2D Inverted Pendulum

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Equations of Motion

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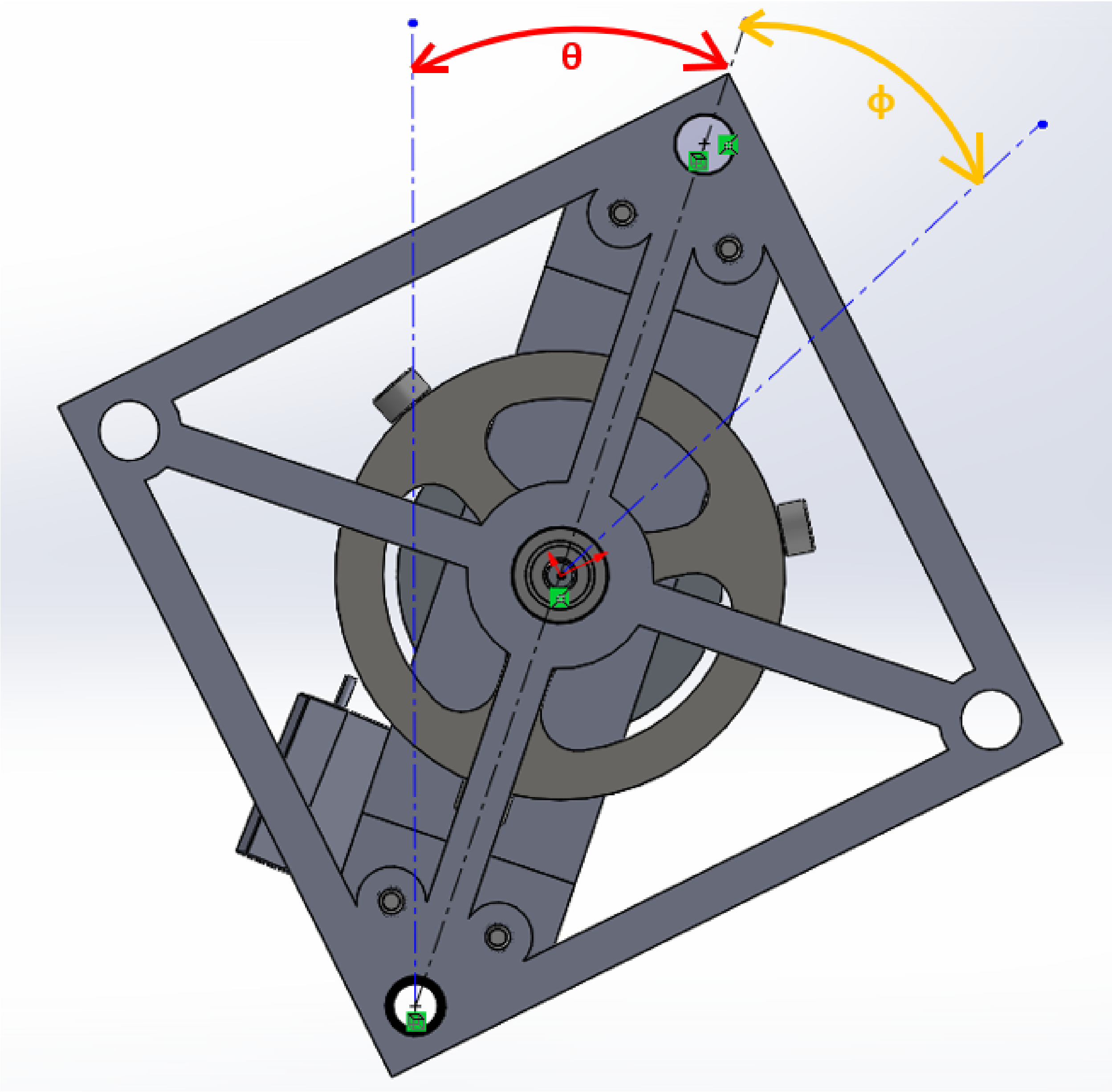
1 Terms

- Constants
  - g: gravity
  - Itot: total mass moment of inertia of the inverted pendulum
  - Im: mass moment of inertia of the just rotor
  - Ib: mass moment of inertia of just the body
  - m: mass of the related system
- Variables
  - omega
- Subscripts
  - **b**: body
  - **m**: motor
  - **g**: gravity, IE a force or moment due strictly to gravity
  - **f**: friction, frictional forces or moments
  - **o**: origin point of the system

2 2D System



3 Lagrangian Equation of Motion

3.1 Lagrangian

$$\mathcal{L} = \frac{1}{2}I_b\omega_b^2 + \frac{1}{2}I_m(\omega_b + \omega_m)^2 - m_t g cos(\theta) \quad (1)$$

$$\mathcal{L} = \frac{1}{2}I_b\dot{\theta}^2 + \frac{1}{2}I_m(\dot{\theta} + \dot{\phi})^2 - m_t g cos(\theta) \quad (2)$$

3.2 Lagrangian Momentum

$$p_\theta = \frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} \tag{3}$$

$$p_\phi = \frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} \tag{4}$$

$$\mathcal{L} = \frac{1}{2} I_b \dot{\theta}^2 + \frac{1}{2} I_m (\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2) - m_t g cos(\theta) \tag{5}$$

$$\mathcal{L} = \frac{1}{2} I_b \dot{\theta}^2 + \frac{1}{2} I_m \dot{\theta}^2 + \frac{2}{2} I_m \dot{\theta}\dot{\phi} + \frac{1}{2} I_m \dot{\phi}^2 - m_t g cos(\theta) \tag{6}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = I_b \dot{\theta} + I_m \dot{\theta} + I_m \dot{\phi} \tag{7}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = \dot{\theta}(I_b + I_m) + I_m \dot{\phi},\ I_b = I_{tot} - I_m \tag{8}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = \dot{\theta}(I_{tot} - I_m + I_m) + I_m \dot{\phi} \tag{9}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = I_{tot} \dot{\theta} + I_m \dot{\phi} \tag{10}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} = I_m \dot{\theta} + I_m \dot{\phi} \tag{11}$$

$$\frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} = I_m (\dot{\theta} + \dot{\phi}) \tag{12}$$

$$\boxed{P_\theta = I_{tot} \dot{\theta} + I_m \dot{\phi}} \tag{13}$$

$$\boxed{P_\phi = I_m (\dot{\theta} + \dot{\phi})} \tag{14}$$

3.3 Equations of Motion

$$\frac{d}{dt} \frac{\sigma \mathcal{L}}{\sigma \dot{\theta}} = \frac{\sigma \mathcal{L}}{\sigma \theta} => \frac{d}{dt} P_\theta = \frac{\sigma \mathcal{L}}{\sigma \theta} \tag{15}$$

$$\frac{d}{dt}(I_{tot} \dot{\theta} + I_m \dot{\phi}) = m_t g sin(\theta) \tag{16}$$

$$I_{tot} \ddot{\theta} + I_m \ddot{\phi} = m_t g sin(\theta) \tag{17}$$

$$\frac{d}{dt} \frac{\sigma \mathcal{L}}{\sigma \dot{\phi}} = \frac{\sigma \mathcal{L}}{\sigma \phi} => \frac{d}{dt} P_\phi = \frac{\sigma \mathcal{L}}{\sigma \phi} \tag{18}$$

$$\frac{d}{dt}(I_m (\dot{\theta} + \dot{\phi})) = T \tag{19}$$

$$I_m (\ddot{\theta} + \ddot{\phi}) = T \tag{20}$$

$$\boxed{I_{tot} \ddot{\theta} + I_m \ddot{\phi} = m_t g sin(\theta)} \tag{21}$$

$$\boxed{I_m (\ddot{\theta} + \ddot{\phi}) = T} \tag{22}$$

4 Equations Of Motion

$$\sum M = I\alpha \tag{23}$$

$$-M_{bg} + M_{bf} - M_{mg} + T_m - M_{mf} = I_{tot}\ddot{\theta} \tag{24}$$

$$M_{bg} = m_b g * l_b sin(\theta), M_{mg} = m_m g * l_m sin(\theta) \tag{25}$$

$$M_{bf} = C_b\dot{\theta}, M_{mf} = C_m\dot{\phi} \tag{26}$$

$$I_{tot} = I_{bo} + I_{mo} \tag{27}$$

$$I_{mo} = I_m + m_w l_m^2 \tag{28}$$

$$-m_b g * l_b sin(\theta) + C_b\dot{\theta}_b - m_m g * l_m sin(\theta) + T_m - C_m\dot{\phi} = I_{tot}\ddot{\theta} \tag{29}$$

$$\ddot{\theta} = \frac{m_b g * l_b sin(\theta) - C_b\dot{\theta} + m_m g * l_m sin(\theta) - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \tag{30}$$

$$\ddot{\theta} = \frac{g sin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta} - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \tag{31}$$

$$-T_m + M_{mf} = I_m(\ddot{\theta} + \ddot{\phi}) \tag{32}$$

$$-T_m + C_m\dot{\phi} = I_m\ddot{\theta} + I_m\ddot{\phi} \tag{33}$$

$$I_m\ddot{\phi} = -I_m\ddot{\theta} - T_m + C_m\dot{\phi} \tag{34}$$

$$\ddot{\phi} = -\ddot{\theta} + \frac{-T_m + C_m\dot{\phi}}{I_m} \tag{35}$$

$$\ddot{\phi} = -\left(\frac{g sin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta} - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2}\right) + \left(\frac{-T_m + C_m\dot{\phi}}{I_m}\right) \tag{36}$$

$$\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m\dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{g sin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \tag{37}$$

$$\boxed{\ddot{\theta} = \frac{g sin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta} - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2}} \tag{38}$$

$$\boxed{\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m\dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{g sin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2}} \tag{39}$$

$$\boxed{T_m = K_m u} \tag{40}$$

5 Control System

5.1 State Space Model

To start, use a linear state space model of the system to design the control algorithm around

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{41}$$

$$x(t) = [\theta,\ \dot{\theta},\ \dot{\phi}] \tag{42}$$

$$u(t) = T_m => K_m u \tag{43}$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} ii & ij & ik \\ ji & jj & jk \\ ki & kj & kk \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} K_m u(t) \tag{44}$$

$$\ddot{\theta} = \frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta} - T_m + C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \tag{45}$$

$$\ddot{\phi} = \frac{(I_{bo} + m_m l_m^2)(T_m - C_m\dot{\phi})}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{gsin(\theta)(m_b l_b + m_m l_m) - C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \tag{46}$$

assuming small angles

$$sin(\theta) \approx \theta \tag{47}$$

$$\ddot{\theta} = \frac{g\theta(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} - \frac{C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2} - \frac{T_m}{I_{bo} + I_m + m_m l_m^2} + \frac{C_m\dot{\phi}}{I_{bo} + I_m + m_m l_m^2} \tag{48}$$

$$\ddot{\phi} = -\frac{C_m\dot{\phi}(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} + \frac{T_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} - \frac{g\theta(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2} + \frac{C_b\dot{\theta}}{I_{bo} + I_m + m_m l_m^2} \tag{49}$$

$$\dot{\theta} = ii\theta + ij\dot{\theta} + ik\dot{\phi} \tag{50}$$

$$ii = 0,\ ij = 1,\ ik = 0 \tag{51}$$

$$\ddot{\theta} = ji\theta + jj\dot{\theta} + jk\dot{\phi} \tag{52}$$

$$ji = \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2},\ jj = -\frac{C_b}{I_{bo} + I_m + m_m l_m^2},\ jk = \frac{C_m}{I_{bo} + I_m + m_m l_m^2} \tag{53}$$

$$\ddot{\phi} = ki\theta + kj\dot{\theta} + kk\dot{\phi} \tag{54}$$

$$ki = -\frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2},\ kj = \frac{C_b}{I_{bo} + I_m + m_m l_m^2},\ kk = -\frac{C_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \tag{55}$$

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g(m_b l_b + m_m l_m)}{I_{bo}+I_m+m_m l_m^2} & -\frac{C_b}{I_{bo}+I_m+m_m l_m^2} & \frac{C_m}{I_{bo}+I_m+m_m l_m^2} \\ -\frac{g(m_b l_b + m_m l_m)}{I_{bo}+I_m+m_m l_m^2} & \frac{C_b}{I_{bo}+I_m+m_m l_m^2} & -\frac{C_m(I_{bo}+m_m l_m^2)}{I_m(I_{bo}+I_m+m_m l_m^2)} \end{bmatrix}} \tag{56}$$

$$\boxed{B = \begin{bmatrix} 0 \\ -\frac{K_m}{I_{bo}+I_m+m_m l_m^2} \\ \frac{K_m(I_{bo}+m_m l_m^2)}{I_m(I_{bo}+I_m+m_m l_m^2)} \end{bmatrix}} \tag{57}$$

5.2 Model Stability

Use eigenvalues of A to determine stability of the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g(m_b l_b + m_m l_m)}{I_{bo}+I_m+m_m l_m^2} & -\frac{C_b}{I_{bo}+I_m+m_m l_m^2} & \frac{C_m}{I_{bo}+I_m+m_m l_m^2} \\ -\frac{g(m_b l_b + m_m l_m)}{I_{bo}+I_m+m_m l_m^2} & \frac{C_b}{I_{bo}+I_m+m_m l_m^2} & -\frac{C_m(I_{bo}+m_m l_m^2)}{I_m(I_{bo}+I_m+m_m l_m^2)} \end{bmatrix} \tag{58}$$

$$|A - \lambda| = 0 \tag{59}$$

$$\left| \begin{bmatrix} 0 & 1 & 0 \\ \frac{g(m_b l_b + m_m l_m)}{I_{bo}+I_m+m_m l_m^2} & -\frac{C_b}{I_{bo}+I_m+m_m l_m^2} & \frac{C_m}{I_{bo}+I_m+m_m l_m^2} \\ -\frac{g(m_b l_b + m_m l_m)}{I_{bo}+I_m+m_m l_m^2} & \frac{C_b}{I_{bo}+I_m+m_m l_m^2} & -\frac{C_m(I_{bo}+m_m l_m^2)}{I_m(I_{bo}+I_m+m_m l_m^2)} \end{bmatrix} - \begin{bmatrix} \lambda_A & 0 & 0 \\ 0 & \lambda_A & 0 \\ 0 & 0 & \lambda_A \end{bmatrix} \right| = 0 \tag{60}$$

$$a_1 = \frac{g(m_b l_b + m_m l_m)}{I_{bo} + I_m + m_m l_m^2},\ a_2 = \frac{C_b}{I_{bo} + I_m + m_m l_m^2} \tag{61}$$

$$a_3 = \frac{C_m}{I_{bo} + I_m + m_m l_m^2},\ a_4 = -\frac{C_m(I_{bo} + m_m l_m^2)}{I_m(I_{bo} + I_m + m_m l_m^2)} \tag{62}$$

$$\left| \begin{bmatrix} -\lambda & 1 & 0 \\ a_1 & -a_2 - \lambda_A & a_3 \\ -a_1 & a_2 & a_4 - \lambda_A \end{bmatrix} \right| = 0 \tag{63}$$

$$-\lambda^3 - a_2\lambda^2 + a_4\lambda^2 + a_2 a_4 \lambda + a_2 a_3 \lambda + a_1 \lambda - a_1 a_4 - a_1 a_3 = 0 \tag{64}$$

Determine stability from roots of the quadratic below. This will be determined from real values for all constants in A after the 2D inverted pendulum is designed.

$$\boxed{\begin{gathered} |A - \lambda| = 0 \\ -\lambda^3 - a_2\lambda^2 + a_4\lambda^2 + a_2 a_4 \lambda + a_2 a_3 \lambda + a_1 \lambda - a_1 a_4 - a_1 a_3 = 0 \end{gathered}} \tag{65}$$