Limit sketches and presentability

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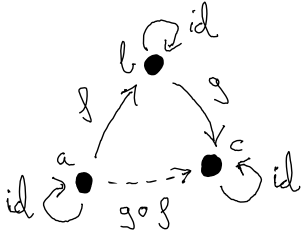
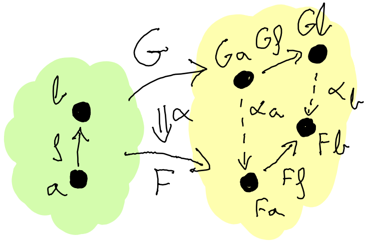
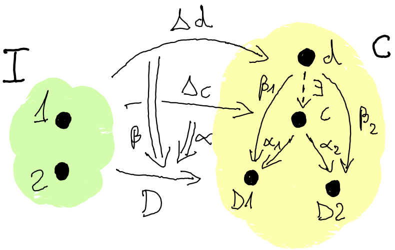
Categories



Functors and natural transformations





Cones and limits

- Diagram $D : I \to C$
- Cone $\alpha : \Delta c \Rightarrow D$
- α is a limit if for all $d \in C$

$$\mathrm{Hom}(d, c) \cong \mathrm{Cones}(D, d)$$

Presentability

A category has two types of collections: the objects, and the morphisms. Then, a category is:

- **Small** if it has a set of objects and sets of morphisms.
- **Locally small** if it has a (maybe large) collection of objects and sets of morphisms.
- **Large** if it has (maybe large) collections of objects and morphisms.

A **(locally) presentable** category is a locally small category which contains a set S of *small objects* such that every object is a *nice* colimit over S.

Examples. Set, Grp, sSet, . . . **Non example.** Top

Limit sketches

A **limit sketch** (Bastiani and Ehresmann 1972) is a pair (A, C) of a small category A and a set of cones C over A.

A **model** of a limit sketch is a functor $F : A \to \mathbf{Set}$ which sends cones of C to limits of **Set**. A category is **limit-sketchable** if it is equivalent to the category of models of some limit sketch.

Example. Let A be the small category generated by the square (a).

Let c be the cone with apex and diagram

Then $(A, \{c\})$ is a sketch

$$A \xrightarrow{\quad \mathbf{F} \quad} \mathbf{Set}$$

$$\begin{tikzcd} 0 \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "(a)"] & 1 \arrow[d] \\ 2 \arrow[r] & 3 \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

A model $\mathbf{F}$ of the sketch $(A, \{c\})$ is a pullback of sets

Representation theorem

Theorem (Adamek and Rosicky 1994)

The following are equivalent:

(i) *Presentable categories.*

(ii) *Limit-sketchable categories.*

Goal

Presentable ∞-categories $\overset{?}{\simeq}$ Limit-sketchable ∞-categories

Plan

Presentable ∞-categories

Limit ∞-sketches

Representation theorem

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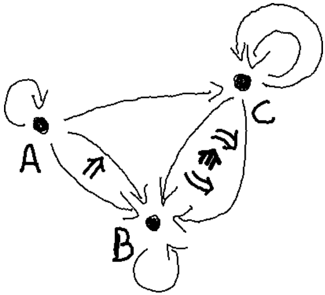
Informal higher categories

A **higher category** has objects and:

- n-morphisms between $(n-1)$-morphisms for all $n \geq 1$,
- Composition, identities and associativity of n-morphisms weakly up to a $(n+1)$-morphism for all $n \geq 1$.



A higher category is an (∞, m)-**category** if for any $n > m$, the n-morphisms are invertible up to a $(n+1)$-morphism.

- ∞-**category** $:= (\infty, 1)$-category
- ∞-**groupoid** $:= (\infty, 0)$-category

Limits and colimits

Let $\mathcal{C}$ be an ∞-category, and I be a small ∞-category. Given any object $x \in \mathrm{Obj}(\mathcal{C})$, the **constant diagram** $\Delta x : I \to \mathcal{C}$ sends all objects of I to x, and all higher morphisms to higher identities.

Let $D : I \to \mathcal{C}$ be a diagram and $y \in \mathrm{Obj}(\mathcal{C})$ be an object of $\mathcal{C}$. A natural transformation $\alpha : \Delta y \Rightarrow D$ **exhibits** y **as a limit of** D if, for all $x \in \mathrm{Obj}(\mathcal{C})$, α induces an equivalence

$$\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{\sim} \mathrm{Cones}(D, x) := \mathrm{Map}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta x, D).$$

Cocones and **colimit cocones** are defined as cones and limit cones in the opposite ∞-category.

Accessibility

Let κ denote a regular cardinal and $\mathcal{C}$ an ∞-category.

- An ∞-category $\mathcal{K}$ is κ-**filtered** if, for every κ-small ∞-category I, every diagram $D : I \to \mathcal{K}$ admits a cocone $\alpha : D \Rightarrow \Delta x$.

- $\mathcal{C}$ admits κ-**filtered colimits** if it admits $\mathcal{K}$-indexed colimits, for every κ-filtered ∞-category $\mathcal{K}$.

- An object $x \in \mathrm{Obj}(\mathcal{C})$ is called κ-**compact** if the mapping space functor $\mathrm{Map}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{S}$ preserves κ-filtered colimits.

An ∞-category $\mathcal{C}$ is **accessible** if it is locally small and there is a regular cardinal κ such that:

- ☑ $\mathcal{C}$ admits κ-filtered colimits.

- ☑ There is some essentially small sub-∞-category of κ-compact objects which generates $\mathcal{C}$ under κ-filtered colimits.

Presentability

Definition

An ∞-category is **presentable** if it is accessible and cocomplete.

Example

(a) The ∞-category of homotopy types $\mathcal{S}$ is presentable.

(b) Any ∞-topos is presentable.

(c) The nerve of any presentable 1-category is presentable.

(d) If $\mathcal{A}$ is a small ∞-category and $\mathcal{C}$ is a presentable ∞-category, then $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is presentable.

Plan

Presentable ∞-categories

Limit ∞-sketches

Representation theorem

Limit ∞-sketches

A **limit ∞-sketch** (Joyal 2008) $\mathcal{T} = (\mathcal{K}, \mathfrak{L})$ is a small ∞-category $\mathcal{K}$ equip with a set $\mathfrak{L}$ of cones.

Let $\mathcal{C}$ be a complete ∞-category. A functor $F : \mathcal{K} \to \mathcal{C}$ is a **model** of a limit ∞-sketch $\mathcal{T} = (\mathcal{K}, \mathfrak{L})$ in $\mathcal{C}$ if it sends each cone in $\mathfrak{L}$ to a limit cone in $\mathcal{C}$.

$$\mathrm{Mod}(\mathcal{T}, \mathcal{C}) \coloneqq \infty\text{-category of models of } \mathcal{T} \text{ in } \mathcal{C}$$
$$\mathrm{Mod}(\mathcal{T}) \coloneqq \infty\text{-category of models of } \mathcal{T} \text{ in } \mathcal{S}$$

We say that an ∞-category is **limit ∞-sketchable** (or **essentially ∞-algebraic**) if it is equivalent to the ∞-category of models of some limit ∞-sketch.

Examples: ∞-algebraic theories

An ∞-**algebraic theory** (or ∞-**Lawvere theory**) is a small ∞-category with finite products. A **model** (or **algebra**) for an ∞-algebraic theory $\mathcal{A}$ is a functor $\mathcal{A} \to \mathcal{S}$ that preserves products.

Any ∞-algebraic theory is an ∞-sketch with only product cones

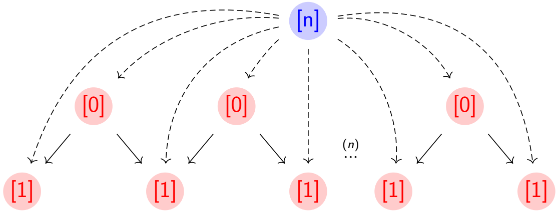
Example. Monoid objects (A_∞-spaces), commutative monoid objects (E_∞-spaces), group objects (∞-groups), R-modules, ...

Theorem (Rosicky 2007 and Lurie 2009)

The ∞-category of models of an ∞-algebraic theory is presentable.

Examples: Internal precategories

Let $\mathcal{C}$ be a complete ∞-category, $\mathcal{A}$ be the nerve of Δ^{op}, and c_n be the cone with apex and diagram for all $n \in \mathbb{N}$:



Then $\mathcal{T} = (\mathcal{A}, \{c_n \mid n \in \mathbb{N}\})$ is a limit ∞-sketch, and a model $F : \mathcal{A} \to \mathcal{C}$ is a simplicial object in $\mathcal{C}$ such that

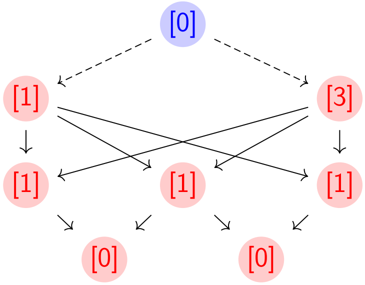
$$F_n \xrightarrow{\sim} F_1 \times_{F_0} F_1 \times_{F_0} \cdots \times_{F_0} F_1. \qquad \textit{(Segal condition)}$$

$$\mathrm{Mod}(\mathcal{T}, \mathcal{C}) \simeq \textbf{Internal precategories}$$

$$\mathrm{Mod}(\mathcal{T}) \simeq \textbf{Segal spaces}$$

Examples: Internal univalent categories

Let $\mathcal{A}$ be as before, $\mathfrak{L}_S$ be the set of cones of the previous sketch, and d_n be the cone with apex and diagram for all $n \in \mathbb{N}$:



Then $\mathcal{T} = (\mathcal{A}, \mathfrak{L}_1 \cup \{d_n \mid n \in \mathbb{N}\})$ is a limit ∞-sketch, and a model $F : \mathcal{A} \to \mathcal{C}$ is an internal precategory in $\mathcal{C}$ such that

$$\begin{tikzcd} F_0 \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & F_3 \arrow[d] \\ F_1 \arrow[r] & F_1 \times_{F_0}^{d_1,d_1} F_1 \times_{F_0}^{d_0,d_0} F_1 & \end{tikzcd}$$

$$\mathrm{Mod}(\mathcal{T}, \mathcal{C}) \simeq \textbf{Internal univalent categories}$$

$$\mathrm{Mod}(\mathcal{T}) \simeq \textbf{Complete Segal spaces}$$

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Limit ∞-sketches

Representation theorem

Representation theorem

Theorem (M.)

An ∞-category is presentable $\iff$ it is limit ∞-sketchable.

Corollary

The ∞-category of models of a limit ∞-sketch in a presentable ∞-category is presentable.

Future work

- **Generalization:** A ∞-category is accessible if, and only if, it is equivalent to the ∞-category of models of an ∞-sketch.

  A **sketch** is a limit sketch with a set of cocones which are sent to colimit cocones by any model.

- Accessibility, presentability, sketches, and representation theorem for ∞-**cosmoi** (Riehl and Verity 2022)

  $\implies$ **Model-independent** version of this presentation!

- Formalize this work with a proof assistant which supports synthetic ∞-categories like `rzk`.

References

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Thank you for listening!

This work is supported by the MCIN/ AEI/10.13039/501100011033/
under the I+D+i grant PID2020-117971GB-C22.

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