GEOMETRY

Through Algebra

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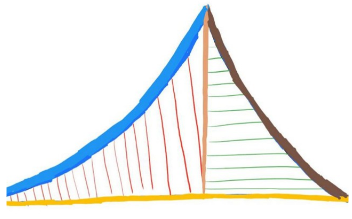

Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -6 \\ 4 \end{pmatrix} \tag{1.1}$$

1.1. Vectors

1.1.1. the direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Find the direction vectors of AB, BC and CA.

Solution:

(a) The Direction vector of AB is

$$= \mathbf{B} - \mathbf{A} \tag{1.1.1.2}$$

$$:: = \begin{pmatrix} -1-(-5) \\ -1-(-2) \end{pmatrix} \tag{1.1.1.3}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.1.1.4}$$

(b) The Direction vector of BC

$$= \mathbf{C} - \mathbf{B} \tag{1.1.1.5}$$

$$= \begin{pmatrix} -6-(-1) \\ 4-(-1) \end{pmatrix} \tag{1.1.1.6}$$

$$= \begin{pmatrix} -5 \\ 5 \end{pmatrix} \tag{1.1.1.7}$$

(c) The Direction vector of CA

$$= \mathbf{A} - \mathbf{C} \tag{1.1.1.8}$$

$$= \begin{pmatrix} -5-(-6) \\ -2-(4) \end{pmatrix} \tag{1.1.1.9}$$

$$= \begin{pmatrix} 1 \\ -6 \end{pmatrix} \tag{1.1.1.10}$$

1.1.2. The length of side AB, BC and AC is **Solution:** Given,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}, \tag{1.1.2.1}$$

$$\mathbf{B} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \tag{1.1.2.2}$$

$$\mathbf{C} = \begin{pmatrix} -6 \\ 4 \end{pmatrix} \tag{1.1.2.3}$$

Now solving for AB,

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B})} \tag{1.1.2.4}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{1.1.2.5}$$

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{\begin{pmatrix} -4 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ -1 \end{pmatrix}} \tag{1.1.2.6}$$

$$= \sqrt{(4)^2 + (1)^2} \tag{1.1.2.7}$$

$$\implies \|\mathbf{A} - \mathbf{B}\| = \sqrt{17} \tag{1.1.2.8}$$

Now solving for BC,

$$\|\mathbf{B}-\mathbf{C}\| = \sqrt{(\mathbf{B}-\mathbf{C})^{\top}(\mathbf{B}-\mathbf{C})} \tag{1.1.2.9}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \tag{1.1.2.10}$$

$$\|\mathbf{B}-\mathbf{C}\| = \sqrt{\begin{pmatrix} 5 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ -5 \end{pmatrix}} \tag{1.1.2.11}$$

$$= \sqrt{(5)^2 + (5)^2} \tag{1.1.2.12}$$

$$\implies \|\mathbf{B}-\mathbf{C}\| = \sqrt{50} \tag{1.1.2.13}$$

Now solving for AC,

$$\|\mathbf{A}-\mathbf{C}\| = \sqrt{(\mathbf{A}-\mathbf{C})^{\top}(\mathbf{A}-\mathbf{C})} \tag{1.1.2.14}$$

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} 1 \\ -6 \end{pmatrix} \tag{1.1.2.15}$$

$$\|\mathbf{A}-\mathbf{C}\| = \sqrt{\begin{pmatrix} 1 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ -6 \end{pmatrix}} \tag{1.1.2.16}$$

$$= \sqrt{(1)^2 + (6)^2} \tag{1.1.2.17}$$

$$\implies \|\mathbf{A}-\mathbf{C}\| = \sqrt{37} \tag{1.1.2.18}$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be colliner if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{1.1.3.1}$$

Solution:

Given that,

$$\mathbf{A} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -6 \\ 4 \end{pmatrix} \tag{1.1.3.2}$$

Given that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} < 3 \tag{1.1.3.3}$$

Let

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ -5 & -1 & -6 \\ -2 & -1 & 4 \end{pmatrix} \tag{1.1.3.4}$$

$$\tag{1.1.3.5}$$

The matrix $\mathbf{R}$ can be row reduced as follows,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ -2 & -1 & -4 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow 2R_1 + R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 1 & 6 \end{pmatrix} \tag{1.1.3.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{4}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & \frac{25}{4} \end{pmatrix} \tag{1.1.3.7}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{4}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{-1}{4} \\ 0 & 0 & \frac{25}{4} \end{pmatrix} \tag{1.1.3.8}$$

There are no zero rows. So,

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{1.1.3.9}$$

Hence, from (**??**) the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear.

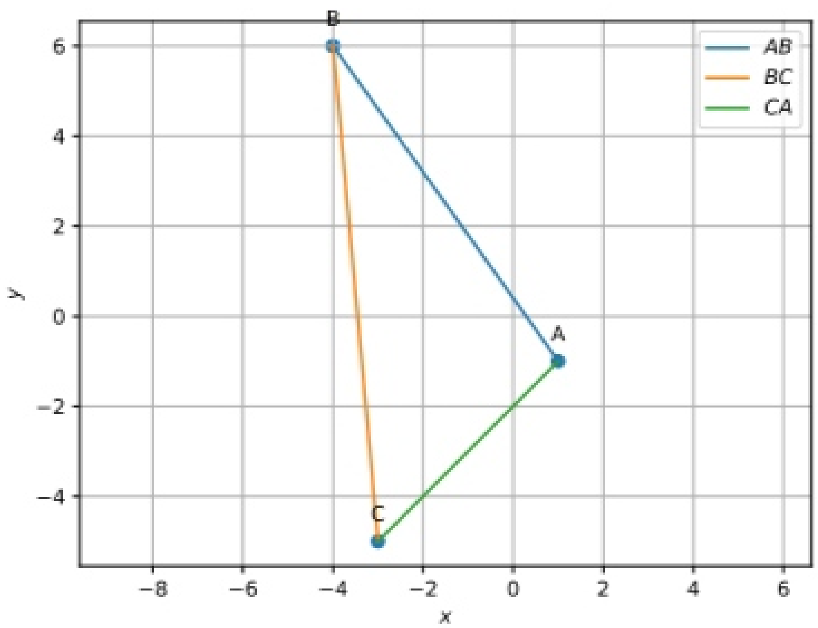

Figure 1.1: $\triangle ABC$

Figure 1.1: $\mathbf{A}, \mathbf{B}, \mathbf{C}$ plot

From Fig. 1.1, We can see that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear .

1.1.4. The parametric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

is the direction vector of AB. Find the parametric equations of AB, BC and CA. **Solution:**

The parametric equation for AB is given by

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.3}$$

$$\text{where, } \mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.4}$$

$$= \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} -5 \\ -2 \end{pmatrix} \tag{1.1.4.5}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.1.4.6}$$

Hence we get,

$$\mathbf{AB} : \mathbf{x} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} + k \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.1.4.7}$$

Similarly,

$$\mathbf{BC} : \mathbf{x} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + k \begin{pmatrix} 5 \\ -5 \end{pmatrix} \tag{1.1.4.8}$$

$$\mathbf{CA} : \mathbf{x} = \begin{pmatrix} -6 \\ 4 \end{pmatrix} + k \begin{pmatrix} 1 \\ -6 \end{pmatrix} \tag{1.1.4.9}$$

1.1.5. The normal form of the equation of **AB** is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.1.5.1}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top \left(\mathbf{B} - \mathbf{A}\right) = 0 \tag{1.1.5.2}$$

or,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.3}$$

then find the normal form of the equations of **AB BC** and **CA Solution:** :

The normal equation for the side AB is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.1.5.4}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \tag{1.1.5.5}$$

Now our task is to find the $\mathbf{n}$ so that we can find $\mathbf{n}^\top$. As given.

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.6}$$

Here $\mathbf{m} = \mathbf{B} - \mathbf{A}$ for side $\mathbf{AB}$

$$\implies \mathbf{m} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} -5 \\ -2 \end{pmatrix} \tag{1.1.5.7}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{1.1.5.8}$$

Now as we have obtained vector $\mathbf{m}$.we can use this to obtain vector $\mathbf{n}$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{1.1.5.9}$$

The transpose of $\mathbf{n}$ is

$$\mathbf{n}^\top = \begin{pmatrix} 1 & 4 \end{pmatrix} \tag{1.1.5.10}$$

Hence the normal equation of side AB is

$$\begin{pmatrix} 1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 4 \end{pmatrix} \begin{pmatrix} -5 \\ -2 \end{pmatrix} \tag{1.1.5.11}$$

$$\implies \begin{pmatrix} 1 & -4 \end{pmatrix} \mathbf{x} = -13 \tag{1.1.5.12}$$

Figure 1.2: The line $\mathbf{AB}$ plotted

Similarly

$$\implies \mathbf{BC} : \begin{pmatrix} 5 & -5 \end{pmatrix} \mathbf{x} = -50 \tag{1.1.5.13}$$

Figure 1.3: The line **BC** plotted

$$\implies \mathbf{CA} : \begin{pmatrix} -6 & 1 \end{pmatrix} \mathbf{x} = -32 \tag{1.1.5.14}$$

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Figure 1.4: The line **CA** plotted

1.1.6. Find the area of the $\triangle ABC$ **Solution:**

Given

$$\mathbf{A} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}; \mathbf{C} = \begin{pmatrix} -6 \\ 4 \end{pmatrix} \tag{1.1.6.1}$$

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}-5\\-2\end{pmatrix}-\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}-4\\-1\end{pmatrix} \tag{1.1.6.2}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}-5\\-2\end{pmatrix}-\begin{pmatrix}-6\\4\end{pmatrix}=\begin{pmatrix}1\\-6\end{pmatrix} \tag{1.1.6.3}$$

$$\therefore (\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})=\begin{vmatrix}-4 & -1\\1 & -6\end{vmatrix} \tag{1.1.6.4}$$

$$= -4\times -6-1\times(1) \tag{1.1.6.5}$$

$$= 24-1 \tag{1.1.6.6}$$

$$= 23 \tag{1.1.6.7}$$

$$\implies \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\|=\frac{1}{2}\left\|23\right\|=\frac{23}{2} \tag{1.1.6.8}$$

1.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B}-\mathbf{A})^\top \mathbf{C}-\mathbf{A}}{\left\|\mathbf{B}-\mathbf{A}\right\|\left\|\mathbf{C}-\mathbf{A}\right\|} \tag{1.1.7.1}$$

Solution:

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$,

(a) Finding the value of angle A

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}4\\1\end{pmatrix} \tag{1.1.7.2}$$

and

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}-1\\6\end{pmatrix} \tag{1.1.7.3}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{A}\|=\sqrt{17} \tag{1.1.7.4}$$
$$\|\mathbf{C}-\mathbf{A}\|=\sqrt{37} \tag{1.1.7.5}$$

and by doing matrix multiplication we get,

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})=\begin{pmatrix}4 & 1\end{pmatrix}\begin{pmatrix}-1\\6\end{pmatrix}=2 \tag{1.1.7.6}$$

So, we get

$$\cos A=\frac{2}{\sqrt{17}\sqrt{37}} \tag{1.1.7.7}$$
$$=\frac{2}{\sqrt{629}} \tag{1.1.7.8}$$
$$\implies A=\cos^{-1}\frac{2}{\sqrt{629}} \tag{1.1.7.9}$$

(b) Finding the value of angle B

$$\mathbf{C}-\mathbf{B} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \tag{1.1.7.10}$$

and

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{1.1.7.11}$$

also calculating the values of norms

$$\|\mathbf{C}-\mathbf{B}\| = \sqrt{50}\,\|\mathbf{A}-\mathbf{B}\| \qquad = \sqrt{17} \tag{1.1.7.12}$$

and by doing matrix multiplication we get,

$$(\mathbf{C}-\mathbf{B})^\top(\mathbf{A}-\mathbf{B}) = \begin{pmatrix} -5 & 5 \end{pmatrix} \begin{pmatrix} -4 \\ -1 \end{pmatrix} = -25 \tag{1.1.7.13}$$

So, we get

$$\cos B = \frac{-25}{50\sqrt{17}} \tag{1.1.7.14}$$

$$= \frac{-25}{\sqrt{850}} \tag{1.1.7.15}$$

$$\implies B = \cos^{-1}\frac{-25}{\sqrt{850}} \tag{1.1.7.16}$$

(c) Finding the value of angle C

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}1\\-6\end{pmatrix} \tag{1.1.7.17}$$

and

$$\mathbf{B}-\mathbf{C}=\begin{pmatrix}5\\-5\end{pmatrix} \tag{1.1.7.18}$$

also calculating the values of norms

$$\|\mathbf{A}-\mathbf{C}\|=\sqrt{37} \tag{1.1.7.19}$$

$$\|\mathbf{B}-\mathbf{C}\|=\sqrt{50} \tag{1.1.7.20}$$

and by doing matrix multiplication we get,

$$\begin{aligned}(\mathbf{A}-\mathbf{C})^{\top}(\mathbf{B}-\mathbf{C})&=\begin{pmatrix}1 & -6\end{pmatrix}\begin{pmatrix}5\\-5\end{pmatrix}\\&=35\end{aligned} \tag{1.1.7.21}$$

so

$$\cos C=\frac{35}{37\sqrt{50}} \tag{1.1.7.22}$$

$$=\frac{35}{\sqrt{1850}} \tag{1.1.7.23}$$

$$\implies C=\cos^{-1}\frac{35}{\sqrt{1850}} \tag{1.1.7.24}$$