General Insurance Seminar

Insuring Tomorrow

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A Unified Approach to Reserving and Pricing

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Unified approach to reserving and pricing

- Chain ladder as Poisson regression
- Standard pricing models
- Policy level chain ladder
- Hybrid models.

Motivation

- Actuarial profession in GI split into reserving / pricing sub-specialisations.
- Relatively little methodological overlap, even when done by the same person.
- Use of risk factors in reserving can really help in situations when the portfolio composition is changing.
- Availability of policy level IBNR simplifies much of monitoring / reporting.

Chain ladder as Poisson regression

Consider the usual claims triangle:

$$
\begin{array}{cccc}
y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\
y_{2,1} & y_{2,2} & y_{2,3} & \\
y_{3,1} & y_{3,2} & & \\
y_{4,1} & & &
\end{array}
$$

where $y_{i,j}$ denotes realised incremental claims experience (count, paid or incurred cost movements etc) in accident period i and development period j.

Chain ladder as Poisson regression - cont'd

We then recast this into regression form where each entry in the claims triangle gets its own row in the dependent variable vector $\mathbf{y}$ and the design matrix X:

$$\mathbf{y} = \begin{bmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ y_{1,4} \\ y_{2,1} \\ y_{2,2} \\ y_{2,3} \\ y_{3,1} \\ y_{3,2} \\ y_{4,1} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

Chain ladder as Poisson regression - cont'd

Now take $\mathbf{y}_i \in \mathbb{Z}^+$ to be a random variable with Poisson distribution and mean $\exp(\mathbf{x}_i\mathbf{w})$, where $\mathbf{x}_i$ denotes the i^{th} row of X and $\mathbf{w}$ the model parameters:

$$P(\mathbf{y}_i \,|\, \mathbf{x}_i\mathbf{w}) = \frac{\exp\big(-\exp(\mathbf{x}_i\mathbf{w})\big)\exp(\mathbf{x}_i\mathbf{w})^{\mathbf{y}_i}}{\mathbf{y}_i!}.$$

This can also be rewritten without referencing X and $\mathbf{w}$ but only components of $\mathbf{w}$ directly

$$P(y_{j,k} \,|\, \alpha_k, \beta_k) = \frac{\exp\big(-\exp(\alpha_j + \beta_k)\big)\exp(\alpha_j + \beta_k)^{y_{j,k}}}{y_{j,k}!}.$$

Chain ladder as Poisson regression - cont'd

Now assuming that all observations $(\mathbf{y}_i, \mathbf{x_i}),\ i = 1, \ldots, n$ are independent, the likelihood function is as follows:

$$P(\mathbf{y} \,|\, X\mathbf{w}) = \prod_{i=1}^{n} \frac{\exp\big(-\exp(\mathbf{x}_i\mathbf{w})\big)\exp(\mathbf{x}_i\mathbf{w})^{\mathbf{y}_i}}{\mathbf{y}_i!},$$

with the corresponding log-likelihood:

$$\mathcal{L}(\mathbf{y} \,|\, X\mathbf{w}) = \sum_{i=1}^{n} \big(-\exp(\mathbf{x}_i\mathbf{w}) + \mathbf{y}_i\mathbf{x}_i\mathbf{w} - \log(\mathbf{y}_i!)\big).$$

Chain ladder as Poisson regression - cont'd

A maximum likelihood estimate of $\mathbf{w}$ can then be obtained by solving the following concave maximisation problem:

$$\underset{\mathbf{w}}{\text{maximise}} \ \sum_{i=1}^{n} \big(-\exp(\mathbf{x}_i\mathbf{w}) + \mathbf{y}_i\mathbf{x}_i\mathbf{w} \big),$$

Note that the above problem does not have a unique solution, but this can be overcome by introducing a constraint $\alpha_1 = \log\left(\sum_k y_{1,k}\right)$.

For the resulting unique solution $\exp \alpha_j^*$ corresponds to the CL ultimate for the j^{th} accident period and $\exp \beta_k^*$ to the proportion of the ultimate observed during the k^{th} development period.

Standard pricing models

For policy i denote the vector describing risk factors as $\mathbf{x}^{(i)}$, the weight of the observation as $\epsilon^{(i)}$ and the response value of interest as $y^{(i)}$. For Poisson model we have:

$$P(y^{(i)} \,|\, \mathbf{x}^{(i)}\mathbf{w}) = \frac{\exp\big(-\epsilon^{(i)}\exp(\mathbf{x}^{(i)}\mathbf{w})\big)\big(\epsilon^{(i)}\exp(\mathbf{x}^{(i)}\mathbf{w})\big)^{y^{(i)}}}{y^{(i)}!},$$

which in turn yields the below maximum likelihood estimation problem:

$$\underset{\mathbf{w}}{\text{maximise}} \;\; \sum_{i=1}^{n} \bigg(-\epsilon^{(i)}\exp\big(\mathbf{x}^{(i)}\mathbf{w}\big) + y^{(i)}\mathbf{x}^{(i)}\mathbf{w}\bigg).$$

Standard pricing models - observation

Note that if $\mathbf{x}^{(i)} = \mathbf{x}^{(j)} = \mathbf{x}$ for all $i, j \in K$, we have:

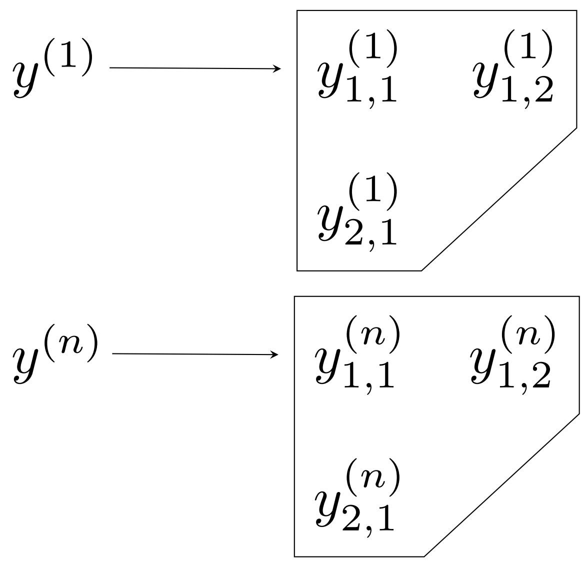
$$\sum_{i\in K}\Big(-\epsilon^{(i)}\exp(\mathbf{x}^{(i)}\mathbf{w}) + y^{(i)}\mathbf{x}^{(i)}\mathbf{w}\Big) = -\Big(\sum_{i\in K}\epsilon^{(i)}\Big)\exp(\mathbf{x}\mathbf{w}) + \Big(\sum_{i\in K}y^{(i)}\Big)\mathbf{x}\mathbf{w},$$

i.e. one gets the same solution $\mathbf{w}^*$ if the rows for the policies with identical exposure characteristics are combined together and weights added up.

Incorporating claims development

Splitting each row (i) into multiple rows, sub-indexed by (i, j), in such a way that:

$$y^{(i)} = \sum_k \sum_j y^{(i)}_{k,j}.$$



Incorporating claims development

This representation is then transformed back to a Poisson regression:

$$\mathbf{y} = \begin{bmatrix} y_{1,1}^{(1)} \\ y_{1,2}^{(1)} \\ y_{2,1}^{(1)} \\ \vdots \\ y_{1,1}^{(n)} \\ y_{1,2}^{(n)} \\ y_{2,1}^{(n)} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1^{(1)} \\ \epsilon_1^{(1)} \\ \epsilon_2^{(1)} \\ \vdots \\ \epsilon_1^{(n)} \\ \epsilon_1^{(n)} \\ \epsilon_2^{(n)} \end{bmatrix}.$$

where $\sum_k \epsilon_k^{(i)} = \epsilon^{(i)}$.

Exposure weighted chain ladder

By applying the earlier observation, it directly follows that the above problem can be simplified to:

$$\mathbf{y} = \begin{bmatrix} \sum_{k=1}^{n} y_{1,1}^{(k)} \\ \sum_{k=1}^{n} y_{1,2}^{(k)} \\ \sum_{k=1}^{n} y_{2,1}^{(k)} \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \sum_{k=1}^{n} \epsilon_{1}^{(k)} \\ \sum_{k=1}^{n} \epsilon_{1}^{(k)} \\ \sum_{k=1}^{n} \epsilon_{2}^{(k)} \end{bmatrix},$$

which is a regression where the MLE is equivalent to the weighted chain ladder algorithm.

Allowing for claim development in pricing models

Now that we've derived chain ladder from policy level data, it is straightforward to reintroduce risk factors:

$$X' = \begin{bmatrix} 1 & 0 & 1 & 0 & \mathbf{x}^{(1)} \\ 1 & 0 & 0 & 1 & \mathbf{x}^{(1)} \\ 0 & 1 & 1 & 0 & \mathbf{x}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 & \mathbf{x}^{(n)} \\ 1 & 0 & 0 & 1 & \mathbf{x}^{(n)} \\ 0 & 1 & 1 & 0 & \mathbf{x}^{(n)} \end{bmatrix}.$$

The new problem will no longer have a closed form solution, but will require the use of GLM estimation software.

Other comments

- Possible to introduce interactions - e.g. a full interaction with a categorical variable denoting risk type will yield independent chain ladder estimates for each type of risk, if no other risk factors are present.
- The basic approach can be readily extended to encompass other popular reserving methods, e.g. PPCI.
- Can apply smoothing to additional parameters to control overfitting.
- No requirement to be restricted to the Poisson distribution.