Presentabilidad en el contexto de infinito categorías

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Supervisors: Carles Casacuberta & Javier J. Gutiérrez

X Encuentro de Jóvenes Topólogos

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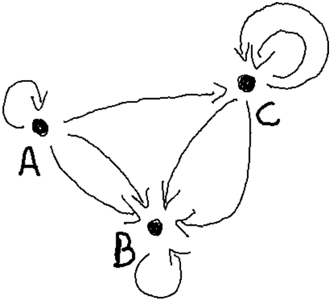

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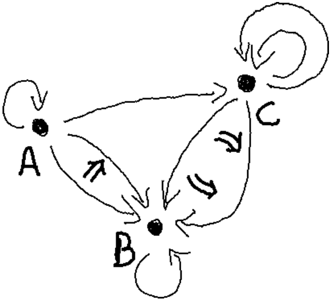

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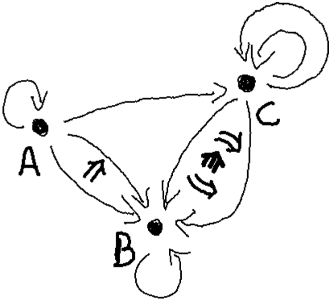

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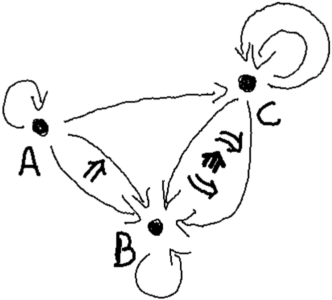

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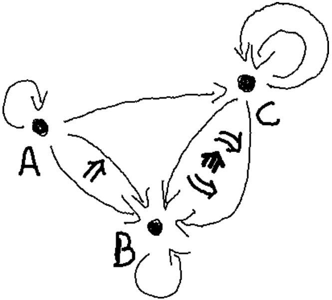

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$$f \circ (g \circ h) \Rightarrow (f \circ g) \circ h$$

- For any n-morphism f there is $(n+1)$-morphisms

$$f \circ \mathrm{Id} \Rightarrow f \Leftarrow \mathrm{Id} \circ f$$

Infinity categories and Infinity groupoids

A higher category is a (∞, m)-category if for any $n > m$, the n-morphisms are invertible up to a $(n+1)$-morphism.

Then, define:

- An ∞-**category** as a $(\infty, 1)$-category.
- An ∞-**groupoid** is a $(\infty, 0)$-category.

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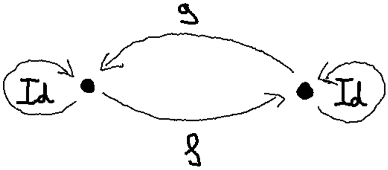

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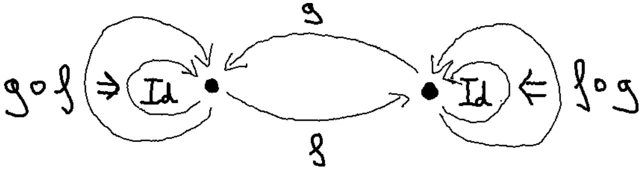

Models of higher categories

Historically, there have been many definitions for higher categories, and each one is called a **model**:

- Globular models [Grothendieck, Batanin, Berger, etc.].
- Topological (or simplicial) categories [Bergner, Lurie, etc.].
- Quasicategories [Joyal, Lurie].
- Segal categories or complete Segal spaces [Segal, Rezk, etc.]
- Relative categories [Dwyer-Kan, Barwick-Kan]

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Simplicial sets

The **simplex category** Δ is the category with objects the linearly ordered sets $[n] := \{0, 1, \ldots, n\}$ for all $n \geq 0$, and morphisms all set functions $[n] \to [m]$ which are order-preserving.

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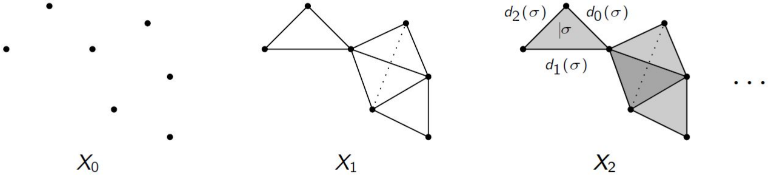
For each $[n]$, a simplicial set X has a set denoted $X[n]$ or X_n. In addition to those sets, X is determined by its **faces** $d_i : X_n \to X_{n-1}$ and **degeneracies** $s_i : X_n \to X_{n+1}$, which satisfy the simplicial identities.

Idea of simplicial sets



$$
\begin{tikzcd}
X_0 & X_1 \arrow[l, shift left=2, "d_0"'] \arrow[l, "d_1"'] & X_2 \arrow[l, shift left=4, "d_0"'] \arrow[l, shift left=2, "d_1"'] \arrow[l, "d_2"'] & \cdots
\arrow[from=1-1, to=1-2, shift right=2, "s_0"']
\arrow[from=1-2, to=1-3, shift right=2, "s_0"']
\arrow[from=1-2, to=1-3, shift right=4, "s_1"']
\end{tikzcd}
$$

Figure: A simplicial set $X : \Delta^{\text{op}} \to \textbf{Set}$, from nLab wiki.

Standard simplices and horns

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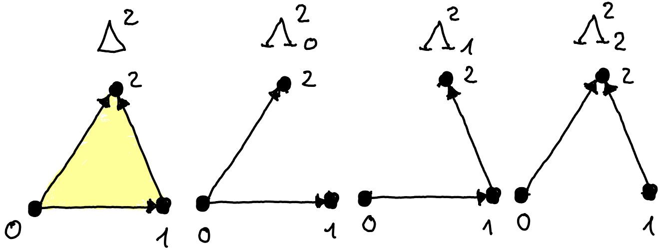

Quasi-categories and Kan complexes

$X \in \mathbf{sSet}$ has the **k-th horn extension property** if for every $n \in \mathbb{N}$ and every map $f : \Lambda^n_k \to X$, there exists a map $\tilde{f} : \Lambda^n_k \to X$ making the following diagram commute:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "f"] \arrow[d, hook, "i"'] & X \\
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A simplicial set is:

- A **quasicategory** if it has the k-th horn extension property for all $0 < k < n$.
- A **Kan complex** is a simplicial set that has the k-th horn extension property for all $0 \leq k \leq n$.

Higher structure

There exists a model of higher categories where quasicategories correspond to ∞-categories, and Kan complexes correspond to ∞-groupoids.

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In a quasicategory $\mathcal{C}$ the 0-simplices represent the objects, and the n-simplices represent n-morphisms.

It can be shown that for any two quasicategories $\mathcal{C}$ and $\mathcal{D}$, there is a quasicategory of ∞-**functors** between them, defined by:

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) := \mathbf{sSet}(\mathcal{C} \times \Delta^{\bullet}, \mathcal{D})$$

Here, the 0-simplices are functors, and the 1-simplices are natural transformations

Mapping space

In an abstract ∞-category, the set of morphisms between two objects is replaced by an ∞-groupoid of morphisms, which includes the information about the higher structure.

All the higher structure between two objects x and y inside a quasicategory $\mathcal{C}$ can be represented as a Kan complex, usually called the **mapping space** $\mathrm{Map}_{\mathcal{C}}(x, y)$, and defined as the following pullback:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "p"] \\
\{(x,y)\} \arrow[r, hook] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

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Limits and colimits

Let K be any simplicial set, $y \in \mathcal{C}_0$ an object, and $F : K \to \mathcal{C}$ any functor. Given any object $x \in \mathcal{C}_0$, the **constant functor** on x $\underline{x} : K \to \mathcal{C}$ sends all objects of K to x, and all higher morphisms to higher identities.

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A natural transformation $\alpha : \underline{y} \Rightarrow F$ **exhibits** y **as a limit of** F if α induces a homotopy equivalence of Kan complexes

$$\mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{x}, F).$$

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A natural transformation $\beta : F \Rightarrow \underline{y}$ **exhibits** y **as a colimit of** F if β induces a homotopy equivalence of Kan complexes

$$\mathrm{Map}_{\mathcal{C}}(y, z) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(F, \underline{z}).$$

Accessible and presentable quasicategories

A quasicategory $\mathcal{C}$ is **accessible** if there is a regular cardinal κ such that:

- $\mathcal{C}$ is locally small.
- $\mathcal{C}$ admits κ-filtered colimits
- The full subcategory $\mathcal{C}_\kappa \subset \mathcal{C}$ of κ-compact objects is essentially small.
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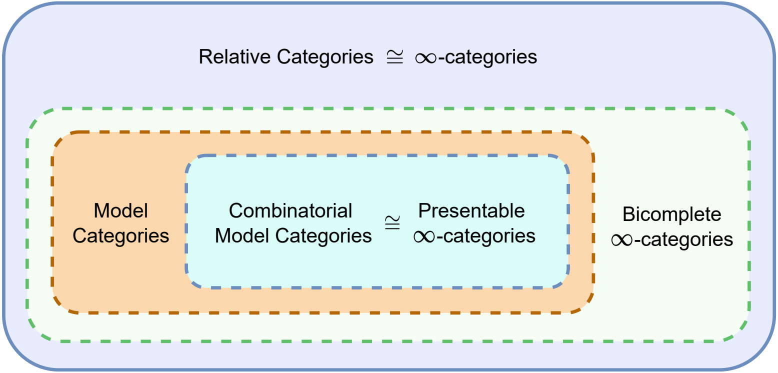
Definition

A quasicategory $\mathcal{C}$ is **(locally) presentable** if $\mathcal{C}$ is accessible and has all small colimits.

Characterization of presentable quasicategories

Theorem (Lurie, Pavlov)

A quasicategory is presentable if, and only if, it is presented by a combinatorial model category.



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Sketches

Recall that a **cone** (resp. **cocone**) of a functor $F : K \to C$ at an object y is a natural transformation $\alpha : \underline{y} \Rightarrow F$ (resp. $\beta : F \Rightarrow \underline{y}$).

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Definition (Ehresmann)

A **sketch** $\mathcal{S} = (K, \mathfrak{L}, \mathfrak{C})$ is a small category K equip with a set $\mathfrak{L}$ of cones and a set $\mathfrak{C}$ of cocones. If $\mathfrak{C} = \emptyset$, we call it a **limit sketch**.

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A **model** $M : K \to \mathbf{Set}$ of a sketch $\mathcal{S} = (K, \mathfrak{L}, \mathfrak{C})$ is a functor sending each cone in $\mathfrak{L}$ to a limit cone, and each cocone in $\mathfrak{C}$ to a colimit cocone. The category of all models of $\mathcal{S}$ is denoted $\mathrm{Mod}(\mathcal{S})$.

Representation theorem

Theorem (Adamek-Rosicky)

Let C be a category. The following are equivalent:

(i) *C is a locally presentable category.*

(ii) *C is equivalent to an accessible localization of the category of presheaves $\mathrm{PSh}(K)$ on a small category K.*

(iii) *C is equivalent to the category of models of limit sketches.*

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Theorem (Simpson, Lurie)

A quasicategory $\mathcal{C}$ is presentable if, and only if, there exists an accessible localization of the quasicategory of presheaves $\mathsf{PSh}(\mathcal{K})$ *on a small quasicategory $\mathcal{K}$.*

Generalization to higher categories

Context	Localization of presheaves	Sketchable
Categories	Adamek-Rosicky	Adamek-Rosicky
Model Categories	Dugger	?
∞-categories	Simpson, Lurie	?

Table: Table of the equivalent characterizations seen in the representation theorem but in more general contexts.

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