# Inductive Conformal Martingales for Change-Point Detection


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# Quickest Change-Point Detection

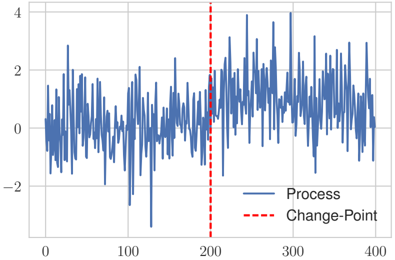
# Change-Point Problem

The **change-point detection problem** is formulated as following:

- $z_1, z_2, \ldots, z_n, \ldots$ are independent random variables;
- $z_1, z_2, \ldots, z_{\theta-1}$ are each distributed according to a distribution $f_0(z)$;
- $z_\theta, z_{\theta+1}, \ldots$ are each distributed according to a distribution $f_1(z)$;
- Change-Point (CP) $2 \leq \theta \leq \infty$ is unknown;

The task is to find the **stopping time** $\tau$, such that

- **Probability of False Alarm:** $\mathbb{P}(\tau \leq \theta) \leq \alpha$ for all $\theta$
- **Mean Delay:** $\mathbb{E}(\tau - \theta \mid \tau > \theta) \to \min_\tau$



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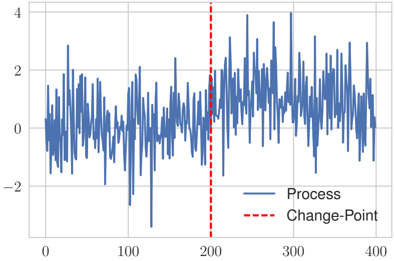

# Quickest Change-Point Detection

- **Important applications:** healthcare, security, production, equipment maintenance, Internet of things, traffic routing, etc.
- **Challenges**:
  - Large volumes of time-series data;
  - Complex, intractable or unknown dynamic models;
  - Apparent non-stationary and quasi-periodicity.

Automatic Change-Point detection is critical in today's world where the sheer volume of data makes it impossible to tag change-point manually

# EEG for a human

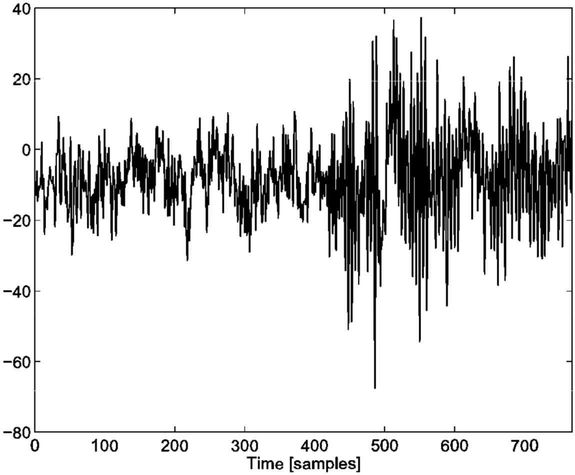


Figure: EEG for a human in a room, where the light is turned of at time 387

# Eathquake Prediction

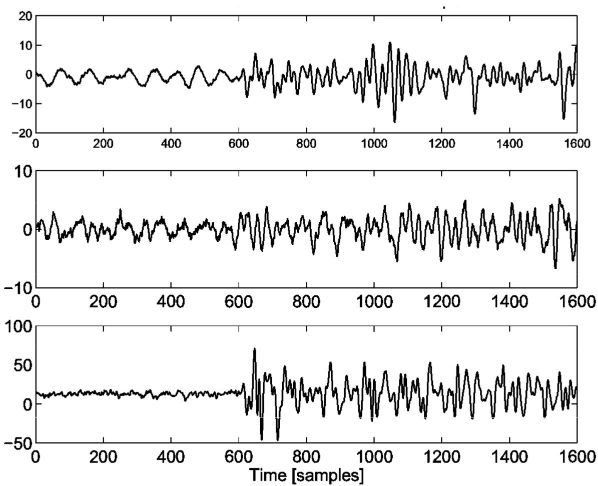


Figure: Seismological signals. Earthquake starts at time 600

# Standard Approaches

**Standard algorithms** for quickest Change-Point detection: Cumulative Sum (CUSUM), Shiryaev-Roberts, Posterior Probability statistics.

- Based on **likelihood ratio** $\frac{L_n^\theta}{L_n}$, where

$$L_n^\theta = \prod_{i=1}^{\theta-1} f_0(z_i) \prod_{i=\theta}^{n} f_1(z_i) \tag{1}$$

  the likelihood of observations $z_1, \ldots, z_n$ when $\theta \in [1, n]$, and by

$$L_n = \prod_{i=1}^{n} f_0(z_i) \tag{2}$$

  the likelihood of observations $z_1, \ldots, z_n$ without Change-Point.
- **Our goal** is to provide distribution-free algorithm for Change-Point detection, that is based on Conformal Martingales.

# Conformal Martingales

# Conformal Predictors

**Conformal martingales:**

- **Non-Conformity measure** (the measure of strangeness)
  $\alpha_i = A(z_i, \{z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n\})$
  - Example: distance to the nearest neighbor.

- **P-values**

$$p_n = p(z_n, z_{n-1}, \dots, z_1) = \frac{\#\{i : \alpha_i > \alpha_n\} + U\#\{i : \alpha_i = \alpha_n\}}{n}$$

  where $U \sim Uniform[0, 1]$ independent of $z_1, z_2, \dots$
  - Small p-values $\Rightarrow$ strange objects (other distribution)

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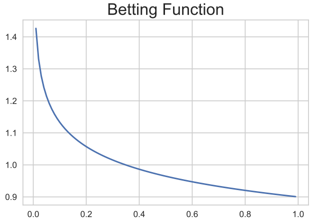
# Conformal Martingales

### Theorem

*If p-values are* **not independent and uniformly distributed** *in* $[0, 1]$*, then* $z_1, \dots, z_n, \dots$ **don't satisfy the i.i.d.** *assumption. [Vovk et al., 2003]*

- **Conformal Martingale:** $S_n = \prod_{i=1}^{n} g_i(p_i)$, $n = 1, 2, \dots,$ where $g_i(p_i) = g_i(p_i \mid p_1, \dots, p_{i-1})$ is what we call a **betting function**.

1. Strange objects (not i.i.d., Change-Point);
2. p-values are not Uniform;
3. penalize with betting function;
4. Martingale grows.



# Betting Functions

- **Constant** Betting function;

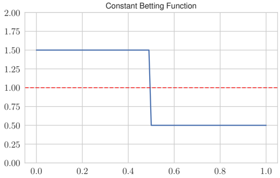
$$g(p) = \begin{cases} 1.5, & \text{if } p \in [0, 0.5), \\ 0.5, & \text{if } p \in [0.5, 1]. \end{cases}$$

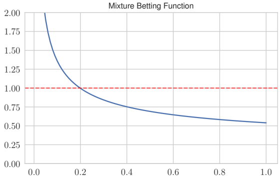
- **Mixture** Betting Function;

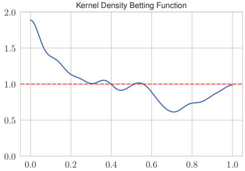
$$g(p) = \int_0^1 \varepsilon p^{\varepsilon - 1} d\varepsilon.$$

- **Kernel** Density Betting Functions;

$$g_n(p_n) = K_{p_{n-L}, \dots, p_{n-1}}(p_n).$$



# Contribution: Inductive Conformal Martingales for Change-Point detection

**Conformal Martingale:**

+ Theoretically valid;
– Computationally very inefficient;
– Wasn't designed for quickest change-point detection.

**Inductive Conformal Martingales:**

- Computation of **non-conformity scores** with fixed training set:

$$\alpha_i = A(z_i, \{z^*_{-(m-1)}, \ldots, z^*_0\});$$

- **Increase efficiency** a lot: no need in *Leave-One-Out*-like method;
- Save validity.

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# Validity of Inductive Conformal Martingales

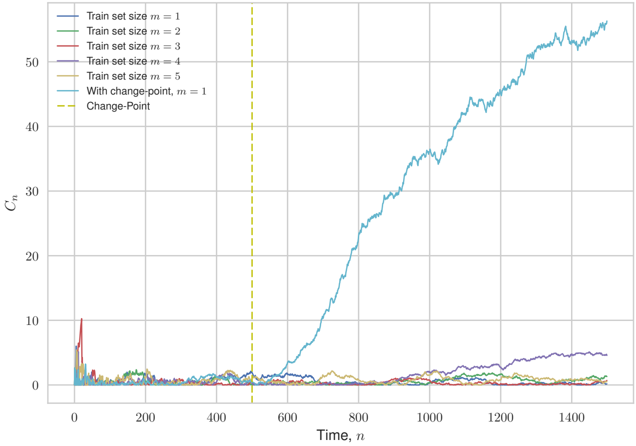


Figure: Validity Test of ICM: case of small train sets

# Contribution: adaptation of ICMs for Change-Point detection

The **stopping rule** is $\tau_{\mathrm{CM}}^{S_n} = \inf\{n : S_n \geq h\}$ for some constant $h$.

**Problems:**

- It takes a lot of time to reach the change-point;
- Martingale can decrease to $-\infty$.

**We propose:**

- "Cut" the martingale:

$$C_n = \max\{0, C_{n-1} + \log(g_n(p_n))\}, \quad (3)$$

  where $p_n$ is a p-value, and $g_n$ is a betting function
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+ This modification performs better in terms of the *mean delay* until Change-Point detection;

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# ICM example

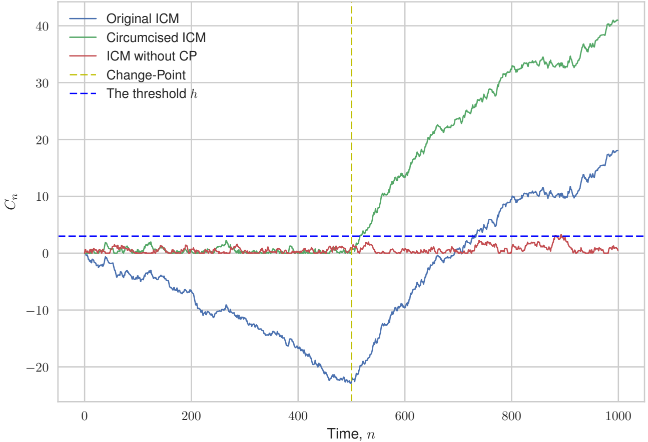
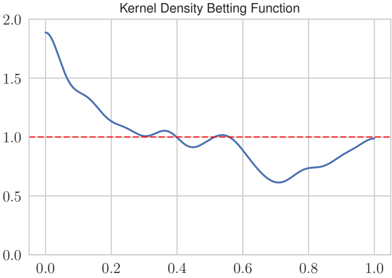



Figure: Example of the ICM in case of data with Change-Point (at $\theta = 500$) and without Change-Point

# Contribution: Kernel Betting Function

**Kernel Density Betting Function** ([Fedorova et al., 2012], estimated from p-values seen before):

- – Computationally inefficient;
- – Need some time to update from uniform distribution after change-point;
- + Theoretically has the best possible growing rate [Fedorova et al., 2012].




We propose to **precompute** it on p-values for data with an example of a change-point:

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- + Faster Change-Point detection.

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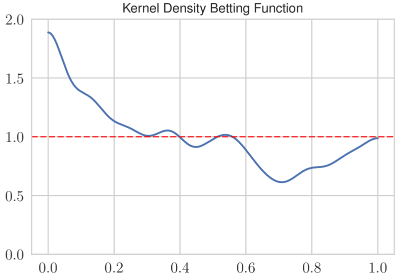


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# Experiments

# Oracles for Change-Point detection

**Classical approach** (CUSUM, Shiryaev-Roberst):

- **Optimal** in terms of Mean Delay;
- Need to know the data model.

**Conformal Martingales:**

- Only i.i.d assumption;
- We expect them to be worse.

**Oracles**, based on standard Change-Point detection algorithms:

- We assume the distribution class to be known;
- We don't know the parameters;
- Likelihood:

$$\overline{L}_n = \int \prod_{i=1}^{n} f(z_i \mid \mathbf{c}_0) q(\mathbf{c}_0) d\mathbf{c}_0. \tag{4}$$

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# Experimental Setup

**Non-Conformity Measures:**

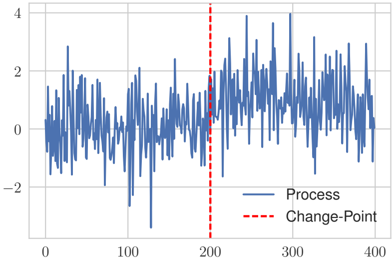
- *k Nearest Neighbors (kNN NCM)* — average distance to $k$ nearest neighbors.
- *Likelihood Ratio (LR NCM)* — the value of likelihood ratio $\frac{f_1(x)}{f_0(x)}$.

**Parameters**:

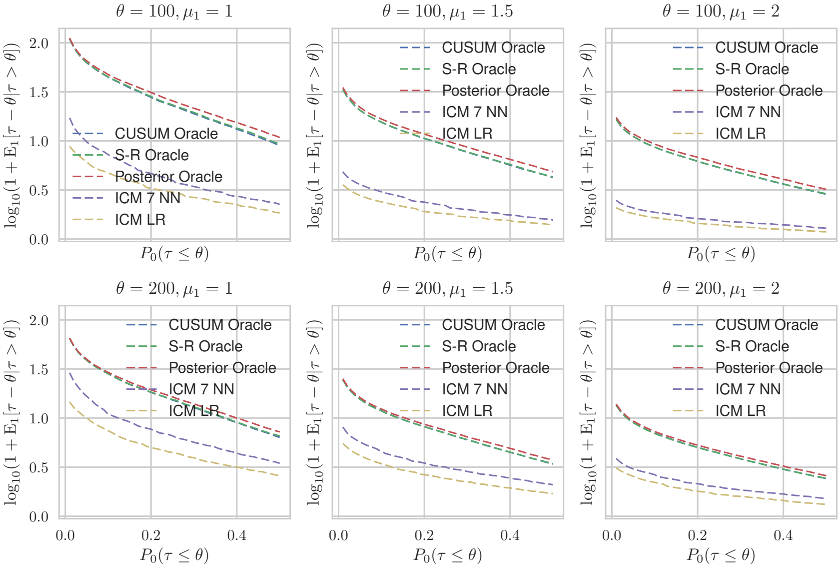
- $f_0 \sim \mathcal{N}(0, 1)$;
- $f_1 \sim \mathcal{N}(m_1, 1)$, $m_1 \in \{1, 1.5, 2\}$;
- $\theta \in [100, 200]$

As a **performance characteristics** we use:

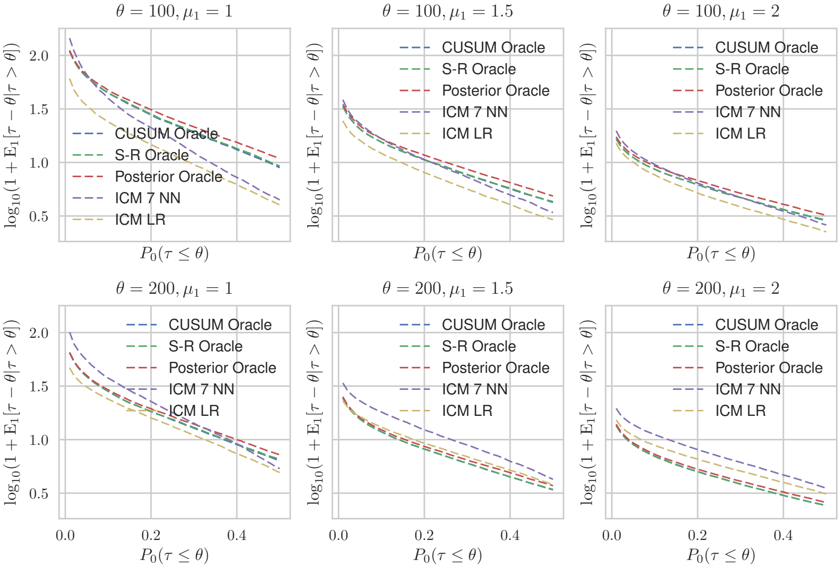
- **Mean delay** until Change-Point detection $\mathbb{E}_1(\tau - \theta \mid \tau > \theta)$,
- **Probability of False Alarm (FA)** $\mathbb{P}_0(\tau \leq \theta)$.



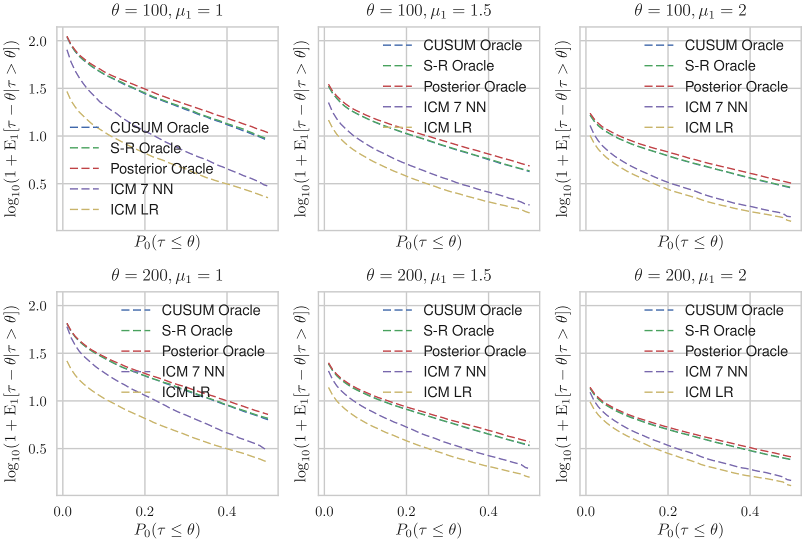
# Mixture Betting Function. Comparison with Oracles



# Kernel Density Betting Function. Comparison with Oracles



# Precomputed Kernel Density Betting Function. Comparison with Oracles



# Kernel and Precomputed Kernel Betting Functions

Table: Mean Delay. Kernel Density Betting Function

| Param.\Probab. of FA | 5% | | | | | 10% | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | ICM LR | ICM kNN | CUSUM Oracle | S-R Oracle | Posterior Oracle | ICM LR | ICM kNN | CUSUM Oracle | S-R Oracle | Posterior Oracle |
| $\theta = 200, \mu_1 = 1$ | **30.06** | 54.14 | 37.78 | 37.80 | 38.73 | **22.90** | 36.57 | 27.24 | 27.24 | 28.25 |
| $\theta = 200, \mu_1 = 1.5$ | 15.44 | 22.02 | 14.62 | **14.52** | 15.16 | 12.08 | 17.13 | 10.85 | **10.81** | 11.36 |
| $\theta = 200, \mu_1 = 2$ | 10.00 | 12.81 | 8.02 | **7.98** | 8.30 | 7.83 | 10.15 | 6.00 | **5.97** | 6.28 |

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| $\theta = 200, \mu_1 = 1$ | **14.14** | 28.70 | 37.78 | 37.80 | 38.73 | **9.65** | 18.91 | 27.24 | 27.24 | 28.25 |
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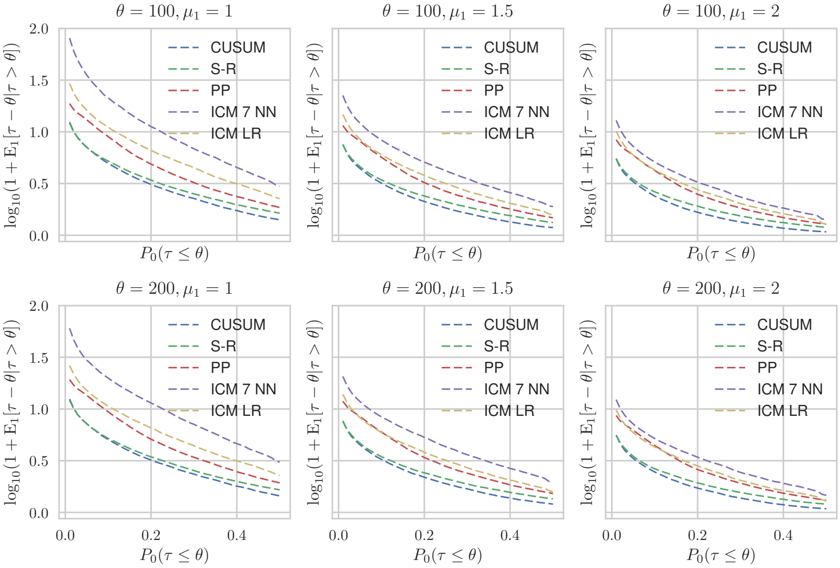
# Comparison with Optimal detectors



# Conclusion

- An **adaptation** of Conformal Martingales for change-point detection problem was proposed;
- We demonstrated the **efficiency** of this approach by comparing it with natural oracles, which are likelihood-based change-point detectors;
- We **proposed and compared** several approaches to calculating a **betting function**;
- We also compared Inductive Conformal Martingales with methods that are optimal for known pre- and post-CP distributions, such as CUSUM, Shiryaev-Roberts and Posterior Probability statistics.
- Paper: [Volkhonskiy et al., 2017]

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