TECHNISCHE UNIVERSITÄT BERLIN
INSTITUT FÜR MATHEMATIK
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WS 2019/20

Numerical Mathematics II for Engineers

Homework assignment 5 : *Domains and boundaries.*

Programming assignments can be solved in MATLAB/Python/Julia.
Use sparse matrices where appropriate!

Deadline : submit before/during the lecture on November 25, 2019.

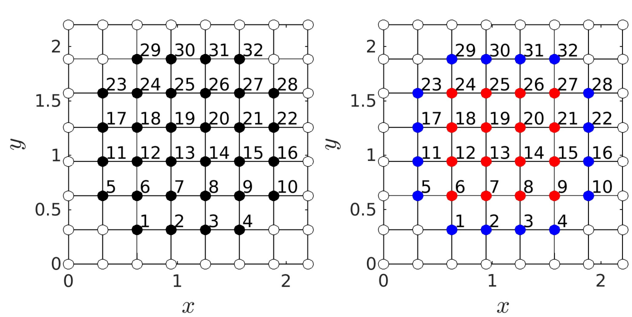
1. Exercise : Poisson problem on general domains **10 points**

For $\bar{\Omega} \subset [0, L]^2 \subset \mathbb{R}^2$ and given functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ consider the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{1}$$

Introduction : Consider a function `b = is_in_domain(x,y,L)`, where x, y are arrays of coordinates and L the box size. The function returns an array of booleans indicating `b(i,j)=true` if `(x(i,j),y(i,j))`$\in \bar{\Omega}$. Assume `is_in_domain(x,y,L)=false` if $x \le 0$ or $y \le 0$ or $x \ge L$ or $y \ge L$ are satisfied.

We define a grid $\bar{\Omega}_h = \bar{\Omega} \cap B_h$ with box $B_h = \{(ih, jh) \in \mathbb{R}^2 : 0 \le i, j \le N+1\}$ and $h = \frac{L}{N+1}$. For the lambda function `is_in_domain = @(x,y,L) (x-L/2).^2 + (y-L/2).^2 <= 1`, $L = 2.2$, $N = 6$ the points in $\bar{\Omega}_h$ are shown below as black dots, while other points $B_h \setminus \bar{\Omega}$ are hollow. If with (ih, jh) also $((i \pm 1)h, jh)$ and $(ih, (j \pm 1)h)$ are in $\bar{\Omega}_h$, then (ih, jh) is in Ω_h (red dots), while other points in $\bar{\Omega}_h$ are in Γ_h (blue dots) so that $\bar{\Omega}_h = \Omega_h \cup \Gamma_h$.



Consider the provided template function `a05ex01_get_laplace` that already creates the underlying mesh. When called with the function, it generates a $N + 2 \times N + 2$ boolean array `bar_omega_h` and provides the lexicographical ordering `ix` as an $N + 2 \times N + 2$ integer array (zero for points outside $\bar{\Omega}_h$). The goal of the following steps is to generate $N + 2 \times N + 2$ boolean arrays for Ω_h, Γ_h (to slice/pick elements from arrays).

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Furthermore, we will generate the index arrays of points and their direct neighbors, to create the 5-point stencil and the equations on the boundary. Therefore, extend the functionality of the function `a05ex01_get_laplace` as follows :

a) Check for points contained in $\bar{\Omega}_h$ if all their direct neighbors are in $\bar{\Omega}_h$. If so, set `omega_h(i,j)=true`, otherwise set `gamma_h(i,j)=true`.

b) Similar to the previous assignment : Create `ix_xpy,ix_xmy,ix_xyp,ix_xym` containing indices of neighbors $i \pm 1, j$ and $i, j \pm 1$ of `ix(i,j)` valid for all points in Ω_h.

c) Based on `ix_xpy,ix_xmy,ix_xyp,ix_xym` and using `omega_h,gamma_h`, create the index arrays `ii,jj` and value arrays `aa` to create the discrete operator `Lh` using the MATLAB command `Lh=sparse(ii,jj,aa)`. Return the results as specified in the function interface.

d) Use `is_in_domain` for $\bar{\Omega} = \{(x,y) \in (0,L)^2 : (x - \frac{L}{2})^2 + (y - \frac{L}{2})^2 \leq 1\}$ for $L = 2.2$. Solve numerically for $f = 1, g = 0$ and plot the solution for $N = 256$. For the sample solution `solve_laplace_domain.m` does the job.

e) Log-log-plot $\|R_h u - u_h\|_{\infty,h}$ vs h and determine the order of convergence.

f) Experiment with more complicated domains $\bar{\Omega}_h$ and solve for different f, g. Plot the corresponding solutions.

2. Exercise : Boundary conditions **8 points**

Let $\Omega = (0,1)$ with $f : [0,1] \to \mathbb{R}$, $c \in \mathbb{R}$ with $c \geq 0$ and $g_i, \alpha_i, \beta_i \in \mathbb{R}$ for $i = 0,1$ given. Consider the Poisson problem

$$-u'' + cu = f \quad \text{in } \Omega, \tag{2}$$

with one of the following sets of boundary conditions

i) Robin boundary conditions $\alpha_0 u(0) + \beta_0 u'(0) = g_0$, $\alpha_1 u(1) + \beta_1 u'(1) = g_1$,

ii) Periodic boundary conditions.

a) Which cases (boundary condition + data) require a solvability condition ?

b) Write a function `[x,u]=a05ex04solvePDE(N,f,data,flag)` returning the grid `x`$= (x_0, .., x_{N+1})^\top \in \mathbb{R}^{N+2}$ with $x_j = hj$ for $j = 0, \ldots, N+1$ and $h = 1/(N+1)$ and the discrete solution `u`$= (u_0, .., u_{N+1})^\top \in \mathbb{R}^{N+2}$. The right-hand-side input is `f` $= (f_0, .., f_{N+1})^\top \in \mathbb{R}^{N+2}$ with $f_j = f(x_j)$ and `data=[c, g0, g1, alpha0, alpha1, beta0, beta1]`. The flag can be `flag=1` for Robin and `flag=2` for periodic boundary conditions. Choose the discretization of the problem and discuss your choice. When necessary, solve the extended system from the lecture.

c) Choose i) with $\alpha_i = 0$, $\beta_i = 1$, $g_0 = -1/2$, $g_1 = 2$ and $f(x) = q$ for $q \in \mathbb{R}$ and solve. Plot the solution for $N = 500$. Which q satisfies the solvability condition ?

d) Choose ii) with $f(x) = 1 + \sin(2\pi x)$ and $c = 1$. Plot the solution for $N = 500$ and determine the order of convergence (experimental or using the exact solution).

3. Exercise (optional) : Stability **4 points**

Consider the alternative 9-point compact difference stencil

$$\Delta_h = \frac{1}{3h^2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3}$$

and assume $u_h : \bar{\Omega}_h \to \mathbb{R}$ with $\Delta_h u_h \geq 0$.

a) Prove the discrete maximum principle for this operator.

b) Show (by example or by proof) that $\Delta_h R_h((x - x_0)^2 + (y - y_0)^2) = 4$. Indicate how you would show stability for this operator.

c) Use the operator from exercise 1 (5-point stencil) and derive a bound for $|||L_h^{-1}|||_{\infty,h}$ with the disc domain (radius 1). Try to experimentally verify/challenge this bound.

d) Write a program for exercise 1 which checks if Ω_h is discretely connected.

total sum : 18 (+4) points