Data driven control schemes in insurance

- optimisation perspective on dynamic GLMs with connections to credibility and graduation
- joint estimation and control via adaptive price testing
- controlling concentration of risk - worst case loss bounds for the exponential premium principle and more general convex risk measures

D. Semenovich, NICTA

Dynamic parameter estimation as convex optimisation

Assume that there are m time periods and in the time period t we obtain n new observations, with i-th observation at time t denoted as y_{ti}.

The sample average is the solution of the following optimisation problem:

$$\underset{w}{\text{minimise}} \sum_{t=1}^{m} \|\mathbf{y}_t - \mathbf{1}w\|_2^2 = \sum_{t=1}^{m}\sum_{i=1}^{n}(y_{ti} - w)^2.$$

This can then be equivalently rewritten with vector $\mathbf{w} \in \mathbb{R}^m$ as the decision variable:

$$\begin{aligned} &\underset{\mathbf{w}}{\text{minimise}} \quad \sum_{t=1}^{m} \|\mathbf{y}_t - \mathbf{1}w_t\|_2^2 = \sum_{t=1}^{m}\sum_{i=1}^{n}(y_{ti} - w_t)^2 \\ &\text{subject to } \; w_{t+1} - w_t = 0, \quad t = 1, \ldots, m-1. \end{aligned}$$

It is then natural to replace hard constraints with e.g. a quadratic penalty term:

$$\underset{\mathbf{w}}{\text{minimise}} \sum_{t=1}^{m} \|\mathbf{y}_t - \mathbf{1} w_t\|_2^2 + \lambda \|D^{(1,m)}\mathbf{w}\|_2^2 = \sum_{t=1}^{m}\sum_{i=1}^{n}(y_{ti} - w_t)^2 + \lambda \sum_{t=1}^{m-1}(w_{t+1} - w_t)^2$$

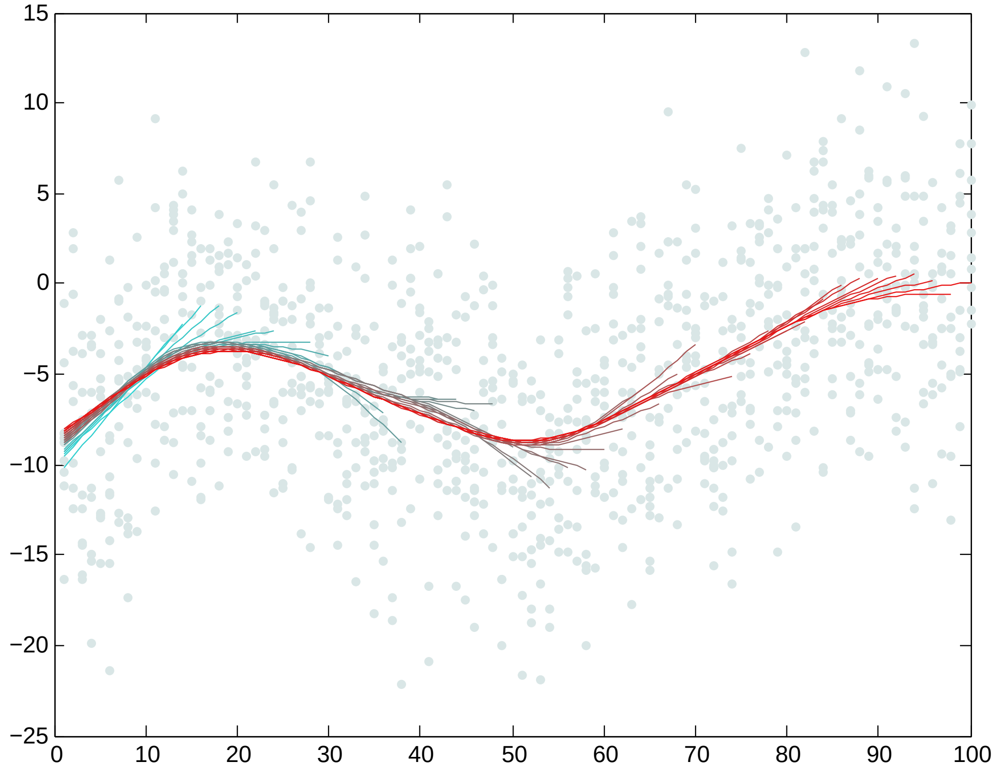
Here $D^{(1,m)}$ is the $(m-1) \times m$ first order finite differences matrix:

$$D^{(1,m)} = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdots & & \\ & & & -1 & 1 \end{bmatrix}$$

and the k-th order finite differences matrix $D^{(k,m)} \in \mathbb{R}^{(m-k)\times m}$ is recursively defined as:

$$D^{(k,m)} = D^{(1,m-k)} D^{(k-1,m)}, \; k = 2, 3, \ldots$$

Dynamic mean estimation - example



This is essentially the same as Jones-Gerber "evolutionary credibility" or, for a fixed t, Bohlmann-Whittaker graduation.

Dynamic parameter estimation - exponential families

Consider a single parameter exponential family of distributions:

$$p(y \mid w_t) = h_0(y) \exp\big(w_t\phi(y) - A(w_t)\big), \tag{1}$$

where A is the log-partition function and ϕ the sufficient statistic.

We can then dynamically update the parameter by re-solving the following penalised maximum likelihood problem for each time period:

$$\underset{\mathbf{w}}{\text{minimise}} \;\; \sum_{t=1}^{m}\sum_{i=1}^{n}\big(A(w_t) - w_t\phi(y_{ti})\big) + \lambda\|D^{(k,m)}\mathbf{w}\|_2^2. \tag{2}$$

Dynamic density estimation - quantile loss

Dynamic density estimation can also be performed without making explicit parametric assumptions, using e.g. quantile loss:

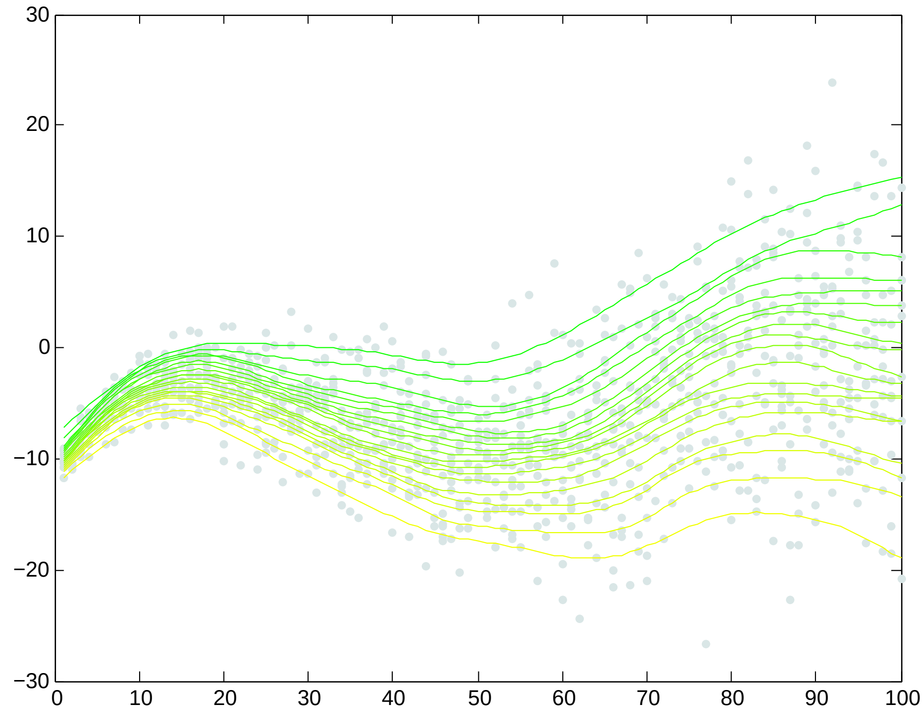
$$\rho_\tau(u) = \begin{cases} \tau u, & u > 0 \\ -(1-\tau)u, & u \leq 0. \end{cases}$$

This results in the following optimisation problem:

$$\underset{\mathbf{w}}{\text{minimise}} \quad \sum_{t=1}^{m} \sum_{i=1}^{n} \rho_\tau(y_{ti} - w_t) + \lambda \| D^{(k,m)} \mathbf{w} \|_2^2, \tag{3}$$

known as quantile splines (Koenker, 1994).

Quantile splines - example



Quantile splines provide time-varying non-parametric density estimation.

Dynamic ordinary least squares

Partition design matrix X and response vector $\mathbf{y}$ into m row blocks:

$$X = \begin{bmatrix} X_1 \\ \cdots \\ X_m \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \cdots \\ \mathbf{y}_m \end{bmatrix}$$

then we can transform the least squares problem:

$$\underset{\mathbf{w}}{\text{minimise}} \quad \|X\mathbf{w} - \mathbf{y}\|_2^2$$

to the following equivalent form with m copies of the parameter vector:

$$\begin{array}{ll} \underset{\mathbf{w}_1,\ldots,\mathbf{w}_m}{\text{minimise}} & \sum_{t=1}^{m} \|X_t\mathbf{w}_t - \mathbf{y}_t\|_2^2 \\ \text{subject to} & \mathbf{w}_t = \mathbf{w}_{t+1}, \quad t = 1,\ldots,m-1 \end{array}$$

State space estimation as an optimisation problem

The above block least squares problem corresponds to this state space model:

$$\mathbf{w}_{t+1} = \mathbf{w}_t \qquad \begin{aligned} \mathbf{y}_t &= X_t \mathbf{w}_t + \boldsymbol{\epsilon}_t \\ \boldsymbol{\epsilon}_t &\sim \mathcal{N}(0, I). \end{aligned}$$

Now add state transition noise (or equivalently, relax the equality constraints on the parameter vectors):

$$\begin{aligned} \mathbf{w}_{t+1} &= \mathbf{w}_t + \boldsymbol{\nu}_t \qquad & \mathbf{y}_t &= X_t \mathbf{w}_t + \boldsymbol{\epsilon}_t \\ \boldsymbol{\nu}_t &\sim \mathcal{N}(0, I) & \boldsymbol{\epsilon}_t &\sim \mathcal{N}(0, I). \end{aligned}$$

We can now write down the resulting estimation problem as follows:

$$\underset{\mathbf{w}_1, \ldots, \mathbf{w}_m}{\text{minimise}} \quad \textstyle\sum_{t=1}^{m} \|X_t \mathbf{w}_t - \mathbf{y}_t\|_2^2 + \sum_{t=1}^{m-1} \|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^2.$$

The above formulation performs both “filtering” and “smoothing” conditional on all the observations up to time m.

If new information becomes available, augment the optimisation problem and solve again to obtain new estimates $\mathbf{w}^* = (\mathbf{w}_1^*, \ldots, \mathbf{w}_m^*, \mathbf{w}_{m+1}^*)$.

General linear Gaussian setting

State equations for the general linear Gaussian state space model are as follows:

$$\begin{aligned} \mathbf{w}_{t+1} &= F\mathbf{w}_t + \boldsymbol{\nu}_t & \mathbf{y}_t &= X_t\mathbf{w}_t + \boldsymbol{\epsilon}_t \\ \boldsymbol{\nu}_t &\sim \mathcal{N}(0, \Sigma_\nu) & \boldsymbol{\epsilon}_t &\sim \mathcal{N}(0, \Sigma_\epsilon). \end{aligned}$$

Denoting $\|\mathbf{a}\|_P = (\mathbf{a}^T P \mathbf{a})^{\frac{1}{2}}$, P-quadratic norm for a positive semidefinite matrix P, the resulting estimation problem is:

$$\underset{\mathbf{w}_1,\ldots,\mathbf{w}_m}{\text{minimise}} \quad \sum_{t=1}^{m} \|X_t\mathbf{w}_t - \mathbf{y}_t\|^2_{\Sigma_\epsilon^{-1}} + \sum_{t=1}^{m-1} \|\mathbf{w}_{t+1} - F\mathbf{w}_t\|^2_{\Sigma_\nu^{-1}}$$

The Kalman filter can be reinterpreted as a recursive solution scheme, exploiting the block diagonal structure in linear system corresponding to the first order optimality condition.

Non-Gaussian observations

We can use any convex loss for the observations, such as quantile, logistic, Poisson, Huber etc:

$$\underset{\mathbf{w}_1,\ldots,\mathbf{w}_m}{\text{minimise}} \quad \sum_{t=1}^{m} \mathcal{L}(X_t\mathbf{w}_t, \mathbf{y}_t) + \sum_{t=1}^{m-1} \|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^2.$$

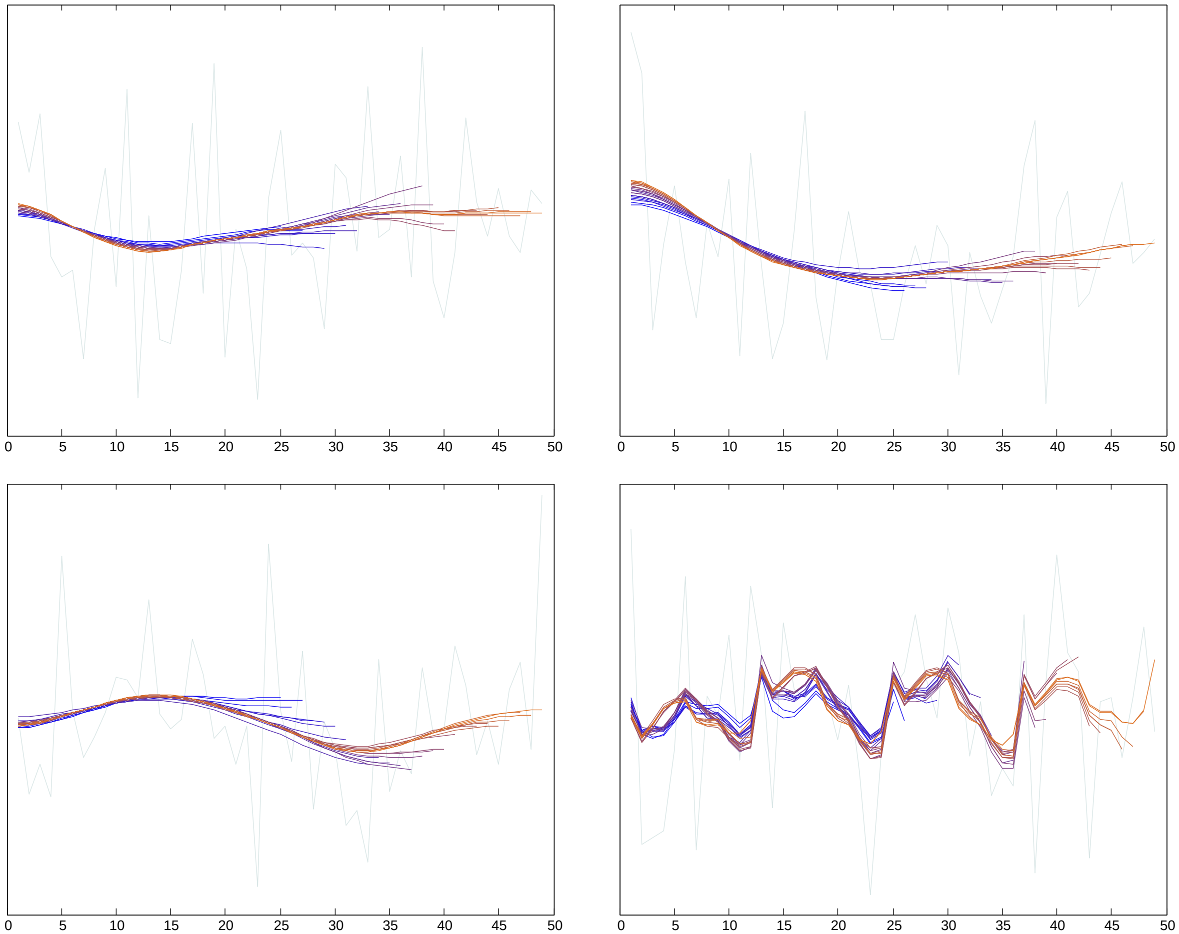
For example for the quantile loss, where τ is the quantile of interest we would have:

$$\mathcal{L}(X_i\mathbf{w}_i, \mathbf{y}_i) = \sum_{i=1}^{n} \rho_\tau(y_{ti} - \mathbf{w}^T\mathbf{x}_{ti})$$

and for conditional exponential families (GLMs):

$$\mathcal{L}(X_t\mathbf{w}_t, \mathbf{y}_t) = \sum_{i=1}^{n} \big(B(\mathbf{w}^T\mathbf{x}_{ti}) - y_{ti}\mathbf{w}^T\mathbf{x}_{ti}\big), \quad B(\theta) = \log\Big(\int_{y\in\mathcal{Y}} \exp\big(\theta y\big) h_0(y)\, dy\Big).$$

Dynamic GLMs - example



Non-Gaussian state noise

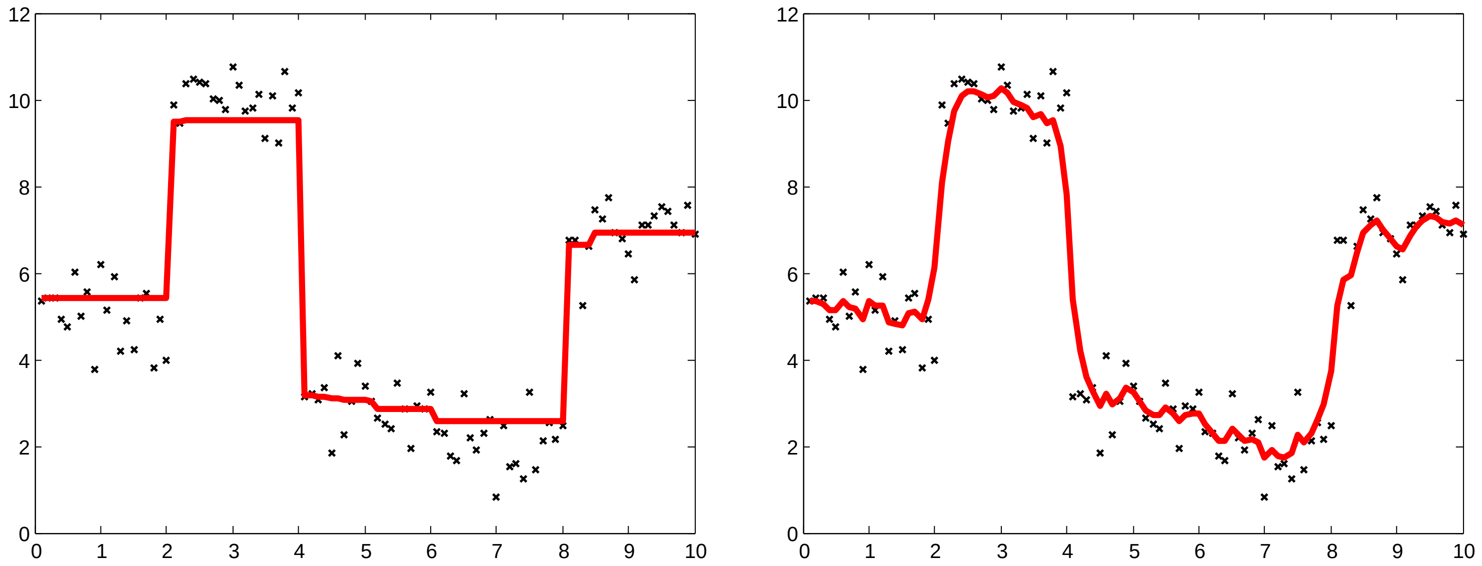
It may be beneficial to apply ℓ_1 norm penalty to state changes provided most of the time parameters stay constant with occasional large jumps:

$$\underset{\mathbf{w}_1,\ldots,\mathbf{w}_m}{\text{minimise}} \quad \sum_{i=1}^{m} \mathcal{L}(X_i \mathbf{w}_i, \mathbf{y}_i) + \sum_{i=1}^{m-1} \|\mathbf{w}_{i+1} - \mathbf{w}_i\|_1.$$

Another possibility is a combination of norms - this will attempt to decompose the state trajectory into a smooth and a piecewise constant component:

$$\underset{\mathbf{w}_1,\ldots,\mathbf{c}_m,}{\text{minimise}} \quad \sum_{i=1}^{m} \mathcal{L}(X_i(\mathbf{w}_i + \mathbf{c}_i), \mathbf{y}_i) + \lambda \sum_{i=1}^{m-1} \|\mathbf{w}_{i+1} - \mathbf{w}_i\|_1 + \mu \sum_{i=1}^{m-1} \|\mathbf{c}_{i+1} - \mathbf{c}_i\|_2^2.$$

ℓ_1-norm vs. squared Euclidean norm regularisation



The effect of using a $\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_1$ penalty term (*left*) versus that of the squared Euclidean norm $\|\mathbf{w}_{t+1} - \mathbf{w}_t\|_2^2$ (*right*).

Other modifications

We can allow linear trends in the parameters (this formulation can be reduced to the standard state space model by expanding the state vector):

$$\underset{\mathbf{w}_1,\ldots,\mathbf{w}_m}{\text{minimise}} \quad \sum_{i=1}^{m} \mathcal{L}(X_i \mathbf{w}_i, \mathbf{y}_i) + \sum_{i=1}^{m-2} \|\mathbf{w}_{i+2} - 2\mathbf{w}_{i+1} + \mathbf{w}_i\|_1.$$

Seasonality adjustments can be handled through the introduction of some equality constraints:

$$\begin{array}{ll} \underset{\mathbf{w}_1,\ldots,\mathbf{c}_m,}{\text{minimise}} & \sum_{i=1}^{m} \mathcal{L}(X_i(\mathbf{w}_i + \mathbf{c}_i), \mathbf{y}_i) + \sum_{i=1}^{m-1} \|\mathbf{w}_{i+1} - \mathbf{w}_i\|_2^2 \\ \text{subject to} & \mathbf{c}_i = \mathbf{c}_{i+k}, \quad i = 1, \ldots, m-k \\ & \sum_{i=1}^{m} \mathbf{c}_i = 0. \end{array}$$

Dynamic GLMs - summary

- All of the models discussed so far result in *convex* optimisation problems, which means that local optimality conditions are sufficient for global optimality.

- Advances in hardware and mathematical programming over the last 20 years have allowed for this type of problems to be solved using general purpose solvers and modelling software, at least for the data set sizes typically found in insurance applications.

- Close relation to mixed effect models - not fully explored.

Taking demand into account

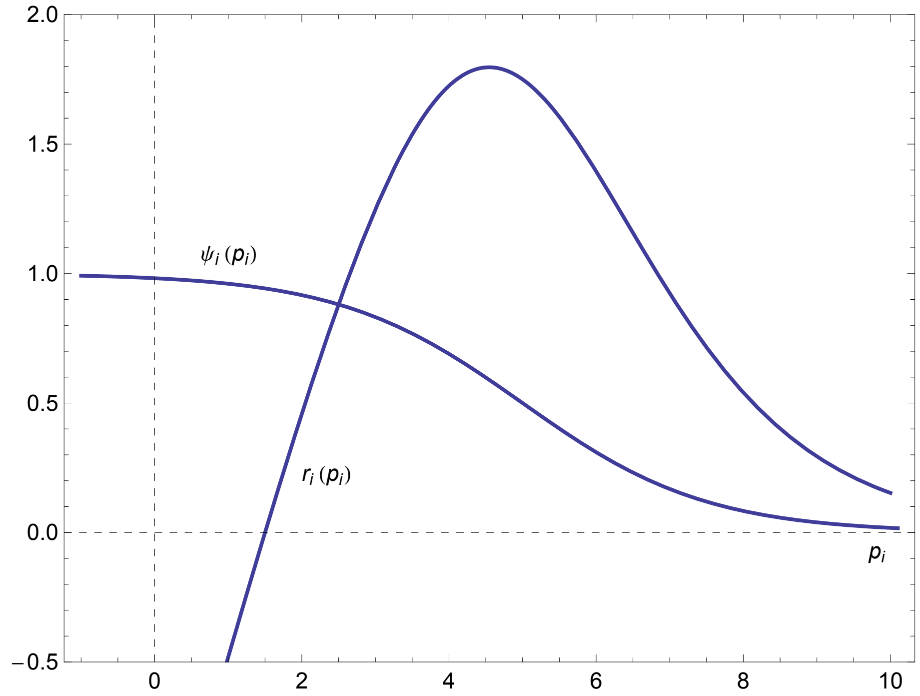
To work out the surplus of an insurance pool with voluntary participation for a given premium schedule, it is necessary to estimate the probabilities of individuals staying in the pool when offered these rates.

A simple single stage control problem notionally maximising surplus subject to a constraint on the size of the pool is as follows:

$$\begin{aligned} \underset{p_1,\dots,p_n}{\text{maximise}} \quad & \sum_{i=1}^{n} (p_i - c_i)\,\psi_i(p_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} \psi_i(p_i) \geq C, \end{aligned}$$

where p_i are premiums, c_i expected costs and $\psi_i(p_i) = Pr(a_i = 1 \,|\, p_i)$ the probability that the i-th member will stay in the pool $\big(a_i = 1\big)$.

Surplus as a function of price



Expected surplus $r_i(p_i) = (p_i - c_i)\,\psi_i(p_i)$ as a function of premium p_i.

Stochastic optimisation

Variations on the model described previously are by now commonly used in practice. It is important to keep in mind that such approaches replace the more realistic stochastic optimisation problem:

$$\begin{aligned} \underset{p_1,\ldots,p_n}{\text{maximize}} \quad & \mathbb{E}_\omega\Big(\sum_{i=1}^{n}(p_i - c_i(\omega_i))\,\psi_i(p_i,\omega_i)\Big) \\ \text{subject to} \quad & \mathbb{E}_\omega\Big(\big[C - \sum_{i=1}^{n}\psi_i(p_i,\omega_i)\big]_+\Big) \le \epsilon \end{aligned}$$

with its "certainty equivalent", where uncertainty in demand and cost estimates is ignored, leading to overly optimistic objective values being obtained "in sample".

Joint control and estimation

Irrespective of the exact model form, an exogenous source of variation in p_i is ideally needed in order to come up with an estimate of $\psi_i(p_i)$, together with a parametric assumption:

$$\begin{aligned}\psi_i(p_i) &= \psi(p_i, \mathbf{x}_i\,;\,\boldsymbol{\theta}) \\ &= Pr(a_i = 1 \,|\, p_i, \mathbf{x}_i, \boldsymbol{\theta}),\end{aligned}$$

which replaces $\psi_i(p_i)$ with a function of observable risk characteristics $\mathbf{x}_i$ chosen from a parametric family $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

One strategy would be to randomly select a proportion ϵ of the portfolio on which randomised "price testing" is carried out. The gathered response and risk characteristics dataset is used to estimate the conditional probability model: $Pr(a = 1 \,|\, p, \mathbf{x}, \hat{\boldsymbol{\theta}})$, e.g. via logistic regression.

Sequential experiments

A more sophisticated strategy, would allocate more tests to the parts of risk characteristic space where demand model improvement is most likely to positively affect surplus (this is a version of the so called "multi-armed bandit" model for sequential experiments).

The following assumes a prior $\pi(\boldsymbol{\theta})$ and a procedure for computing the posterior distribution of $\boldsymbol{\theta}$:

$$Pr\big(\boldsymbol{\theta} \,|\, \{(a_i, p_i, \mathbf{x_i})\}_{i=1,\dots,n}\big),$$

given a collection $\{(a_i, p_i, \mathbf{x_i})\}_{i=1,\dots,n}$ of risk characteristics, quoted premiums and associated outcomes.

Instead of adopting the Bayesian formalism, approximate sampling distribution of $\hat{\boldsymbol{\theta}}$ can be obtained via bootstrap.

Joint control and estimation - an algorithm

Choose $\pi(\boldsymbol{\theta})$; initialize $D = \emptyset$,

for $t = 1, \ldots, T$ **do**

Receive new batch of risk characteristics $\{\mathbf{x}_i\}_{i=1,\ldots n}$.

Draw $\tilde{\boldsymbol{\theta}} \sim Pr(\boldsymbol{\theta} \mid D)$.

$$\begin{aligned} \mathbf{p}^\star = \underset{\mathbf{p}}{\operatorname{argmax}} & \textstyle\sum_{i=1}^{n} (p_i - c_i)\psi(p_i, \mathbf{x}_i\,;\,\tilde{\boldsymbol{\theta}}) \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{n} \psi(p_i, \mathbf{x}_i\,;\,\tilde{\boldsymbol{\theta}}) \geq C_t \end{aligned}$$

Receive responses $\{a_i\}_{i=1,\ldots,n}$.

$D = D \cup \{(a_i, p_i^\star, \mathbf{x}_i)\}_{i=1,\ldots n}$

end for

This algorithm adaptively injects increased premium variability for those risks where the demand model is less certain, aiding in estimation. The basic idea goes back to Thompson (1937).

Controlling risk accumulation

The previous section effectively assumed linear utility for the insurer. It may be desirable to limit risk concentration at the time premiums are quoted, however, making pricing dependent on the existing portfolio.

The existing literature on premium principles and risk measures easily accommodates this conceptually, but it also assumes precise knowledge of the cumulative loss distributions, making direct application difficult.

In the following some results from the recent “prediction market” literature are presented, recast into insurance terminology. Focusing on the finite number of scenarios makes immunising the portfolio against worst case loss tractable.

An automated bookmaker

Consider a discrete space of events Ω with n elements and assume that the book maker has an exponential utility function $u(x) = 1 - e^{-\alpha x},\ \alpha > 0$ and zero starting endowment. It then prices any proposed bet $\mathbf{q} \in \mathbb{R}^n$ via the zero utility premium principle by solving for C:

$$\mathbb{E}\Big(1 - e^{-\alpha(C-\mathbf{q})}\Big) = \sum_{i=1}^{n} p_i \left(1 - e^{-\alpha C - q_i}\right) = 0.$$

We can rewrite the above as follows:

$$e^{-\alpha C} \sum_{i=1}^{n} -p_i e^{\alpha q_i} = -1$$

and obtain an expression for the price of the bet $\mathbf{q}$ given an empty starting inventory:

$$C(\mathbf{q}) = \frac{1}{\alpha} \log \sum_{i=1}^{n} p_i e^{\alpha q_i}.$$

It is easy to check that the cost function is *path independent*, namely, denoting by $C(\mathbf{q} + \mathbf{r} \,|\, \mathbf{r})$ the price charged for the bet $\mathbf{q}$ by the bookmaker with the inventory $\mathbf{r} \in \mathbb{R}^n$, we obtain:

$$C(\mathbf{q} + \mathbf{r} \,|\, \mathbf{r}) = C(\mathbf{q} + \mathbf{r}) - C(\mathbf{r}).$$

This implies that the market participants do not gain by "strategically" splitting their bets.

"Logarithmic market scoring rule"

The algorithm employed by the bookmaker following the "logarithmic scoring rule" is extremely simple:

- given the current inventory of bets $\mathbf{r}$, charge (or pay out) $C(\mathbf{q}+\mathbf{r})-C(\mathbf{r})$ for the proposed bet $\mathbf{q}$.
- if the quote is accepted, update the inventory to be $\mathbf{q}+\mathbf{r}$ and wait for next request.

Assuming the relative size of the bet is small, we can approximate the price vector of n bets paying $\$1$ in state i by $\nabla_{\mathbf{r}}C(\mathbf{r})$:

$$\frac{\partial C}{\partial r_i} = \frac{p_i e^{\alpha r_i}}{\sum_{j=1}^{n} p_j e^{\alpha r_j}}.$$

- this is an exponential family probability distribution with *carrier density* $\mathbf{p}$ or the Esscher transform of $\mathbf{p}$.

Upper bound on the loss of the bookmaker

This would not be of much interest if the above procedure did not guarantee to limit the loss the bookmaker can suffer irrespective of the bets it accept or the actual outcome $\omega \in \Omega$.

We can write down the worst possible loss as follows:

$$\underset{\mathbf{q}, i \in \{1, \ldots, n\}}{\text{maximise}} \quad q_i - (C(\mathbf{q}) - C(\mathbf{0}))$$

Consider the objective for a fixed i:

$$\begin{aligned} q_i - (C(\mathbf{q}) - C(\mathbf{0})) &= \frac{1}{\alpha} \log p_i e^{\alpha q_i} - \frac{1}{\alpha} \log p_i - \left(\frac{1}{\alpha} \log \left(\sum_{j=1}^{n} p_j e^{\alpha q_j} \right) - C(\mathbf{0}) \right) \\ &= \frac{1}{\alpha} \log \left(\frac{p_i e^{\alpha q_i}}{\sum_{j=1}^{n} p_j e^{\alpha q_j}} \right) - \frac{1}{\alpha} \log p_i \end{aligned}$$

Upper bound - continued

- it is easy to see that $\frac{1}{\alpha}\log\left(\frac{p_i e^{\alpha q_i}}{\sum_{j=1}^{n} p_j e^{\alpha q_j}}\right)$ is bounded from above by 0 for any $\mathbf{q} \in \mathbb{R}^n$.
- this leaves us with $\max_i \frac{1}{\alpha}\log\frac{1}{p_i}$ as the upper bound on the loss of the bookmaker.
- this upper bound in turn is minimised if the bookmaker's subjective distribution $\mathbf{p}$ is uniform and is then given by $\frac{1}{\alpha}\log n$.

Convex premium principles as cost functions

In general we can use any convex premium principle C as the cost function, satisfying the following properties:

- **convexity:** $C(\mathbf{q})$ is a convex function of $\mathbf{q}$,
- **increasing monotonicity:** for any $\mathbf{q}$ and $\mathbf{r}$ if $\mathbf{q} \geq \mathbf{r}$ (entrywise), then $C(\mathbf{q}) \geq C(\mathbf{r})$,
- **translation invariance:** for any $\mathbf{q}$ and k, $C(\mathbf{q} + k\mathbf{1}) = C(\mathbf{q}) + k$.

This is identical, up to a minus sign, to the definition of *convex risk measures* (Föllmer and Schied, 2002) and it turns out that we can use their "robust representation":

$$C(\mathbf{r}) = \sup_{\mathbf{p} \in \boldsymbol{\Delta}} \mathbf{p}^T \mathbf{r} - R(\mathbf{p})$$

to derive worst case loss bounds:

$$\sup_{\mathbf{q}, i \in \{1,\ldots,n\}} q_i - (C(\mathbf{q}) - C(\mathbf{0})) = \max_{i \in \{1,\ldots,n\}} R(\mathbf{e}_i) + C(\mathbf{0}).$$

Insurance applications

- Unlike the previous setting, each new member joining an insurance scheme usually results in the multiplicative growth of the relevant outcome space.

- it is, however, possible to construct certain automatic safeguards against unanticipated or risky changes in portfolio composition by making quotes dependent on the current “inventory”, i.e. by limiting exposure to certain cells, analogous to underwriting limits.

- it may also be possible to devise insurance products around the idea directly, e.g. in situations when a diverse group of agents have “insurable interest” in certain events.