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Numerical Mathematics II for Engineers

Homework assignment 2

Deadline: submit before/during the lecture on November 4, 2019.

1. Exercise: Laplace equation with change of coordinates **8 points**

Consider the following boundary value problem on an annulus

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega = \{(x,y) \in \mathbb{R}^2 \, : 1 < \sqrt{x^2+y^2} < 2\} \subset \mathbb{R}^2, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where we use the boundary condition

$$g(x,y) = \begin{cases} x & \text{for } x^2+y^2 = 2^2 \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

a) Transform $(x,y) \in \Omega$ to polar coordinates $(r,\phi) \in \Omega_r$ using $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ with $r \in (0,1]$, $\varphi \in (0,2\pi]$. Let $v : \Omega_r \to \mathbb{R}$ defined by $v(r,\varphi) = u(x,y)$. Show that if u satisfies the Laplace equation (1) then v satisfies the equation

$$-\Big(v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi}\Big) = 0. \tag{3}$$

b) What is the domain Ω_v for v and what boundary conditions should one impose?

c) Assume that $v(r,\varphi) = R(r)\Phi(\varphi)$ for all $r \in (1,2)$, $\varphi \in (0,2\pi)$. Use separation of variables in order to derive ODEs for R and Φ, if v satisfies (3).

d) Solve the problem using a superposition of solutions from c).

e) Write a MATLAB/Python/Julia program `a02e01solution.m/.py/.jl` which returns the solution value `u=a02e01solution(x,y)` at given points $(x,y) \in \Omega$. The function should also accept a list of points and then return a list of values.

f) Write a MATLAB/Python/Julia program `a02e01plot.m/.py/.jl` to visualize u.

Hint: Define Ω_r via `meshgrid` (MATLAB)/`np.meshgrid` (Python), map to annulus Ω, plot using `surf` (MATLAB) or `mplot3d` (Python). In Julia use `r=LinRange(..)` and `u= r' .* sin.(phi)` inside PyPlot's `surf(r' .* cos.(phi), ..., u)` (for example).

Please turn the page!

2. Exercise: 1D heat equation **8 points**

Consider the homogeneous heat equation on $\Omega = (0, 1)$:

$$\begin{cases} u_t - au_{xx} + cu = 0, & (t,x) \in Q_T = (0,\infty) \times \Omega, \\ u(0,x) = 1 - \cos(2\pi x), & x \in \Omega, \end{cases}$$

with constant coefficients $a > 0$ and $c \geq 0$ and with homogeneous i) Dirichlet boundary conditions $u = 0$ or ii) homogeneous Neumann boundary $u_x = 0$ conditions on $\partial\Omega$. Note that the initial condition accidentially satisfies both boundary conditions.

a) Solve the PDE for i) and ii) using a careful separation of variables.

b) What are the stationary solutions in a), i.e., the limit $u_\infty(x) = \lim_{t\to\infty} u(t,x)$?

c) Write a function `a02e02solveheat.m/.py/.jl` with the interface `function u = a02e02solveheat(mode,x,t,a,c)` returning $u(t,x)$ from a) as a vector `u(i)` for the solution evaluated at `x(i)` at time `t` for Dirichlet `mode=1` and Neumann `mode=2` boundary conditions. Truncate the series reasonably.

d) Write a program `a02e02plot.m/.py/.jl` which creates plots of the solution at times $t = \{0, 10^{-3}, 10^{-2}, 10^{-1}, 1\}$ using `a02e02solveheat` for the cases $a = 1, c = 0$ and $a = 1, c = 1$ with boundary conditions i) and ii) (4 plots with a time-series each). Show and explain what happens for $c = 0$ and $a = -1$ for i) and ii).

Hint: $1 - \cos(2\pi x) = \sum b_n \sin(n\pi x)$, $b_n = \frac{16}{\pi(4n-n^3)}$ for odd n, $b_n = 0$ else.

3. Exercise: Difference stencils and Taylor expansions **6 points**

Let $u\colon [0,1] \to \mathbb{R}$ be a sufficiently smooth function.

a) Consider the difference quotients $D^0u(x)$, $D^+u(x)$, $D^-u(x)$, $D^+D^-u(x)$ with a given step size $h \in (0, \frac{1}{2})$. For each quotient determine all $x \in [0,1]$ such that the quotient is defined.

b) Show that the equality $D^0u(x) = \frac{1}{2}\big(D^+u(x) + D^-u(x)\big)$ holds for all $x \in [0,1]$, for which both sides of the equation are defined.

c) Show that the equality $D^+D^-u(x) = D^-D^+u(x)$ holds for all $x \in [0,1]$, for which both sides of the equation are defined.

d) Apply Taylor's formula to show that

$$D^0u(x) = u'(x) + h^2R_0,$$

with $|R_0| \leq \frac{1}{6}\max_{\xi\in[x-h,x+h]} |u'''(\xi)|$, and

$$D^+D^-u(x) = u''(x) + h^2R_1,$$

with $|R_1| \leq \frac{1}{12}\max_{\xi\in[x-h,x+h]} |u^{(4)}(\xi)|$. Explain why $D^-D^0u(x)$ is less suitable for approximating the second derivative.

total sum: 22 points