Self-study: Linear Algebra

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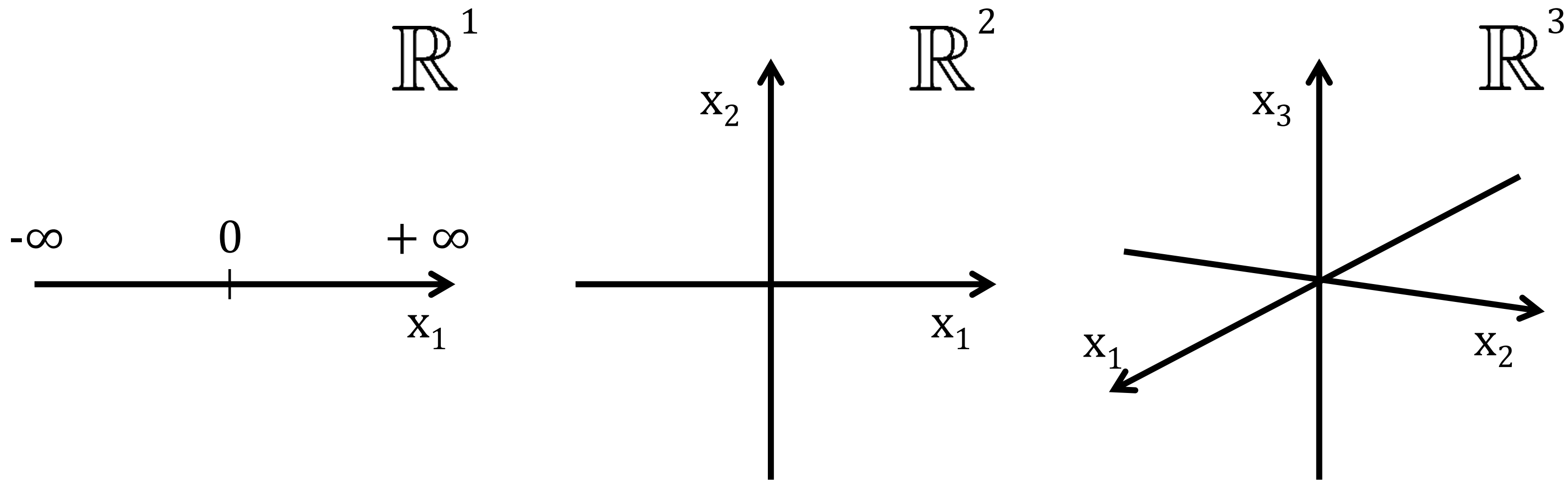
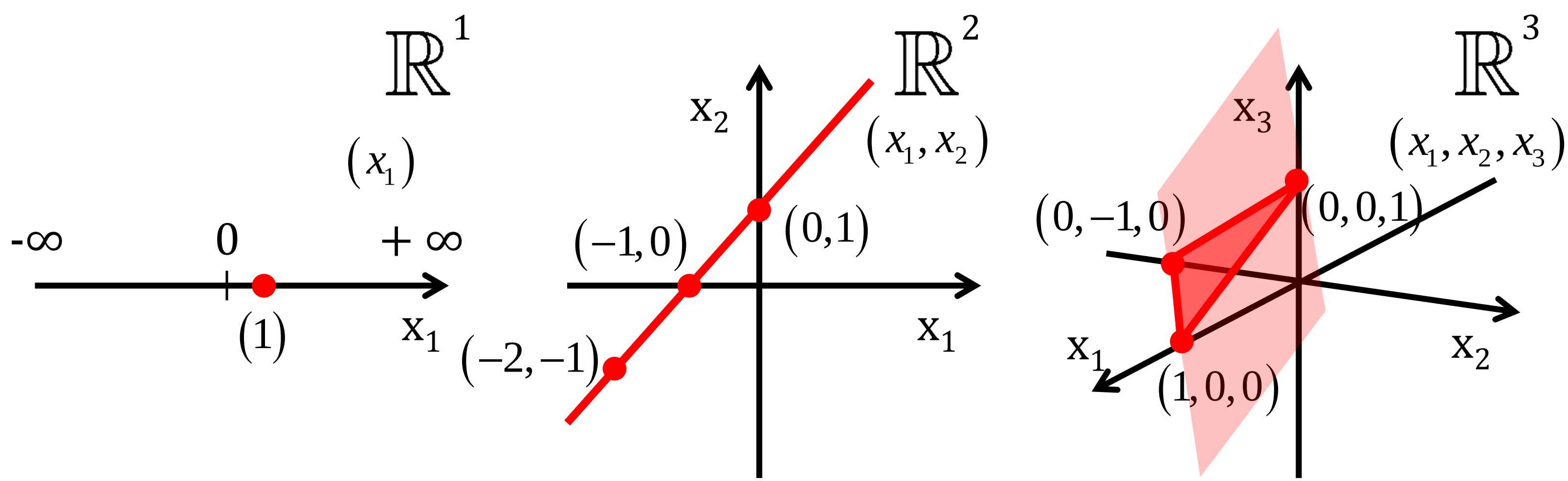
Linear equations and geometry



Equations: $x_1 = 1$ $-x_1 + x_2 = 1$ $x_1 - x_2 + x_3 = 1$



What is a system of equations?

- Set of equations which should be solved **simultaneously**! This means that **a solution to the system must satisfy all the equations in the system**, otherwise it is not a solution to the system.
- Ex.:

Solves the first equation ☺! Second equation: $-3 \neq 3$

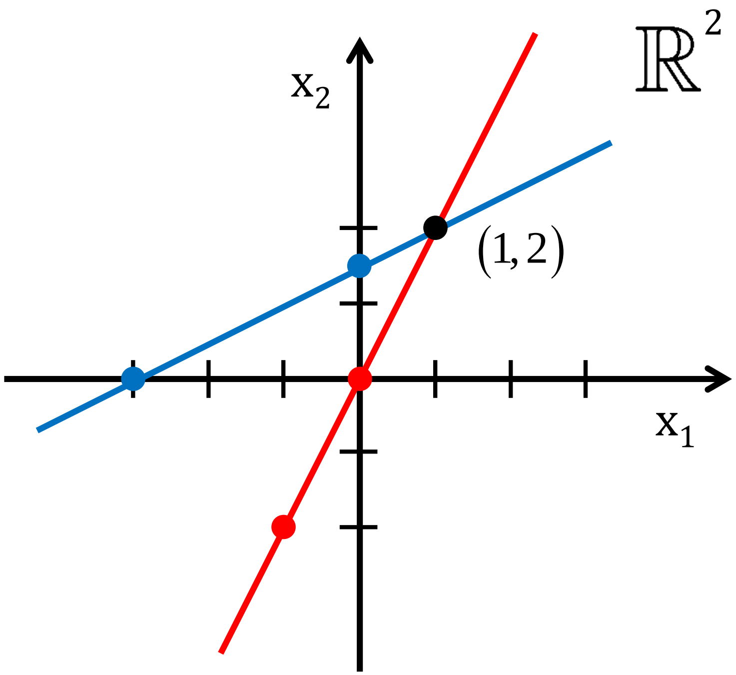
$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

$\rightarrow (-1,-2)$ However, NOT a solution to the system ☹!

$\searrow (1,2)$ Solves both the first and second equation ☺!

The row picture

$$2x_1 - x_2 = 0$$
$$-x_1 + 2x_2 = 3$$



The vector notation

First element == LHS of first equation

$$\boxed{2x_1 - x_2} = 0$$
$$\boxed{-x_1 + 2x_2} = 3$$

$$\Rightarrow \begin{bmatrix} \boxed{2x_1 - x_2} \\ \boxed{-x_1 + 2x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Second element == LHS of second equation

Some notations:

$$c\mathbf{v}_1 = c\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix} \qquad \mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \end{bmatrix}$$

Leads to:

$$\begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ 2x_2 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Inspired by the Video Lectures of Gilbert Strang (MIT Open Courseware):
https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/

Matrix-Vector form

$$\begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ 2x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

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Matrix-Vector form

$$\begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ 2x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Matrix-Vector form

$$A\mathbf{x} = \mathbf{b}$$

Algebraic form

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Matrix-Vector form

$$\begin{array}{cc} & \text{columns/ variables} \rightarrow \\ \text{rows / equations} \downarrow & \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \end{array}$$

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 3 \end{aligned} \quad \Rightarrow \quad A = \left[a_{i,j} \right] = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Matrix: Array of ordered numbers. In this course, they represent the constant coefficients of the linear system.
m denotes the number of equations, where **i** refers to the **i-th** equations.
n refers to the number of variables, where **j** denotes the **j-th** variables.

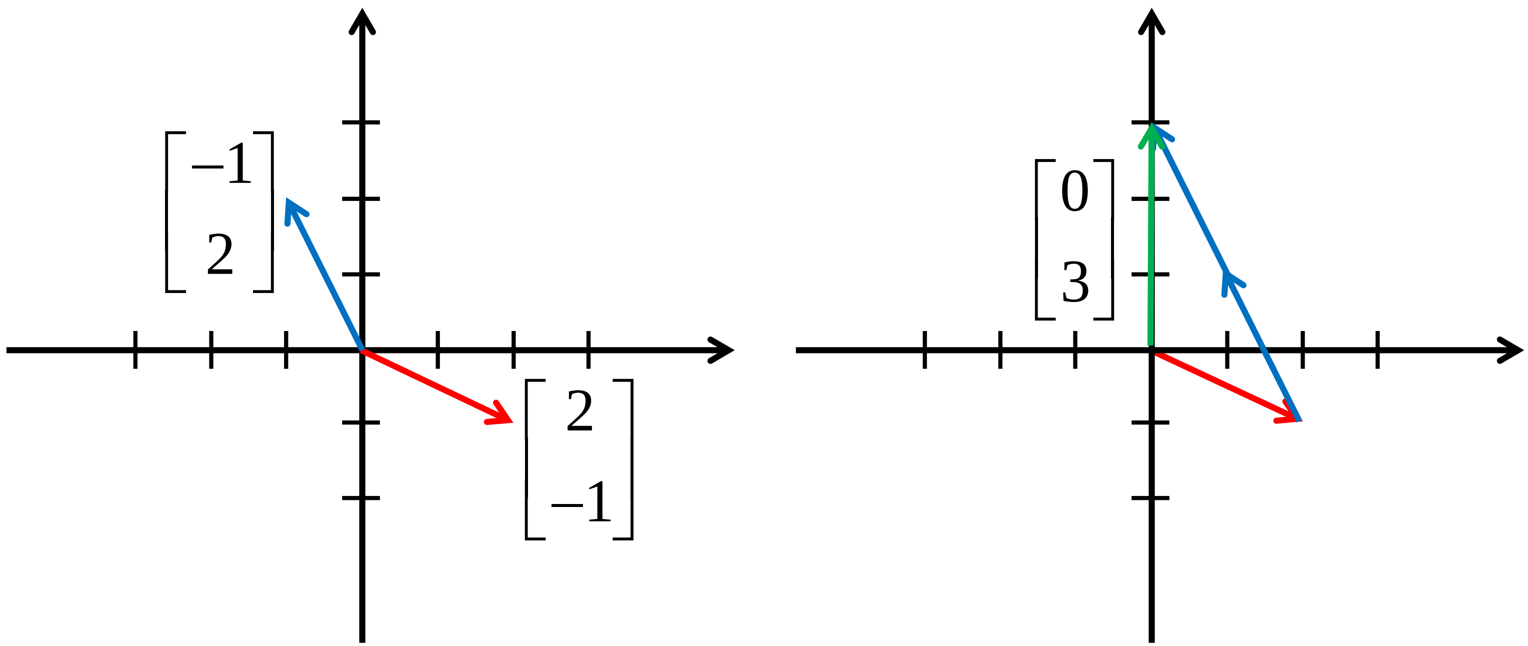
The column picture

RHS

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 3 \end{aligned} \quad \Rightarrow \quad x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Solution was: $(1, 2)$

Solution at the right hand side (RHS) of the equations is simply a **linear combination of the two vectors!**



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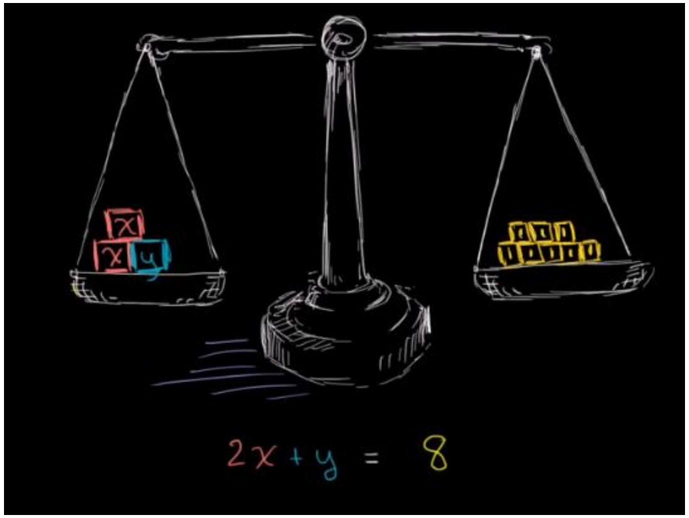
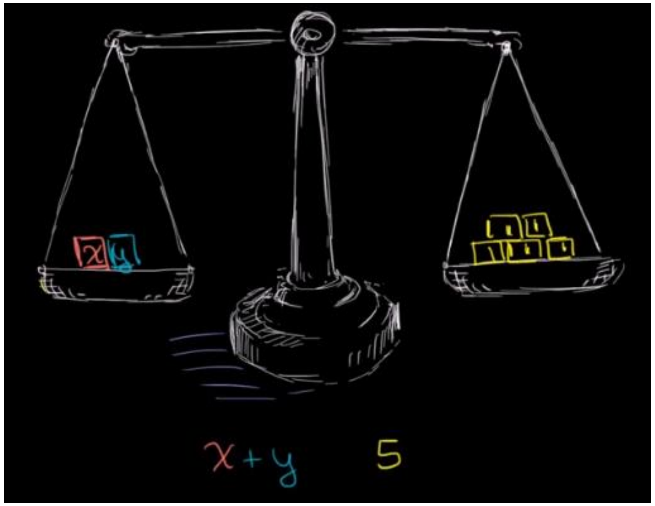
Subtracting equations from another

Scale one:



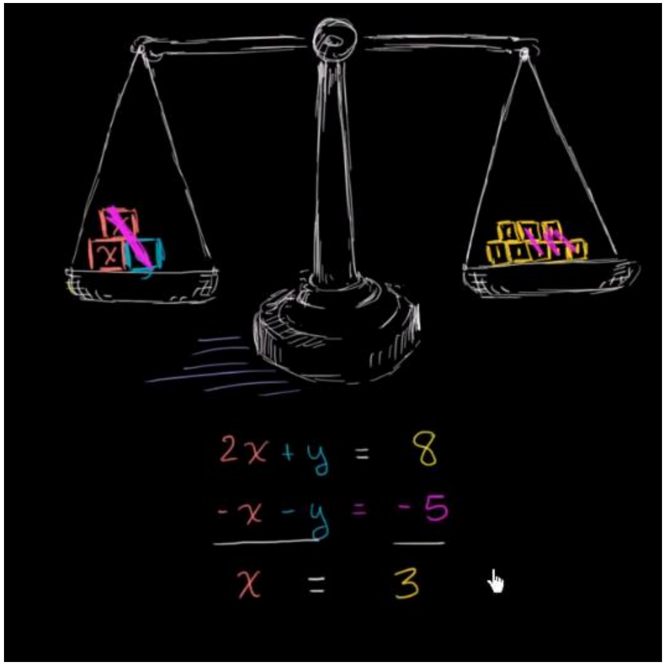
Scale two:



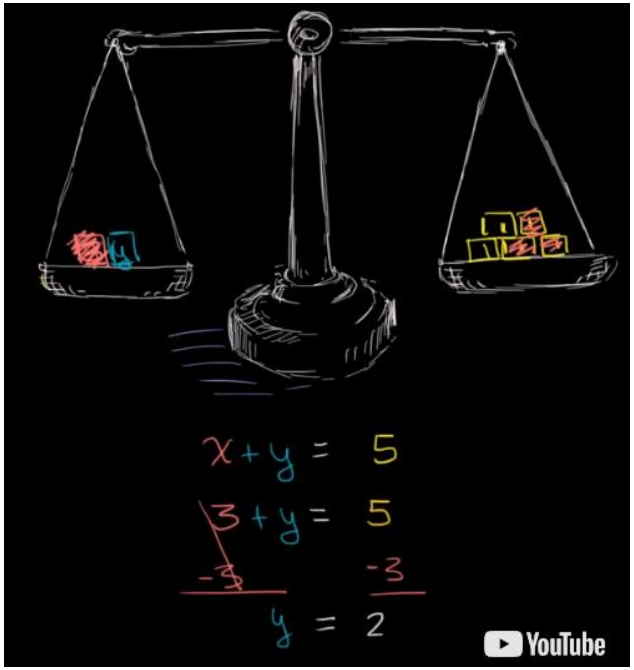
https://www.khanacademy.org/math/algebra/systems-of-linear-equations/equivalent-systems-of-equations/v/why-we-do-the-same-thing-to-both-sides-basic-systems

Subtracting equations from another

Scale one:



Scale two:



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How to solve a linear system?

$$2x_1 - x_2 = 0$$
$$-x_1 + 2x_2 = 3$$

- Subtracting one equation from the other, in order to **eliminate** a variable from the resulting equation → Fundamental process of **Gaussian Elimination**
- Ex.: **Add half of equation one to equation two**

$$\begin{array}{rl} 2x_1 - x_2 = 0 & \times 0.5 \\ -x_1 + 2x_2 = 3 & \times 1 \\ \hline 0x_1 + 1.5x_2 = 0 & + \end{array} \Rightarrow \begin{array}{l} 2x_1 - x_2 = 0 \\ 0x_1 + 1.5x_2 = 3 \end{array} \Rightarrow \begin{array}{l} x_2 = 3/1.5 = 2 \\ x_1 = 2/2 = 1 \end{array}$$

How does that look in Matrix form?

"Stick RHS of equation on to the matrix"

- Augment matrix with RHS:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Add half of equation one to equation two:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1.5 & 3 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Divide equation two by 1.5:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1 & 2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Add equation two to equation one:

$$\left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Divide equation one by 2:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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How does that look in Matrix form?

- Augment matrix with RHS:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Return back to normal system:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Reduced Row Echelon Form (rref)

So solution is:

$$x_1 = 1$$
$$x_2 = 2$$

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How does that look in Matrix form?

- Augment matrix with RHS:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Add half of equation one to equation two:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1.5 & 3 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Divide equation two by 1.5:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1 & 2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Add equation two to equation one:

$$\left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Divide equation one by 2:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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How does that look in Matrix form?

- Augment matrix with RHS:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Add half of equation one to equation two:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1.5 & 3 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Upper Triangular Matrix

$$x_2 = 3/1.5 = 2 \Rightarrow 2x_1 - x_2 = 2x_1 - 2 = 0 \Rightarrow x_1 = 1$$

Backwards substitution

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Matrix-Vector operations

- Addition/Subtraction:

$$\mathbf{v}+\mathbf{u}=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}+\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} v_1+u_1 \\ v_2+u_2 \end{bmatrix}$$

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

- Multiplication of matrix and vector:

$$\mathbf{Ax}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} a_{11}x_1+a_{12}x_2 \\ a_{21}x_1+a_{22}x_2 \end{bmatrix}=x_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}+x_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

- Multiplication of matrices:

$$\mathbf{AB}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

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Matrix-Vector operations

- Matrix and vector transpose

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \qquad \mathbf{v}^T = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

- Vector multiplication:

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{v}^T \mathbf{u} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = v_1 u_1 + v_2 u_2$$ ← **Dot product**

Rank one matrix → $$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{bmatrix}$$

- Matrix inverse: $\begin{bmatrix} \mathbf{A} & | & \mathbf{I} \end{bmatrix} \sim \begin{bmatrix} \mathbf{I} & | & \mathbf{A}^{-1} \end{bmatrix}$ $\qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

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Is there always a solution?

$$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

- Not necessarily!
  - Ex.:

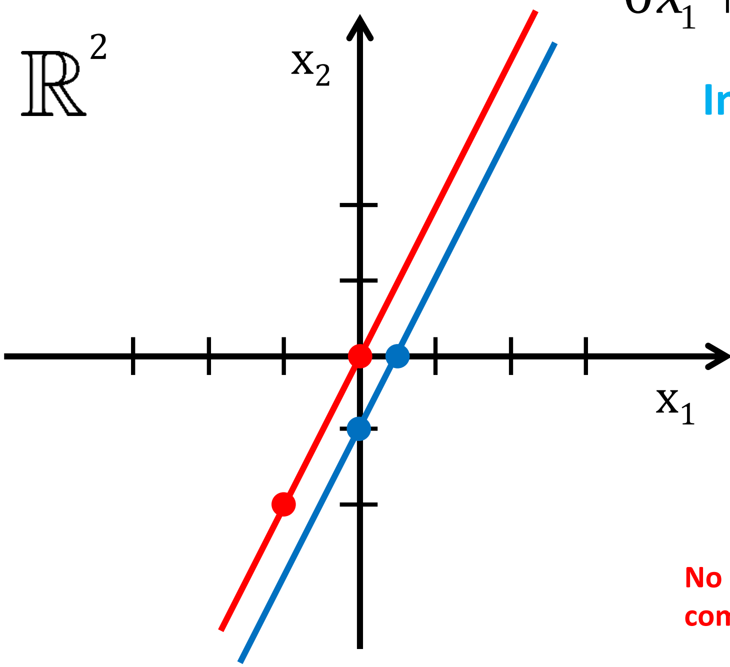
$$\begin{aligned} 2x_1 - x_2 &= 0 & \times 2 \\ -4x_1 + 2x_2 &= -2 & \times 1 \\ \hline 0x_1 + 0x_2 &= -2 & + \end{aligned}$$

$\Rightarrow$ $0 \neq -2$

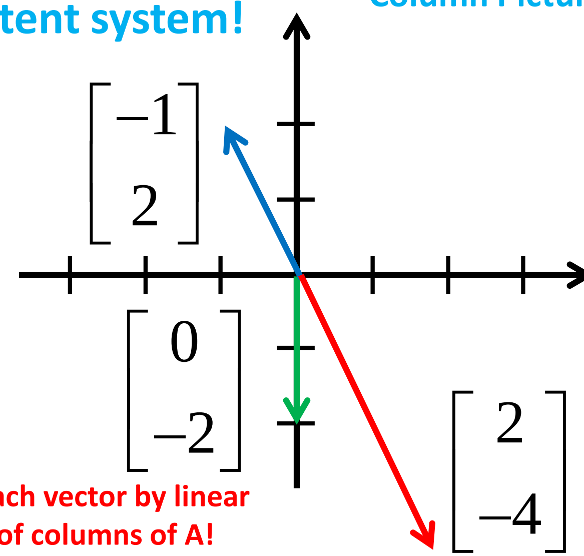
Inconsistent system!

Row Picture:

$\mathbb{R}^2$



Column Picture:



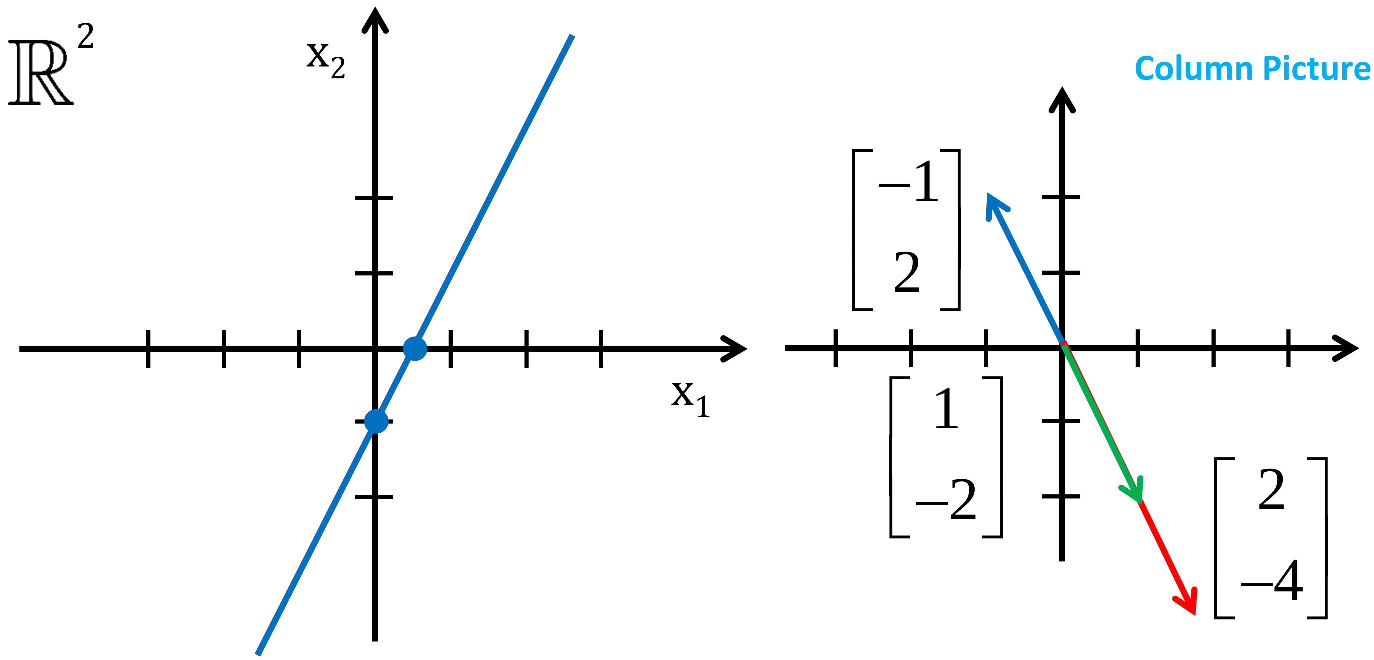
No way to reach vector by linear combination of columns of A!

Is there always a solution?

- What about: $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \boxed{1} \\ -2 \end{bmatrix}$

Row Picture:

Column Picture:

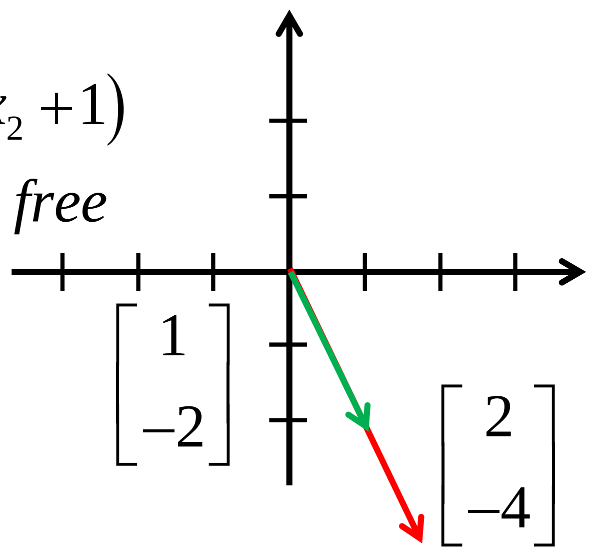


Is there always a solution?

$$\begin{array}{rl} \boxed{2x_1 - x_2 = 1} & \times 2 \\ -4x_1 + 2x_2 = -2 & \times 1 \\ \hline 0x_1 + 0x_2 = 0 & + \end{array}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= 1/2(x_2+1) \\ x_2 &= x_2 \rightarrow \textit{free} \end{aligned}$$

Column Picture:



Solution to Ax=b is then:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2x_2 + 1/2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \boxed{x_2} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Infinitely many solutions!

$$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \left(\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x_2 \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

No matter what value for x_2, solution doesn't change → Vector in Nullspace of A

Summary of solutions

- There exists three cases:
  - No solution

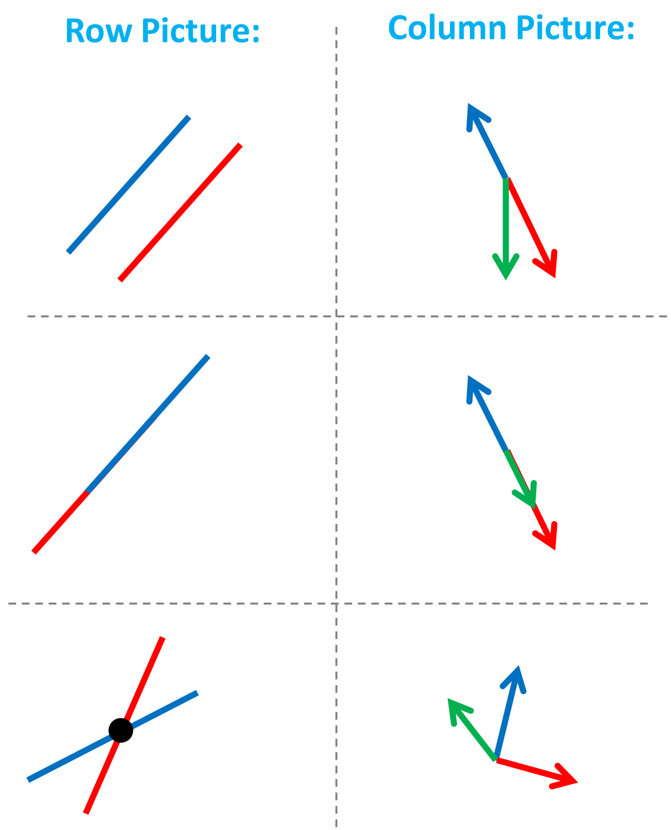
$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 3 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

  - Infinitely many solutions

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

  - Exactly one solution

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Linear combination

- Matrix-vector multiplication: $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

  $$\mathbf{Ax} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \mathbf{x} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2$$

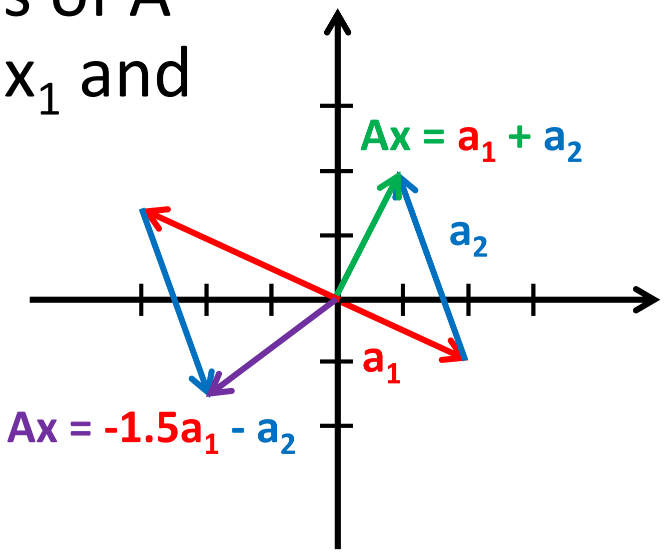
- Multiplication of **Ax** is a linear combination of the columns of A ($\mathbf{a}_1$ and $\mathbf{a}_2$) with coefficient x_1 and x_2 (elements of vector **x**)

Inspired by the Video Lectures of Gilbert Strang (MIT Open Courseware): https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/

Linear combination

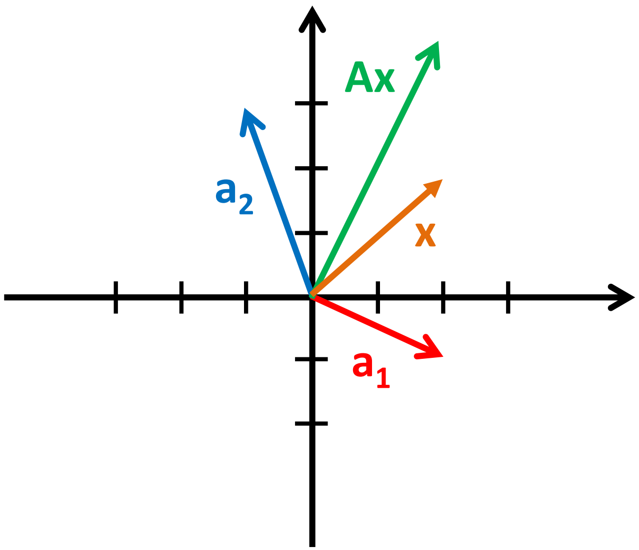
- Matrix-vector multiplication: $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

  $\mathbf{Ax} = [\mathbf{a}_1 \quad \mathbf{a}_2]\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$
- Multiplication of **Ax** is a linear combination of the columns of A ($\mathbf{a}_1$ and $\mathbf{a}_2$) with coefficient x_1 and x_2 (elements of vector **x**)
- All combinations of $\mathbf{a}_1$ and $\mathbf{a}_2$ is called the span(**A**) or span($\{\mathbf{a}_1, \mathbf{a}_2\}$)



Linear (in)dependence

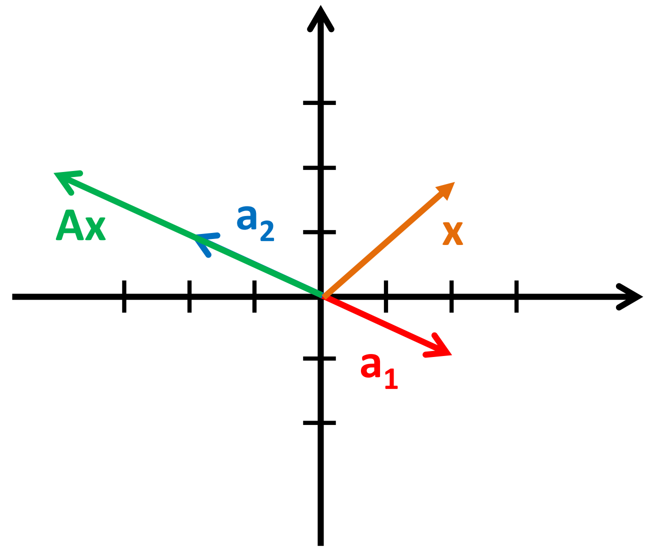
Linear independent:



$$\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\mathbf{A}^{-1}$ exists ☺! → det(A)≠0

Linear dependent:



$$\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix}$$

$\mathbf{A}^{-1}$ doesn't exists ☹! → det(A)=0

$$\mathbf{Ax} = \mathbf{b} \rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

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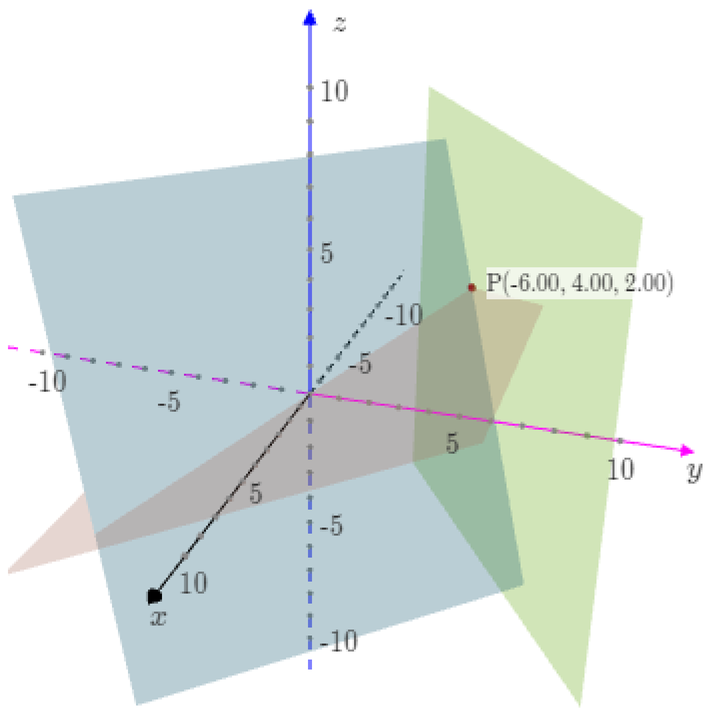
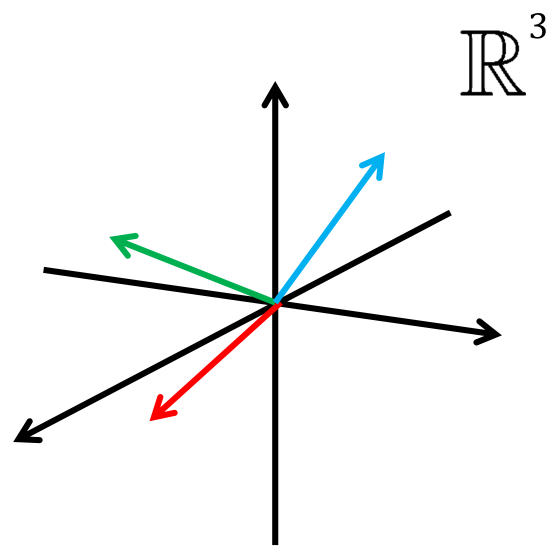
Generalize to higher dimensions

Row picture: Planes in R^3



Column picture: Vectors in R^3



Inspired by the Video Lectures of Gilbert Strang (MIT Open Courseware): https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/

Linear Transformation

- **Additivity:** $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$
- **Homogeneity:** $L(c\mathbf{v}) = cL(\mathbf{v}) \qquad c \in \mathbb{R}$
- **Example:**

$$L(\bullet) = \frac{d}{dx}(\bullet)$$

$$L(cf) = \frac{d}{dx}(cf) = c\frac{d}{dx}(f)$$

$$L(f+g) = \frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

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Linear Transformation

- **Additivity:** $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$
- **Homogeneity:** $L(c\mathbf{v}) = cL(\mathbf{v})$ $\quad c \in \mathbb{R}$
- **Example:** $f = x^2 \quad g = x \quad c = 2$

$$\frac{d}{dx}(cf) = \frac{d}{dx}\left(2x^2\right) = 4x = 2\frac{d}{dx}\left(x^2\right)$$

$$\frac{d}{dx}(f+g) = \frac{d}{dx}\left(x^2+x\right) = 2x+1 = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(x)$$

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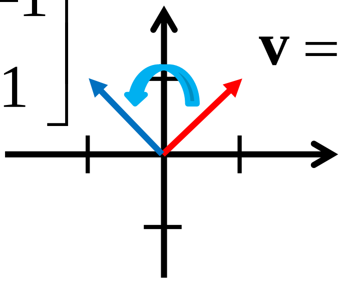
Linear Transformation

- **Additivity:** $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$
- **Homogeneity:** $L(c\mathbf{v}) = cL(\mathbf{v}) \qquad c \in \mathbb{R}$
- **Example:** $L = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \theta = \frac{\pi}{2} \quad L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$L\mathbf{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Linear Transformation

- **Additivity:** $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$
- **Homogeneity:** $L(c\mathbf{v}) = cL(\mathbf{v}) \qquad c \in \mathbb{R}$
- **Example:** $L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad c = 2$

$$L(c\mathbf{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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Linear Transformation

- **Additivity:** $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$

- **Homogeneity:** $L(c\mathbf{v}) = cL(\mathbf{v}) \qquad c \in \mathbb{R}$

- **Example:** $L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad c = 2$

$$L(\mathbf{v}+\mathbf{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

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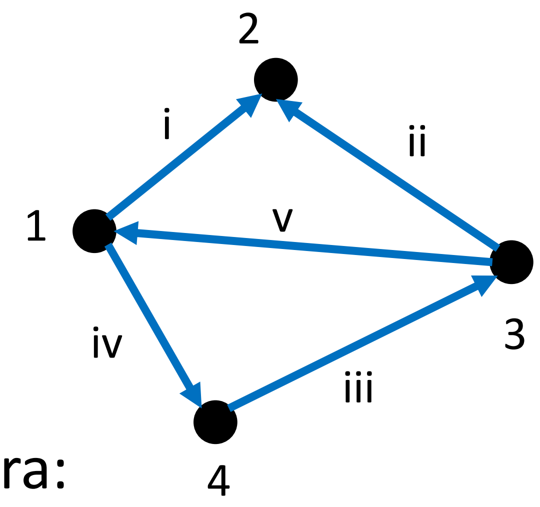
Applications



- Application of (linear) systems/algebra:
  - Conservation of Charge
  - Differential Equations
  - Statistics: linear regression, least squares estimates, principal component analysis
  - **Numerical methods for solving differential equations**
  - Various others such as large data transformations, neural networks, image processing

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End of second part (theory)
Start with Python ☺!

Course contents

- Introduce elementary concepts of programming:
  - Logic behind programming and algorithms;
  - Basic coding flows;
  - Guidelines (commandments) of programming.
- After successfully completing this course you should be able to:
  - Write structured and readable code;
  - Complete the laboratory/computer assignments of **AESM1305** using Python;
  - Debug yours and other peoples code;
  - Have a basic understanding of linear algebra, calculus, statistics and numerical methods.