GEOMETRY

Through Algebra

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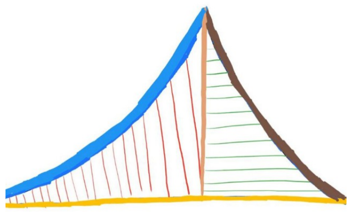

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Chapter 1

Triangle

1.1. Median

1.1.1. If $\mathbf{D}$ divides BC n the ratio $k:1$

$$\mathbf{D} = \frac{kC + B}{k+1} \tag{1.1.1.1}$$

Find the mid-points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.

Solution: Since $\mathbf{D}$ is the mid point of BC,

$$\mathbf{k} = 1\mathbf{D} = \frac{\mathbf{C} + \mathbf{B}}{2} = \frac{1}{2}\begin{pmatrix} 7 \\ \\ 3 \end{pmatrix} \tag{1.1.1.2}$$

Similarly,

$$E = \frac{A + C}{2} = \frac{1}{2}\begin{pmatrix} -11 \\ \\ 2 \end{pmatrix} \tag{1.1.1.3}$$

$$F = \frac{A+B}{2} = \frac{1}{2}\begin{pmatrix} -6 \\ 3 \end{pmatrix} \tag{1.1.1.4}$$

1.1.2. Find the equation of AD, BE and CF.

Solution: The direction vector of AD is

$$m = \mathbf{D} - \mathbf{A} = \left(\tfrac{7}{2} - (-5) - \left(\tfrac{3}{2}\right) - (-2)\right) = \frac{1}{2}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \end{pmatrix} 3 \tag{1.1.2.1}$$

Hence, the normal equation of AD is

$$n^T(x - A) = 0 \implies \begin{pmatrix} 7 & 3 \end{pmatrix} x = \begin{pmatrix} (\end{pmatrix} 73) \begin{pmatrix} -5 \\ -2 \end{pmatrix} = 7 \times -5 + 3 \times -2 = -29 \tag{1.1.2.2}$$

For BE,

$$m = \mathbf{E} - \mathbf{B} = \left(\tfrac{-11}{2} - (-1)\right) \tag{1.1.2.3}$$

$$\left(\tfrac{-2}{2} - (-1)\right) = \begin{pmatrix} \frac{-9}{2} \\ \frac{4}{2} \end{pmatrix} n = \begin{pmatrix} 4 \\ 9 \end{pmatrix} \tag{1.1.2.4}$$

Hence, the normal equation of median BE is

$$n^T(x - B) = 0 \implies \begin{pmatrix} 4 & 9 \end{pmatrix} x = \begin{pmatrix} 4 & 9 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 4 \times -1 + 9 \times -1 = -13 \tag{1.1.2.5}$$

For median CF,

$$m = \mathbf{F} - \mathbf{C} = \begin{pmatrix} -\frac{-6}{2} - (-6) \end{pmatrix} \tag{1.1.2.6}$$

$$\begin{pmatrix} \frac{-3}{2} - (4) \end{pmatrix} = \begin{pmatrix} \frac{6}{2} \\ \frac{-11}{2} \end{pmatrix} = \begin{pmatrix} 6 \\ -11 \end{pmatrix} n = \begin{pmatrix} 11 \\ 6 \end{pmatrix} \tag{1.1.2.7}$$

Hence, the normal equation of median $\begin{pmatrix} CF \end{pmatrix}$ is

$$n^T(x - C) = 0 \implies \begin{pmatrix} 11 & 6 \end{pmatrix} x = \begin{pmatrix} 11 & 6 \end{pmatrix} \begin{pmatrix} -6 \\ 4 \end{pmatrix} = 11 \times -6 + -6 \times 4 = -42 \tag{1.1.2.8}$$

1.1.3. Find the intersection $\begin{pmatrix} B \end{pmatrix}$ of BE and CF.

Solution: From BE and CF,the above equations are:

$$\begin{pmatrix} 49 \end{pmatrix} x = -13 \begin{pmatrix} 116 \end{pmatrix} x = -42 \tag{1.1.3.1}$$

From the above, the augmented matrix is

$$\begin{pmatrix} 4 & 9 & -13 \\ 11 & 6 & -42 \end{pmatrix} \xleftrightarrow{R_1} \leftarrow \frac{R_1}{4} \tag{1.1.3.2}$$

The matrix can be reduced as follows:

$$\begin{pmatrix} 1 & \frac{9}{4} & \frac{-13}{4} \\ 11 & 6 & -42 \end{pmatrix} \xleftrightarrow{R_2} \leftarrow R_2 - 11R_1 \begin{pmatrix} 1 & \frac{9}{4} & \frac{13}{4} \\ 0 & \frac{-75}{4} & \frac{-25}{4} \end{pmatrix} \xleftrightarrow{R_2} \leftarrow \frac{-4R_2}{75} \begin{pmatrix} 1 & \frac{9}{4} & \frac{-13}{4} \\ 0 & 1 & \frac{1}{3} \end{pmatrix} \xleftrightarrow{R_2} \leftarrow R_1 - \tag{1.1.3.3}$$

1.1.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.1.4.1}$$

Solution: In order to find BG, we need to find $\mathbf{G} - \mathbf{B}$ and in order to find GE, we need to find $\mathbf{E} - \mathbf{G}$.

$$BG = \mathbf{G} - \mathbf{B} = \begin{pmatrix} -4 - (-1) \end{pmatrix} \tag{1.1.4.2}$$

$$\begin{pmatrix} \frac{1}{3} - (-1) \end{pmatrix} \tag{1.1.4.3}$$

$$= \begin{pmatrix} -3 \\ \frac{4}{3} \end{pmatrix} \tag{1.1.4.4}$$

$$GE = \mathbf{E} - \mathbf{G} = \begin{pmatrix} \frac{-11}{2} - (-4) \end{pmatrix} \tag{1.1.4.5}$$

$$\begin{pmatrix} \frac{2}{2} - \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ \frac{4}{6} \end{pmatrix} \tag{1.1.4.6}$$

$$Therefore, \frac{BG}{GE} = 2 \tag{1.1.4.7}$$

$$\tag{1.1.4.8}$$

In order to find GF, we need to find $\mathbf{F} - \mathbf{G}$ and in order to find CG , we need to find $\mathbf{G} - \mathbf{C}$.

$$GF = \mathbf{F} - \mathbf{G} = \begin{pmatrix} \frac{-6}{2} - (-4) \end{pmatrix} \tag{1.1.4.9}$$

$$\begin{pmatrix} \frac{-3}{2} - \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{2} \\ \frac{-11}{6} \end{pmatrix} \tag{1.1.4.10}$$

$$CG = \mathbf{G} - \mathbf{C} = \begin{pmatrix} -4 - (-6) \end{pmatrix} \tag{1.1.4.11}$$

$$\begin{pmatrix} \frac{1}{3} - (4) \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{-11}{3} \end{pmatrix} \tag{1.1.4.12}$$

$$Therefore, \frac{CG}{GF} = 2 \tag{1.1.4.13}$$

In order to find AG, we need to find $\mathbf{G} - \mathbf{A}$ and in order to find GD

, we need to find $\mathbf{D} - \mathbf{G}$.

$$AG = \mathbf{G} - \mathbf{A} = \begin{pmatrix} -4 - (-5) \end{pmatrix} \tag{1.1.4.14}$$

$$\begin{pmatrix} \frac{1}{3} - (-2) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{7}{3} \end{pmatrix} \tag{1.1.4.15}$$

$$GD = \mathbf{D} - \mathbf{G} = \begin{pmatrix} \frac{-7}{2} - (-4) \end{pmatrix} \tag{1.1.4.16}$$

$$\begin{pmatrix} \frac{3}{2} - \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{7}{6} \end{pmatrix} \tag{1.1.4.17}$$

$$Therefore, \frac{AG}{GD} = 2 \tag{1.1.4.18}$$

$$\implies \frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.1.4.19}$$

$$Henceverified \tag{1.1.4.20}$$

$$\tag{1.1.4.21}$$

1.1.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear if

Solution: Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \tag{1.1.5.1}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -5 & \frac{-7}{2} & -4 \\ -2 & \frac{3}{2} & \frac{1}{3} \end{pmatrix} \xleftrightarrow{R_2} \leftarrow R_3 + 5R_1 \tag{1.1.5.2}$$

The matrix can be reduced as follows:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{3}{2} & 1 \\ -2 & \frac{3}{2} & \frac{1}{3} \end{pmatrix} \xleftrightarrow{R_3} \leftarrow R_3 + 2R_1 \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{3}{2} & 1 \\ 0 & \frac{7}{2} & \frac{7}{3} \end{pmatrix} \xleftrightarrow{R_3} \leftarrow R_3 - \frac{7R_2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.5.3}$$

1.1.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.1.6.1}$$

Solution:

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.1.6.2}$$

$$\mathbf{G} = \frac{1}{3}\left(\begin{pmatrix} -5 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} -6 \\ 4 \end{pmatrix}\right) \tag{1.1.6.3}$$

$$= \frac{1}{3}\begin{pmatrix} -12 \\ 1 \end{pmatrix} \tag{1.1.6.4}$$

$$= \begin{pmatrix} -4 \\ \frac{1}{3} \end{pmatrix} \tag{1.1.6.5}$$

$$\implies \mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.1.6.6}$$

Hence verifed

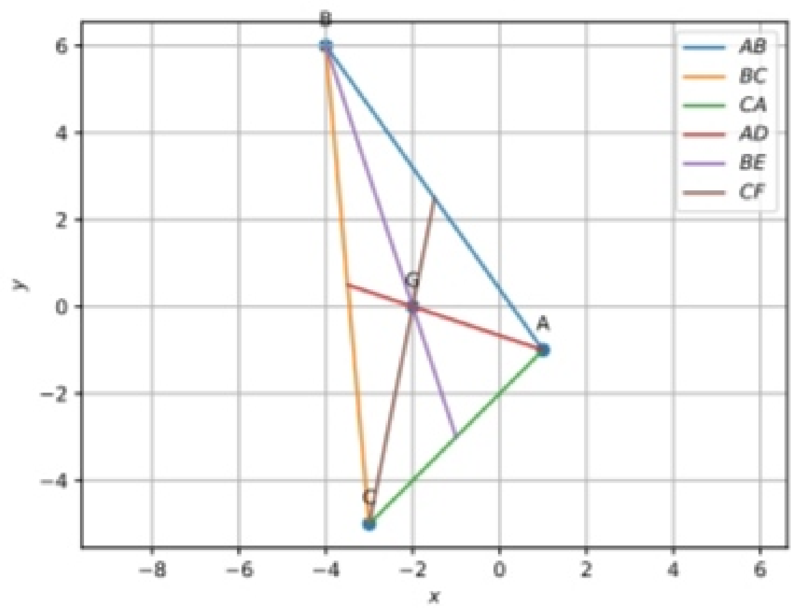

Figure 1.2: Medians of $\triangle ABC$ meet at $\mathbf{G}$.

Figure 1.1: Triangle

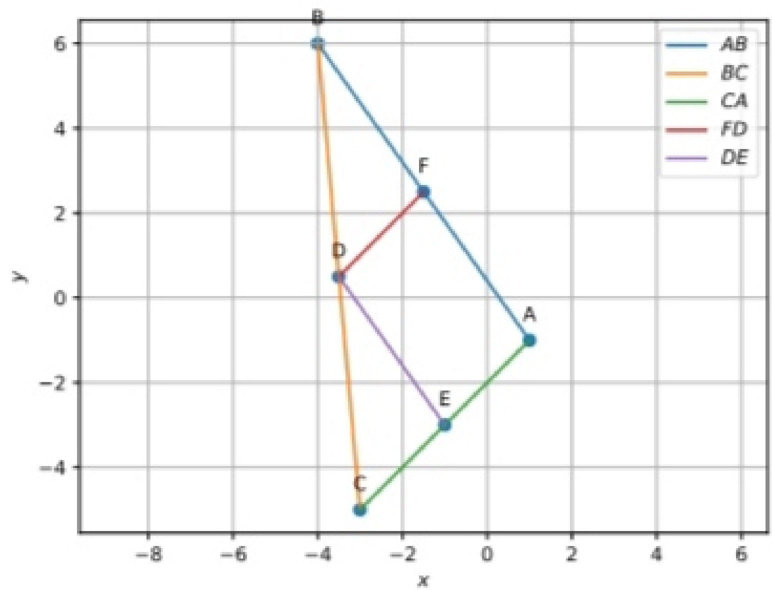

Figure 1.3: $AFDE$ forms a parallelogram in triangle ABC

Figure 1.2: Triangle

1.1.7. Verify that

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.1.7.1}$$

Solution:

$$\mathbf{A} - \mathbf{F} = \left((-5) - \frac{-6}{2} \right) \tag{1.1.7.2}$$

$$\left((-2) - \frac{-3}{2} \right) = \begin{pmatrix} (-2) \\ \\ \frac{-1}{2} \end{pmatrix} \mathbf{E} - \mathbf{D} = \left(\frac{11}{2} - \frac{-7}{2} \right) \tag{1.1.7.3}$$

$$\left(\frac{2}{2} - \frac{3}{2} \right) = \begin{pmatrix} -2 \\ \\ \frac{-1}{2} \end{pmatrix} \implies \mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.1.7.4}$$

Hence verified