No arbitrage, incomplete markets and actuarial pricing

- linear pricing model
- exploiting arbitrage in betting markets
- state prices and no arbitrage principle
- portfolio optimisation as a control problem
- no arbitrage and equivalent utility pricing bounds
- multi-period setting, binomial lattice model as a control problem

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Two stage linear pricing model

- a two period model with the state of the world at $t = 0$ assumed known while at time $t = 1$ there can be n distinct states, $\omega_1, \omega_2, \ldots, \omega_n$ with an associated probabilities vector $\mathbf{p} \in \mathbb{R}^n$.

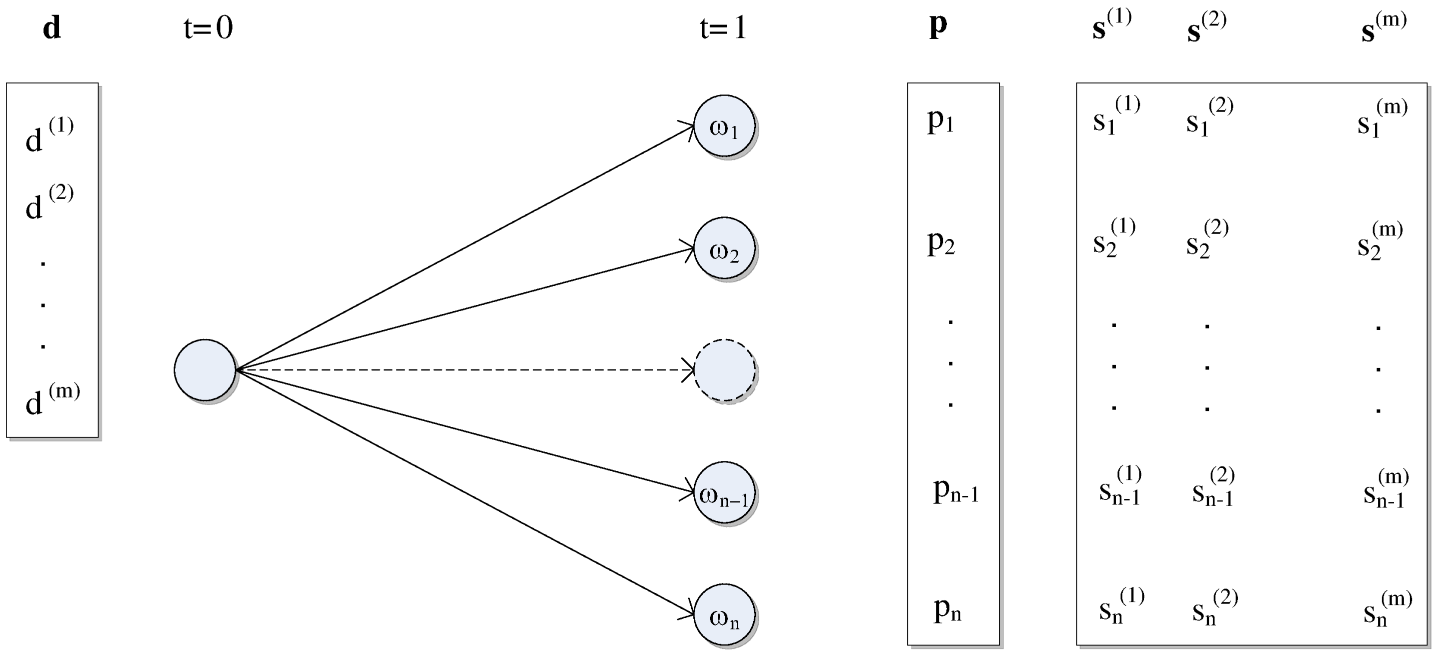
- there are m available securities where the i-th security has a pay-off vector $\mathbf{s}^{(i)} \in \mathbb{R}^n$ at $t = 1$ and a price $d^{(i)} \in \mathbb{R}$ at $t = 0$

- combine prices into a vector $\mathbf{d} \in \mathbb{R}^m$ with:

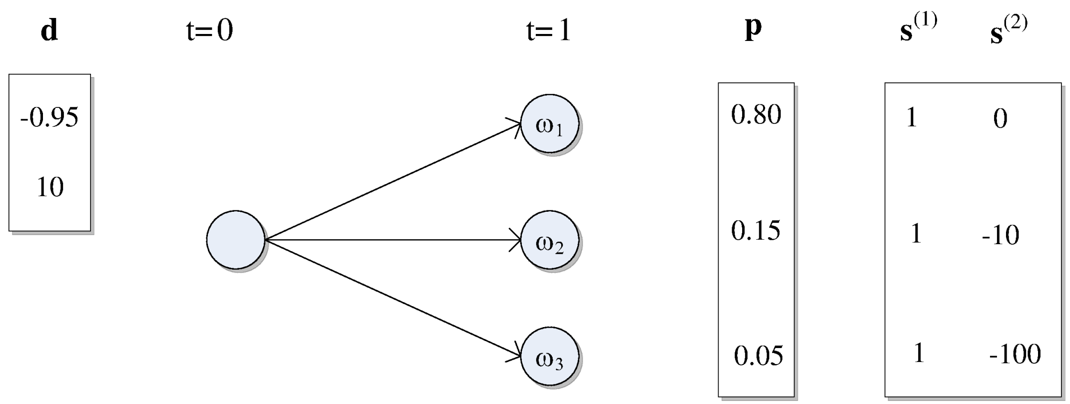
$$\mathbf{d} = \left[d^{(1)}, d^{(2)}, \ldots, d^{(n)}\right]^T$$

- and pay-off vectors into a matrix $S \in \mathbb{R}^{n \times m}$ such that:

$$S = \left[\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \dots, \mathbf{s}^{(m)}\right]$$



- this is how a very simple insurance problem might look in this setting:



Linear pricing - continued

- the key assumption is that the prices are "linear" i.e. if we purchase $\theta^{(i)}$ of the i-th asset at time $t = 0$ the pay-off vector $\mathbf{s} \in \mathbb{R}^n$ at $t = 1$ will be:

$$\mathbf{s} = \sum_{i=1}^{m} \theta^{(i)} \mathbf{s}^{(i)} = S\theta$$

- this is generally far from true - can negotiate better unit prices for big orders, harder to insure large risks etc.

- okay for trades in liquid securities, especially when no issues of control, revealing private information etc.

Betting arbitrage

- can use this set up for "riskless" arbitrage in the betting market where e is the initial outlay:

$$\begin{aligned} \underset{\theta}{\text{maximize}} \quad & E(S\theta) = \mathbf{p}^T S\theta \\ \text{subject to} \quad & \theta \geq \mathbf{0} \\ & \mathbf{d}^T\theta \leq e \\ & S\theta > e. \end{aligned}$$

- every feasible point provides a "riskless" profit and we select the one with highest expected pay-off (using our assessment of probabilities), if there are no arbitrage opportunities the problem is infeasible.

- if we allow borrowing to finance the bets, we can drive the expected pay off to infinity, which is of course a good thing

- the main reason people worry about arbitrage is because they want to avoid it in the prices they quote - otherwise they are on the losing end of an "infinite" expected utility pay off.

- arbitrage between *different* sellers merely helps the market to reach equilibrium prices.

State prices

- any pay-off vector $\mathbf{s} \in \mathbb{R}^n$ for which there exist $\theta \in \mathbb{R}^m$ such that $\mathbf{s} = S\theta$ is called *attainable*.

- it is a basic result in linear algebra that if there are at least n securities with linearly independent pay-offs, *any* new pay-off vector can be represented as a linear combination of existing securities, i.e. become *attainable*.

- the prices for different "portfolios" with the same pay-offs needn't be the same.

- if this is the case and short-selling/borrowing is allowed we can achieve infinite expected return with no chance of loss - "arbitrage".

State prices - continued

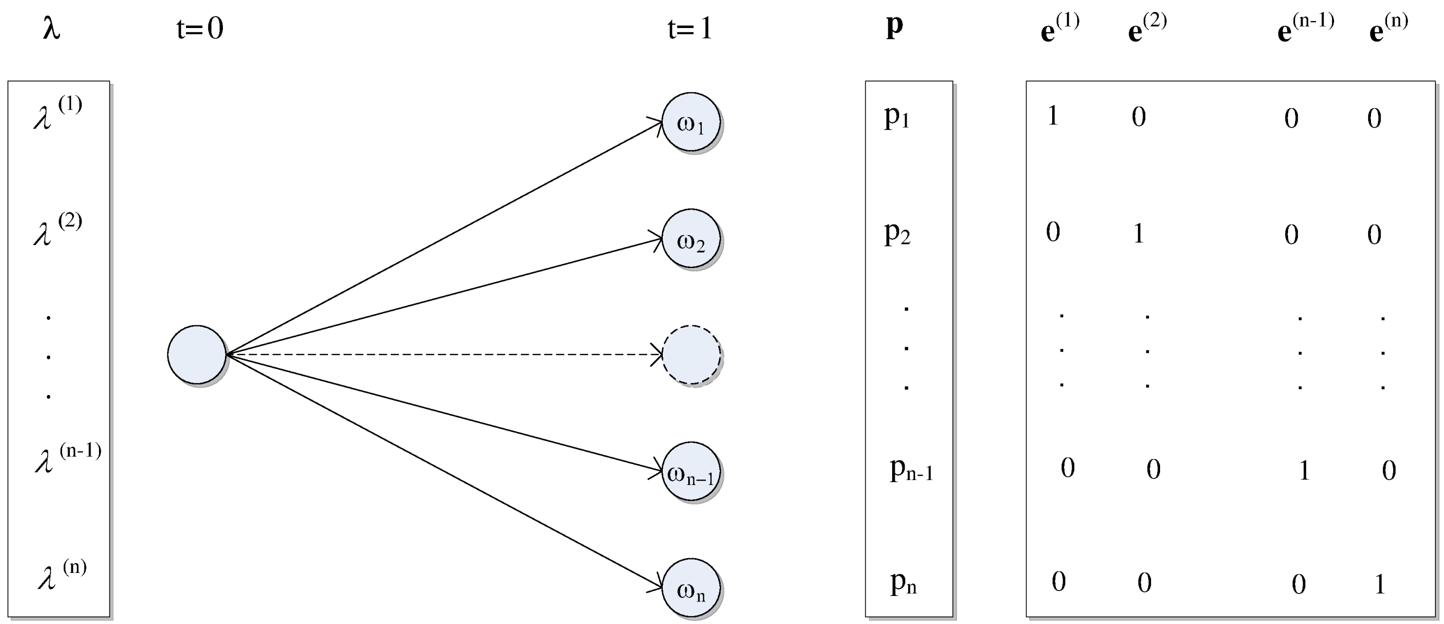
- "fundamental theorem of asset pricing" characterizes when arbitrage opportunities exist - by LP duality the first problem is unbounded (i.e arbitrage exists) if and only if the second problem is infeasible (i.e. there are no λ satisfying the constraints):

$$\begin{array}{llcll} \underset{\theta}{\text{minimize}} & \mathbf{d}^T\theta & \qquad & \underset{\lambda}{\text{maximize}} & \mathbf{0}^T\lambda \\ \text{subject to} & S\theta \geq \mathbf{0} & & \text{subject to} & S^T\lambda = \mathbf{d} \\ & & & & \lambda \geq 0 \end{array}$$

- therefore if we can exhibit λ^* satisfying the constraints of the second problem, we can certify that the structure of prices and pay-offs offers no arbitrage opportunities.

State prices - continued

- λ has a natural interpretation as a vector of "state prices" e.g. prices of simple pay-off vectors $\mathbf{e}^{(i)}$ which return 1 in the i-th state and 0 otherwise (these are known as Arrow-Debreu securities).



- if every pay-off vector is attainable (and there is no arbitrage), state prices exist and are unique - this is called a *complete market*. Price of any pay-off vector $\mathbf{x} \in \mathbf{R}^n$ is given as $\mathbf{x}^T \lambda^*$.
- if there is no arbitrage yet not every pay-off is attainable, there are (infinitely) many sets of state prices λ that satisfy $S^T \lambda = \mathbf{d}$ - in this case the market is called *incomplete*.
- state prices are often normalized to sum to one and subsequently referred to as "risk neutral probabilities" - these do not seem to have a clear economic interpretation.
- most of the classical results in financial economics concern *complete* markets, with completeness derived from extraordinarily strong assumptions about price dynamics.
- insurance market is so far from being complete that the point is seldom brought up.

Portfolio optimisation

- many models in financial economics and insurance can be reduced to a control problem maximizing expected utility (or some related objective) subject to a budget constraint e - these include MPT, CAPM, optimal reinsurance, asset liability matching etc.

$$
\begin{aligned}
\underset{\theta}{\text{maximize}} \quad & E\big(u(\mathbf{c})\big) = \sum_{j=1}^{n} u(c_j) p_j \\
\text{subject to} \quad & c_j = \sum_{i=1}^{m} \theta^{(i)} \left(s_j^{(i)} - d^{(i)} \right), \quad j = 1, 2, \ldots, n \\
& \mathbf{d}^T \theta \leq e.
\end{aligned}
$$

- the market prices are assumed to be given; generally difficult to estimate both the utility function and real world probabilities.

- in complete markets, can find the optimal consumption profile directly and then determine the corresponding replicating portfolio in a separate step.

- can also be used for “marginal” pricing of new securities in incomplete markets, i.e. at what price level does a security drop out of the optimal portfolio - this is specific to the agent and does not necessarily correspond to the market prices.

- marginal pricing only considers acquiring a very small portion of the security where as often it needs to be purchased in its entirety or not at all.

Incomplete markets - no arbitrage bounds

- in an incomplete market we can attempt to construct bounds on the price of a new pay-off $\mathbf{x}$, e.g. it needs to be cheaper than the cheapest portfolio of marketed securities that pays as much or more in every state.

$$\begin{array}{llcll} \underset{\theta}{\text{minimize}} & \mathbf{d}^T\theta & \qquad & \underset{\lambda}{\text{maximize}} & \mathbf{x}^T\lambda \\ \text{subject to} & S\theta \geq \mathbf{x} & & \text{subject to} & S^T\lambda = \mathbf{d} \\ & & & & \lambda \geq 0 \end{array}$$

- this problem can be expressed in terms of portfolio weights (*left*) or state prices (*right*) - the latter searches for the state price vector resulting in the highest price for pay-off $\mathbf{x}$ and yet compatible with prices of marketed securities.

- the two formulations are related via LP duality.

Indifference bounds

- no arbitrage bounds are usually far too wide - for the insurance example they would be 0 and 100, the maximum possible loss.

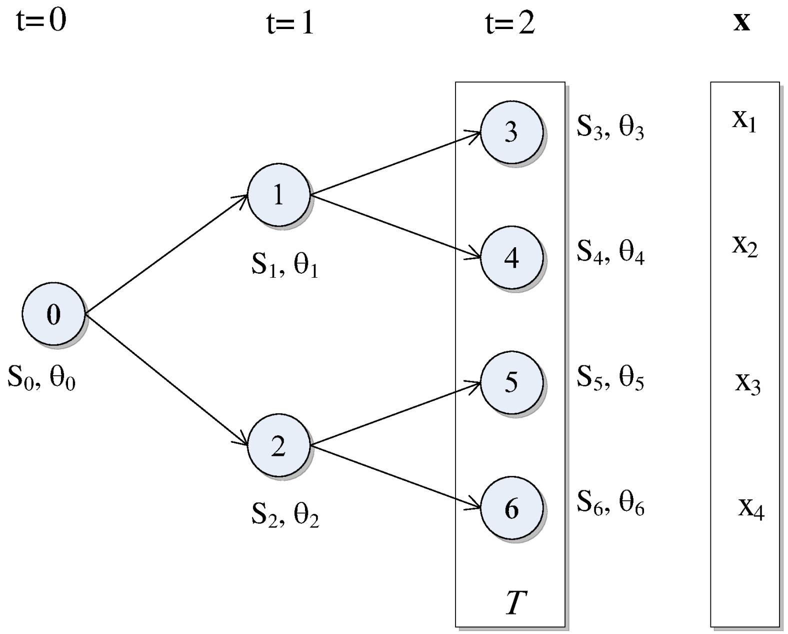
- we can construct a tighter bound on price by requiring that the price we charge for a security results in expected utility at least as high as that of the baseline portfolio.

$$\underset{\theta}{\text{minimize}} \quad \mathbf{d}^T \theta$$
$$\text{subject to} \quad E\big(u(S\theta - \mathbf{x})\big) \geq E\big(u(0)\big)$$

- this is indeed equivalent to the venerable "equivalent utility" pricing principle from the actuarial literature:

$$u(0) = E\big(u(\theta - \mathbf{x})\big)$$

Multistage models



$$\underset{\theta_1,\ldots,\theta_n}{\text{minimize}} \quad S_0\theta_{\mathbf{0}}$$

$$\text{subject to} \quad S_n(\theta_n - \theta_{a(n)}) = 0, \quad n \in \mathcal{N}_t, \quad t \geq 1$$

$$S_n\theta_{\mathbf{n}} \geq \mathbf{x}, \quad n \in \mathcal{N}_T$$

Conclusions

- essentially all problems studied in financial economics can be reduced to maximizing the utility of consumption under a real world probability measure (see J. Cochrane, "Asset Pricing").

- in financial engineering, the objects usually dealt with are state prices, "risk neutral" probability measures or "pricing kernels".

- in incomplete markets these are formally related (via Lagrange duality) to selecting expected utility maximizing portfolios under the real world probability measure.

- "actuarial" approach to pricing risks is essentially the same as the "financial engineering approach", indeed it focuses exclusively on the more difficult setting of "incomplete markets"!