Quantum Mechanics

Postulate 1: State Space $|\psi\rangle \in \mathbb{C}^n$

Associated with any physical system is a complex vector space known as the state space of the system. If the system is isolated, then the system is completely described by its state vector which is a unit vector in the system's state space

Postulate 2: Unitary Dynamics $|\psi'\rangle = U|\psi\rangle$

The evolution of an isolated quantum system is described by a unitary matrix acting on the state space of the system. From $|\psi\rangle$ at time t_1 to $|\psi'\rangle$ at t_2, U: $|\psi'\rangle = U|\psi\rangle$. Matrix U depends on t_1 and t_2, but does not depend on $|\psi\rangle$ and $|\psi'\rangle$. (Works for discrete time)

time rate of change of system

Schrödinger's eq^n: (Continuous time)

$$i \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

A Hermitian matrix = Hamiltonian of the system

From the differential to the discrete :

$$|\psi_{t_2}\rangle = \underbrace{e^{iH(t_2-t_1)}}_{\text{A Unitary matrix}} |\psi_{t_1}\rangle$$

Postulate 3: Measurement

→ Measurement operators: $\{M_m\}$

where m: outcome

→ if $|\psi\rangle$ is the state immediately before measurement, then

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

(square of the amplitude)

The state after measurement, posterior state:

$$= \frac{M_m|\psi\rangle}{\sqrt{p(m)}}$$

→ Completeness relation:

$$\sum_m M_m^\dagger M_m = I.$$

This relation basically leads to probabilities to adding to 1:

$$\sum_m p(m) = \sum_m \langle\psi| M_m^\dagger M_m |\psi\rangle$$

$$= \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1.$$

non-trivial example: $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$

M_0 and M_1 are measurements in the computational basis.

Now,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$p(0) = \langle\psi|M_0^\dagger M_0|\psi\rangle$$

$$= \langle\psi|0\rangle\langle 0|0\rangle\langle 0|\psi\rangle$$

→ Substituting $\langle 0|0\rangle = 1$,

$$p(0) = \langle\psi|0\rangle\langle 0|\psi\rangle = |\langle 0|\psi\rangle|^2$$

$|\alpha|^2$

→ Similarly, $p(1) = |\beta|^2$

→ Back to M_0, the state after measurement:

$$\frac{|0\rangle\langle 0|\psi\rangle}{\sqrt{p(0)}} = \frac{\alpha}{|\alpha|}|0\rangle$$: Global Phase

→ Global phase does not change measurement probability: if $|\psi'\rangle = e^{i\theta}|\psi\rangle$

$$p'(m) = \langle\psi'|M_m^\dagger M_m|\psi'\rangle = \langle\psi|M_m^\dagger M_m|\psi\rangle = p(m)$$

→ X-measurement:

$$M_+ = |+\rangle\langle+| \qquad M_- = |-\rangle\langle-|$$

$$|x\rangle = \gamma|+\rangle + \delta|-\rangle$$

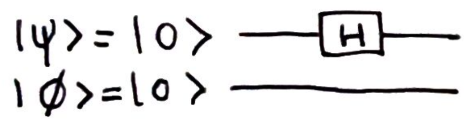
Postulate 4: Composite Systems

→ The composite system of n component systems $|\psi_j\rangle$, $j \in [1, n]$

$$= |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle$$

→ $|0\rangle \otimes |1\rangle = |0\rangle|1\rangle = |01\rangle$

→ The following circuit



$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

can be described by

$$(H \otimes I)(|\psi\rangle \otimes |\phi\rangle) = (H|\psi\rangle) \otimes (I|\phi\rangle)$$

$$= (H|\psi\rangle \otimes |\phi\rangle)$$

——— X ———

* Bell inequality:

→ If a system is locally realistic,

$$\text{Avg}(AC) + \text{Avg}(BC) + \text{Avg}(BD) - \text{Avg}(AD) \leq 2$$

→ Q.M. can describe Bell inequality violations that are derived from real-world experiments.

*→ For a set of measurement operators $\{M_m\}$ The average outcome is

$$\sum_m p(m)m = \sum_m \langle\psi| M_m^\dagger M_m |\psi\rangle m$$

$$\text{Here,} \quad \sum_m M_m^\dagger M_m = M$$

$$\text{Then the average} = \langle\psi|M|\psi\rangle$$

- M is known as the observable

example: If we wanted $+1$ and -1 as outcomes,

$$M_1 = |0\rangle\langle 0|, \quad M_{-1} = |1\rangle\langle 1|:$$

$$M = |0\rangle\langle 0|0\rangle\langle 0| - |1\rangle\langle 1|1\rangle\langle 1| = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \text{Pauli } Z$$

Quantum mechanics in a nutshell

1. States

Every physical system has a state space which is a complex vector space. An isolated system's state can be described by a unit vector in that state space.

$$|\psi\rangle \in \mathbb{C}^n$$

2. Dynamics

An isolated quantum system's evolution is described by a unitary matrix acting on the system's state space.

$$|\psi'\rangle = U|\psi\rangle$$

3. Measurement

Measurement operators describe how we get information out of a quantum system.

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

$$|\psi_m\rangle = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

$$\sum_m M_m^\dagger M_m = I$$

4. Combining systems

Quantum state spaces combine via tensor products.

$$\text{state space } A$$

$$\text{state space } B$$

$$\Longrightarrow$$

$$\text{combined state space } A \otimes B$$