Sequential analysis of convex risk measures

- A tractable mechanism to enable front-line underwriting decisions to reflect risk accumulation across the entire portfolio,
- specification of risk dependencies directly in terms of scenarios,
- worst case analysis with respect to scenario probabilities.

Individual risk model

Consider the classical individual risk model:

$$S = X_1 + X_2 + \ldots + X_n,$$

with X_i representing the loss incurred under the insurance contract i during a given time period.

Assumptions about the joint distribution of $\{X\}_{i=1}^n$ are required in order to make non-trivial statements about S.

If $\{X\}_{i=1}^n$ are taken to be independent, for instance, and further conditions on the moments are satisfied, one can appeal to the Central Limit Theorem to approximate the distribution of S, provided n is sufficiently large.

THE

DOCTRINE

OF

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OR,

A METHOD of Calculating the Probabilities of Events in PLAY.

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By A. DE MOIVRE,

Fellow of the ROYAL SOCIETY, *and Member of the* ROYAL ACADEMIES OF SCIENCES *of* Berlin *and* Paris.

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By A. DE MOIVRE. F. R. S.

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Another approach is to directly apply one of the concentration inequalities, such as the Bernstein inequality, which states that:

$$P\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mathbb{E}(X_i) > \epsilon\right) \leq \exp\left(-\frac{\frac{1}{2}\epsilon^2}{\sum_{i=1}^{n} \mathbb{E}(X_i^2) + \frac{1}{3}M\epsilon}\right),$$

provided that $\{X\}_{i=1}^{n}$ are independent and $X_i \leq M$ almost surely for all i.

This result can be interpreted as saying that excess capital ϵ will be insufficient to meet all claims originating from n policies with the probability no greater than that given by the right hand side.

Modelling dependence via scenarios

Consider a collection of risks $\{1, \ldots, n\}$, with losses from "independent" or "local" causes represented by positive random variables $\{X_i\}_{i=1}^n$ as before. To represent dependencies between risks, we add a finite set of scenarios Ω, together with a family of functions $\psi_i(\omega)$ which denote the loss incurred by the risk i if the scenario $\omega \in \Omega$ is realised.

Finally, we define the loss incurred by the risk i due to both "independent" and systemic causes as the maximum of the two:

$$Z_i = \max\big(X_i, \psi_i(\omega)\big),$$

with the joint distribution of $\big(X_1, \ldots, X_n, \psi_1(\omega), \ldots, \psi_n(\omega)\big)$ determined by a product measure $\left(\bigotimes_{i=1}^n \mathbb{P}_i\right) \otimes \mathbb{P}_\Omega$ on $\mathbb{R}^n \times \Omega$.

Controlling risk accumulation - exponential premium principle

Now assume that an insurer has an exponential utility function $u(x) = 1 - e^{-\alpha x},\ \alpha > 0$ and zero starting endowment. It then prices the part of the cost of a proposed insurance contract i attributable to scenarios in Ω via the zero utility premium principle by solving for C:

$$\mathbb{E}_\omega \left(1 - e^{-\alpha\left(C - \psi_i(\omega)\right)}\right) = \sum_{\omega \in \Omega} p_\omega \left(1 - e^{-\alpha\left(C - \psi_i(\omega)\right)}\right) = 0.$$

The expectation is with respect to the insurer's subjective assessment of scenario probabilities.

We can rewrite the above as follows:

$$e^{-\alpha C} \sum_{\omega \in \Omega} p_\omega e^{\alpha \psi_i(\omega)} = 1$$

and obtain an expression for the price of the exposure ψ_i given an empty starting “inventory” of risks:

$$C(\psi_i) = \frac{1}{\alpha} \log \sum_{\omega \in \Omega} p_\omega e^{\alpha \psi_i(\omega)}.$$

obtaining the exponential premium principle (Gerber, 1974).

It is easy to check that the sequential application of the exponential premium principle has the property that $C(\psi_j + \xi \,|\, \xi)$, the price charged for additional exposure ψ_j, resulting from accepting risk j when insurer's total portfolio exposure is given by $\xi \in \mathbb{R}_+^{|\Omega|}$, decomposes as:

$$C(\psi_j + \xi \,|\, \xi) = C(\psi_j + \xi) - C(\xi).$$

This implies that there is no benefit for an insured to split their desired cover into multiple contracts if all are to be placed at the same time.

The "algorithm" employed by the insurer sequentially applying the exponential premium principle is extremely simple:

Algorithm 1 Sequential application of exponential premium principle

Require: parameter α, permutation π.

1: initialize $\xi \leftarrow 0$
2: **for** $j = 1, \ldots, t$ **do**
3: quote $C(\psi_{\pi(j)} + \xi) - C(\xi)$ for the insurance contract $\pi(j)$.
4: $\xi \leftarrow \psi_{\pi(j)} + \xi$ (assume every quote is accepted without loss of generality).
5: **end for**

Output: Scenario $\omega \in \Omega$ is realised. Claims of the amount $\sum_{j=1}^{t} \psi_{\pi(j)}(\omega)$ paid out.

Upper bound on the loss

We can write down the worst possible loss for the insurer following the algorithm in the previous subsection as follows:

$$\sup_{\xi,\omega} \xi(\omega) - \big(C(\xi) - C(0)\big).$$

Consider the objective for a fixed ω:

$$\xi(\omega) - \big(C(\xi) - C(0)\big) = \frac{1}{\alpha}\log p_\omega e^{\alpha\xi(\omega)} - \frac{1}{\alpha}\log p_\omega - \left(\frac{1}{\alpha}\log\Big(\sum_{\omega'\in\Omega} p_{\omega'}e^{\alpha\xi(\omega')}\Big) - C(0)\right)$$

$$= \frac{1}{\alpha}\log\left(\frac{p_\omega e^{\alpha\xi(\omega)}}{\sum_{\omega'\in\Omega} p_{\omega'}e^{\alpha\xi(\omega')}}\right) - \frac{1}{\alpha}\log p_\omega.$$

It is easy to see that:

$$\frac{1}{\alpha}\log\left(\frac{p_\omega e^{\alpha\xi(\omega)}}{\sum_{\omega'\in\Omega} p_{\omega'} e^{\alpha\xi(\omega')}}\right) \leq 0$$

for any $\xi \in \mathbb{R}^{|\Omega|}$. This leaves us with $\sup_\omega \frac{1}{\alpha}\log\frac{1}{p_\omega}$ as the upper bound on the loss of the insurer.

This upper bound in turn is minimised if the insurer's subjective distribution $p(\omega)$ is uniform and is then given by:

$$\left(\sup_{\xi,\omega} \xi(\omega) - \big(C(\xi) - C(0)\big)\right) \leq \frac{1}{\alpha}\log|\Omega|.$$

Note that the bound is independent of the real world probabilities of the scenarios (Hanson, 2007).

Convex premium principles

Recall that convex premium principles satisfy the following properties:

convexity: $C(\xi)$ is a convex function of ξ,

increasing monotonicity: for any ξ and ξ' if $\xi \geq \xi'$ (entrywise), then $C(\xi) \geq C(\xi')$,

translation invariance: for any ξ and k, $C(\xi + k\mathbf{1}) = C(\xi) + k$.

This is identical, up to a minus sign, to the definition of *convex risk measures* (Föllmer and Schied, 2002).

We notice that the Fenchel conjugate of a convex premium principle C defined over a finite dimensional outcome space Ω is given by:

$$C^*(\nu) = \sup_{\xi} \ \ \nu^T \xi - C(\xi),$$

which can be further elaborated as:

$$C^*(\nu) = \begin{cases} \sup\limits_{\xi} \ \ \nu^T \xi - C(\xi) & \text{for } \nu^T \mathbf{1} = 1, \ \nu \geq 0 \\ \infty & \text{otherwise,} \end{cases}$$

where the condition $\nu^T \mathbf{1} = 1$ follows from translation invariance and $\nu \geq 0$ from increasing monotonicity.

Now we can derive worst case loss bounds for convex premium principles in terms of their conjugates:

$$\sup_{\xi,\omega} \ \xi(\omega) - (C(\xi) - C(0)) = \sup_{\omega} C^*\big(\delta_\omega\big) + C(0),$$

where δ_ω is the Kronecker delta.

Note that by using the fact that $C^{**} = C$, we recover the "dual representation" (Föllmer and Schied, 2002) in the finite dimensional setting:

$$C(\xi) = C^{**}(\xi) = \sup_{\nu \in \Delta} \xi^T \nu - C^*(\nu),$$

where Δ is the $|\Omega|$ dimensional probability simplex.

Modifying the individual risk model

Consider a sequence of policies $1, \ldots, t$ being issued in order from a pool of n available risks. Denote the identity of the risk in position j, $1 \leq j \leq t$ as $\pi(j)$ with π being a permutation and recall that:

$$Z_i = \max\big(X_i, \psi_i(\omega)\big).$$

Then for any choice of measures $\mathbb{P}_i$ such that $X_i < M$ almost surely, and any choice of $\mathbb{P}_\Omega$:

$$P\left(\sum_{i=1}^{t} Z_{\pi(i)} - C_t \geq \epsilon\right) \leq \exp\left(\frac{-\frac{1}{2}\big(\epsilon - \frac{1}{\alpha}\log(|\Omega|)\big)^2}{\sum_{i=1}^{t}\mathbb{E}X_{\pi(i)}^2 + \frac{1}{3}M\big(\epsilon - \frac{1}{\alpha}\log(|\Omega|)\big)}\right),$$

where C_t corresponds to the total premium charged to accept the collection of risks $\{\pi(i)\}_{i=1}^{t}$:

$$C_t = \sum_{i=1}^{t} \mathbb{E} X_{\pi(i)} + \frac{1}{\alpha} \log \Bigg(\sum_{\omega \in \Omega} \exp \Big(\alpha \sum_{i=1}^{t} \psi_{\pi(i)}(\omega) \Big) \Bigg).$$

This follows directly from combining Bernstein bound and the worst case loss bound in the previous section.

Applications

- The proposed scheme makes it practical to incorporate risk accumulations into operational decision making, whether pricing or placement of facultative reinsurance,

- it may also provide a viable complement to purely statistical approaches that rely largely on historical data.

- finally it demonstrates a setting where it might be beneficial to depart from independent "scoring".