Generalizing classifying spaces via the homotopy coherent nerve

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Métodos Categóricos y Homotópicos en Álgebra, Geometría, Topología y Análisis Funcional

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Classifying space

Given a topological group G, a *classifying space* BG is the quotient of a weakly contractible space EG by a proper free action of G.

Example

- For any discrete group G, $BG = K(G, 1)$.
- $B\mathbb{Z} = \mathbb{S}^1, E\mathbb{Z} = \mathbb{R}$.
- $B\mathbb{S}^1 = \mathbb{C}P^\infty, E\mathbb{S}^1 = \mathbb{S}^\infty$.

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Some expected properties of the classifying space:

- $\pi_{n+1}(BG) \cong \pi_n(G)$
- $G \simeq \Omega BG$

What exactly the classifying space classifies?

A principal G-bundle over a topological space X is a continuous map $p : P \to X$ with a G-action $e : P \times G \to P$ such that p is locally trivial and p is isomorphic to the quotient map $P \to P/G$.

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Proposition

For any topological space X there is an natural isomorphism

$$G\text{-}Bundles(X) \cong \pi_0(\mathbf{Top}(X, BG))$$

where G-$Bundles(X)$ is the set of principal G-bundle over x up to isomorphism.

Functorial classifying space

A *functorial classifying space* is a pair of functors

$$B : \mathbf{tGrp} \to \mathbf{Top}, \quad E : \mathbf{tGrp} \to \mathbf{Top}$$

such that EG is weakly contractible space, has a proper free action of G and $EG/G \cong BG$.

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- Segal (1968) generalize the construction to groupoids.

Simplicial objects

The *simplex category* Δ is the category with objects the linearly ordered sets $[n] := \{0, 1, \ldots, n\}$ for all $n \geq 0$, and morphisms all set functions $[n] \to [m]$ which are order-preserving.

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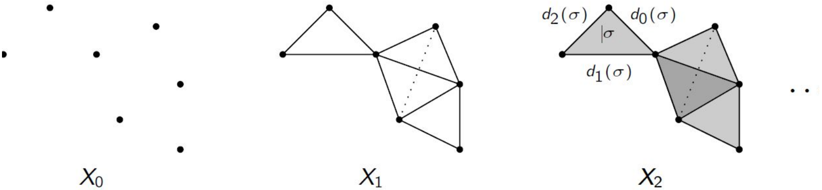
A *simplicial object* in a category $\mathcal{C}$ is a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$, and together with natural transformations form a category $[\Delta^{\mathrm{op}}, \mathcal{C}]$.



Example

Simplicial sets $\mathbf{sSet} := [\Delta^{\mathrm{op}}, \mathbf{Set}]$ are simplicial objects in **Set**, and simplicial spaces $\mathbf{sTop} := [\Delta^{\mathrm{op}}, \mathbf{Top}]$ are simplicial objects in **Top**.

Bar construction

Let G be a topological group, X a right G-space and Y a left G-space. The *bar construction* $\widetilde{\mathrm{B}} : \mathbf{Top} \times \mathbf{tGrp} \times \mathbf{Top} \to \mathbf{sTop}$ defines the following simplicial space

$$\widetilde{\mathrm{B}}_0(X, G, Y) = X \times Y, \quad \widetilde{\mathrm{B}}_n(X, G, Y) = X \times G^n \times Y,$$

with faces and degeneracies

$$d_i(x, g_1, \dots, g_n, y) = \begin{cases} (x \cdot g_1, g_2, \dots, g_n, y) & i = 0 \\ (x, g_1, \dots, g_i \cdot g_{i+1}, \dots, g_n, y) & 0 < i < n \\ (x, g_1, \dots, g_{n-1}, g_n \cdot y) & i = n \end{cases}$$

$$s_i(x, g_1, \dots, g_n, y) = (x, g_1, \dots, g_{i-1}, \mathrm{Id}, g_{i+1}, \dots, g_n, y)$$

Milnor's classifying space

The *geometric realization* $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ sends a simplicial set X to

$$|X| = \int^{[n]\in\Delta} X_n \times \Delta^n.$$

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There exists an analogous construction to the geometric realization, called the *topological geometric realization* $|\cdot|_t : \mathbf{sTop} \to \mathbf{Top}$ sends a simplicial space X to

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where in this case the product is between two topological spaces.

Let $*$ bet the trivial topological space. Then, *Milgram's functorial classifying space* is defined by

$$BG := |\widetilde{\mathrm{B}}(*, G, *)|_t, \quad EG := |\widetilde{\mathrm{B}}(*, G, G)|_t$$

Simplicial groups

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There is an equivalence of homotopy theories between **Top** and **sSet**, given by the geometrical realization and the *singular chain complex* $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ which is defined by

$$\mathrm{Sing}_n(X) := \mathbf{Top}(\Delta^n, X),$$

where $\Delta^n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1,\ \sum_{i=0}^{n} t_i = 1\}$.

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This equivalence also works in the case of topological groups and simplicial groups. Therefore, any topological group G can be seen as a simplicial one $\mathrm{Sing}(G)$.

Classifying complex functor

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The *classifying complex functor* $\overline{\mathrm{W}} : \mathbf{sGrp} \to \mathbf{sSet}$ is defined for each simplicial group G as the simplicial set with one single vertex, and for all $n \geq 1$

$$(\overline{\mathrm{W}}\, G)_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0,$$

and with faces and degeneracies

$$d_i(g_{n-1}, \dots, g_0) = \begin{cases} (g_{n-2}, \dots, g_0) & i = 0 \\ (d_i(g_{n-1}), \dots, d_1(g_{n-i-1}), g_{n-i-1} d_0(g_{n-i}), g_{n-i-2}, \dots, g_0) & i > 0 \end{cases}$$

$$s_i(g_{n-1}, \dots, g_0) = \begin{cases} (1, g_{n-1}, \dots, g_0) & i = 0 \\ (s_{i-1}(g_{n-1}), \dots, s_0(g_{n-i}), 1, g_{n-i-1}, \dots, g_0) & i > 0 \end{cases}$$

Nerve of a category

The *nerve* $\mathsf{N} : \mathbf{Cat} \to \mathbf{sSet}$ sends any category $\mathcal{C}$ to the simplicial set such that $\mathsf{N}_0(\mathcal{C}) := \mathsf{Obj}(\mathcal{C})$ and

$$\mathsf{N}_n(\mathcal{C}) := \overbrace{\mathsf{Mor}(\mathcal{C}) \times_{\mathsf{Obj}(\mathcal{C})} \cdots \times_{\mathsf{Obj}(\mathcal{C})} \mathsf{Mor}(\mathcal{C})}^{n}.$$

The faces are generated by composing two morphisms, and the degeneracies by introducing identities.

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Given a simplicial group G, any of its groups G_n can be seen as a category with one object and the group as the only homset.

Then, we can apply to each level of G the nerve functor, obtaining a bisimplicial set $\mathbf{bSet} := [\Delta^{\mathsf{op}}, \mathbf{sSet}]$, i.e., a simplicial object in the category of simplicial sets.

Homotopy equivalence between $\widetilde{\mathrm{B}}$ and $\overline{\mathrm{W}}$

The stronger proof can be found in [Berger and Huebschmann, 98], but I will explain the one found in [Stevenson, 11]:

$$\begin{tikzcd}
\mathbf{tGrp} \arrow[rr, "\mathrm{Sing}"] \arrow[d, "\widetilde{\mathrm{B}}"'] & & \mathbf{sGrp} \arrow[dl, "\mathrm{N}"'] \arrow[d, "\overline{\mathrm{W}}"] \\
\mathbf{sTop} \arrow[r, "\mathrm{Sing}"'] \arrow[dr, "|\cdot|_t"'] & \mathbf{bSet} \arrow[r, shift left, "T"] \arrow[r, shift right, "\mathrm{d}"'] & \mathbf{sSet} \arrow[dl, "|\cdot|"] \\
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- $|\widetilde{\mathrm{B}}(G)|_t \simeq |\,\mathrm{d}\,\mathrm{Sing}\,\widetilde{\mathrm{B}}(G)|$.

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- $T \simeq \mathrm{d}$ (Cegarra-Remedios, Stevenson)

$$\Longrightarrow |\,\widetilde{\mathrm{B}}(G)|_t \simeq |\,\overline{\mathrm{W}}\,\mathrm{Sing}(G)|$$

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In this case, which functor substitutes the classifying complex $\overline{\mathrm{W}}$?

Enriched categories

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Example

Topological categories are **Top**-enriched categories, and simplicial categories are **sSet**-enriched categories.

Generalized bar construction

Let $\mathcal{C}$ be a small topological category, and $X : \mathcal{C}^{op} \to \textbf{Top}$, $Y : \mathcal{C} \to \textbf{Top}$ topologically enriched functors. The *general bar construction* $\widetilde{\mathrm{B}}(X, \mathcal{C}, Y)$ as the simplicial space given by

$$\widetilde{\mathrm{B}}_0(X, \mathcal{C}, Y) := \coprod_{A \in \mathcal{C}} X(A) \times X(A)$$
$$\widetilde{\mathrm{B}}_n(X, \mathcal{C}, Y) := \coprod_{A, B \in \mathcal{C}} X(B) \times \mathcal{C}_n(A, B) \times Y(A)$$

where $\mathcal{C}_n(A, B)$ is the space of n-tuples of morphisms $(f_1, \dots, f_n)$ that are composable and $f_1 \circ \cdots \circ f_n \in \mathcal{C}(A, B)$; with boundaries and degeneracies very similar to the previous construction.

Segal's nerve

For every topological category $\mathcal{C}$, the *homotopy category* $\mathrm{h}\mathcal{C}$ has the same objects as $\mathcal{C}$ and $\mathrm{h}\mathcal{C}(X, Y) = \pi_0(\mathcal{C}(X, Y))$. A topological category $\mathcal{C}$ is an ∞-*groupoid* if $\mathrm{h}\mathcal{C}$ is a groupoid.

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Using the previous general bar construction, we can define a functorial classifying space for any ∞-groupoid:

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$$BG := |\,\widetilde{\mathrm{B}}(*, G, *)|_t, \quad EG := |\,\widetilde{\mathrm{B}}(*, G, G)|_t.$$

This coincides with the Segal nerve defined in 1968, but the original definition uses a modification to the nerve of categories to preserve the topological information.

Formal nerve

Given any functor $Q : \Delta \to \mathcal{C}$, define the *Q-nerve* $\mathsf{N}^Q : \mathcal{C} \to \mathbf{sSet}$ as mapping an object $A \in \mathcal{C}$ to the simplicial set defined for every $[n] \in \Delta$ by

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If $\mathcal{C}$ is cocomplete, there is a left adjoint to N^Q, the *Q-realization* functor $|\cdot|_Q : \mathbf{sSet} \to \mathcal{C}$ defined for all $X \in \mathbf{sSet}$ as:

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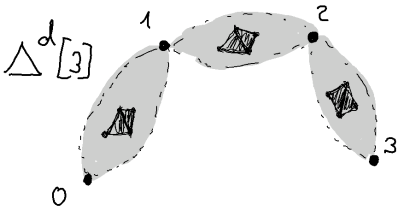
Example

- The nerve of a category N, with $Q[n] = [n]$.
- Sing and $|\cdot|$, with $Q[n] = \Delta^n$.

Diagonal nerve

There is a functor defined for each $[n] \in \Delta$ as the simplicial category $\Delta^d[n]$ with:

- $\mathrm{Obj}(\Delta^d[n]) = [n]$.
- Morphisms of $\Delta^d[n]$ are freely generated by the n-simplices $a_i \in \mathrm{Hom}(i-1, i)$ for all $i = 1, \ldots, n$.



This functor defines a Δ^d-nerve, which we call the *diagonal simplicial nerve* N^d : **sSet-Cat** $\to$ **sSet**.

Relation between diagonal nerve and bar construction

$$
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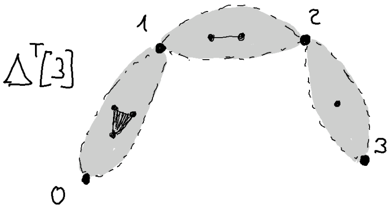
$$

$$
\left.\begin{array}{c} \mathrm{N}^d = \mathrm{d}\,\mathrm{N} \\ \mathrm{N}\,\mathrm{Sing} \cong \mathrm{Sing}\,\widetilde{\mathrm{B}} \end{array}\right\} \implies \mathrm{d}\,\mathrm{Sing}\,\widetilde{\mathrm{B}} \cong \mathrm{N}^d\,\mathrm{Sing}
$$

Total nerve

There is a functor defined for each $[n] \in \Delta$ as the simplicial category $\Delta^T[n]$ with:

- $\mathrm{Obj}(\Delta^T[n]) = [n]$.
- Morphisms of $\Delta^T[n]$ are freely generated by $(n-i)$-simplices $g_i \in \mathrm{Hom}(i-1, i)$ for $i = 1, \ldots, n$.



This functor defines a Δ^T-nerve, which we call the *total simplicial nerve* N^T : **sSet-Cat** $\to$ **sSet**.

Equivalence between total and diagonal

Follows from the equivalence between T and d by [Stevenson, 11]:

$$\begin{tikzcd}
\mathbf{Top\text{-}Cat} \arrow[rr, "\mathrm{Sing}"] \arrow[d, "\widetilde{\mathrm{B}}"'] & & \mathbf{sSet\text{-}Cat} \arrow[dl, "\mathrm{N}"'] \arrow[d, shift right, "\mathrm{N}^d"'] \arrow[d, shift left, "\mathrm{N}^T"] \\
\mathbf{sTop} \arrow[r, "\mathrm{Sing}"'] & \mathbf{bSet} \arrow[r, shift left, "T"] \arrow[r, shift right, "\mathrm{d}"'] & \mathbf{sSet}
\end{tikzcd}$$

$$\implies \widetilde{\mathrm{B}}(G) \cong \mathrm{N}^d\,\mathrm{Sing}(G) \simeq \mathrm{N}^T\,\mathrm{Sing}(G)$$

Topological group-like monoids

Proposition

If X is a good simplicial space, then

$$|\operatorname{d}(\operatorname{Sing}(X))| \simeq |X|_t,$$

which is equivalent to the commutativity up to weak equivalence of the following diagram:

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\mathbf{sTop} \arrow[r, "\mathrm{Sing}"] \arrow[dr, "|\cdot|_t"'] & \mathbf{bSet} \arrow[r, "\mathrm{d}"] & \mathbf{sSet} \arrow[dl, "|\cdot|"] \\
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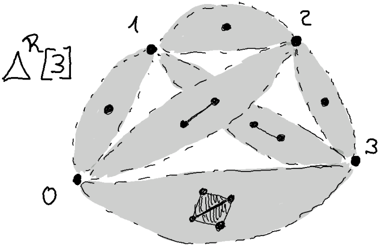
Corollary

If M is a well-pointed monoid, $|\,\widetilde{\mathrm{B}}(M)|_t \simeq |\,\mathrm{N}^d\,\mathrm{Sing}(M)|$.

Homotopy coherent nerve

There is a functor defined for each $[n] \in \Delta$ as the simplicial category $\Delta^{\mathfrak{R}}[n]$ with:

- $\mathrm{Obj}(\Delta^{\mathfrak{R}}[n]) = [n] = \{0, \ldots, n\}$
- For every $i, j \in \mathrm{Obj}(\Delta^{\mathfrak{R}}[n])$, $\mathrm{Hom}(i, j) = (\Delta[1])^{(j-i-1)}$



This functor defines a $\Delta^{\mathfrak{R}}$-nerve, which we call the *homotopy coherent nerve* $\mathrm{N}^{\mathfrak{R}} : \mathbf{sSet\text{-}Cat} \to \mathbf{sSet}$.

Model via the homotopy coherent nerve

Our goal is to prove that $|\,\mathsf{N}^{\mathfrak{R}}\,\mathsf{Sing}\,\mathcal{M}|$ is homotopy equivalent to the classifying space $|\,\widetilde{\mathsf{B}}(\mathcal{M})|_t$, for any topological group-like monoid $\mathcal{M}$.

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$$
\begin{tikzcd}
\mathbf{Top\text{-}Cat} \arrow[rr, "\mathrm{Sing}"] \arrow[d, "\widetilde{\mathrm{B}}"'] & & \mathbf{sSet\text{-}Cat} \arrow[dl, dashed, "\mathrm{N}"'] \arrow[d, dashed, shift right=4, "\mathrm{N}^d"'] \arrow[d, dashed, "\mathrm{N}^T"] \arrow[d, shift left=4, "\mathrm{N}^{\mathfrak{R}}"] \\
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- The $\overline{\mathrm{W}}$ functor had good model theoretic properties, but neither N^d or N^T have similar properties.

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Why homotopy coherent nerve?

- The $\overline{\mathrm{W}}$ functor had good model theoretic properties, but neither N^d or N^T have similar properties.
- Instead, the homotopy coherent nerve $\mathrm{N}^{\mathfrak{R}}$ forms a Quillen equivalence between simplicial sets and simplicial categories.
- By forming a Quillen equivalence, the coherent nerve preserves the model theoretic structure (fibrations and weak equivalences).

Equivalence between nerves

Theorem

For any ∞-groupoid $\mathcal{C}$, there is a weak equivalence:

$$\mathsf{N}^T(\mathcal{C}) \simeq \mathsf{N}^{\mathfrak{R}}(\mathcal{C}).$$

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Idea of the proof inspired by (Hinnich, 2007).

- Construct an explicit map from $\mathsf{N}^T(\mathcal{C})$ to $\mathsf{N}^{\mathfrak{R}}(\mathcal{C})$.

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Equivalence between nerves

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For any ∞-groupoid $\mathcal{C}$, there is a weak equivalence:

$$\mathrm{N}^{T}(\mathcal{C}) \simeq \mathrm{N}^{\Re}(\mathcal{C}).$$

Idea of the proof inspired by (Hinnich, 2007).

- Construct an explicit map from $\mathrm{N}^{T}(\mathcal{C})$ to $\mathrm{N}^{\Re}(\mathcal{C})$.
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- $\mathrm{N}^{d}(\mathcal{M}) \simeq \mathrm{N}^{T}(\mathcal{M})$ implies that $\pi_n(\mathrm{N}^{T}(\mathcal{M})) \cong \pi_{n-1}(\mathcal{M})$.
- Show that $\pi_n(\mathrm{N}^{\Re}(\mathcal{C})) \cong \pi_{n-1}(\mathcal{M})$ by explicit calculation.

$\square$

Contents

Classifying spaces of topological groups

Generalizing the classifying space
- Simplicial nerves
- Homotopy coherent nerve

Application to Moore path categories

Moore path categories

For each topological space X, define the *Moore path category* $\Pi_\infty^M(X)$ as the ∞-grupoid such that:

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- The composition is defined by

$$\begin{aligned} \circ : \ P_{x,y}^M X \times P_{y,z}^M X &\longrightarrow \quad P_{x,z}^M X \\ ((f, r), (g, s)) &\longmapsto (f * g, r + s) \end{aligned}$$

$$(f * g)(t) = \begin{cases} f(t) & \text{if } 0 \leq t < r \\ g(t - r) & \text{if } t \geq r \end{cases}$$

Topological categories as a model for homotopy types

Each topological category models an ∞-category, with the higher homotopies of the map spaces as higher morphisms.

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Then, there is a zigzag of weak equivalences between spaces and topological categories

$$\mathbf{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{sSet}_Q \underset{k^!}{\overset{k_!}{\rightleftarrows}} \mathbf{sSet}_J \underset{\mathrm{N}^{\mathfrak{R}}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathbf{sSet}\text{-}\mathbf{Cat} \underset{\mathrm{Sing}}{\overset{|\cdot|_e}{\rightleftarrows}} \mathbf{Top}\text{-}\mathbf{Cat}$$

where the functors $k_!$ and $k^!$ a localization adjunction between the Joyal and Quillen model structures.

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where the functors $k_!$ and $k^!$ a localization adjunction between the Joyal and Quillen model structures.

Finally, given any topological space X, the fundamental ∞-groupoid associated to X is $(|\cdot|_e \circ \mathfrak{C} \circ k_! \circ \mathrm{Sing})(X)$.

The fundamental ∞-groupoid as a Moore path category

Proposition

Let $\Omega^M_x(X)$ be the topological group-like monoid $P^M_{x,x}X$. For every path-connected pointed topological space (X, x), $B\Omega^M_x(X) \simeq X$.

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Theorem (Martínez Carpena)

Let (X, x) be a path-connected well-pointed topological space. The topological space $|\,\mathsf{N}^{\mathfrak{R}}\,\mathsf{Sing}\,\Omega^M_x(X)|$ is homotopy equivalent to the classifying space for $\Omega^M_x(X)$ and, as a consequence,

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$$|\,\mathsf{N}^{\mathfrak{R}}\,\mathsf{Sing}\,\Omega^M(X)| \simeq X.$$

Corollary (McGarry-Martínez Carpena)

The ∞-groupoid $\Pi^M_\infty(X)$ is weakly homotopy equivalent to the fundamental ∞-groupoid $(|\cdot|_e \circ \mathfrak{C} \circ k_! \circ \mathsf{Sing})(X)$.

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Generalizing classifying spaces via the homotopy coherent nerve

David Martínez Carpena

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Métodos Categóricos y Homotópicos en Álgebra, Geometría, Topología y Análisis Funcional

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