As1 q1

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Problem 1

$(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space

show $\mathbb{P}$ continuous from below

- Let's define the set:

$$B_1, B_2, \dots, B_k, \dots, B_l, \dots B_n$$

where $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$

- $B_k \cap B_l = \emptyset$ (The sets are disjoint where $k \neq l$

proof:

assume $B_k \cap B_l$ not $= \emptyset$

implies: $\exists x \in B_k \cap B_l$

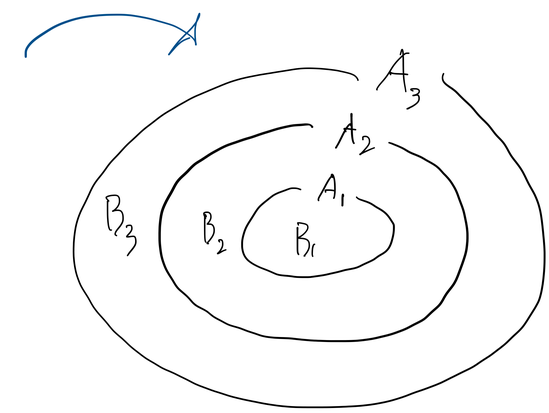
$\hookrightarrow x \in B_k \equiv x = k$

$\hookrightarrow x \in B_l \equiv x = l$

but $k \neq l$

$\therefore$ The sets are disjoint

based on the diagram



Lets say $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right)$

$$= \sum_{n=1}^{\infty} \mathbb{P}(B_n)$$

$$= \lim_{n \to \infty} \left(\sum_{n=1}^{N} \mathbb{P}(B_n) \right)$$

$$= \lim_{n \to \infty} \left(\sum_{n=1}^{N} \mathbb{P}\left(A_n \setminus \bigcup_{k=1}^{n-1} A_k\right) \right)$$

$$= \lim_{n \to \infty} \left(\sum_{n=1}^{N} \left(\underbrace{\mathbb{P}(A_n)}_{\text{remains the same ...}} - \underbrace{\mathbb{P}\left(\bigcup_{k=1}^{n-1} A_k\right)}_{\to 0\ ?} \right) \right)$$

$$= \lim_{n \to \infty} \mathbb{P}(A_n)$$

$$\therefore \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mathbb{P}(A_n)$$

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question 2

urn contains 5 white balls, 7 black balls.

One ball is drawn and not put back. Second ball is drawn

1)

$$\Omega = \{ \omega = (\omega_1, \omega_2) \mid \omega_1, \omega_2 \in \{WH \cup BL\}, \ \omega_1 \neq \omega_2 \}$$

where $WH = \{ h_1, \dots, h_5 \}$

where h_i are the white balls where $i = 1, 2, \dots, 5$

$BL = \{ b_1, \dots, b_7 \}$

where b_j are the black balls where $j = 1, 2, \dots, 7$

$$\mathcal{A} = \mathcal{P}(\Omega)$$

$$P\{\omega\} = \frac{1}{\#\Omega} = \frac{1}{N(N-1)}$$

where N = # of balls in the urn

$N-1$ = # of balls in the urn after one ball is taken out

So

$$P\{\omega\} = \frac{1}{N(N-1)} = \frac{1}{12(12-1)} = \frac{1}{(12)(11)} = \frac{1}{132}$$

chance of pulling two different balls

2)

$$W_1 = \{ \omega = (\omega_1, \omega_2) \mid \omega_1 \in \{WH\}, \ \omega_1 \neq \omega_2 \}$$

$$W_2 = \{ \omega = (\omega_1, \omega_2) \mid \omega_2 \in \{WH\}, \ \omega_1 \neq \omega_2 \}$$

$$B_1 = \{ \omega = (\omega_1, \omega_2) \mid \omega_1 \in \{BL\}, \ \omega_1 \neq \omega_2 \}$$

$$B_2 = \{ \omega = (\omega_1, \omega_2) \mid \omega_2 \in \{BL\}, \ \omega_1 \neq \omega_2 \}$$

3) note: intuitively, the events for the second ball depend on the first ball

$$P(W_2 \mid W_1) = \frac{P(W_2) \cap P(W_1)}{P(W_1)}$$ (probability of the second ball being white given the first ball was white)

well:

$$P(W_1) = \frac{\#\ WH}{\#\ \text{total balls}} = \frac{5}{12}$$

$$P(W_1 \mid W_2) = \frac{\#\ WH - 1 \text{ white ball that was removed}}{\#\ \text{total balls} - 1} = \frac{5-1}{12-1} = \frac{4}{11} = \frac{4}{11} = 0.36$$

$P(W_2 \mid B_1)$ probability of 2nd ball being white given 1st is black

$$P(B_1) = \frac{\#\ BL}{\#\ \text{total balls}} = \frac{7}{12}$$

$$P(W_2 \mid B_1) = \frac{\#\ WH}{\#\ \text{total balls} - 1} = \frac{5}{11} = 0.45$$

$$P(W_2) = \frac{4}{11} \text{ or } \frac{5}{11}$$

4) are W_1 and W_2 independent?

No, W_1 and W_2 are not independent

We know $P(W_1) = \frac{5}{12}$ $\longrightarrow$ no dependency on W_2

(as mentioned before)

For $P(W_2)$, it's value is determined by the event that preceeds it

if $P(W_2 \mid W_1)$

then $P(W_2) = \frac{4}{11}$

if $P(W_2 \mid B_1)$

then $P(W_2) = \frac{5}{11}$

The probability changes as a diffent dependency is used, $\therefore$ W_1 and W_2 are not independent events

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Problem 3)

Show that the given $\underline{P}$ is not a probability measure

$P(\mathbb{N}) = 1$

probability of natural numbers $= 1$

$A_n = \{1, \dots, n\}$

A_n is like natural numbers

each element has an equally likely chance to occur

The natural numbers is a growing set

Using continuity we will derive a contradiction.

If we assume $\underline{P}$ is a probability measure, then

$$\boxed{\underline{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{k \to \infty} P(A_k)}$$

Should hold

$\underline{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \underline{P}(\mathbb{N}) = 1$ (because A_n is really just a subset of natural numbers)

Because of how probability of natural numbers is equal to 1

however, RS sugests this

$\lim_{k \to \infty} P(A_k) = P(A_\infty) = \infty \neq 1$

The ∞^{th} natural number

$\therefore$ by contradiction, The described $\underline{P}$ is not a probability measure.

As1 q4

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Question 4

- probability of child with brown eyes $= 3/4$
- 4 children
- $B_1, \dots, B_i, \dots, B_4$ are independent events say the ith child has Brown eyes

Given: $P(B_i) = \frac{3}{4}$

probability of blue eyes

$\underbrace{i = 1 \dots 4}$

assumed independent events (events won't influence each other)

Probability space:

$\Omega = \{B_1, B_2, B_2, B_4\}$

$\mathcal{A} = \mathcal{P}(\Omega)$

$P(B_i) = \frac{3}{4}$

$P(B_i^c) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$

1) the event in which at least 3 children have brown eyes $:= B_{3\geq}$

$B_{3\geq} = B_4^c \cup B_3^c \cup B_2^c \cup B_1^c$

2) $(B_{3\geq} \mid B_1)$

— B_i are independent events which implies:

$P(C \mid A) = P(C)$ if C, A independent

which means:

$P(B_{3\geq} \mid B_1) = P(B_{3\geq}) = P(B_4^c \cup B_3^c \cup B_2^c \cup B_1^c)$

Via inclusion exclusion:

Singletons

pairs

triples

quadruples

$$= \underbrace{P(B_4^c) + P(B_3^c) + P(B_2^c) + P(B_1^c)} - \Big[\underbrace{P(B_4^c) \cap P(B_3^c)} + \underbrace{P(B_4^c) \cap P(B_2^c)} + \underbrace{P(B_4^c) \cap P(B_1^c)} + \underbrace{P(B_3^c) \cap P(B_2^c)} + \underbrace{P(B_3^c) \cap P(B_1^c)}\Big] + \underbrace{P(B_4^c) \cap P(B_3^c) \cap P(B_2^c)} + \underbrace{P(B_4^c) \cap P(B_3^c) \cap P(B_1^c)} + \underbrace{P(B_4^c) \cap P(B_2^c) \cap P(B_1^c)} + \underbrace{P(B_3^c) \cap P(B_2^c) \cap P(B_1^c)} - \Big[\underbrace{P(B_4^c) \cap P(B_3^c) \cap P(B_2^c) \cap P(B_1^c)}\Big]$$

$$= 4 \cdot \frac{27}{64} - \left[5\left(\frac{27}{64}\right)^2\right] + 4\left(\frac{27}{64}\right)^3 - \left[\left(\frac{27}{64}\right)^5\right]$$

≈ 1

As1 q5

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Problem 5

- 3 cards, with 2 sides
- 1 card, green on both sides
- 1 card, red on both sides

1) before defining our events, lets define our probability space:

$$\Omega = \{ \omega_i \mid S_j \in \omega_i \mid S_1 \in \omega_1 = G,\ S_2 \in \omega_1 = G,\ S_1 \in \omega_2 = R,\ S_2 \in \omega_2 = R,\ S_1 \in \omega_3 = G,\ S_2 \in \omega_3 = R \}$$

where : ω_i is the ith card

S_j is the jth side of a card

$i = 1, 2, 3$

$j = 1, 2$

R = the outcome of the colour red

G = the outcome of the colour green

$\mathcal{F}(\Omega)$

defining events:

$$C_i = \{ \omega_i \}$$

$$G_1 = \{ S_1 \in \omega_1 \}$$

$$G_2 = \{ S_2 \in \omega_1,\ S_2 \in \omega_3 \}$$

Conditional probability values

$P(G_j \mid C_i) \longrightarrow$ chance of green on j^{th} side of a card, given i^{th} card being drawn

$P(G_j)$ will vary based on which chard C_i is pulled

if $i = 1$ (card 1 is pulled)

Side 1 is picked $\longrightarrow j = 1 \longrightarrow P(G_j \mid C_i) = \frac{1}{2} \longleftarrow$ one out of two chances to pull green on 1 first side of remaining cards

$j = 2 \longrightarrow P(G_j \mid C_i) = 0$

if $i = 2$ (card 2 is pulled)

$j = 1 \longrightarrow P(G_j \mid C_i) = 1$

$j = 2 \longrightarrow \ldots = \frac{1}{2}$

if $i = 3$ (card 3 is pulled)

$j = 1 \longrightarrow \ldots = \frac{1}{2}$

$j = 2 \longrightarrow \ldots = \frac{1}{2}$

2) $P(G_2 \mid G_1) \longrightarrow$ two cases : C_1 or C_3 are pulled for G_1 (does not include C_2, as we only observe possible green events)

if C_1 pulled :

$$P(G_2 \mid G_1) = 0$$

based on how I described Ω:

When ω_1 is removed from the set Ω, ω_2 and ω_3 are left.

ω_3 is defined as having a green value only on the first side $(j = 1)$.

If we only pick side two for our outcome, we will never get green.

If C_3 is pulled

$$P(G_2 \mid G_1) = \frac{1}{2}$$

as the remaining set would contain only G for $S_1 \in \omega_1$ and R for $S_1 \in \omega_2$