

LIMIT SKETCHES AND PRESENTABILITY

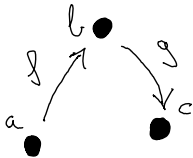
David Martínez Carpena

Carles Casacuberta Javier J. Gutiérrez

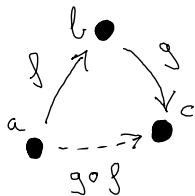


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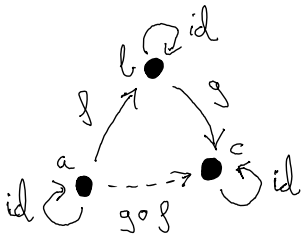
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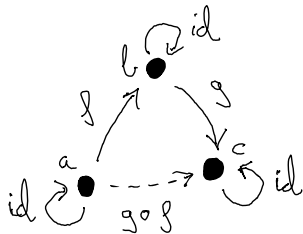
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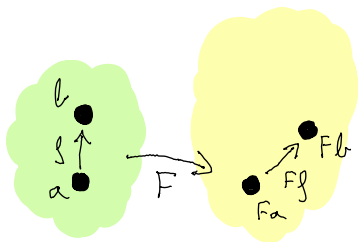
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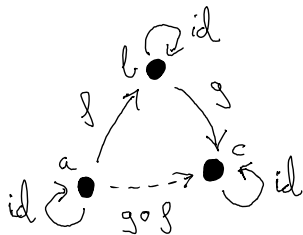
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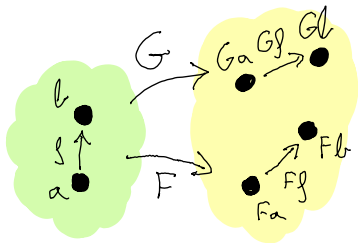
Functors and natural transformations



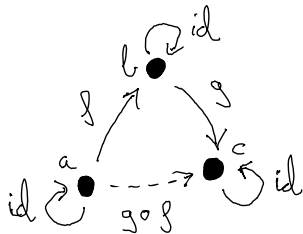
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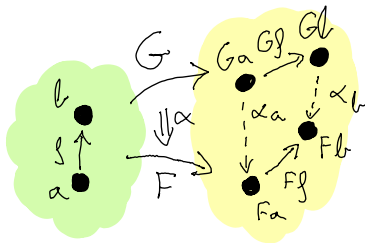
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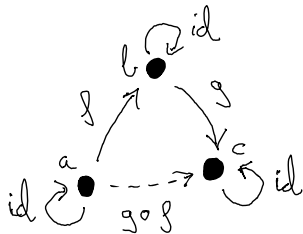
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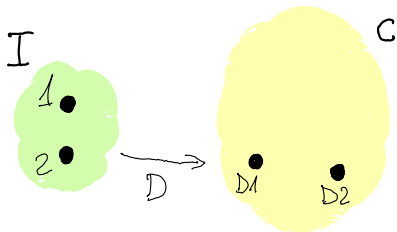
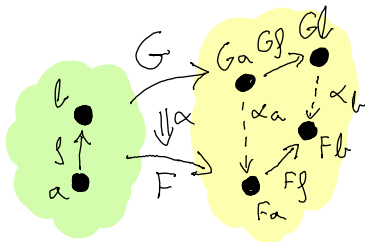
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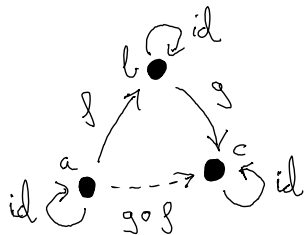
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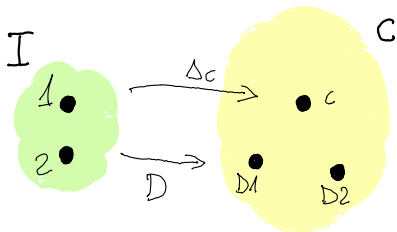
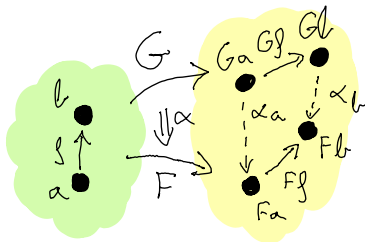
Cones and limits

> Diagram $D : I \rightarrow C$

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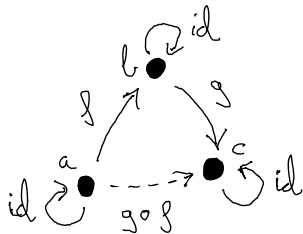
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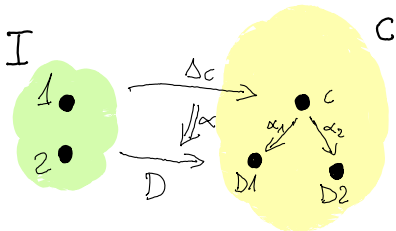
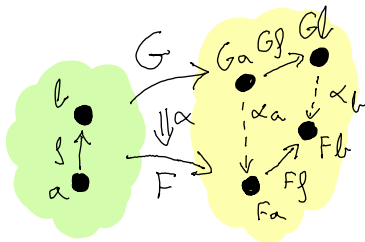
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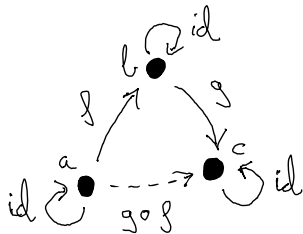
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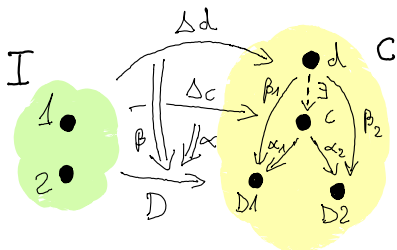
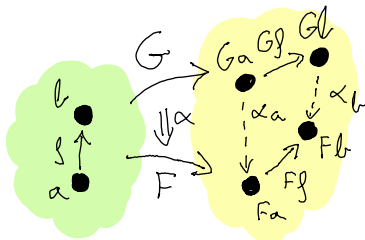
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- > Cone $\alpha : \Delta c \Rightarrow D$

Categories



Functors and natural transformations



Cones and limits

- > Diagram $D : I \rightarrow C$
- > Cone $\alpha : \Delta c \Rightarrow D$
- > α is a limit if for all $d \in C$


$$\text{Hom}(d, c) \cong \text{Cones}(D, d)$$

Presentability

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Examples. Set, Grp, sSet, ...

Non example. Top

Limit sketches

A **limit sketch** (Bastiani and Ehresmann 1972) is a pair (A, C) of a small category A and a set of cones C over A .

Example. Let A be the small category generated by the square (a).

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & (a) & \downarrow \\ 2 & \longrightarrow & 3 \end{array}$$

Limit sketches

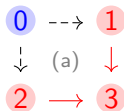
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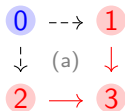
A **model** of a limit sketch is a functor $F : A \rightarrow \mathbf{Set}$ which sends cones of C to limits of **Set**. A category is **limit-sketchable** if it is equivalent to the category of models of some limit sketch.

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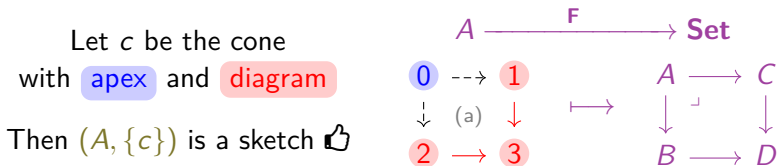


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A model F of the sketch $(A, \{c\})$ is a pullback of sets 👍

Representation theorem

Theorem (Adamek and Rosicky 1994)

The following are equivalent:

- (i) *Presentable categories.*
- (ii) *Limit-sketchable categories.*

Representation theorem

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Goal

Presentable ∞ -categories $\stackrel{?}{\simeq}$ Limit-sketchable ∞ -categories

Plan

Presentable ∞ -categories

Limit ∞ -sketches

Representation theorem

Plan

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Informal higher categories

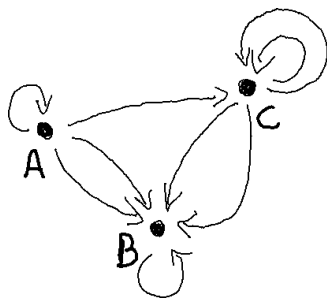
A **higher category** has objects and:

- ✓ n -morphisms between $(n - 1)$ -morphisms for all $n \geq 1$,
- ✓ Composition, identities and associativity of n -morphisms weakly up to a $(n + 1)$ -morphism for all $n \geq 1$.

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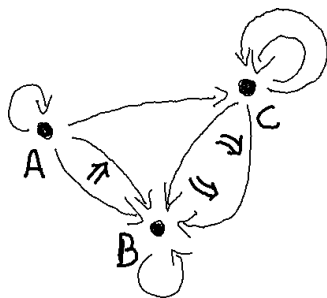
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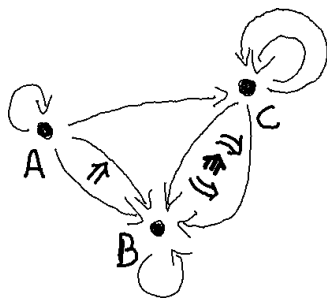
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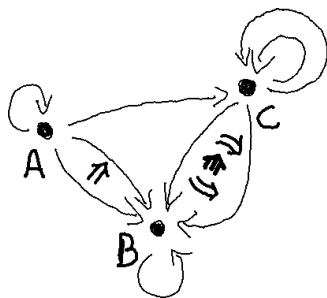
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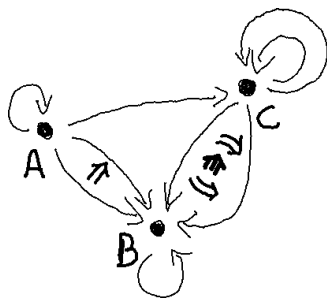


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- > ∞ -**category** $:= (\infty, 1)$ -category
- > ∞ -**groupoid** $:= (\infty, 0)$ -category

Limits and colimits

Let \mathcal{C} be an ∞ -category, and I be a small ∞ -category. Given any object $x \in \text{Obj}(\mathcal{C})$, the **constant diagram** $\Delta_x : I \rightarrow \mathcal{C}$ sends all objects of I to x , and all higher morphisms to higher identities.

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Cocones and **colimit cocones** are defined as cones and limit cones in the opposite ∞ -category.

Accessibility

Let κ denote a regular cardinal and \mathcal{C} an ∞ -category.

- An ∞ -category \mathcal{K} is κ -**filtered** if, for every κ -small ∞ -category I , every diagram $D : I \rightarrow \mathcal{K}$ admits a cocone $\alpha : D \Rightarrow \Delta_X$.

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An ∞ -category \mathcal{C} is **accessible** if it is locally small and there is a regular cardinal κ such that:

- ✓ \mathcal{C} admits κ -filtered colimits.
- ✓ There is some essentially small sub- ∞ -category of κ -compact objects which generates \mathcal{C} under κ -filtered colimits.

Presentability

Definition

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Example

- (a) The ∞ -category of homotopy types \mathcal{S} is presentable.
- (b) Any ∞ -topos is presentable.
- (c) The nerve of any presentable 1-category is presentable.
- (d) If \mathcal{A} is a small ∞ -category and \mathcal{C} is a presentable ∞ -category, then $\mathrm{Fun}(\mathcal{A}, \mathcal{C})$ is presentable.

Plan

Presentable ∞ -categories

Limit ∞ -sketches

Representation theorem

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Let \mathcal{C} be a complete ∞ -category. A functor $F : \mathcal{K} \rightarrow \mathcal{C}$ is a **model** of a limit ∞ -sketch $\mathcal{T} = (\mathcal{K}, \mathfrak{L})$ in \mathcal{C} if it sends each cone in \mathfrak{L} to a limit cone in \mathcal{C} .

$\mathrm{Mod}(\mathcal{T}, \mathcal{C}) := \infty\text{-category of models of } \mathcal{T} \text{ in } \mathcal{C}$

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We say that an ∞ -category is **limit ∞ -sketchable** (or **essentially ∞ -algebraic**) if it is equivalent to the ∞ -category of models of some limit ∞ -sketch.

Examples: ∞ -algebraic theories

An **∞ -algebraic theory** (or **∞ -Lawvere theory**) is a small ∞ -category with finite products. A **model** (or **algebra**) for an ∞ -algebraic theory \mathcal{A} is a functor $\mathcal{A} \rightarrow \mathcal{S}$ that preserves products.

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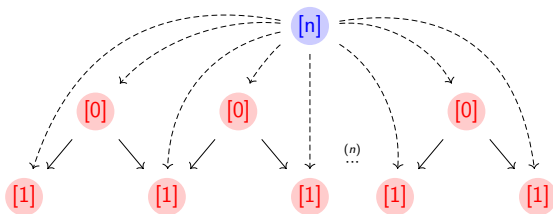
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Theorem (Rosicky 2007 and Lurie 2009)

The ∞ -category of models of an ∞ -algebraic theory is presentable.

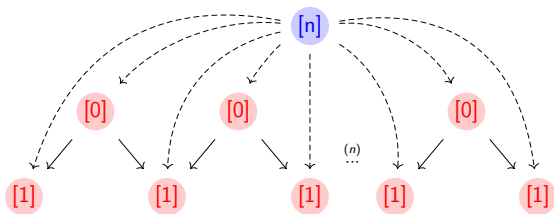
Examples: Internal precategories

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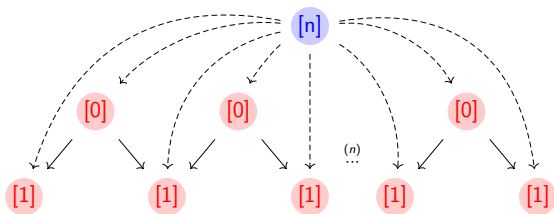


Then $\mathcal{T} = (\mathcal{A}, \{c_n \mid n \in \mathbb{N}\})$ is a limit ∞ -sketch, and a model $F : \mathcal{A} \rightarrow \mathcal{C}$ is a simplicial object in \mathcal{C} such that

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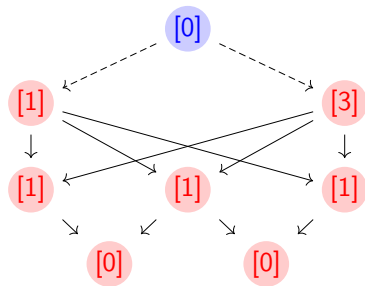
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$\text{Mod}(\mathcal{T}, \mathcal{C}) \simeq$ **Internal precategories**

$\text{Mod}(\mathcal{T}) \simeq$ **Segal spaces**

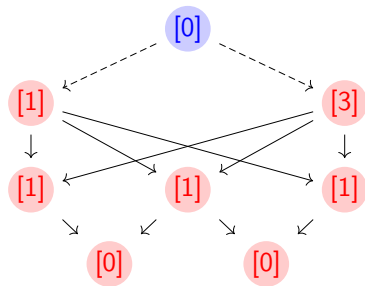
Examples: Internal univalent categories

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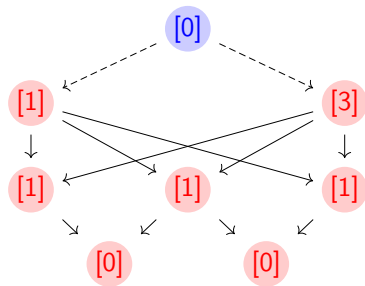


Then $\mathcal{T} = (\mathcal{A}, \mathcal{L}_S \cup \{d_n \mid n \in \mathbb{N}\})$ is a limit ∞ -sketch, and a model $F : \mathcal{A} \rightarrow \mathcal{C}$ is an internal precategory in \mathcal{C} such that

$$\begin{array}{ccc}
 F_0 & \xrightarrow{\quad} & F_3 \\
 \downarrow & \lrcorner & \downarrow \\
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$\text{Mod}(\mathcal{T}, \mathcal{C}) \simeq$ **Internal univalent categories**

$\text{Mod}(\mathcal{T}) \simeq$ **Complete Segal spaces**

Plan

Presentable ∞ -categories

Limit ∞ -sketches

Representation theorem

Representation theorem

Theorem (M.)

An ∞ -category is presentable \iff it is limit ∞ -sketchable.

Corollary

The ∞ -category of models of a limit ∞ -sketch in a presentable ∞ -category is presentable.

Future work

💡 **Generalization:** A ∞ -category is accessible if, and only if, it is equivalent to the ∞ -category of models of an ∞ -sketch.

A **sketch** is a limit sketch with a set of cocones which are sent to colimit cocones by any model.

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⇒ **Model-independent** version of this presentation!

💡 Formalize this work with a proof assistant which supports synthetic ∞ -categories like `rzk`.

References

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Thank you for listening!

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LIMIT SKETCHES AND PRESENTABILITY

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