PCV - Exercise 1: Part 1: Theory

1) Compute intersection between two parallel $l_1 \times l_2$ using homogenous coords:

$l_1 = x\cos\varphi + y\sin\varphi = d_1$ | $l_1 = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -d_1 \end{pmatrix}$ $l_2 = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -d_2 \end{pmatrix}$

$l_2 = x\cos\varphi + y\sin\varphi = d_2$

$$l_1 \times l_2 = \frac{\begin{matrix} \cos\varphi & & \cos\varphi \\ \sin\varphi & \times & \sin\varphi \\ -d_1 & & -d_2 \end{matrix}}{\begin{matrix} \cos\varphi & & \cos\varphi \\ \sin\varphi & \times & \sin\varphi \\ -d_1 & & -d_2 \end{matrix}} = \begin{pmatrix} -d_2\sin\varphi - (-d_1\sin\varphi) \\ -d_1\cos\varphi - (-d_2\cos\varphi) \\ \cos\varphi\sin\varphi - \cos\varphi\sin\varphi \end{pmatrix}$$

$$= \begin{pmatrix} -d_2\sin\varphi + d_1\sin\varphi \\ -d_1\cos\varphi + d_2\cos\varphi \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\varphi\,(d_1 - d_2) \\ \cos\varphi\,(d_2 - d_1) \\ 0 \end{pmatrix} = p_1$$

How to interpret this point?

- We have obtained an ideal point out of the parallel lines' intersection, meaning that they both intersect at infinity

2) $\neq \varphi_1$ & φ_2. Calculate ideal points of 1.) What line goes through both points? How is this line to be interpreted?

$l_3 : x\cos\varphi_2 + y\sin\varphi_2 = d_3$ | $l_3 = \begin{pmatrix} \cos\varphi_2 \\ \sin\varphi_2 \\ -d_3 \end{pmatrix}$ $l_4 = \begin{pmatrix} \cos\varphi_2 \\ \sin\varphi_2 \\ -d_4 \end{pmatrix}$

$l_4 : x\cos\varphi_2 + y\sin\varphi_2 = d_4$

$$l_3 \times l_4 = \frac{\begin{matrix} \cos\varphi_2 & & \cos\varphi_2 \\ \sin\varphi_2 & \times & \sin\varphi_2 \\ -d_3 & & -d_4 \end{matrix}}{\begin{matrix} \cos\varphi_2 & & \cos\varphi_2 \\ \sin\varphi_2 & \times & \sin\varphi_2 \\ -d_3 & & -d_4 \end{matrix}} = \begin{pmatrix} -d_4\sin\varphi_2 + d_3\sin\varphi_2 \\ -d_3\cos\varphi_2 + d_4\cos\varphi_2 \\ \cos\varphi_2\sin\varphi_2 - \cos\varphi_2\sin\varphi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta_2 \, (d_3 - d_4) \\ \cos\theta_2 \, (d_4 - d_3) \\ 0 \end{pmatrix} = p_2$$

line that goes through points p_1 & p_2 =

$$p_1 \times p_2 = \frac{\begin{pmatrix} \sin\theta \, (d_1 - d_2) \\ \cos\theta \, (d_2 - d_1) \\ 0 \end{pmatrix} \times \begin{pmatrix} \sin\theta_2 \, (d_3 - d_4) \\ \cos\theta_2 \, (d_4 - d_3) \\ 0 \end{pmatrix}}{\begin{pmatrix} \sin\theta \, (d_1 - d_2) \\ \cos\theta \, (d_2 - d_1) \\ 0 \end{pmatrix} \times \begin{pmatrix} \sin\theta_2 \, (d_3 - d_4) \\ \cos\theta_2 \, (d_3 - d_4) \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \sin\theta(d_1 - d_2)(\cos\theta_2(d_3 - d_4)) - \sin\theta_2(d_3 - d_4)\cos\theta(d_2 - d_1) \end{pmatrix}$$

Interpretation: the line connecting two ideal points is a line at infinity (difficult to imagine?)

3. Where does general line $x\cos\theta + y\sin\theta = d$ intersect the line $(0, 0, 1)^T$ given in homog. coords? How to interpret the point?

$$gl = \begin{pmatrix} \cos\theta \\ \sin\theta \\ -d \end{pmatrix} \qquad il = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$gl \times il = \frac{\begin{pmatrix} \cos\theta \\ \sin\theta \\ -d \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} \cos\theta \\ \sin\theta \\ -d \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$

Interpretation: A general line intersects an ideal one at an ideal point $\rightarrow$ a point at infinity

1. Exercise - Part II: Practical

1) $x = [5, 6]^T$

$y = [-3, 7]^T$

a) Determine connecting line (cross product of vector points)

$$l = \begin{bmatrix} 5 & -3 \\ 6 & 7 \\ 1 & 1 \\ 5 & -3 \\ 6 & 7 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6-7 \\ -3-5 \\ 35+18 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 53 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -53 \end{bmatrix}$$

b) translation

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\boxed{t = \begin{bmatrix} 6 \\ -7 \end{bmatrix}}$$

$$x' = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix}$$

$$y' = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -3 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Rotation

$$\begin{matrix} u' \\ v' \\ w' \end{matrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\boxed{\theta = 135°}$$

$135° = \frac{3\pi}{4} \text{ rad} \Rightarrow \cos(135°) = -\frac{\sqrt{2}}{2}$; $\sin(135°) = \frac{\sqrt{2}}{2}$

$$x'' = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix} = \begin{bmatrix} -\frac{11\sqrt{2}}{2} & + \frac{\sqrt{2}}{2} & + 0 \\ \frac{11\sqrt{2}}{2} & + \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x'' = \begin{pmatrix} -\frac{10\sqrt{2}}{2} \\ \frac{12\sqrt{2}}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -5\sqrt{2} \\ 6\sqrt{2} \\ 1 \end{pmatrix}$$

$$y'' = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3\sqrt{2}/2 \\ 3\sqrt{2}/2 \\ 1 \end{pmatrix}$$

Scaling

$$\boxed{\lambda = 6}$$

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$x''' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -5\sqrt{2} \\ 6\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -30\sqrt{2} \\ 36\sqrt{2} \\ 1 \end{pmatrix}$$

$$y''' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -3\sqrt{2}/2 \\ 3\sqrt{2}/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -18\sqrt{2}/2 \\ 18\sqrt{2}/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -9\sqrt{2} \\ 9\sqrt{2} \\ 1 \end{pmatrix}$$

c) $l = \begin{pmatrix} 1 \\ 8 \\ -53 \end{pmatrix} \Rightarrow \boxed{l' = H^{-T} l}$

$$l' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 7 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 8 \\ -53 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix}$$

$$l'' = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 8 \\ -3 \end{pmatrix} = \begin{pmatrix} -9/\sqrt{2} \\ -7/\sqrt{2} \\ -3 \end{pmatrix}$$

$$l''' = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -9/\sqrt{2} \\ -7/\sqrt{2} \\ -3 \end{pmatrix} = \begin{pmatrix} -9/6\sqrt{2} \\ -7/6\sqrt{2} \\ -3 \end{pmatrix} = \begin{pmatrix} +3/2\sqrt{2} \\ +7/6\sqrt{2} \\ +3 \end{pmatrix}$$

2) Check whether transformed points lie on the transformed line

$$\frac{3}{2\sqrt{2}}(x) + \frac{7}{6\sqrt{2}}(y) + 3 = 0$$

for $x''' = \begin{pmatrix} -30\sqrt{2} \\ 36\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow \frac{3(-30\sqrt{2})}{2\sqrt{2}} + \frac{7(36\sqrt{2})}{6\sqrt{2}} + 3 = 0$

$$-\frac{\cancel{90}^{45}\cancel{\sqrt{2}}}{\cancel{2\sqrt{2}}} + \frac{252\cancel{\sqrt{2}}}{6\cancel{\sqrt{2}}} + 3 = 0$$

$$-45 + 42 + 3 = 0 \Rightarrow 0 = 0$$

x''' lies on l'

for $y''' = \begin{pmatrix} -9\sqrt{2} \\ +9\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow \frac{3(-9\sqrt{2})}{2\sqrt{2}} + \frac{7(9\sqrt{2})}{6\sqrt{2}} + 3 = 0$

$$-\frac{27}{2} + \frac{63}{6} + 3 = 0$$

$$-13{,}5 + 10{,}5 + 3 = 0 \Rightarrow 0 = 0$$

y''' lies on l'

3) $l = (6, -2, 2)^T$

a) Hessian normal form

$$x\cos\theta + y\sin\theta - d = 0$$

$$\theta = \tan^{-1}(-2/6) = 1{,}249° = 0{,}0217 \text{ rad}$$

$$x\cos(0{,}0217) + y\sin(0{,}0217) - 2 = 0$$

$$0{,}99x + 0{,}0217y - 2 = 0$$

b) Axis intercept

$$\frac{x}{x_0} + \frac{y}{y_0} - 1 = 0$$

$$-\frac{x}{0,3} + y - 1 = 0$$

$-c/a = -0,3$

$-c/b = 1$

C