2D Inverted Pendulum

Davis Benz
Equations of Motion

July 18, 2021

1 Terms

- Constants

  - g: gravity
  - Itot: total mass moment of inertia of the inverted pendulum
  - Im: mass moment of inertia of the just rotor
  - Ib: mass moment of inertia of just the body
  - m: mass of the related system

- Variables

  - omega

- Subscripts

  - **b**: body
  - **m**: motor
  - **g**: gravity, IE a force or moment due strictly to gravity
  - **f**: friction, frictional forces or moments
  - **o**: origin point of the system

2   2D System
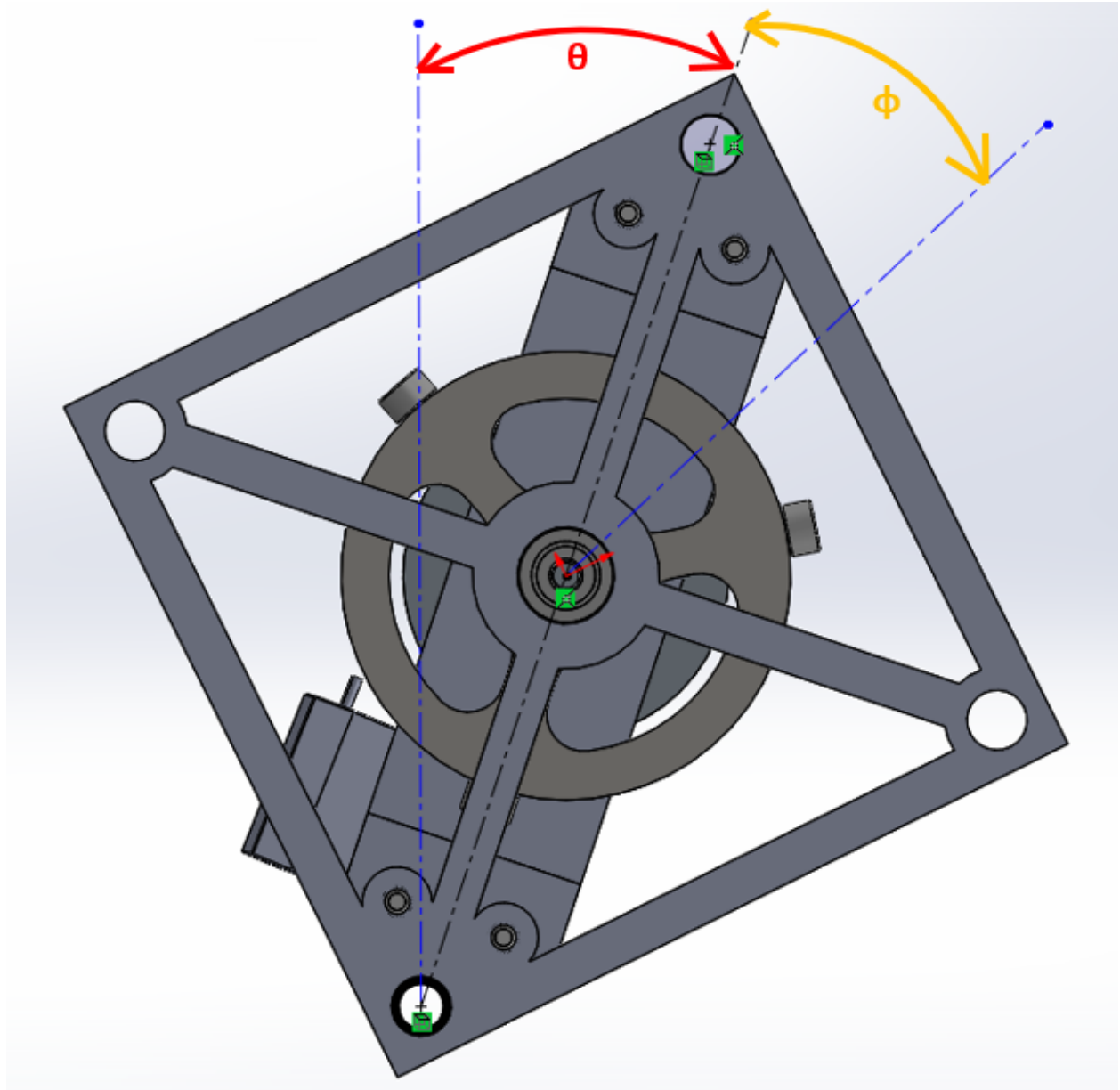
3   Lagrangian Equation of Motion

3.1   Lagrangian

$$\mathcal{L} = \frac{1}{2}I_b\omega_b^2 + \frac{1}{2}I_m(\omega_b + \omega_m)^2 - m_t g cos(\theta) \tag{1}$$

$$\mathcal{L} = \frac{1}{2}I_b\dot{\theta}^2 + \frac{1}{2}I_m(\dot{\theta} + \dot{\phi})^2 - m_t g cos(\theta) \tag{2}$$

3.2 Lagrangian Momentum

$$p_\theta = \frac{\sigma \mathcal{L}}{\sigma \dot\theta} \tag{3}$$

$$p_\phi = \frac{\sigma \mathcal{L}}{\sigma \dot\phi} \tag{4}$$

$$\mathcal{L} = \frac{1}{2}I_b\dot\theta^2 + \frac{1}{2}I_m(\dot\theta^2 + 2\dot\theta\dot\phi + \dot\phi^2) - m_t g cos(\theta) \tag{5}$$

$$\mathcal{L} = \frac{1}{2}I_b\dot\theta^2 + \frac{1}{2}I_m\dot\theta^2 + \frac{2}{2}I_m\dot\theta\dot\phi + \frac{1}{2}I_m\dot\phi^2 - m_t g cos(\theta) \tag{6}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot\theta} = I_b\dot\theta + I_m\dot\theta + I_m\dot\phi \tag{7}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot\theta} = \dot\theta(I_b + I_m) + I_m\dot\phi, \ I_b = I_{tot} - I_m \tag{8}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot\theta} = \dot\theta(I_{tot} - I_m + I_m) + I_m\dot\phi \tag{9}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot\theta} = I_{tot}\dot\theta + I_m\dot\phi \tag{10}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot\phi} = I_m\dot\theta + I_m\dot\phi \tag{11}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot\phi} = I_m(\dot\theta + \dot\phi) \tag{12}$$

$$\boxed{P_\theta = I_{tot}\dot\theta + I_m\dot\phi} \tag{13}$$

$$\boxed{P_\phi = I_m(\dot\theta + \dot\phi)} \tag{14}$$

3.3 Equations of Motion

$$\frac{d}{dt}\frac{\sigma \mathcal{L}}{\sigma \dot\theta} = \frac{\sigma \mathcal{L}}{\sigma \theta} => \frac{d}{dt}P_\theta = \frac{\sigma \mathcal{L}}{\sigma \theta} \tag{15}$$

$$\frac{d}{dt}(I_{tot}\dot\theta + I_m\dot\phi) = m_t g sin(\theta) \tag{16}$$

$$I_{tot}\ddot\theta + I_m\ddot\phi = m_t g sin(\theta) \tag{17}$$

$$\frac{d}{dt}\frac{\sigma \mathcal{L}}{\sigma \dot\phi} = \frac{\sigma \mathcal{L}}{\sigma \phi} => \frac{d}{dt}P_\phi = \frac{\sigma \mathcal{L}}{\sigma \phi} \tag{18}$$

$$\frac{d}{dt}(I_m(\dot\theta + \dot\phi)) = T \tag{19}$$

$$I_m(\ddot\theta + \ddot\phi) = T \tag{20}$$

$$\boxed{I_{tot}\ddot\theta + I_m\ddot\phi = m_t g sin(\theta)} \tag{21}$$

$$\boxed{I_m(\ddot\theta + \ddot\phi) = T} \tag{22}$$

3

4   Equations Of Motion

$$\sum M = I\alpha \tag{23}$$

$$-M_{bg} + M_{bf} - M_{mg} + T_m - M_{mf} = I_{tot}\ddot{\theta} \tag{24}$$

$$M_{bg} = m_b g * l_b sin(\theta), M_{mg} = m_m g * l_m sin(\theta) \tag{25}$$

$$M_{bf} = C_b\dot{\theta}, M_{mf} = C_m\dot{\phi} \tag{26}$$

$$I_{tot} = I_{bo} + I_{mo} \tag{27}$$

$$I_{mo} = I_m + m_w l_m^2 \tag{28}$$

$$-m_b g * l_b sin(\theta) + C_b\dot{\theta}_b - m_m g * l_m sin(\theta) + T_m - C_m\dot{\phi} = I_{tot}\ddot{\theta} \tag{29}$$

$$\ddot{\theta} = \frac{m_b g * l_b sin(\theta) - C_b\dot{\theta} + m_m g * l_m sin(\theta) - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \tag{30}$$

$$\ddot{\theta} = \frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta} - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \tag{31}$$

$$-T_m + M_{mf} = I_m(\ddot{\theta} + \ddot{\phi}) \tag{32}$$

$$-T_m + C_m\dot{\phi} = I_m\ddot{\theta} + I_m\ddot{\phi} \tag{33}$$

$$I_m\ddot{\phi} = -I_m\ddot{\theta} - T_m + C_m\dot{\phi} \tag{34}$$

$$\ddot{\phi} = -\ddot{\theta} + \frac{-T_m + C_m\dot{\phi}}{I_m} \tag{35}$$

$$\ddot{\phi} = -\left(\frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta} - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2}\right) + \left(\frac{-T_m + C_m\dot{\phi}}{I_m}\right) \tag{36}$$

$$\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m\dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \tag{37}$$

$$\boxed{\ddot{\theta} = \frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta} - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2}} \tag{38}$$

$$\boxed{\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m\dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2}} \tag{39}$$

$$\boxed{T_m = K_m u} \tag{40}$$

4

5   Control System

To start, use a linear state space model of the system to design the control algorithm around

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{41}$$

$$x(t) = [\theta, \ \dot{\theta}, \ \dot{\phi}] \tag{42}$$

$$u(t) = T_m => K_m u \tag{43}$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} ii & ij & ik \\ ji & jj & jk \\ ki & kj & kk \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} K_m u(t) \tag{44}$$

$$\ddot{\theta} = \frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta} - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \tag{45}$$

$$\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m\dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \tag{46}$$

assuming small angles

$$sin(\theta) \approx \theta \tag{47}$$

$$\ddot{\theta} = \frac{g\theta(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} - \frac{C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2} - \frac{T_m}{I_{bo} + I_m + m_m l_m^2} + \frac{C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \tag{48}$$

$$\ddot{\phi} = -\frac{C_m\dot{\phi}(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} + \frac{T_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{g\theta(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} + \frac{C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \tag{49}$$

$$\dot{\theta} = ii\theta + ij\dot{\theta} + ik\dot{\phi} \tag{50}$$

$$ii = 0, \ ij = 1, \ ik = 0 \tag{51}$$

$$\ddot{\theta} = ji\theta + jj\dot{\theta} + jk\dot{\phi} \tag{52}$$

$$ji = \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2}, \ jj = -\frac{C_b}{I_{bo} + I_m + m_m l_m^2}, \ jk = \frac{C_m}{I_{bo} + I_m + m_m l_m^2} \tag{53}$$

$$\ddot{\phi} = ki\theta + kj\dot{\theta} + kk\dot{\phi} \tag{54}$$

$$ki = -\frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2}, \ kj = \frac{C_b}{I_{bo} + I_m + m_m l_m^2}, \ kk = -\frac{C_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \tag{55}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} & -\frac{C_b}{I_{bo} + I_m + m_m l_m^2} & \frac{C_m}{I_{bo} + I_m + m_m l_m^2} \\ -\frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} & \frac{C_b}{I_{bo} + I_m + m_m l_m^2} & -\frac{C_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \end{bmatrix} \tag{56}$$

$$B = \begin{bmatrix} 0 \\ -\frac{K_m}{I_{bo} + I_m + m_m l_m^2} \\ \frac{K_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \end{bmatrix} \tag{57}$$

5