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- Class roll no. : 26
- Subject : Design and Analysis of Algorithms
- Subject code : TCS 505

Tutorial - 4

Q. For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply:

1) $T(n) = 3T(n/2) + n^2$

→ According to Master's Theorem,

If $T(n) = aT(n/b) + f(n)$; $a \geq 1$, $b > 1$

and $c = \log_b a$, then if:

- $f(n) < n^c \Rightarrow T(n) = \Theta(n^c)$
- $f(n) = n^c \Rightarrow T(n) = \Theta(n^c \log n)$
- $f(n) > n^c \Rightarrow T(n) = \Theta(f(n))$

Here, $a = 3$, $b = 2$

$c = \log_2 3 = 1.58$

$f(n) = n^2$, $n^c = n^{1.58}$

$\because f(n) > n^c \Rightarrow \boxed{T(n) = \Theta(n^2)}$

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2) $T(n) = 4T(n/2) + n^2$

→ Here, $a = 4$, $b = 2$

$c = \log_2 4 = 2$

$f(n) = n^2$, $n^c = n^2$

$\because f(n) = n^c \Rightarrow \boxed{T(n) = \Theta(n^2 \log n)}$

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3) $T(n) = T(n/2) + 2^n$

→ Here, $f(n) = 2^n$ which is not a polynomial.

$\therefore$ Master's Theorem does not apply.

4) $T(n) = 2^n T(n/2) + n^n$

→ Here, $a = 2^n$ which is not a constant.

$\therefore$ Master's Theorem does not apply.

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5) $T(n) = 16\,T(n/4) + n$

→ Here, $a = 16$, $b = 4$

$c = \log_4 16 = 2$

$f(n) = n$, $n^c = n^2$

$\because f(n) < n^c \Rightarrow \boxed{T(n) = \theta(n^2)}$

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6) $T(n) = 2T(n/2) + n \log n$

→ According to extended Master's Theorem,

If $T(n) = aT(n/b) + \theta(n^k \log^p n)$; $a \geq 1$, $b > 1$, $k \geq 0$, $p \in R$

then if:

- $a > b^k \Rightarrow T(n) = \theta(n^{\log_b a})$
- $a = b^k$, then if:
  - ✓ $p > -1 \Rightarrow T(n) = \theta(n^{\log_b a} \log^{p+1} n)$
  - ✓ $p = -1 \Rightarrow T(n) = \theta(n^{\log_b a} \log \log n)$.
  - ✓ $p < -1 \Rightarrow T(n) = \theta(n^{\log_b a})$
- $a < b^k$, then if:
  - ✓ $p \geq 0 \Rightarrow T(n) = \theta(n^k \log^p n)$
  - ✓ $p < 0 \Rightarrow T(n) = \theta(n^k)$

Here, $a = 2$, $b = 2$, $k = 1$, $p = 1$

Now, $b^k = 2^1 = 2 = a$

and $p > -1$

$\Rightarrow T(n) = \theta(n^{\log_2 2} \log^{1+1} n)$

$\Rightarrow \boxed{T(n) = \theta(n \log^2 n)}$

7) $T(n) = 2T(n/2) + n/\log n$

→ Using extended Master's Theorem,

$T(n) = aT(n/b) + \theta(n^k \log^p n)$

Here, $a = 2$, $b = 2$, $k = 1$, $p = -1$

Now, $b^k = 2^1 = 2 = a$

and $p = -1$

$\Rightarrow T(n) = \theta(n^{\log_2 2} \log \log n)$

$\Rightarrow \boxed{T(n) = \theta(n \log\log n)}$

8) $T(n) = 2T(n/4) + n^{0.51}$

→ Here, $a = 2$, $b = 4$

$c = \log_4 2 = 0.5$

$f(n) = n^{0.51}$, $n^c = n^{0.5}$

$\because f(n) > n^c \Rightarrow \boxed{T(n) = \theta(n^{0.51})}$

9) $T(n) = 0.5\, T(n/2) + 1/n$

→ Here, $a < 1$

$\therefore$ Master's Theorem does not apply.

10) $T(n) = 16\, T(n/4) + n!$

→ Here, $a = 16$, $b = 4$

$c = \log_4 16 = 2$

$f(n) = n!$, $n^c = n^2$

$\because f(n) > n^c \Rightarrow \boxed{T(n) = \theta(n!)}$

11) $T(n) = 4T(n/2) + \log n$

$\rightarrow$ Using extended Master's Theorem,

$$T(n) = aT(n/b) + \theta(n^k \log^p n)$$

Here, $a=4$, $b=2$, $k=0$, $p=1$

Now, $b^k = 2^0 = 1 < a$

$\Rightarrow T(n) = \theta(n^{\log_2 4})$

$\Rightarrow \boxed{T(n) = \theta(n^2)}$

12) $T(n) = \sqrt{n}\, T(n/2) + \log n$

$\rightarrow$ Here, $a = \sqrt{n}$ which is not a constant.

$\therefore$ Master's Theorem does not apply.

13) $T(n) = 3T(n/2) + n$

$\rightarrow$ Here, $a=3$, $b=2$

$c = \log_2 3 = 1.58$

$f(n) = n$, $n^c = n^{1.58}$

$\because f(n) < n^c \Rightarrow \boxed{T(n) = \theta(n^{1.58})}$

14) $T(n) = 3T(n/3) + \sqrt{n}$

$\rightarrow$ Here, $a=3$, $b=3$

$c = \log_3 3 = 1$

$f(n) = \sqrt{n}$, $n^c = n$

$\because f(n) < n^c \Rightarrow \boxed{T(n) = \theta(n)}$

15) $T(n) = 4T(n/2) + kn$

$\rightarrow$ Here, $a=4$, $b=2$

$c = \log_2 4 = 2$

$f(n) = kn$, $n^c = n^2$

$\because f(n) < n^c \Rightarrow \boxed{T(n) = \theta(n^2)}$

16) $T(n) = 3T(n/4) + n \log n$

$\rightarrow$ Using extended Master's Theorem,

$$T(n) = aT(n/b) + \theta(n^k \log^p n)$$

Here, $a=3$, $b=4$, $k=1$, $p=1$

Now, $b^k = 4^1 = 4 > a$

and $p \geq 0$

$\Rightarrow T(n) = \theta(n^1 \log^1 n)$

$\Rightarrow \boxed{T(n) = \theta(n \log n)}$

17) $T(n) = 3T(n/3) + n/2$

$\rightarrow$ Here, $a=3$, $b=3$

$c = \log_3 3 = 1$

$f(n) = n/2 \approx n$, $n^c = n$

$\because f(n) = n^c \Rightarrow \boxed{T(n) = \theta(n \log n)}$

18) $T(n) = 6T(n/3) + n^2 \log n$

$\rightarrow$ Using extended Master's Theorem,

$$T(n) = aT(n/b) + \theta(n^k \log^p n)$$

Here, $a=6$, $b=3$, $k=2$, $p=1$

Now, $b^k = 3^2 = 9 > a$

and $p \geq 0$

$\Rightarrow T(n) = \theta(n^2 \log^1 n)$

$\Rightarrow \boxed{T(n) = \theta(n^2 \log n)}$

19) $T(n) = 4T(n/2) + n/\log n$

$\rightarrow$ Using extended Master's Theorem,

$$T(n) = aT(n/b) + \theta(n^k \log^p n)$$

Here, $a=4$, $b=2$, $k=1$, $p=-1$

Now, $b^k = 2^1 = 2 < a$

$\Rightarrow T(n) = \theta(n^{\log_2 4}) \Rightarrow \boxed{T(n) = \theta(n^2)}$

20) $T(n) = 64T(n/8) - n^2 \log n$

$\rightarrow$ Here, $f(n) = -n^2 \log n$ which is negative.

$\therefore$ Master's Theorem does not apply.

21) $T(n) = 7T(n/3) + n^2$

$\rightarrow$ Here, $a = 7$, $b = 3$

$c = \log_3 7 = 1.77$

$f(n) = n^2$, $n^c = n^{1.77}$

$\because f(n) > n^c \Rightarrow \boxed{T(n) = \Theta(n^2)}$

22) $T(n) = T(n/2) + n(2 - \cos n)$

$\rightarrow$ Here, the Regularity condition is violated.

$\{a f(n/b) \leq c f(n) ; c < 1\}$

$\therefore$ Master's Theorem does not apply.