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WS 2019/20

Numerical Mathematics II for Engineers

Homework assignment 11 : *Advanced finite elements.*

Programming assignments can be solved in MATLAB/Python/Julia.

Deadline : submit until January, Monday 27, 2020.

1. Exercise : Regularity of solutions **6 (+2) points**

Consider the disc with a corner

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, x < 0 \text{ or } y > 0\},$$

and solve $-\Delta u(x, y) = 0$ in Ω with Dirichlet boundary conditions

$$u(\cos\varphi, \sin\varphi) = \sin(\tfrac{2}{3}\varphi)$$

and $u = 0$ on the remaining boundary $\partial\Omega \setminus \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

a) Show that $u(x = r\cos\varphi, y = r\sin\varphi) = r^{2/3}\sin(\tfrac{2}{3}\varphi)$ solves the PDE.

b) Check the following properties :
1. Is the solution in $L^2(\Omega)$? Compute $\|u\|_{L^2}$.
2. Is the solution in $H^1(\Omega)$? Estimate $\|\nabla u\|_{L^2}$.
3. Is the solution in $H^2(\Omega)$? Estimate $\|\partial^2 u/\partial r^2\|_{L^2}$.

c) Solve the problem on the provided domain and compare $\|u\|_{L^2}$ with the norm of the numerical solution. Write a script/function `a11ex01norm.{m,py,jl}` which solves this task.

d) (optional) Use the extra provided meshes and study the convergence order in the L^2 or H^1 or L^∞ (max) norm. Where is the error $u(x) - u_h(x)$ largest ?

Hint : Polar coordinates might be helpful.

2. Exercise : Eigenfunctions and values using FEM **8 (+2) points**

Consider the eigenvalue problem with homogeneous Neumann boundary conditions on the disc

$$-\Delta u = \lambda u \qquad \text{on} \qquad \Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

a) Show that the discretized weak formulation is a generalized eigenvalue problem $Au = \lambda Mu$ with the Galerkin matrix $A_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \,\mathrm{d}x$ and $M_{ij} = \int_\Omega \varphi_i\varphi_j \,\mathrm{d}x$.

b) Solve and visualize the first 6 eigenfunctions and print the resulting eigenvalues in a function/script `a11ex02evpb.{m,py,jl}`.

c) Modify the problem, so that you solve with homogeneous Dirichlet boundary conditions by reduction (slice) and compare with eigenvalues/functions that you already know in `a11ex02evpc.{m,py,jl}`.

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d) (optional) How do you determine the eigenvalues for homogeneous Neumann boundary conditions ? Compute the first 6 eigenvalues analytically (you do not need to plot the eigenfunctions again) in `a11ex02evpd.{m,py,jl}`.

3. Exercise : Extended variational form in 1D **6 (+2) points**

Consider the 1D variational form : Seek $u \in H^1(0,1)$ such that

$$a(u,v) = \int_0^1 au'(x)v'(x) + cu(x)v(x)\,\mathrm{d}x + \alpha_0 u(0)v(0) + \alpha_1 u(1)v(1),$$
$$F(v) = \int_0^1 f(x)v(x)\,\mathrm{d}x + g_0 v(0) + g_1 v(1).$$

for all $v \in H^1(0,1)$ with given $a, c, \alpha_0, \alpha_1, g_0, g_1 \in \mathbb{R}$

a) What problem and what boundary conditions are we solving ?

b) For $a, c > 0$ and $\alpha_i = g_i = 0$ check the assumptions of the Lax-Milgram theorem.

c) Implement and solve the problem for $a = 2$, $c = 1$, $\alpha_0 = 1$, $\alpha_1 = 0$, $g_0 = 1$, $g_1 = 0$ and $f(x) = 1$ and determine the and compare with the exact solution.

d) (optional) For $a, c, \alpha_i > 0$ check the assumptions of the Lax-Milgram theorem.

4. Exercise : Convection-diffusion problem **(optional !)** **(+4 points)**

Consider the homogeneous parabolic problem

$$\partial_t u + Lu = 0 \qquad \text{in} \qquad \Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

with homogeneous Neumann boundary conditions and initial data $u(t = 0, x) = u_0(x)$. The elliptic operator is $Lu(x) = -\nabla \cdot a(x)\nabla u + \mathbf{b}(x) \cdot \nabla u(x) + c(x)u(x)$. The variational problem is

$$(\partial_t u(t), v)_{L^2} + a(u(t), v) = F(v) \qquad \text{for all } v \in V,$$

and is discretized in time using the Crank-Nicolson scheme, i.e.,

$$(M + \tfrac{\tau}{2}Au^{k+1}) = (M - \tfrac{\tau}{2}A)u^k.$$

For the disc domain let $\mathbf{b}(x) = (-x_2, x_1)$

a) In addition to the previously constructed (local) matrices consider

$$C_{ij}(\mathbf{b}) = \int_\Omega \varphi_j \mathbf{b}(x) \cdot \nabla\varphi_i \mathrm{d}x.$$

Assume each component of $\mathbf{b}$ is in V_h. Map the integral to the reference and integrate by substitution. Write a function `cloc = localconv(b,Fdet,Finv)` which determines the local integral for a vector field `b(i,d)` on a given triangle in the node $i = 1, 2, 3$ and in direction $d = 1, 2$.

b) Let $a(x) = 0.1$ and $c(x) = 0.1$ and solve the problem for $T = 12$. Write a function `a11e04solve.{m,py,jl}` which solves the problem and plots the solution at $t = 0, 3, 6, 9, 12$.

total sum : 20 (+10) points