x/L) \leq 0 & \wedge \\ & (y - L/12) & - & \quad 0.1 \sin(10 \cdot \pi \cdot x/L) \geq 0 & \wedge \\ & 0.8(x - L/2)^2 & + & \quad 1.3(y - L/2)^2 \geq 0 & \}, \end{aligned} \tag{4}$$

the layout of the sub-domain is illustrated in Figure 4. The outer boundaries of the sub-domain were modeled with sine functions, as the inner boundary was shaped using an elliptic formulation.

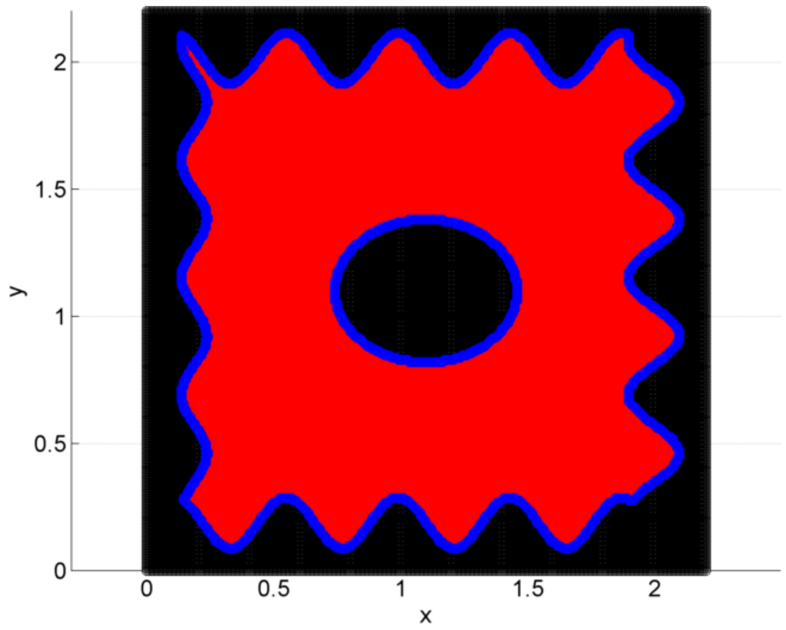


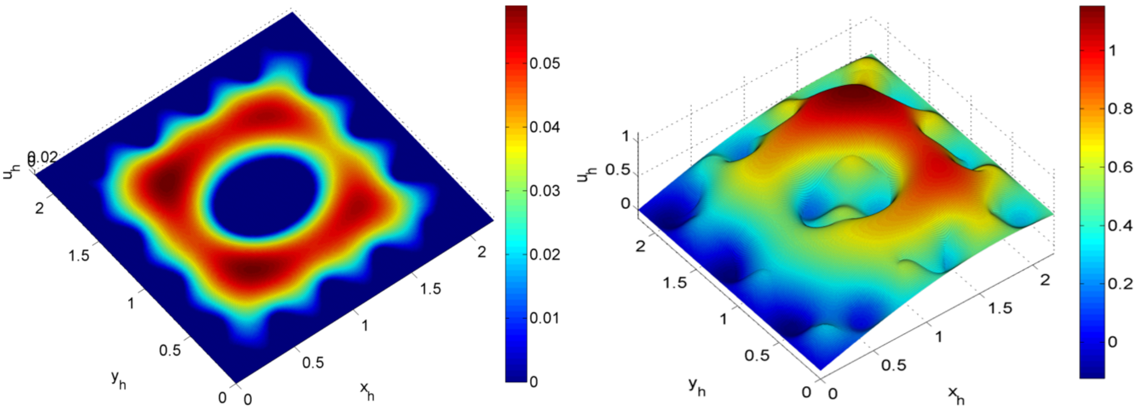
Figure 4 | The layout of the sub-domain of Equation (4). The definitions of the edges are corresponding in following sequence: First line for left border, Second line for right border, Third line for upper border, fourth line for lower border, fifth line for the inner elliptic border.

Two types of problems are to be considered for this topology:

$$f = 1, \qquad g = 0 \tag{5}$$

$$f = 5x + 3y, \qquad g = 0.2 \cdot \sin(6 \cdot \pi \cdot x/L) \cdot \sin(6 \cdot \pi \cdot y) + 0.5 \cdot \sin(2 \cdot x/L) \tag{6}$$

The former problem is chosen exactly same in **d)** for checking the quality of the solution, while the latter problem refers more complex boundary conditions. The surface plots of the solutions are displayed in Figure 5.



(a) First Problem **(b)** Second Problem

Figure 5 | Numerical solutions of the above stated problems, where the first problem refers to Equation (5) and second problem to Equation (6).

Exercise 2

a) For type i) a solvability condition is required if $\alpha_i = c = 0$, since in this case the problem coincides with a von Neumann condition which we know has a non-invertible matrix. As soon as an alpha component is added, the Robin condition is well conditioned, as it is a linear combination of the von Neumann condition and the well conditioned Dirichlet boundary condition.

For ii) we need a boundary condition in any case.

b) Please refer to the program `a05e02solvePDE.py` to see an implementation of the BVP. Analogously as with the symmetric difference stencil for the von Neumann boundary seen in class, we have chosen to use a symmetric difference stencil here too, since it leads to a better convergence rate than asymmetric stencils D^+ or D^- As we can see in Figure 6, the resulting function with the von Neumann boundary conditions has a positive curvature, which coincides with the value $q = 1.497$ from the last entry in the expanded column vector. The value coincides with the curvature of the function over the domain and also the difference $g_1 - g_0$, which gives a more concrete meaning to the equation II.26 seen in the lecture.

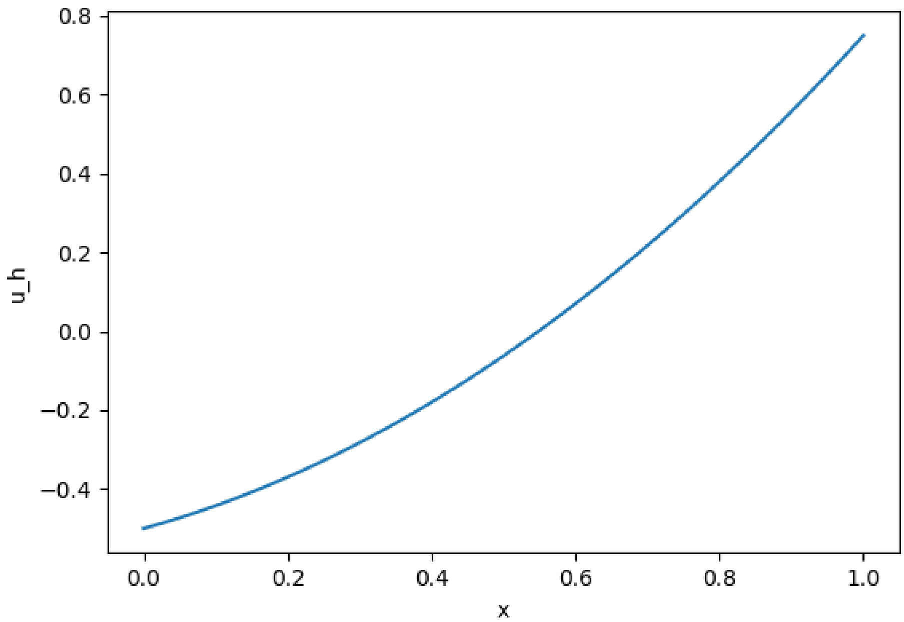


Figure 6 | Plot of the BVP. Notice the upwards curve that indicates that a positive second order derivative is needed

c) Figure 7 shows a set of solutions regarding to i) with $\alpha_i = 0$, $\beta_i = 1$, $g_0 = -1/2$, $g_1 = 2$ and $f(x) = q$ for $q \in \mathbb{R}$.

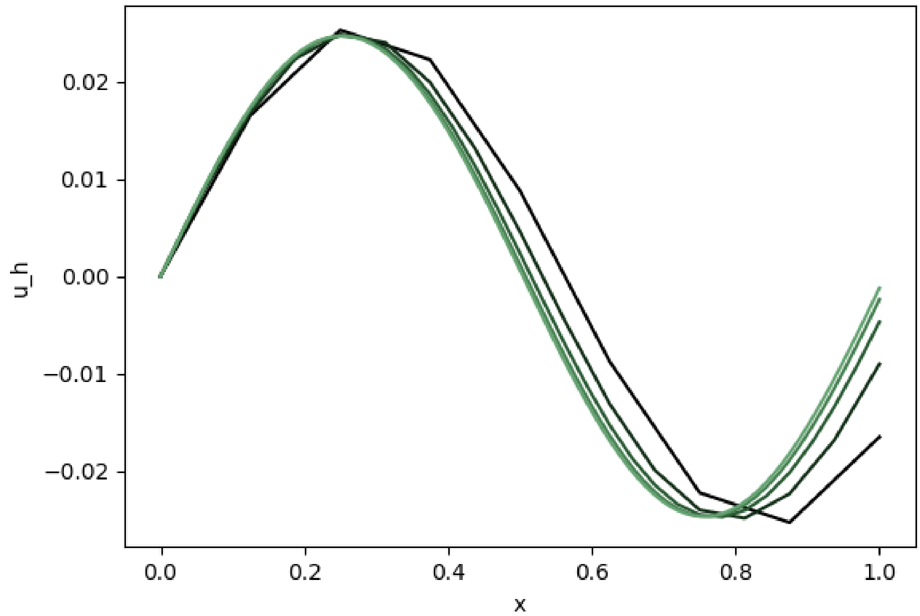


Figure 7 | Family of plots showing convergence as the mesh gets more refined

d) As illustrated in Figure 8, the log-log plot of the error with respect to a fine mesh size of $N = 2^{10}$ for a given mesh size up to $2^9 = 512 \approx 500$ gives us a regression coefficient of $r^2 = -0.99$ leading to an order of convergence of $p \approx 1$

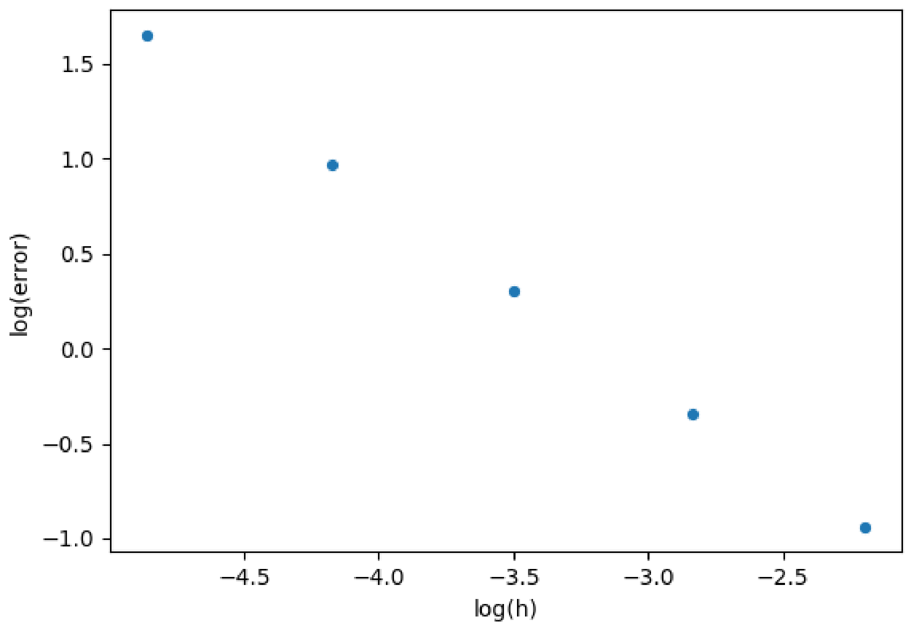


Figure 8 | log-log plot of the error versus the solution with very fine mesh $N = 2^{10}$ versus the step size h_i

Exercise 3

a) The discrete maximum principle states that if the operator is positive for every x_0, than the maximum must be on the boundary. Substituting the operator in the inequality, we have:

$$\begin{aligned} &\Delta_h u_h \geq 0 \\ &-8u(x_0) + \sum_{n=1}^{8} u(x_n) \geq 0 \\ &u(x_0) \leq \frac{1}{8} \sum_{n=1}^{8} u(x_n) \end{aligned} \tag{7}$$

Where x_n are the neighboring points of x_0. By the Equation (**??**) we know that x_0 must be less or equal than the average of the points on its neighborhood.

This implies that for every point x_0 in the domain i) either at least one neighbor x_n is larger than x_0, or ii) all neighbors are equal to x_0. Therefore the maximum of the function must lay on the boundary of the domain, since it is the only location where it could have a larger neighbor (outside the boundary) and still be the maximum of the domain.

b)

c)

d)