

2D Inverted Pendulum

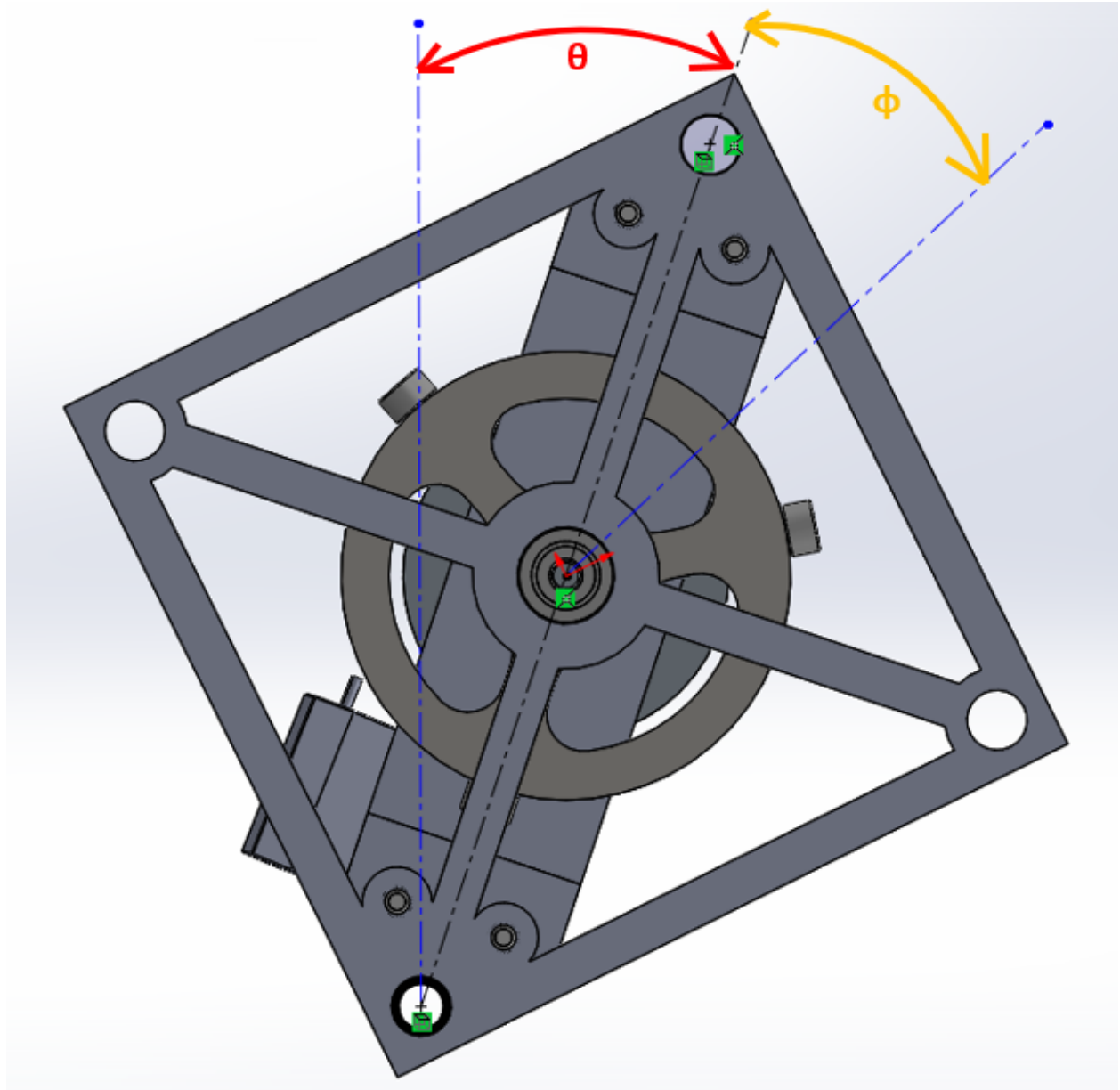
Davis Benz
Equations of Motion

July 18, 2021

1 Terms

- Constants
 - g : gravity
 - I_{tot} : total mass moment of inertia of the inverted pendulum
 - I_m : mass moment of inertia of the just rotor
 - I_b : mass moment of inertia of just the body
 - m : mass of the related system
- Variables
 - ω : omega
- Subscripts
 - **b**: body
 - **m**: motor
 - **g**: gravity, IE a force or moment due strictly to gravity
 - **f**: friction, frictional forces or moments
 - **o**: origin point of the system

2 2D System



3 Lagrangian Equation of Motion

3.1 Lagrangian

$$\mathcal{L} = \frac{1}{2}I_b\omega_b^2 + \frac{1}{2}I_m(\omega_b + \omega_m)^2 - m_t g \cos(\theta) \quad (1)$$

$$\mathcal{L} = \frac{1}{2}I_b\dot{\theta}^2 + \frac{1}{2}I_m(\dot{\theta} + \dot{\phi})^2 - m_t g \cos(\theta) \quad (2)$$

3.2 Lagrangian Momentum

$$p_\theta = \frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} \quad (3)$$

$$p_\phi = \frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} \quad (4)$$

$$\mathcal{L} = \frac{1}{2}I_b\dot{\theta}^2 + \frac{1}{2}I_m(\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2) - m_t g \cos(\theta) \quad (5)$$

$$\mathcal{L} = \frac{1}{2}I_b\dot{\theta}^2 + \frac{1}{2}I_m\dot{\theta}^2 + \frac{2}{2}I_m\dot{\theta}\dot{\phi} + \frac{1}{2}I_m\dot{\phi}^2 - m_t g \cos(\theta) \quad (6)$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = I_b\dot{\theta} + I_m\dot{\theta} + I_m\dot{\phi} \quad (7)$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = \dot{\theta}(I_b + I_m) + I_m\dot{\phi}, \quad I_b = I_{tot} - I_m \quad (8)$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = \dot{\theta}(I_{tot} - I_m + I_m) + I_m\dot{\phi} \quad (9)$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = I_{tot}\dot{\theta} + I_m\dot{\phi} \quad (10)$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} = I_m\dot{\theta} + I_m\dot{\phi} \quad (11)$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} = I_m(\dot{\theta} + \dot{\phi}) \quad (12)$$

$$\boxed{P_\theta = I_{tot}\dot{\theta} + I_m\dot{\phi}} \quad (13)$$

$$\boxed{P_\phi = I_m(\dot{\theta} + \dot{\phi})} \quad (14)$$

3.3 Equations of Motion

$$\frac{d}{dt} \frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = \frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} \Rightarrow \frac{d}{dt} P_\theta = \frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} \quad (15)$$

$$\frac{d}{dt} (I_{tot}\dot{\theta} + I_m\dot{\phi}) = m_t g \sin(\theta) \quad (16)$$

$$I_{tot}\ddot{\theta} + I_m\ddot{\phi} = m_t g \sin(\theta) \quad (17)$$

$$\frac{d}{dt} \frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} = \frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} \Rightarrow \frac{d}{dt} P_\phi = \frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} \quad (18)$$

$$\frac{d}{dt} (I_m(\dot{\theta} + \dot{\phi})) = T \quad (19)$$

$$I_m(\ddot{\theta} + \ddot{\phi}) = T \quad (20)$$

$$\boxed{I_{tot}\ddot{\theta} + I_m\ddot{\phi} = m_t g \sin(\theta)} \quad (21)$$

$$\boxed{I_m(\ddot{\theta} + \ddot{\phi}) = T} \quad (22)$$

4 Equations Of Motion

$$\sum M = I\alpha \quad (23)$$

$$-M_{bg} + M_{bf} - M_{mg} + T_m - M_{mf} = I_{tot}\ddot{\theta} \quad (24)$$

$$M_{bg} = m_bg * l_b \sin(\theta), M_{mg} = m_m g * l_m \sin(\theta) \quad (25)$$

$$M_{bf} = C_b \dot{\theta}, M_{mf} = C_m \dot{\phi} \quad (26)$$

$$I_{tot} = I_{bo} + I_{mo} \quad (27)$$

$$I_{mo} = I_m + m_w l_m^2 \quad (28)$$

$$-m_bg * l_b \sin(\theta) + C_b \dot{\theta} - m_m g * l_m \sin(\theta) + T_m - C_m \dot{\phi} = I_{tot}\ddot{\theta} \quad (29)$$

$$\ddot{\theta} = \frac{m_bg * l_b \sin(\theta) - C_b \dot{\theta} + m_m g * l_m \sin(\theta) - T_m + C_m \dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \quad (30)$$

$$\ddot{\theta} = \frac{g \sin(\theta)(m_b l_b + m_m l_m) - C_b \dot{\theta} - T_m + C_m \dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \quad (31)$$

$$-T_m + M_{mf} = I_m(\ddot{\theta} + \ddot{\phi}) \quad (32)$$

$$-T_m + C_m \dot{\phi} = I_m \ddot{\theta} + I_m \ddot{\phi} \quad (33)$$

$$I_m \ddot{\phi} = -I_m \ddot{\theta} - T_m + C_m \dot{\phi} \quad (34)$$

$$\ddot{\phi} = -\ddot{\theta} + \frac{-T_m + C_m \dot{\phi}}{I_m} \quad (35)$$

$$\ddot{\phi} = -\left(\frac{g \sin(\theta)(m_b l_b + m_m l_m) - C_b \dot{\theta} - T_m + C_m \dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \right) + \left(\frac{-T_m + C_m \dot{\phi}}{I_m} \right) \quad (36)$$

$$\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m \dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{g \sin(\theta)(m_b l_b + m_m l_m) - C_b \dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \quad (37)$$

$$\boxed{\ddot{\theta} = \frac{g \sin(\theta)(m_b l_b + m_m l_m) - C_b \dot{\theta} - T_m + C_m \dot{\phi}}{I_{bo} + I_m + m_m l_m^2}} \quad (38)$$

$$\boxed{\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m \dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{g \sin(\theta)(m_b l_b + m_m l_m) - C_b \dot{\theta}}{I_{bo} + I_m + m_m l_m^2}} \quad (39)$$

$$\boxed{T_m = K_m u} \quad (40)$$

5 Control System

5.1 State Space Model

To start, use a linear state space model of the system to design the control algorithm around

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (41)$$

$$x(t) = [\theta, \dot{\theta}, \dot{\phi}] \quad (42)$$

$$u(t) = T_m \Rightarrow K_m u \quad (43)$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} ii & ij & ik \\ ji & jj & jk \\ ki & kj & kk \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} K_m u(t) \quad (44)$$

$$\ddot{\theta} = \frac{g \sin(\theta)(m_b l_b + m_m l_m) - C_b \dot{\theta} - T_m + C_m \dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \quad (45)$$

$$\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m \dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{g \sin(\theta)(m_b l_b + m_m l_m) - C_b \dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \quad (46)$$

assuming small angles

$$\sin(\theta) \approx \theta \quad (47)$$

$$\ddot{\theta} = \frac{g\theta(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} - \frac{C_b \dot{\theta}}{I_{bo} + I_m + m_m l_m^2} - \frac{T_m}{I_{bo} + I_m + m_m l_m^2} + \frac{C_m \dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \quad (48)$$

$$\ddot{\phi} = -\frac{C_m \dot{\phi}(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} + \frac{T_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{g\theta(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} + \frac{C_b \dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \quad (49)$$

$$\dot{\theta} = ii\theta + ij\dot{\theta} + ik\dot{\phi} \quad (50)$$

$$ii = 0, \quad ij = 1, \quad ik = 0 \quad (51)$$

$$\ddot{\theta} = ji\theta + jj\dot{\theta} + jk\dot{\phi} \quad (52)$$

$$ji = \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2}, \quad jj = -\frac{C_b}{I_{bo} + I_m + m_m l_m^2}, \quad jk = \frac{C_m}{I_{bo} + I_m + m_m l_m^2} \quad (53)$$

$$\ddot{\phi} = ki\theta + kj\dot{\theta} + kk\dot{\phi} \quad (54)$$

$$ki = -\frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2}, \quad kj = \frac{C_b}{I_{bo} + I_m + m_m l_m^2}, \quad kk = -\frac{C_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \quad (55)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} & -\frac{C_b}{I_{bo} + I_m + m_m l_m^2} & \frac{C_m}{I_{bo} + I_m + m_m l_m^2} \\ -\frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} & \frac{C_b}{I_{bo} + I_m + m_m l_m^2} & -\frac{C_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \end{bmatrix} \quad (56)$$

$$B = \begin{bmatrix} 0 \\ -\frac{K_m}{I_{bo} + I_m + m_m l_m^2} \\ \frac{K_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \end{bmatrix} \quad (57)$$

5.2 Model Stability

Use eigenvalues of A to determine stability of the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} & -\frac{C_b}{I_{bo} + I_m + m_m l_m^2} & \frac{C_m}{I_{bo} + I_m + m_m l_m^2} \\ -\frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} & \frac{C_b}{I_{bo} + I_m + m_m l_m^2} & -\frac{C_m(I_{bo} + I_m + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \end{bmatrix} \quad (58)$$

$$|A - \lambda| = 0 \quad (59)$$

$$\left| \begin{bmatrix} 0 & 1 & 0 \\ \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} & -\frac{C_b}{I_{bo} + I_m + m_m l_m^2} & \frac{C_m}{I_{bo} + I_m + m_m l_m^2} \\ -\frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} & \frac{C_b}{I_{bo} + I_m + m_m l_m^2} & -\frac{C_m(I_{bo} + I_m + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \end{bmatrix} - \begin{bmatrix} \lambda_A & 0 & 0 \\ 0 & \lambda_A & 0 \\ 0 & 0 & \lambda_A \end{bmatrix} \right| = 0 \quad (60)$$

$$a_1 = \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2}, \quad a_2 = \frac{C_b}{I_{bo} + I_m + m_m l_m^2} \quad (61)$$

$$a_3 = \frac{C_m}{I_{bo} + I_m + m_m l_m^2}, \quad a_4 = -\frac{C_m(I_{bo} + I_m + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \quad (62)$$

$$\left| \begin{bmatrix} -\lambda & 1 & 0 \\ a_1 & -a_2 - \lambda_A & a_3 \\ -a_1 & a_2 & a_4 - \lambda_A \end{bmatrix} \right| = 0 \quad (63)$$

$$-\lambda^3 - a_2 \lambda^2 + a_4 \lambda^2 + a_2 a_4 \lambda + a_2 a_3 \lambda + a_1 \lambda - a_1 a_4 - a_1 a_3 = 0 \quad (64)$$

Determine stability from roots of the quadratic below

$$|A - \lambda| = 0$$

$$-\lambda^3 - a_2 \lambda^2 + a_4 \lambda^2 + a_2 a_4 \lambda + a_2 a_3 \lambda + a_1 \lambda - a_1 a_4 - a_1 a_3 = 0$$

(65)