Intersection-free spatial embeddings of graphs

Gabay, Jeremie Arad, Dvir Pr. Levit, Vadim

1 Introduction

In this paper, we will provide proof that any given graph containing edges crossing each others can be embedded in the space without said edges crossing any other.

2 Definitions

2.1 Planar/Non-planar graph:

A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other [1].

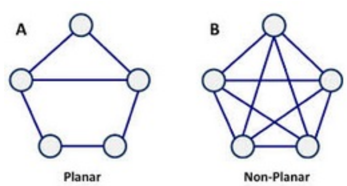


Figure 1: Planar & Non-planar Graphs

2.2 Crossing Number:

The crossing number of a graph $G = (V, E)$ is the minimum number of crossings between edges of G among all the ways to draw G in the plane. It is denoted $cr(G)$.

The edges in a drawing of G need not be line segments, they are allowed to be arbitrary continuous curves. If one restricts to the straight-line drawings, then one obtains the rectilinear crossing number $lin - cr(G)$.

The space crossing number of G, denoted $cr_4(G)$, is the least number of crossings in any spatial drawing of G.

As in the planar case, the spatial rectilinear crossing number $lin - cr_4(G)$ is obtained by restricting to straight-line spatial drawings. [2]

2.3 Complete Graph:

A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. [3]

$$|V| = n \qquad |E| = \frac{n(n+1)}{2}$$

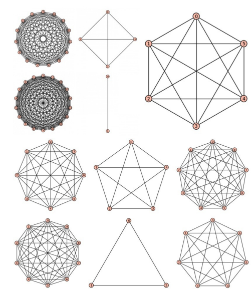


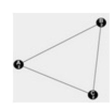
Figure 2: A few complete graphs

3 Proposition

Any given graph with $lin - cr(G) \geq 0$ crossings can be embedded in the euclidean space with no crossing edges,
i.e. $lin - cr_4(G) = 0$.

4 Proof using Planes

A few lemmas needed for our proof.



Lemma 4.1 *3 points in general position in space defines a plane.*

Lemma 4.2 *The volume V of any given plane (2 dimensional space) is $V = 0$.*

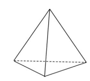
Lemma 4.3 *The number of 2 dimensional subspaces in 3 dimensional space is infinite.*

4.1 Proof by induction:

We will prove by induction on n that for any given graph $G(V, E)$ embedded in space such as $|V(G)| = n$ and $lin - cr_4(G) = 0$, we can add another vertex e', thus creating G' such as $lin - cr_4(G') = lin - cr_4(G) = 0$.

Base Case:

1. Let's assume the complete $H(V,E)$ such as $V(H) = 3$ embedded in space (it is trivial that $lin - cr_4(H) = 0$).
2. Using ***Lemma 1***, we can notice that H is also embedded into a single plane P.
3. Let's add a 4^{th} vertex e so that $e \notin P$.



Transition Step:
Let's assume the proposition is true for the complete spatial graph $G(V,E)$ such as $V(G) = n$ and $lin - cr_4(G) = 0$.

Induction Step: *Let's see that the proposition is also true for the complete spatial graph $G'(V,E)$ such as $V(G') = n + 1$*

1. Let's find a $(n+1)^{th}$ point e' in space. Using ***Lemma 2*** & ***Lemma 3***, e' can be found on a plane not yet populated by 3 already existing vertices.
2. By connecting e' to all existing vertex of G, we obtain G'.

Hence, we created a graph in which no 4 vertices are coplanar, thus avoiding any crossing edges in the resulting graph.

5 Conclusion

By means of decomposition and recomposition, any given graph can be embedded in space without any crossing edges. $\blacksquare$

References

[1] Trudeau, Richard J. (1993) *Introduction to Graph Theory.* New York: Dover Pub. p.64. "Thus a planar graph, when drawn on a flat surface, either has no edge-crossings or can be redrawn without them."

[2] Bukh, Boris & Hubard, Alfredo (2011) *Space Crossing Numbers.* arXiv:1102.1275v2. p.1.

[3] Diestel, Reinhard (2010) *Graph Theory, Fourth Electronic Edition.* Springer. "If all the vertices of G are pairwise adjacent, then G is complete."