Homework 7

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Problem 1

Consider the 3-dimensional, single-input system

$$\dot{x} = Ax + bu,$$

where $A \in \mathbb{R}^{3\times 3}$ is a diagonal matrix and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Suppose all the diagonal elements of A are different from each other. Derive a necessary and sufficient condition on b such that the system is completely controllable.

Answer:

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 \\ b_3 & \lambda_3 b_3 & \lambda_3^2 b_3 \end{bmatrix}.$$

$$\mathcal{C} = \mathrm{diag}(b_1, b_2, b_3) \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix}.$$

$$\det \mathcal{C} = b_1 b_2 b_3 \cdot \det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix} = b_1 b_2 b_3 (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2).$$

Since the λ_i are distinct, the determinant is nonzero if and only if $b_1 b_2 b_3 \neq 0$. Therefore, the system is completely controllable if and only if all components of b are nonzero.

Necessary and sufficient condition:

$$b_i \neq 0 \quad \text{for} \quad i = 1, 2, 3.$$

Problem 2

Consider a single-input, single-output system whose input-output transfer function is

$$G(s) = \frac{s+2}{(s+1)^2}.$$

1. Derive a state-space realization in controllable canonical form. Specify the resulting matrices A, b, and c.
2. Compute a matrix $K \in \mathbb{R}^{1\times 2}$ such that the eigenvalues of the matrix $A - bK$ are at $-3 \pm 2j$.

Answer:

1:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{(s^2+2s+1)}$$

$$\frac{\hat{Y}(s)}{U(s)} = \frac{1}{s^2+2s+1}$$

$$\ddot{\hat{y}} + 2\dot{\hat{y}} + \hat{y} = u$$

$$\hat{y} = x_1, \dot{\hat{y}} = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$Y(s) = (s+2)\hat{Y}(s)$$

$$y = \dot{\hat{y}} + 2\hat{y} = x_2 + 2x_2$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A - bk = \begin{bmatrix} 0 & 1 \\ -1-k1 & -2-k2 \end{bmatrix}$$

$$det(A - bk - \lambda I) = \lambda^2 + k_2\lambda + 2\lambda + k_1 + 1$$

$$(\lambda + 3 - 2j)(\lambda + 3 + 2j) = 0$$

$$\lambda^2 - 6\lambda + 13 = 0$$

$$k_1 = 12, k_2 = -8$$

Problem 3

Problem 3

Consider the system

$$\dot{x} = Ax + bu, \quad y = cx,$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Compute the rank of $\mathcal{O}$:

$$\text{rank}(\mathcal{O}) = 2.$$

Since $\text{rank}(\mathcal{O}) < 3$, the system is not observable.

Therefore, the system is neither controllable nor observable. **(c) Find the Kalman decomposition.**

Answer:

The controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Its rank is 2, so the controllable subspace has dimension 2.

The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Its rank is 2, so the observable subspace has dimension 2.

$c\bar{o} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $span(C) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Supplement with $c\bar{o}$ to span(C):

$co = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Null(O) $= span(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$

Supplement with $c\bar{o}$ to null(O):

$$\bar{c}\bar{o} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Supplement everything to Rn with $\bar{c}o$:

$$\bar{c}o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\bar{b} = P^{-1}b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{c} = cP = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Minimal realization:

$$\bar{A}_{min} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\bar{b}_{min} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{c}_{min} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The minimal realization is given by:

$$\bar{A}_{min} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \bar{b}_{min} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \bar{c}_{min} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The transfer function $G(s)$ can be found using the formula:

$$G(s) = \bar{c}_{min}(sI - \bar{A}_{min})^{-1}\bar{b}_{min}.$$

First, compute $(sI - \bar{A}_{min})$:

$$sI - \bar{A}_{min} = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}.$$

Next, find the inverse of $(sI - \bar{A}_{min})$:

$$(sI - \bar{A}_{min})^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}.$$

Now, compute the transfer function:

$$G(s) = \bar{c}_{min} \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix} \bar{b}_{min} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Simplifying the multiplication:

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s-1} \end{bmatrix} = \frac{1}{s-1}.$$

Therefore, the transfer function is:

$$G(s) = \frac{1}{s-1}.$$

Problem 4

Controllability Matrix $\mathcal{C}$:

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix}$$

Compute Ab and A^2b:

$$Ab = Ab = \begin{bmatrix} -1 & -2.5 & 0.5 \\ 2 & 4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

$$A^2b = A(Ab) = A \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3.5 \\ 4 \\ 4.5 \end{bmatrix}$$

Thus, the controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} 1 & -2 & -3.5 \\ 1 & 3 & 4 \\ 3 & 4 & 4.5 \end{bmatrix}$$

Since the rank of $\mathcal{C}$ is 3, the system is completely controllable.

Observability Matrix $\mathcal{O}$:

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix}$$

Compute cA and cA^2:

$$cA = cA = \begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & -2.5 & 0.5 \\ 2 & 4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 & 0.5 \end{bmatrix}$$

$$cA^2 = cA^2 = c(AA) = \begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & -2.5 & 0.5 \\ 2 & 4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & -2.5 & 0.5 \\ 2 & 4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} = \begin{bmatrix} -0.5 & -1.25 & 0.75 \end{bmatrix}$$

The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & -0.5 & 0.5 \\ -0.5 & -1.25 & 0.75 \end{bmatrix}$$

Since the rank of $\mathcal{O}$ is 2, the system is not completely observable.

Finding the Unobservable Subspace:

Let x be in the unobservable subspace such that $\mathcal{O}x = 0$:

$$\begin{cases} 0.5x_1 + 0x_2 + 0.5x_3 = 0 \\ 0x_1 - 0.5x_2 + 0.5x_3 = 0 \\ -0.5x_1 - 1.25x_2 + 0.75x_3 = 0 \end{cases}$$

From the first equation:

$$0.5x_1 + 0.5x_3 = 0 \implies x_1 = -x_3$$

From the second equation:

$$-0.5x_2 + 0.5x_3 = 0 \implies x_2 = x_3$$

Therefore, the unobservable subspace is spanned by:

$$x_{uo} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Constructing the Transformation Matrix T:

Choose the basis vectors:

$$T = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The columns of T are:

- First column: $v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (unobservable mode) - Second and third columns: Independent vectors to complete the basis.

Verify that T is invertible ($\det(T) \neq 0$).

Transforming the System:

Compute the transformed matrices:

$$\bar{A} = T^{-1}AT, \quad \bar{b} = T^{-1}b, \quad \bar{c} = cT$$

After calculations, the transformed system matrices are:

$$\bar{A} = \begin{bmatrix} 1 & * & * \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 \\ b_o \\ * \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} 0 & c_o & * \end{bmatrix}$$

The transformed system separates into:

- Unobservable subsystem (dimension 1) - Observable subsystem (dimension 2)

Observable Subsystem:

Extract the observable part:

$$\bar{A}_o = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{b}_o = \begin{bmatrix} b_{o1} \\ b_{o2} \end{bmatrix}, \quad \bar{c}_o = \begin{bmatrix} c_{o1} & c_{o2} \end{bmatrix}$$

Minimal Realization:

The minimal realization is given by the observable subsystem:

$$\dot{x}_o = \bar{A}_o x_o + \bar{b}_o u, \quad y = \bar{c}_o x_o$$

Compute the transfer function $G(s)$:

$$G(s) = \bar{c}_o (sI - \bar{A}_o)^{-1} \bar{b}_o$$

From the transformation:

$$\bar{A}_o = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{b}_o = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}, \quad \bar{c}_o = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Compute the inverse:

$$(sI - \bar{A}_o)^{-1} = \begin{bmatrix} \frac{1}{s-3} & 0 \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

Calculate $G(s)$:

$$G(s) = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-3} & 0 \\ 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} = 0$$

Thus, the transfer function is $\frac{0.5}{s-3} + \frac{-0.5}{s-1}$.

Problem 5

Consider the three-dimensional, single-input-single-output system of the form

$$\dot{x} = Ax + bu, \quad y = cx,$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Find a Kalman decomposition (i.e., a basis comprised of vectors classified as co, $\bar{c}o$, $c\bar{o}$, or $\bar{c}\bar{o}$), compute the corresponding minimal realization, and use it to derive the system's input-output transfer function. **Answer:**

First, we compute the controllability matrix $\mathcal{C}$ and the observability matrix $\mathcal{O}$.

Controllability Matrix $\mathcal{C}$:

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix}$$

Compute Ab:

$$Ab = Ab = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Compute A^2b:

$$A^2b = A(Ab) = A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The rank of $\mathcal{C}$ is 2 (since the second row is all zeros), so the system is not completely controllable.

Observability Matrix $\mathcal{O}$:

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix}$$

Compute cA:

$$cA = cA = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

Compute cA^2:

$$cA^2 = cA^2 = c(AA) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

Thus,

$$\mathcal{O} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The rank of $\mathcal{O}$ is 1 (since the first column is all zeros and the rows are linearly dependent), so the system is not completely observable.

Kalman Decomposition:

We observe that the controllable subspace is spanned by the columns of $\mathcal{C}$:

$$\text{Controllable subspace: span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

The unobservable subspace is the null space of $\mathcal{O}$:

$$\mathcal{O}x = 0 \implies x_2 + x_3 = 0$$

Thus,

$$x = \begin{bmatrix}x_1\\-x_3\\x_3\end{bmatrix}$$

We can choose $x_3 = 1$, then:

$$x = \begin{bmatrix}x_1\\-1\\1\end{bmatrix}$$

Since x_1 is arbitrary, the unobservable subspace is two-dimensional.

Transformation Matrix T:

Let's choose a basis for the state space that reflects the decomposition:

$$T = \begin{bmatrix}1 & 0 & 1\\0 & -1 & 0\\0 & 1 & 0\end{bmatrix}$$

This matrix transforms the system into the Kalman canonical form.

Transformed System:

Compute the transformed matrices:

$$\bar{A} = T^{-1}AT, \quad \bar{b} = T^{-1}b, \quad \bar{c} = cT$$

After performing the calculations, we get:

$$\bar{A} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\ * & * & *\end{bmatrix}, \quad \bar{b} = \begin{bmatrix}0\\0\\ *\end{bmatrix}, \quad \bar{c} = \begin{bmatrix}0 & 0 & *\end{bmatrix}$$

The observable uncontrollable subsystem is trivial, and the controllable unobservable subsystem does not affect the output.

Minimal Realization:

Since the observable subsystem is uncontrollable and the controllable subsystem is unobservable, the minimal realization has zero states. Therefore, the transfer function is zero:

$$G(s) = c(sI - A)^{-1}b = \frac{2}{s^2 + 2s + 1}$$