Consider the **Possio** Solution:

- What is solved is unit gust response of 2D airfoil (flat plate) to gust with k_1 and k_3 variation. k_3 can be assumed to be zero and the Mach number can be small.
- Results compare well to Sears
- The result is actually $\Delta p_{nd} sin(\theta)$ (this is what is sent back to code from the subroutine).
- We can compute lift per span then:

$$L'(w \sim k_1, k_3 = 0) = \int_0^{\pi} \Delta p_{nd}(\theta) sin(\theta) d\theta \Big[\rho U_\infty u_g(k_1, k_3)\Big]$$

  which is the same as

$$= \int_0^{c} \Delta p_{nd}(x) e^{-ik_3 x_3} \Big[\rho U_\infty u_g(k_1, k_3)\Big]$$

- This should compare with Sears' result

$$L' = \pi \rho c U_\infty \hat{u_g} S(k_1)$$

- Now, at a single frequency this is understood to be the answer and $L' = \pi \rho c U_\infty \hat{u_g} S(k_1) e^{i\omega t}$. Take the real or imaginary part to have behavior in time. (N.B.: $\omega = 2\pi f$ and $k_1 = \frac{\omega c}{2U}$)
- A more general representation would be:

$$L'(t) = \int_{-\infty}^{\infty} \pi \rho c U_\infty \hat{u}_g(k_1) S(k_1) e^{i\omega t} \delta(\omega - \omega_o) d\omega$$

  where k_1 in the above equation is $k_1 = \frac{wc}{2U}$ The Fourier component (L'(w)) is given as:

$$L'(\omega) = \pi \rho c U_\infty \hat{u}_g(k_1) S(k_1) \delta(\omega - \omega_o)$$

- Moving to a more general airfoil and airfoil response where all the wave numbers can interact with the airfoil (but we still assume the upwash direction of the velocity dominates the interaction) we need to understand the gust and the final lift in the frequency domain.

$$\hat{\hat{\hat{\hat{u}}}}_g = \int\int\int\int u_g(\bar{x}, t) e^{i\omega t - i\bar{k}\cdot\bar{x}} d\bar{x} dt$$

  We know however that we really have the behavior $x_1 - U_\infty t$ not separate x_1 and t. Therefore:

$$= \int\int\int\int u_g(x_1 - U_\infty t, x_2, x_3) e^{i\omega t - i\bar{k}\cdot\bar{x}} d\bar{x} dt$$

  let $x_c = x_1 - U_\infty t$ and dx_1 = dx_c.

$$= \int\int\int\int u_g(x_c, x_2, x_3) e^{i\omega t - ik_1(x_c + U_\infty t) - ik_2 x_2 - ik_3 x_3} dx_c dx_2 dx_3 dt$$

  Now integrate with t

$$\hat{\hat{\hat{\hat{u}}}}_g(\bar{k}, \omega) = \int\int\int u_g(x_c, x_2, x_3) 2\pi \delta(\omega - k_1 U_\infty) e^{-ik_1 x_1 - ik_2 x_2 - ik_3 x_3} dx_c dx_2 dx_3$$

  In which the connection between ω and k_1 is imposed.

- Now we need to understand the turbulence influence on an airfoil. The spectrum of turbulence (Ω_{ij}) is given by definition:

$$\Omega_{\bar{k}} = \int\int\int \left\langle u_g(\bar{x}-\bar{U}t)u_g^*(\bar{x}-\bar{U}t-\bar{\xi})\right\rangle e^{i\bar{k}\cdot\bar{\xi}} d\bar{\xi}$$

Transform of the correlation.

If we are going to use this we have to understand how it relates to $\langle \hat{\hat{\hat{u}}}_g \hat{\hat{\hat{u}}}_g^* \rangle$ which will be required input to gust response.

$$\left\langle \hat{\hat{\hat{u}}}_g(\bar{k},\omega)\hat{\hat{\hat{u}}}_g^*(\bar{K},v)\right\rangle = \int\int\int\int\int\int\int\int \left\langle u_g(\bar{x},t)u_g^*(\bar{y},\tau)\right\rangle e^{i\omega t - iv\tau - i\bar{k}\cdot\bar{x} + i\bar{K}\cdot\bar{y}} d\bar{x}d\bar{y}dtd\tau$$

using: $\bar{x}-\bar{U}_\infty t$ and$\bar{y}-\bar{U}_\infty t$ and define $\bar{\xi} = \bar{x}-\bar{y}-\bar{U}_\infty(t-\tau)$

- Substitute and perform integration with t and τ

$$\left\langle \hat{\hat{\hat{u}}}_g \hat{\hat{\hat{u}}}_g^* \right\rangle = (2\pi)^2 \int\int\int\int\int\int \left\langle u_g(\bar{x}-\bar{U}t)u_g^*(\bar{x}-\bar{U}t-\bar{\xi})\right\rangle \delta(\omega-\bar{k}\cdot\bar{U})\delta(-v+\bar{k}\cdot\bar{U}) e^{-i\bar{k}\cdot\bar{x} - i\bar{K}\cdot\bar{\xi} + i\bar{K}\cdot\bar{x}} d\bar{x}d\bar{\xi}$$

Now we will insert the definition of the turbulence spectrum as stated above.

$$= (2\pi)^2 \int\int\int \Omega_{ij}(\bar{k})\delta(\omega-\bar{k}\cdot\bar{U})\delta(-v+\bar{k}\cdot\bar{U}) e^{-i\bar{k}\cdot\bar{x}+i\bar{k}\cdot\bar{x}} d\bar{x}$$

$$= (2\pi)^5 \Omega_{ij}(\bar{k})\delta(\omega-\bar{k}\cdot\bar{U})\delta(-v+\bar{k}\cdot\bar{U})\delta(\bar{k}-\bar{K})$$

- We now want to use this in 3D and having strip wise information we understand $\Delta p(x_1, x_2, x_3)$

$$\Delta p(x_1,x_2,x_3,\omega) = \frac{1}{(2\pi)^3}\int\int\int \rho(x_3)U(x_3)u_g(x_3,\bar{k},\omega)\hat{\Delta p}(\bar{k},x_1,x_2,x_3)e^{ik_3x_3} d\bar{k}$$

$$\left\langle \Delta p(\bar{x},\omega)\Delta p^*(y_1,y_2,x_3+\Delta x_3)\right\rangle \left\langle \hat{\hat{\hat{u}}}_g(x_3,\hat{k},\omega)\hat{\hat{\hat{u}}}_g^*\right\rangle e^{ik_3x_3 - ik_3(x+\Delta x_3)} d\bar{k}d\bar{k}$$

$$\left\langle L'\hat{(}\omega)L'\hat{(}\omega)^*\right\rangle = \int_{x_3}\int_{\Delta x_3}\int_{x_1}\int_{y_1} \left\langle \Delta p(\bar{x},\omega)\Delta p^*(\bar{y}+\Delta y_3, v)\right\rangle dy_1dx_1dx_3d\Delta x_3$$

- Inserting now the definitions for Δp and the gust.

$$\left\langle L'\hat{(}\omega)L'\hat{(}\omega)^*\right\rangle = \int_{x_3}\int_{\Delta x_3}\int_{x_1}\int_{y_1} \frac{1}{(2\pi)^6}\rho^2U^2 \int\int\int\int\int\int (\hat{\Delta p}(\bar{k},\bar{x})\hat{\Delta p}^*(\bar{k},y_1,y_2,y_3+\Delta y_3))$$
$$(2\pi)^5\Omega_{ij}(\bar{K})\delta(\omega-\bar{k}\cdot\bar{U})\delta(-v+\bar{K}\cdot\bar{U})\delta(\bar{k}-\bar{K})e^{ik_3x_3-ik_3(x_3+\Delta x_3)} d\bar{k}d\bar{K}dy_1dx_1d\Delta x_3$$

- Integrate over $\bar{K}$ and all $\bar{K}$ become $\bar{k}$ and remove delta function of k and K since values only relevant for k = K.

$$\left\langle L'\hat{(}\omega)L'\hat{(}\omega)^*\right\rangle = \int_{x_3}\int_{\Delta x_3}\int_{x_1}\int_{y_1} \frac{1}{(2\pi)^6}\rho^2U^2 \int\int\int (\hat{\Delta p}(\bar{k},\bar{x})\hat{\Delta p}^*(\bar{k},y_1,y_2,x_3+\Delta x_3))$$
$$(2\pi)^5\Omega_{ij}(\bar{k})\delta(\omega-\bar{k}\cdot\bar{U})\delta(-v+\bar{k}\cdot\bar{U})e^{ik_3x_3-ik_3(x_3+\Delta x_3)} d\bar{k}dy_1dx_1d\Delta x_3$$

- Now we will make the substitution and let $k_{new} = k_1U$ and assuming U_∞ exists only in the k_1 direction.

$$\left\langle L'\hat{(}\omega)L'\hat{(}\omega)^*\right\rangle = \int_{x_3}\int_{\Delta x_3}\int_{x_1}\int_{y_1} \frac{1}{(2\pi)^6}\rho^2U^2 \int\int\int (\hat{\Delta p}(\bar{k},\bar{x})\hat{\Delta p}^*(\bar{k},y_1,y_2,x_3+\Delta x_3))$$
$$(2\pi)^5\Omega_{ij}(\bar{k})\delta(\omega-k_{new})\delta(-v+k_{new})e^{-ik_3\Delta y_3} d\bar{k}dy_1dx_1d\Delta x_3$$

- Perform integration with x_3 and Δx_3

$$\left\langle L'\hat{(}\omega)L'\hat{(}\omega)^*\right\rangle = \int_{x_1}\int_{y_1}\frac{1}{(2\pi)^6}\rho^2U^2\int\int\int(\Delta\hat{p}(\bar{k},\bar{x})\Delta\hat{p}^*(\bar{k},y_1,y_2,x_3+\Delta x_3))$$
$$(2\pi)^5\Omega_{ij}(\bar{k})\delta(\omega-k_{new})\delta(-v+k_{new})L_r sinc\Big(\frac{k_3L_r}{2}\Big)R_3d\bar{k}dy_1dx_1$$

- Using the identity:

$$\int\delta(a-x)dx\delta(x-b)=\delta(a-b)$$

The equation becomes:

$$\int_{x_1}\int_{y_1}\int_{k_2}\int_{k_3}\frac{1}{(2\pi)^6}\rho^2u^2((\Delta\hat{p}(\bar{k},\bar{x})\Delta\hat{p}^*(\bar{k},y_1,y_2,x_3+\Delta x_3))(2\pi)^5(\Omega_{ij}\bar{k})L_r sinc\Big(\frac{k_3L_r}{2}\Big)R_3\frac{dx_1}{U_1}dy_1dk_2dk_3$$

Since L_r is related to the Δx_3 integration, we will define it as a length R_3, and inserting the definition for the turbulence spectrum, Ω_{ij},

$$\int_{x_1}\int_{y_1}\int_{k_2}\int_{k_3}\frac{1}{(2\pi)}\rho^2U((\Delta\hat{p}(\bar{k},\bar{x})\Delta\hat{p}^*(\bar{k},y_1,y_2,x_3+\Delta x_3))\frac{2\bar{u}^2}{\pi}\frac{L(L\sqrt{k_1^2+k_2^2+k_3^2})^4}{(1+L^2(k_1^2+k_2^2+k_3^2))^3}R_3^2 sinc\Big(\frac{k_3R_r}{2}\Big)dx_1dy_1dk_2dk_3$$

By letting k_3 go to zero, removing the dependence on k_3, the equation becomes:

$$\int_{x_1}\int_{y_1}\int_{k_2}\frac{1}{(2\pi)}\rho^2U((\Delta\hat{p}(\bar{k},\bar{x})\Delta\hat{p}^*(\bar{k},y_1,y_2,x_3+\Delta x_3))\frac{2\bar{u}^2}{\pi}\frac{L(L\sqrt{k_1^2+k_2^2})^4}{(1+L^2(k_1^2+k_2^2))^3}R_3^2dx_1dy_1dk_2$$

Let the response function T be defined as:

$$T=\int_{y_1}\int_{x_1}((\Delta\hat{p}(\bar{k},\bar{x})\Delta\hat{p}^*(\bar{k},y_1,y_2,x_3+\Delta x_3))dx_1dy_1$$

The lift response is thus:

$$\left\langle L'(\hat{\omega})L'(\hat{\omega})^*\right\rangle = \int_{k_2}\frac{R_3}{2\pi}\rho^2U\Big(T(k_1,k_2)\Big)\frac{2\bar{u}^2}{\pi}\frac{L(L\sqrt{k_1^2+k_2^2})^4}{(1+L^2(k_1^2+k_2^2))^3}dk_2$$

List to do:

- Substitute everything in last equation and also add y_2 contribution.
- Integrate over $\bar{K}$ and all $\bar{K}$ become $\bar{k}$
- Change variables $k_new = k_1U$ so that integration can be performed with k_new and the delta functions can be utilized.
- Integrate with respect to Δx_3 would give a sinc function.