Prediction markets and implications for insurance

- subjective probabilities,
- traditional betting market mechanisms,
- "automated market makers", exponential utility premium principle and the "logarithmic market scoring rule",
- remarkable bounds on worst case performance and a connection with convex risk measures,
- speculative implications for insurance.

Subjective probabilities

PROBABILITY DOES NOT EXIST. The abandonment of superstitious beliefs about the existence of Phlogiston, the Cosmic Ether, Absolute Space and Time...or Fairies and Witches, was an essential step along the road to scientific thinking. Probability, too, if regarded as something endowed with some kind of objective existence, is no less a misleading conception, an illusory attempt to exteriorize or materialize our true probabilistic beliefs.

Bruno de Finetti, *Theory of Probability*, 1974

The proposal of de Finetti was to define “probabilities” as prices one would place on certain lottery tickets.

Subjective probabilities - continued

Denote $p(E)$ as the price at which one is indifferent between buying and selling a lottery ticket that pays $\$1$ if the event E occurred and $\$0$ otherwise. Similarly $p(E|F)$ is the indifference price for a ticket paying $\$1$ if $E \cap F$ occurs, $\$0$ if F occurs without E and $\$-1$ if F does not occur.

Following assumptions make the prices "coherent", i.e. eliminate "arbitrage":

- $p(E) \geq \$0$,
- $p(E) + p(E^C) = \$1$,
- $p(E \cup F) = p(E) + p(F)$ if $E \cap F = \varnothing$,
- $p(E \cap F) = p(E|F)p(F)$.

This is identical to the "no arbitrage" assumptions in finance.

Betting or prediction markets

The above are subjective probabilities. Neither de Finetti's formalism nor classical Bayesian updating say anything about how to generate consensus estimates. An appealing (to some) approach is to defer to a market. Most popular *mechanisms* include:

- **parimutuel betting:** the total pool is allocated to the participants betting on the winning outcome in proportion to the size of their original bets.
- **fixed odds betting:** a bookmaker is the counterparty in every transaction and is responsible for updating the odds.
- **continuous double auction:** every market participant can "buy" and "sell" bets. A transaction occurs when bid and ask prices overlap.

Limitations of traditional mechanisms

- for parimutuel, the odds are not known until the betting has concluded, which ca discourage a sophisticated participant from betting just before closure.
- fixed odds betting requires a sophisticated bookmaker, who can be exposed to potentially unlimited losses.
- double auction mechanisms can suffer from low liquidity.

Turns out it is possible to design a mechanism that is no more complicated than parimutuel and yet both quotes odds at any time and limits the worst case loss.

An automated (prediction) market maker

Consider a discrete space of events Ω with n elements and assume that the market maker has an exponential utility function $u(x) = 1 - e^{-\alpha x},\ \alpha > 0$ and zero starting endowment. It then prices any proposed bet $\mathbf{q} \in \mathbb{R}^n$ via the equivalent utility (premium) principle by solving for C:

$$\mathbb{E}(1 - e^{-\alpha(C-\mathbf{q})}) = \sum_{i=1}^{n} p_i \left(1 - e^{-\alpha C - q_i}\right) = 0.$$

We can rewrite the above as follows:

$$e^{-\alpha C} \sum_{i=1}^{n} -p(\omega_i) e^{\alpha q_i} = -1$$

and obtain an expression for the price of the bet $\mathbf{q}$ given an empty starting inventory:

$$C(\mathbf{q}) = \frac{1}{\alpha} \log \sum_{i=1}^{n} p_i e^{\alpha q_i}.$$

It is easy to check that the cost function is *path independent*, namely, denoting by $C(\mathbf{q} + \mathbf{r} \,|\, \mathbf{r})$ the price charged for the bet $\mathbf{q}$ by the market maker with the inventory $\mathbf{r} \in \mathbb{R}^n$, we obtain:

$$C(\mathbf{q} + \mathbf{r} \,|\, \mathbf{r}) = C(\mathbf{q} + \mathbf{r}) - C(\mathbf{r}).$$

This implies that the market participants do not gain by "strategically" splitting their bets.

Logarithmic market scoring rule

The algorithm employed by the market maker following the “logarithmic scoring rule” is extremely simple:

- given the current inventory of bets $\mathbf{r}$, charge (or pay out) $C(\mathbf{q}+\mathbf{r}) - C(\mathbf{r})$ for the proposed bet $\mathbf{q}$.
- if the quote is accepted, update the inventory to be $\mathbf{q}+\mathbf{r}$ and wait for next request.

Assuming the relative size of the bet is small, we can approximate the price vector of n bets paying $\$1$ in state i by $\nabla_{\mathbf{r}} C(\mathbf{r})$:

$$\frac{\partial C}{\partial r_i} = \frac{p_i e^{\alpha r_i}}{\sum_{j=1}^{n} p_j e^{\alpha r_j}}.$$

- this is an exponential family probability distribution with *carrier density* $\mathbf{p}$ or the Esscher transform of $\mathbf{p}$.

Upper bound on the loss of the market maker

None of this would be of any interest if the above procedure did not guarantee to limit the loss the market maker can suffer irrespective of the bets it accept or the actual outcome $\omega \in \Omega$.
We can write down the worst possible loss as follows:

$$\underset{\mathbf{q}, i \in \{1,\ldots,n\}}{\text{maximize}} \quad q_i - (C(\mathbf{q}) - C(\mathbf{0}))$$

Consider the objective for a fixed i:

$$q_i - (C(\mathbf{q}) - C(\mathbf{0})) = \frac{1}{\alpha}\log p_i e^{\alpha q_i} - \frac{1}{\alpha}\log p_i - \left(\frac{1}{\alpha}\log\left(\sum_{j=1}^{n} p_j e^{\alpha q_j}\right) - C(\mathbf{0})\right)$$

$$= \frac{1}{\alpha}\log\left(\frac{p_i e^{\alpha q_i}}{\sum_{j=1}^{n} p_j e^{\alpha q_j}}\right) - \frac{1}{\alpha}\log p_i$$

Upper bound - continued

- it is easy to see that $\frac{1}{\alpha}\log\left(\frac{p_i e^{\alpha q_i}}{\sum_{j=1}^{n} p_j e^{\alpha q_j}}\right)$ is bounded from above by 0 for any $\mathbf{q} \in \mathbb{R}^n$.
- this leaves us with $\max_i \frac{1}{\alpha}\log\frac{1}{p_i}$ as the upper bound on the loss of the market maker.
- this upper bound in turn is minimised if the market maker's subjective distribution $\mathbf{p}$ is uniform and is then given by $\frac{1}{\alpha}\log n$.

Some additional properties of the cost function

We can (but won't) verify additional properties of C:

- **convexity:** $C(\mathbf{q})$ is a convex function of $\mathbf{q}$,
- **increasing monotonicity:** for any $\mathbf{q}$ and $\mathbf{r}$ if $\mathbf{q} \geq \mathbf{r}$ (entrywise), then $C(\mathbf{q}) \geq C(\mathbf{r})$,
- **translation invariance:** for any $\mathbf{q}$ and k, $C(\mathbf{q} + k\mathbf{1}) = C(\mathbf{q}) + k$.

This is identical (up to a minus sign) to the definition of *convex risk measures* Föllmer and Schied (2002) and it turns out that we can use their "representation theorem" to derive worst case bounds for any convex risk measure used as the cost function C.

Some speculative insurance applications

- is it possible to incorporate certain automatic safeguards against unanticipated or risky changes in portfolio composition by making quotes dependent on the current "inventory", e.g. by limiting exposure to certain factors?

- it may also be possible to devise insurance products around this idea directly, e.g. in situations when a diverse group of agents have "insurable interest" in certain events.