

See discussions, stats, and author profiles for this publication at:
<https://www.researchgate.net/publication/235008656>

Extensions of Lipschitz maps into a Hilbert space

Article · January 1984

DOI: 10.1090/conm/026/737400

CITATIONS

1,185

READS

4,721

2 authors, including:



[William B Johnson](#)

Texas A&M University

177 PUBLICATIONS 6,482 CITATIONS

SEE PROFILE

EXTENSIONS OF LIPSCHITZ MAPPINGS INTO A HILBERT SPACE

William B. Johnson¹ and Joram Lindenstrauss²

INTRODUCTION

In this note we consider the following extension problem for Lipschitz functions: Given a metric space X and $n = 2, 3, 4, \dots$, estimate the smallest constant $L = L(X, n)$ so that every mapping f from every n -element subset of X into ℓ_2 extends to a mapping \tilde{f} from X into ℓ_2 with

$$\|\tilde{f}\|_{\ell_1 p} \leq L \|f\|_{\ell_1 p}.$$

(Here $\|g\|_{\ell_1 p}$ is the Lipschitz constant of the function g .) A classical result of Kirszbraun's [14, p. 48] states that $L(\ell_2, n) = 1$ for all n , but it is easy to see that $L(X, n) \rightarrow \infty$ as $n \rightarrow \infty$ for many metric spaces X .

Marcus and Pisier [10] initiated the study of $L(X, n)$ for $X = L_p$. (For brevity, we will use hereafter the notation $L(p, n)$ for $L(L_p(0,1), n)$.) They prove that for each $1 < p < 2$ there is a constant $C(p)$ so that for $n = 2, 3, 4, \dots$,

$$L(p, n) \leq C(p) (\log n)^{1/p - 1/2}.$$

The main result of this note is a verification of their conjecture that for some constant C and all $n = 2, 3, 4, \dots$,

$$L(X, n) \leq C(\log n)^{1/2}$$

for all metric spaces X . While our proof is completely different from that of Marcus and Pisier, there is a common theme: Probabilistic techniques developed for linear theory are combined with Kirszbraun's theorem to yield extension theorems.

The main tool for proving Theorem 1 is a simply stated elementary geometric lemma, which we now describe: Given n points in Euclidean space, what

¹Supported in part by NSF MCS-7903042.

²Supported in part by NSF MCS-8102714.

is the smallest $k = k(n)$ so that these points can be moved into k -dimensional Euclidean space via a transformation which expands or contracts all pairwise distances by a factor of at most $1 + \varepsilon$? The answer, that $k \leq C(\varepsilon) \log n$, is a simple consequence of the isoperimetric inequality for the n -sphere in the form studied in [2].

It seems likely that the Marcus-Pisier result and Theorem 1 give the right order of growth for $L(p, n)$. While we cannot verify this, in Theorem 3 we get the estimate

$$L(p, n) \geq \delta \left(\frac{\log n}{\log \log n} \right)^{1/p - 1/2} \quad (1 \leq p < 2)$$

for some absolute constant $\delta > 0$. (Throughout this paper we use the convention that $\log x$ denotes the maximum of 1 and the natural logarithm of x .) This of course gives a lower estimate of

$$\delta \left(\frac{\log n}{\log \log n} \right)^{1/2}$$

for $L(\infty, n)$. That our approach cannot give a lower bound of $\delta(\log n)^{1/p - 1/2}$ for $L(p, n)$ is shown by Theorem 2, which is an extension theorem for mappings into ℓ_2 whose domains are ε -separated.

The minimal notation we use is introduced as needed. Here we note only that $B_Y(y, \varepsilon)$ (respectively, $b_Y(y, \varepsilon)$) is the closed (respectively, open) ball in Y about y of radius ε . If $y = 0$, we use $B_Y(\varepsilon)$ and $b_Y(\varepsilon)$, and we drop the subscript Y when there is no ambiguity. $S(Y)$ is the unit sphere of the normed space Y . For isomorphic normed spaces X and Y , we let

$$d(X, Y) = \inf \|T\| \|T^{-1}\|,$$

where the \inf is over all invertible linear operators from X onto Y . Given a bounded Banach space valued function f on a set K , we set

$$\|f\|_\infty = \sup_{x \in K} \|f(x)\|.$$

1. THE EXTENSION THEOREMS

We begin with the geometrical lemma mentioned in the introduction.

LEMMA 1. For each $1 > \tau > 0$ there is a constant $K = K(\tau) > 0$ so that if $A \subset \ell_2^n$, $\bar{A} = n$ for some $n = 2, 3, \dots$, then there is a mapping f from A onto a subset of ℓ_2^k ($k \equiv [K \log n]$) which satisfies

$$\|f\|_{\ell_{ip}} \|\tilde{f}^{-1}\|_{\ell_{ip}} \leq \frac{1+\tau}{1-\tau}.$$

PROOF. The proof will show that if one chooses at random a rank k orthogonal projection on ℓ_2^n , then, with positive probability (which can be made arbitrarily close to one by adjusting k), the projection restricted to A will satisfy the condition on \tilde{f} . To make this precise, we let Q be the projection onto the first k coordinates of ℓ_2^n and let σ be normalized Haar measure on $O(n)$, the orthogonal group on ℓ_2^n . Then the random variable

$$f : (O(n), \sigma) \rightarrow L(\ell_2^n)$$

defined by

$$f(u) = U^* Q U$$

determines the notion of "random rank k projection." The applications of Levy's inequality in the first few self-contained pages of [2] make it easy to check that $f(u)$ has the desired property. For the convenience of the reader, we follow the notation of [2].

Let $|||\cdot|||$ denote the usual Euclidean norm on \mathbb{R}^n and for $1 \leq k \leq n$ and $x \in \mathbb{R}^n$ set

$$r(x) = r_k(x) = \sqrt{n} \left(\sum_{i=1}^k x(i)^2 \right)^{1/2},$$

which is equal to

$$\sqrt{n} |||Qx|||$$

for our eventual choice of $k = [K \log n]$. Thus $r(\cdot)$ is a semi-norm on ℓ_2^n which satisfies

$$r(x) \leq \sqrt{n} |||x||| \quad (x \in \ell_2^n).$$

(In [2], $r(\cdot)$ is assumed to be a norm, but inasmuch as the left estimate $a|||x||| \leq r(x)$ in formula (2.5) of [2] is not needed in the present situation, it is okay that $r(\cdot)$ is only a semi-norm.)

Setting

$$B = \left\{ \frac{x-y}{|||x-y|||} : x, y \in A; x \neq y \right\} \subset S^{n-1},$$

we want to select $U \in O(n)$ so that for some constant M ,

$$M(1 - \tau) \leq r(Ux) \leq M(1 + \tau) \quad (x \in B) .$$

Let M_r be the median of $r(\cdot)$ on S^{n-1} , so that

$$\mu_{n-1}[x \in S^{n-1} : r(x) \geq M_r] \geq 1/2$$

and

$$\mu_{n-1}[x \in S^{n-1} : r(x) \leq M_r] \leq 1/2$$

where μ_{n-1} is normalized rotationally invariant measure on S^{n-1} .

We have from page 58 of [2] that for each $y \in S^{n-1}$ and $\varepsilon > 0$,

$$\sigma[U \in O(n) : M_r - \sqrt{n} \varepsilon \leq r(Uy) \leq M_r + \sqrt{n} \varepsilon] \geq 1 - 4 \exp \left(\frac{-n\varepsilon^2}{2} \right).$$

Hence

$$(1.1) \quad \sigma[U \in O(n) : M_r - \sqrt{n} \varepsilon \leq r(Uy) \leq M_r + \sqrt{n} \varepsilon \text{ for all } y \in B] \geq \\ \geq 1 - 2n(n+1) \exp \left(\frac{-n\varepsilon^2}{2} \right).$$

By Lemma 1.7 of [2], there is a constant

$$C \leq 4 \sum_{m=1}^{\infty} (m+1) e^{-m^2/2}$$

so that

$$(1.2) \quad \left| \int_{S^{n-1}} r(x) d\mu_{n-1}(x) - M_r \right| < C .$$

We now repeat a known argument for estimating $\int_{S^{n-1}} r(x) d\mu_{n-1}(x)$ which uses only Khintchine's inequality.

For $1 \leq k \leq n$ we have:

$$\begin{aligned} & \text{Av} \int_{S^{n-1}} \left| \sum_{i=1}^k \pm x(i) \right| d\mu_{n-1}(x) = \\ & = \text{Av} \int_{S^{n-1}} \left| \langle x, \sum_{i=1}^k \pm \delta_i \rangle \right| d\mu_{n-1}(x) \\ & = \sqrt{k} \int_{S^{n-1}} \left| \langle x, \delta_1 \rangle \right| d\mu_{n-1}(x) \quad \left[\begin{array}{l} \text{by the rotational} \\ \text{invariance of } \mu_{n-1} \end{array} \right] . \end{aligned}$$

Setting

$$\alpha_n = \int_{S^{n-1}} \left| \langle x, \delta_1 \rangle \right| d\mu_{n-1}(x) ,$$

we have from Khintchine's inequality that for each $1 \leq k \leq n$,

$$\sqrt{nk} \alpha_n \leq \int_{S^{n-1}} r_k(x) d\mu_{n-1}(x) \leq \sqrt{2nk} \alpha_n.$$

(We plugged in the exact constant of $\sqrt{2}$ in Khintchine's inequality calculated in [5] and [13], but of course any constant would serve as well.)

Since obviously $r_n(x) = \sqrt{n}$, we conclude that for $1 \leq k \leq n$

$$(1.3) \quad \sqrt{k/} \leq \int_{S^{n-1}} r_k(x) d\mu_{n-1}(x) \leq \sqrt{k}.$$

Specializing now to the case $k = [K \log n]$, we have from (1.2) and (1.3) that

$$\sqrt{k/3} \leq M_r$$

at least for $K \log n$ sufficiently large. Thus if we define

$$\varepsilon = \tau \sqrt{k/3n}$$

we get from (1.1) that

$$\begin{aligned} \sigma [U \in O(n) : (1 - \tau)M_r &\leq r(Uy) \leq (1 + \tau)M_r \text{ for all } y \in B] \\ &\geq 1 - 2n(n+1) \exp \left(-\frac{\tau^2 k}{18} \right) \\ &\geq 1 - 2n(n+1) \exp \left(-\frac{\tau^2 K \log n}{18} \right) \end{aligned}$$

which is positive if, say,

$$K \geq (10/\tau)^2.$$

□

It is easily seen that the estimate $K \log n$ in Lemma 1 cannot be improved. Indeed, in a ball of radius 2 in ℓ_2^k there are at most 4^k vectors $\{x_i\}$ so that $\|x_i - x_j\| \geq 1$ for every $i \neq j$ (see the proof of Lemma 3 below). Hence for τ sufficiently small there is no map F which maps an orthonormal set with more than 4^k vectors into a k -dimensional subspace of ℓ_2 with

$$\|F\|_{\ell_{ip}} \|F^{-1}\|_{\ell_{ip}} \leq \frac{1 + \tau}{1 - \tau}.$$

We can now verify the conjecture of Marcus and Pisier [10].

THEOREM 1. $\sup_{n=2, 3, \dots} (\log n)^{-1/2} L(\infty, n) < \infty$. In other words: there is a constant K so that for all metric spaces X and all finite subsets M of X (card $M = n$, say) every function f from M into ℓ_2 has a Lipschitz extension $\tilde{f} : X \rightarrow \ell_2$ which satisfies

$$\|\tilde{f}\|_{\ell_{ip}} \leq K \sqrt{\log n} \|f\|_{\ell_{ip}}.$$

PROOF. Given $X, M \subset X$ with card $M = n$, and $f : M \rightarrow \ell_2$, set $A = f[M]$. We apply Lemma 1 with $\tau = 1/2$ to get a one-to-one function g^{-1} from A onto a subset $g^{-1}[A]$ of ℓ_2^k (where $k \leq K \log n$) which satisfies

$$\|g^{-1}\|_{\ell_{ip}} \leq 1; \quad \|g\|_{\ell_{ip}} \leq 3.$$

By Kirszbraun's theorem, we can extend g to a function $\tilde{g} : \ell_2^k \rightarrow \ell_2$ in such a way that

$$\|\tilde{g}\|_{\ell_{ip}} \leq 3.$$

Let $I : \ell_2^k \rightarrow \ell_\infty^k$ denote the formal identity map, so that

$$\|I\| = 1, \quad \|I^{-1}\| = \sqrt{k}.$$

Then

$$h \equiv Ig^{-1}f, \quad h : M \rightarrow \ell_\infty^k$$

has Lipschitz norm at most $\|f\|_{\ell_{ip}}$, so by the non-linear Hahn-Banach theorem (see, e.g., p. 48 of [14]), h can be extended to a mapping

$$\tilde{h} : X \rightarrow \ell_\infty^k$$

which satisfies

$$\|\tilde{h}\|_{\ell_{ip}} \leq \|f\|_{\ell_{ip}}.$$

Then

$$\tilde{f} \equiv \tilde{g} I^{-1} \tilde{h}; \quad \tilde{f} : X \rightarrow \ell_2$$

is an extension of f and satisfies

$$\|\tilde{f}\|_{\ell_{ip}} \leq 3 \sqrt{k} \|f\|_{\ell_{ip}} \leq 3K \sqrt{\log n} \|f\|_{\ell_{ip}}.$$

□

Next we outline our approach to the problem of obtaining a lower bound for $L(\infty, n)$. Take for f the inclusion mapping from an ε -net for S^{N-1} into ℓ_2^N , and consider ℓ_2^N isometrically embedded into L_∞ . A Lipschitz extension of f to a mapping $\tilde{f} : L_\infty \rightarrow \ell_2$ should act like the identity ℓ_2^N , so the techniques of [8] should yield a linear projection from L_∞ onto ℓ_2^N whose norm is of order $\|f\|_{\ell_{ip}}$. Since ℓ_2^N is complemented in L_∞ only of order \sqrt{N} and there are ε -nets for S^{N-1} of cardinality $n \approx [4/\varepsilon]^N$, we should get that

$$L(\infty, n) \geq \sqrt{N} \geq \delta \left(\frac{\log n}{-\log \varepsilon} \right)^{1/2}.$$

In Theorem 2 we make this approach work when ε is of order N^{-2} , so we get

$$L(\infty, n) \geq \delta' \left(\frac{\log n}{\log \log n} \right)^{1/2}.$$

That the difficulties we incur with the outlined approach for larger values of ε are not purely technical is the gist of the following extension result.

(*)THEOREM 2. Suppose that X is a metric space, $A \subset X$, $f : A \rightarrow \ell_2$ is Lipschitz and $d(x, y) \geq \varepsilon > 0$ for all $x \neq y \in A$. Then there is an extension $\tilde{f} : X \rightarrow \ell_2$ of f so that

$$\|\tilde{f}\|_{\ell_{ip}} \leq \frac{6D}{\varepsilon} \|f\|_{\ell_{ip}},$$

where D is the diameter of A .

PROOF. We can assume by translating f that there is a point $0 \in A$ so that $f(0) = 0$. Set $B = A \setminus \{0\}$ and define

$$F : A \rightarrow \ell_1^B \text{ by}$$

$$F(b) = \begin{cases} \delta_b, & b \neq 0 \\ 0, & b = 0 \end{cases}.$$

Define

$$G : \ell_1^B \rightarrow \ell_2$$

by

$$G\left(\sum_{b \in B} \alpha_b \delta_b\right) = \sum_{b \in B} \alpha_b f(b).$$

(*) See the appendix for a generalization of Theorem 2 proved by Yoav Benyamini.

Then

$GF = f$, G is linear with

$$\|G\| \leq D \|f\|_{\ell_{ip}}, \text{ and } \|F\|_{\ell_{ip}} \leq 2/\varepsilon.$$

A weakened form of Grothendieck's inequality (see section 2.6 in [9]) yields that G (as any bounded linear operator from an L_1 space into a Hilbert space) factors through an $\ell_\infty(\mathcal{N})$ space:

$$G = HJ, \quad \|J\| = 1, \quad \|H\| \leq 3 \|G\|,$$

$$J : \ell_1^B \rightarrow \ell_\infty(\mathcal{N}), \quad H : \ell_\infty(\mathcal{N}) \rightarrow \ell_2.$$

By the non-linear Hahn-Banach Theorem the mapping JF has an extension

$E : X \rightarrow \ell_\infty(\mathcal{N})$ which satisfies

$$\|E\|_{\ell_{ip}} \leq \|JF\|_{\ell_{ip}} \leq 2/\varepsilon.$$

Then $\tilde{f} \equiv HE$ extends f and $\|\tilde{f}\| \leq \frac{6D}{\varepsilon} \|f\|_{\ell_{ip}}$, as desired. \square

For the proof of Theorem 3, we need three well known facts which we state as lemmas.

LEMMA 2. Suppose that Y, X are normed spaces and $f : S(Y) \rightarrow X$ is Lipschitz with $f(0) = 0$. Then the positively homogeneous extension of f , defined for $y \in Y$ by

$$\tilde{f}(y) = \|y\| f\left(\frac{y}{\|y\|}\right), \quad (y \neq 0); \quad \tilde{f}(0) = 0$$

is Lipschitz and

$$\|\tilde{f}\|_{\ell_{ip}} \leq 2 \|f\|_{\ell_{ip}} + \|f\|_\infty.$$

PROOF. Given $y_1, y_2 \in Y$ with $0 < \|y_1\| \leq \|y_2\|$,

$$\begin{aligned} \|\tilde{f}(y_1) - \tilde{f}(y_2)\| &\leq \left| \|y_1\| f\left(\frac{y_1}{\|y_1\|}\right) - \|y_2\| f\left(\frac{y_1}{\|y_1\|}\right) \right| + \|y_2\| \left| f\left(\frac{y_1}{\|y_1\|}\right) - f\left(\frac{y_2}{\|y_2\|}\right) \right| \\ &\leq \left(\|y_2\| - \|y_1\| \right) \left\| f\left(\frac{y_1}{\|y_1\|}\right) \right\| + \|y_2\| \|f\|_{\ell_{ip}} \left| \frac{y_1}{\|y_1\|} - \frac{y_2}{\|y_2\|} \right| \\ &\leq \|y_1 - y_2\| \|f\|_\infty + \|f\|_{\ell_{ip}} \left| \frac{\|y_2\|}{\|y_1\|} y_1 - y_2 \right| \end{aligned}$$

$$\begin{aligned} &\leq \|f\|_{\infty} \|y_1 - y_2\| + \|f\|_{\ell^1 p} \left[\left(\frac{\|y_2\|}{\|y_1\|} - 1 \right) \|y_1\| + \|y_1 - y_2\| \right] \\ &\leq \left(\|f\|_{\infty} + 2 \|f\|_{\ell^1 p} \right) \|y_1 - y_2\|. \quad \square \end{aligned}$$

LEMMA 3. If Y is an n -dimensional Banach space and $0 < \varepsilon$, then $S(Y)$ admits an ε -net of cardinality at most $(1 + 4/\varepsilon)^n$.

PROOF. Let M be a subset of $S(Y)$ maximal with respect to " $\|x - y\| \geq \varepsilon$ for all $x \neq y \in M$ ".

Then the sets

$$b(y, \varepsilon/2) \cap S(Y), \quad (y \in M)$$

are pairwise disjoint hence so are the sets

$$b(y, \varepsilon/4), \quad (y \in M).$$

Since these last sets are all contained in $b(1 + \varepsilon/4)$, we have that

$$\text{card } M \cdot \text{vol } b(\varepsilon/4) \leq \text{vol } b(1 + \varepsilon/4)$$

so that

$$\text{card } M \leq \left[\frac{4}{\varepsilon} (1 + \varepsilon/4) \right]^n. \quad \square$$

LEMMA 4. There is a constant $\delta > 0$ so that for each $1 \leq p < 2$ and each $N = 1, 2, \dots$, L_p contains a subspace E such that

$$d(E, \ell_2^N) \leq 2$$

and every projection from L_p onto E has norm at least

$$\delta N^{1/p - 1/2}.$$

PROOF. Given a finite dimensional Banach space X and $1 \leq p < \infty$, let

$$\gamma_p(X) = \inf \{ \|T\| \|S\| : T : X \rightarrow L_p, S : L_p \rightarrow X, ST = I_X \}.$$

So $\gamma_{\infty}(X)$ is the projection constant of X , hence by [4], [12]

$$\gamma_1(\ell_2^N) = \gamma_{\infty}(\ell_2^N) = \sqrt{2n/\pi}.$$

This gives the $p = 1$ case.

For $1 < p < 2$ we reduce to the case $p = 1$ by using Example 3.1 of [2], which asserts that there is a constant $C < \infty$ so that for $1 \leq p < 2$ ℓ_p^{CN} contains a subspace E with $d(E, \ell_2^{\text{N}}) \leq 2$. Since, obviously,

$$d(\ell_p^{\text{CN}}, \ell_1^{\text{CN}}) \leq (\text{CN})^{1 - 1/p}$$

we get that if E is K -complemented in ℓ_p^{CN} , then

$$\begin{aligned} \pi^{-1/2} (2n)^{1/2} = \gamma_1(\ell_2^{\text{N}}) &\leq d(E, \ell_2^{\text{N}}) d(\ell_p^{\text{CN}}, \ell_1^{\text{CN}}) K \\ &\leq 2 (\text{CN})^{1 - 1/p} K. \end{aligned}$$

□

The next piece of background information we need for Theorem 3 is a linearization result which is an easy consequence of the results in [8].

PROPOSITION 1. Suppose $X \subset Y$ and Z are Banach spaces, $f : Y \rightarrow Z$ is Lipschitz, and $U : X \rightarrow Z$ is bounded, linear. Then there is a linear operator $G : Z^* \rightarrow Y^*$ so that $\|G\| \leq \|f\|_{\ell_{ip}}$ and

$$\|R_2 G - U^*\| \leq \|f|_X - U\|_{\ell_{ip}},$$

where R_2 is the natural restriction map from Y^* onto X^* .

REMARK. Note that if Z is reflexive, the mapping $F \equiv G^*|_Y : Y \rightarrow Z$ satisfies $\|F\| \leq \|f\|_{\ell_{ip}}$ and $\|F|_X - U\| \leq \|f|_X - U\|_{\ell_{ip}}$.

PROOF. We first recall some notation from [8]. If Y is a Banach space, $Y^\#$ denotes the Banach space of all scalar valued Lipschitz functions $y^\#$ from Y for which $y^\#(0) = 0$, with the norm $\|y^\#\|_{\ell_{ip}}$. There is an obvious isometric inclusion from Y^* into $Y^\#$. For a Lipschitz mapping $f : Y \rightarrow Z$, Z a normed space, we can define a linear mapping

$$f^\# : Z^* \rightarrow Y^\# \text{ by}$$

$$f^\# z^* = z^* f.$$

Given Banach spaces $X \subset Y$, Theorem 2 of [8] asserts that there are norm one linear projections

$$P_Y : Y^\# \rightarrow Y^*, \quad P_X : X^\# \rightarrow X^*$$

so that

$$P_X R_1 = R_2 P_Y,$$

where R_1 is the restriction mapping from $Y^\#$ onto $X^\#$. Thus if $X \subset Y$, f , U , Z are as in the hypothesis of Proposition 1, the linear mapping $P_Y f^\#$ satisfies

$$\|P_Y f^\#\| \leq \|f\|_{\ell_{1p}}, \quad R_2 P_Y f^\# = P_X R_1 f^\#.$$

Since $U: X \rightarrow Z$ is linear,

$$U^* = P_X U^\#$$

so

$$\begin{aligned} \|R_2 P_Y f^\# - U^*\| &= \|P_X(R_1 f^\# - U^\#)\| \\ &\leq \|R_1 f^\# - U^\#\| = \sup_{z^* \in S(Z^*)} \|R_1 f^\# z^* - U^\# z^*\| \\ &= \sup_{z^* \in S(Z^*)} \|(z^* f)|_X - z^* U\| \leq \|f|_X - U\|_{\ell_{1p}}. \quad \square \end{aligned}$$

The final lemma we use in the proof of Theorem 3 is a smoothing result for homogeneous Lipschitz functions.

LEMMA 5. Suppose $X \subset Y$ and Z are Banach spaces with $\dim X = k < \infty$, $F: Y \rightarrow Z$ is Lipschitz with F positively homogeneous (i.e. $F(\lambda y) = \lambda F(y)$ for $\lambda \geq 0$, $y \in Y$) and $U: X \rightarrow Z$ is linear. Then there is a positively homogeneous Lipschitz mapping

$\tilde{F}: Y \rightarrow Z$ which satisfies

- (1) $\|\tilde{F}|_X - U\|_{\ell_{1p}} \leq (8k + 2) \|F|_{S(X)} - U|_{S(X)}\|_\infty$
- (2) $\|\tilde{F}\|_{\ell_{1p}} \leq 4 \|F\|_{\ell_{1p}}$.

PROOF. For $y \in S(Y)$ define

$$\hat{F}y = \int_{B_X(1)} F(y+x) d\mu(x)$$

where $\mu(\cdot)$ is Haar measure on $X (= \mathbb{R}^k)$ normalized so that

$$\mu(B_X(1)) = 1.$$

For $y_1, y_2 \in S(Y)$ we have

$$\begin{aligned} \|\hat{F}y_1 - \hat{F}y_2\| &\leq \int_{B_X(1)} \|F(y_1 + x) - F(y_2 + x)\| d\mu(x) \\ &\leq \|F\|_{\ell_{ip}} \|y_1 - y_2\| \end{aligned}$$

so

$$\|\hat{F}\|_{\ell_{ip}} \leq \|F\|_{\ell_{ip}}.$$

For $x_1, x_2 \in S(X)$ with $\|x_1 - x_2\| = \delta > 0$ we have, since U is linear, that

$$\begin{aligned} &\|(\hat{F} - U)x_1 - (\hat{F} - U)x_2\| = \\ &\left\| \int_{B_X(1)} F(x_1 + x) d\mu(x) - \int_{B_X(1)} U(x_1 + x) d\mu(x) - \int_{B_X(1)} F(x_2 + x) d\mu(x) + \right. \\ &\quad \left. \int_{B_X(1)} U(x_2 + x) d\mu(x) \right\| \leq \\ &\leq \int_{B_X(x_1; 1) \Delta B_X(x_2; 1)} \|Fx - Ux\| d\mu(x) \leq \\ &\leq \sup_{x \in B_X(2)} \|Fx - Ux\| \mu[B_X(x_1; 1) \Delta B_X(x_2; 1)] \\ &= 2 \sup_{x \in B_X(1)} \|Fx - Ux\| \mu[B_X(x_1; 1) \Delta B_X(x_2; 1)] \quad \left[\begin{array}{l} \text{since } F \text{ is posi-} \\ \text{tively homogeneous} \end{array} \right] \end{aligned}$$

Since

$$B_X(x_1; 1) \Delta B_X(x_2; 1) \subset [B_X(x_1; 1) \sim B_X(x_1; 1-\delta)] \cup [B_X(x_2; 1) \sim B_X(x_2; 1-\delta)]$$

we have if $\delta \leq 1$ that

$$\mu[B_X(x_2; 1) \Delta B_X(x_2; 1-\delta)] \leq 2[1 - (1-\delta)^k]$$

$$\leq 2k\delta$$

and hence for all $x_1, x_2 \in S(X)$ that

$$\|(\hat{F} - U)x_1 - (\hat{F} - U)x_2\| \leq 4k \|F|_{S(X)} - U|_{S(X)}\| \|x_1 - x_2\|$$

whence

$$\|\hat{F}|_{S(X)} - U|_{S(X)}\|_{\ell_{ip}} \leq 4k \|F|_{S(X)} - U|_{S(X)}\|_{\infty}.$$

Finally, note that the positive homogeneity of F implies that

$$\|\hat{F}\|_{\infty} \leq 2 \|F\|_{\ell_{ip}} \quad \text{and} \quad \|\hat{F}|_{S(X)} - U|_{S(X)}\|_{\infty} \leq 2 \|F|_{S(X)} - U|_{S(X)}\|_{\infty}.$$

It now follows from Lemma 2 that the positively homogeneous extension \tilde{F} of \hat{F} satisfies the conclusions of Lemma 5. \square

THEOREM 3. There is a constant $\tau > 0$ so that for all $n = 2, 3, 4, \dots$ and all $1 \leq p < 2$,

$$L(p, n) \geq \tau \left(\frac{\log n}{\log \log n} \right)^{1/p - 1/2}.$$

REMARK. Since $L(\infty, n) \geq L(1, n)$, we get the lower estimate for $L(\infty, n)$ mentioned in the introduction.

PROOF. Given p and n , for a certain value of $N = N(n)$ to be specified later choose a subspace E of L_p with $d(E, \ell_2^N) \leq 2$ and E only $\delta N^{1/p - 1/2}$ -complemented in L_p (Lemma 4). For a value $\varepsilon = \varepsilon(n) > 0$ to be specified later, let A be a minimal ε -net of $S(E)$, so, by Lemma 3,

$$\text{card } A \leq (1 + 4/\varepsilon)^N.$$

One relation among n, N, ε we need is

$$(1.4) \quad (1 + 4/\varepsilon)^N + 1 \leq n.$$

Let $f : A \cup \{0\} \rightarrow E$ be the identify map. Since $d(E, \ell_2^N) \leq 2$, we can by Lemma 2 get a positively homogeneous extension $\tilde{f} : L_p \rightarrow E$ of f so that

$$\|\tilde{f}\|_{\ell_{ip}} \leq 6 L(p, n).$$

Since $\tilde{f}(a) = f(a) = a$ for $a \in A$ and A is an ε -net for $S(E)$, we get that for $x \in S(E)$,

$$\|\tilde{f}(x) - x\| \leq (6 L(p, n) + 1) \varepsilon.$$

Therefore, from Lemma 5 we get a Lipschitz mapping $\hat{f} : L_p \rightarrow E$ which satisfies

$$\|\hat{f}\|_{\ell_{ip}} \leq 24 L(p, n)$$

$$(1.5) \quad \|\hat{f}|_E - I_E\| \leq (8N + 2)(6 L(p, n) + 1)\varepsilon.$$

Note that if

$$(1.6) \quad (8N + 2)(6 L(p, n) + 1)\varepsilon \leq 1/2,$$

(1.5) implies that there is a linear projection from L_p onto E with norm at most $48 L(p, n)$, so we can conclude that

$$L(p, n) > \delta/48 N^{1/p - 1/2}.$$

Finally, we just need to observe that (1.4) and (1.6) are satisfied (at least for sufficiently large n) if we set

$$\varepsilon = \text{Log}^{-2} n, \quad N = \frac{\text{Log } n}{2 \text{ Log Log } n}. \quad \square$$

2. OPEN PROBLEMS.

Besides the obvious question left open by the preceding discussion (i.e. whether the estimate for $L(\infty, n)$ given in Theorem 1 is indeed the best possible), there are several other problems which arise naturally in the present context. We mention here only some of them.

PROBLEM 1. Is it true that for $1 < p < 2$, every subset X of $L_p(0, 1)$, and every Lipschitz map f from X into ℓ_2^k there is an extension \tilde{f} of f from $L_p(0, 1)$ into ℓ_2^k with

$$(2.1) \quad \|\tilde{f}\|_{\ell_{ip}} \leq C(p) \|f\|_{\ell_{ip}} k^{1/p - 1/2}$$

where $C(p)$ depends only on p ?

A positive answer to problem 1 combined with Lemma 1 above will of course provide an alternative proof to the result of Marcus and Pisier [10] mentioned in the introduction. The linear version of problem 1 (where X is a subspace and f a linear operator) is known to be true (cf. [7] and [3]).

PROBLEM 2. What happens in the Marcus-Pisier theorem if $2 < p < \infty$? Is the Lipschitz analogue of Maurey's extension theorem [11] (cf. also [3]) true? In other words, is it true that for $2 < p < \infty$ there is a $c(p)$ such that for every Lipschitz map f from a subset X of $L_p(0, 1)$ into ℓ_2 there is a Lipschitz extension \tilde{f} from $L_p(0, 1)$ into ℓ_2 with

$$\|\tilde{f}\|_{\ell_{ip}} \leq c(p) \|f\|_{\ell_{ip}}?$$

PROBLEM 3. What are the analogues of Lemma 1 in the setting of Banach spaces different from Hilbert spaces? The most interesting special case seems to be concerning the spaces ℓ_∞^n . It is well known that every finite metric space $X = \{x_i\}_{i=1}^n$ embeds isometrically into ℓ_∞^n (the point x_1 is mapped to the n -tuple $\{d(x_1, x_1), d(x_2, x_1), \dots, d(x_n, x_1)\}$ in ℓ_∞^n). Hence in view of Lemma 1 it is quite natural to ask the following. Does there exist for all $\varepsilon > 0$ (or alternatively for some $\varepsilon > 0$) a constant $K(\varepsilon)$ so that for every metric space X with cardinality n there is a Banach space Y with $\dim Y \leq K(\varepsilon) \log n$ and a map f from X into Y so that

$$\|f\|_{\ell_{ip}} \|f^{-1}\|_{\ell_{ip}} \leq 1 + \varepsilon?$$

A weaker version of Problem 3 is

PROBLEM 4. It is true that for every metric space X with cardinality n there is a subset \tilde{X} in ℓ_2 and a Lipschitz map F from X onto \tilde{X} so that

$$(2.2) \quad \|F\|_{\ell_{ip}} \|F^{-1}\|_{\ell_{ip}} \leq K \sqrt{\log n}$$

for some absolute constant K ?

Since for every Banach space Y with $\dim Y = k$ we have $d(Y, \ell_2^k) \leq \sqrt{k}$ (cf. [6]) it is clear that a positive answer to problem 3 implies a positive answer to problem 4. V. Milman pointed out to us that it follows easily from an inequality of Enflo (cf. [1]) that (2.2), if true, gives the best possible estimate. (In the notation of [1], observe that the "m-cube"

$$x_\theta = (\theta_1, \theta_2, \dots, \theta_m) \quad (\theta \in \{-1, 1\}^m)$$

in ℓ_1^m has all "diagonals" of length $2m$ and all "edges" of length 2, so that if F is any Lipschitz mapping from these 2^m points in ℓ_1^m into a Hilbert space, the corollary in [1] implies that

$$\|F\|_{\ell_{ip}} \|F^{-1}\|_{\ell_{ip}} \geq m^{1/2}.)$$

3. APPENDIX.

After this note was written, Yoav Benyamini discovered that Theorem 2 remains valid if ℓ_2 is replaced with any Banach space. He kindly allowed us to reproduce here his proof. The main lemma Benyamini uses is:

LEMMA 6. Let Γ be an indexing set and let $\{e_\gamma\}_{\gamma \in \Gamma}$ be the unit vector basis for $c_0(\Gamma)$. Set

$$A = \{\alpha e_\gamma : 0 \leq \alpha \leq 1; \gamma \in \Gamma\}$$

$$B = \overline{\text{conv } A} \text{ (= positive part of } B_{\ell_1}(\Gamma)).$$

Then

(i) there is a retraction G from $\ell_\infty(\Gamma)$ onto B which satisfies
 $\|G\|_{\text{lip}} \leq 2$

(ii) there is a mapping H from $\ell_\infty(\Gamma)$ into A which satisfies
 $\|H\|_{\text{lip}} \leq 4$ and $He_\gamma = e_\gamma$ for all $\gamma \in \Gamma$.

PROOF. Since the mapping $x \rightarrow x^+$ is a contractive retraction from $\ell_\infty(\Gamma)$ onto its positive cone, $\ell_\infty(\Gamma)^+$; to prove (i) it is enough to define G only on $\ell_\infty(\Gamma)^+$.

For $y \in \ell_\infty(\Gamma)^+$, let

$$g(y) = \inf \{t : \|(y - te)^+\|_1 \leq 1\}$$

where $e \in \ell_\infty(\Gamma)$ is the function identically equal to one and $\|\cdot\|_1$ is the usual norm in $\ell_1(\Gamma)$. Clearly the inf is actually a minimum and $0 \leq g(y) \leq \|y\|_\infty$. Note that

$$|g(y) - g(z)| \leq \|y - z\|_\infty.$$

Indeed, assume that $g(y) \geq g(z)$. Then

$$y - [g(z) + \|y - z\|_\infty e] \leq y - g(z)e + z - y \leq z - g(z)e$$

and hence

$$\|(y - [g(z) + \|y - z\|_\infty e]^+\|_1 \leq 1;$$

that is

$$g(y) \leq g(z) + \|y - z\|_\infty.$$

Now set for $y \in \ell_\infty(\Gamma)^+$

$$G(y) = (y - g(y)e)^+.$$

To prove (ii), it is enough, in view of (i), to define H on B with $\|H\|_B \leq 2$. For $y \in B$, $y = \{y(\gamma)\}_{\gamma \in \Gamma}$, defined H_y by

$$H_y(\gamma) = (2y(\gamma) - 1)^+.$$

For $y \in B$, there is at most one $\gamma \in \Gamma$ for which $y(\gamma) > \frac{1}{2}$, hence $HB \subset A$. Evidently $He_\gamma = e_\gamma$ for $\gamma \in \Gamma$ and $\|H\|_{\mathcal{B}} \leq 2$.

THEOREM 2 (Y. Benyamini). Suppose that X is a metric space, Y is a subset of X with $d(x, y) \geq \varepsilon > 0$ for all $x \neq y \in Y$, Z is a Banach space, and $f : Y \rightarrow Z$ is Lipschitz. Then there is an extension $\tilde{f} : X \rightarrow Z$ of f so that

$$\|\tilde{f}\|_{\text{lip}} \leq (4D/\varepsilon) \|f\|_{\text{lip}}$$

where D is the diameter of Y .

PROOF. Represent

$$Y = \{0\} \cup \{y_\gamma : \gamma \in \Gamma\}$$

and assume, by translating f , that $f(0) = 0$. We can factor f through the subset $C = \{0\} \cup \{e_\gamma : \gamma \in \Gamma\}$ of $\ell_\infty(\Gamma)$ by defining $g : Y \rightarrow C$, $h : C \rightarrow Z$ by

$$g(y_\gamma) = e_\gamma, \quad g(0) = 0$$

$$h(e_\gamma) = f(y_\gamma), \quad h(0) = 0.$$

Evidently,

$$\|g\|_{\text{lip}} \leq 1/\varepsilon, \quad \|h\|_{\text{lip}} \leq D\|f\|_{\text{lip}}.$$

By the non-linear Hahn-Banach theorem, g has an extension to a function $\tilde{g} : X \rightarrow \ell_\infty(\Gamma)$ with $\|\tilde{g}\|_{\text{lip}} = \|g\|_{\text{lip}}$, so to complete the proof, it suffices to extend h to a function $\tilde{h} : B \rightarrow Z$ with $\|\tilde{h}\|_{\text{lip}} = \|h\|_{\text{lip}}$ and apply Lemma 6(ii).

Define for $0 \leq t \leq 1$ and $\gamma \in \Gamma$

$$\tilde{h}(te_\gamma) = th(e_\gamma).$$

If $1 \geq t \geq s \geq 0$ and $\gamma \neq \Delta \in \Gamma$ then

$$\|\tilde{h}(te_\gamma) - \tilde{h}(se_\Delta)\| \leq (t-s)\|h(e_\gamma)\| + s\|h(e_\Delta) - h(e_\gamma)\|$$

$$\leq (t-s)\|h\|_{\text{lip}} + s\|h\|_{\text{lip}} = \|h\|_{\text{lip}} \|te_\gamma - se_\Delta\|_\infty,$$

so $\|\tilde{h}\|_{\text{lip}} = \|h\|_{\text{lip}}$. □

REFERENCES

1. P. Enflo, On the non-existence of uniform homeomorphisms between L_p spaces, Arkiv for Matematik 8 (1969), 195-197.
2. T. Figiel, J. Lindenstrauss and V. Milman, The dimension of almost spherical sections of convex bodies, Acta. Math. 139 (1977), 53-94.
3. T. Figiel and N. Tomczak-Jaegermann, Projections onto Hilbertian subspaces of Banach spaces, Israel J. Math. 33 (1979), 155-171.
4. D. J. H. Garling and Y. Gordon, Relations between some constants associated with finite-dimensional Banach spaces, Israel J. Math. 9 (1971), 346-361.
5. U. Haagerup, The best constant in the Khintchine inequality, Studia Math. 70 (1982), 231-283.
6. F. John, Extremum problems with inequalities as subsidiary conditions, Courant anniversary volume, Interscience N.Y. (1948), 187-204.
7. D. Lewis, Finite dimensional subspaces of L_p , Studia Math. 63 (1978), 207-212.
8. J. Lindenstrauss, On non-linear projections in Banach spaces, Mich. Math. J. 11 (1964), 263-287.
9. J. Lindenstrauss and L. Tzafriri, Classical Banach spaces Vol. I sequence spaces, Ergebnisse n. 92, Springer Verlag, 1977.
10. M. B. Marcus and G. Pisier, Characterizations of almost surely continuous p -stable random Fourier series and strongly stationary processes, to appear.
11. B. Maurey, Un theoreme de prolongment, C. R. Acad. Paris 279 (1974), 329-332.
12. D. Rutovitz, some parameters associated with finite dimensional Banach spaces, J. London Math. Soc. 40 (1965), 241-255.
13. S. J. Szarek, On the best constant in the Khintchine inequality, Studia Math. 58 (1978), 197-208.
14. J. H. Wells and L. R. Williams, Embeddings and Extensions in Analysis, Ergebnisse n. 84 Springer Verlag 1975.

William B. Johnson
 The Ohio State University and
 Texas A & M University

Joram Lindenstrauss
 The Hebrew University of Jerusalem,
 Texas A & M University, and
 The Ohio State University