1

1. See attached code, includes explanations & output.

   $f(n) = n^2 + n + 17$ smallest composite @ $n = 16$

   $g(n) = n^2 + 21n + 1$ smallest composite @ $n = 18$

   $h(n) = 3n^2 + 3n + 23$ smallest composite @ $n = 22$

2. See attached code, includes explanation of algorithm

   Counter example was verified

3. See attached code, includes output & explanations

   Confirmed for all odd integers $3 \leq n \leq 75$

4. Lemma: For all pairs of odd integers, the only way to get a product of the form $6k+5$ is via the factors of forms $6k+1$ and $6k+5$

   a. By div algorithm, all integers can be represented via forms below, however only $6k+1$, $6k+3$ and $6k+5$ are odd.

$6k$	$6k+1$	$6k+2$	$6k+3$	$6k+4$	$6k+5$
$2 \mid 2 \cdot 3k$	odd	$2 \mid 2(3k+1)$	odd	$2 \mid 2(3k+2)$	odd

   b. Their products are as follows

$(6k+1)(6k'+1) = 36kk' + 6k + 6k' + 1 = 6(6kk'+k+k')+1 \in 6k+1$

$(6k+1)(6k'+3) = 36kk'+18k+6k'+3 = 6(6kk'+3k+k')+3 \in 6k+3$

$(6k+3)(6k'+3) = 36kk'+18k+18k'+9 = 6(6kk'+3k+3k')+9 = 6(6kk'+3k+3k'+1)+3 \in 6k+3$

$(6k+3)(6k'+5) = 36kk'+30k+18k'+15 = 6(6kk'+5k+3k')+15 = 6(6kk'+5k+3k'+2)+3 \in 6k+3$

$(6k+5)(6k'+5) = 36kk'+30k+30k'+25 = 6(6kk'+5k+5k')+25 = 6(6kk'+5k+5k'+4)+1 \in 6k+1$

$(6k+5)(6k'+1) = 36kk'+6k+30k'+5 = 6(6kk'+k+5k')+5 \in 6k+5$

As you can see an integer of the form $6k+5$ with only odd factors must be the product of integers of form $6k+1$ and $6k+5$

Continued →

4. proof: (BWOC) consider the positive integer N where $q_1 q_2 \cdots q_r$ is the product of the presumed finite set of primes having the form $6k+5$.

$$N = 6 q_1 q_2 \cdots q_r - 1 = 6(q_1 q_2 \cdots q_r - 1) + 5$$

By FTA, N has a prime factorization $N = r_1 r_2 \cdots r_s$
For all $k \leq s$ $r_k \neq 2$ as N is not even

$$6k+5 = 2 \cdot 3k + 5 = 2(3k+2)+1 \in 2k+1$$

Thus by the lemma, we know that at least one factor, r_i must be of the form $6k+5$, as the final product has the form $6k+5$, but r_i cannot be in the set q as this leads to a contradiction that $r_i \mid 1$!!! Therefore we must conclude that there are an infinite number of primes of form $6k+5$.

5. 13 is largest prime to divide consecutive integers of form $a_n = n^2 + 3$

let p be the divisor in

$$p \mid a_{n+1} \Rightarrow p \mid (n+1)^2 + 3 \Rightarrow p \mid n^2 + 2n + 4$$
$$p \mid a_n \Rightarrow p \mid n^2 + 3$$
$$\overline{}$$
$$p \mid 2n+1$$

implied via Cramer's rule

$$n^2 + 3 \equiv 0 \pmod{p}$$
$$n^2 \equiv -3 \pmod{p}$$
$$4n^2 \equiv -12 \pmod{p}$$

$$2n + 1 \equiv 0 \pmod{p}$$
$$(2n)^2 \equiv (-1)^2 \pmod{p}$$
$$4n^2 \equiv 1 \pmod{p}$$
$$-12 \equiv 1 \pmod{p}$$
$$13 \equiv 0 \pmod{p}$$
$$\therefore \underline{\underline{p \mid 13}}$$

~~Because the divisor, p, must be~~
~~divisible by 13~~ 13 must be the largest prime divisor as all larger values must be divisible by 13 and thus composite

6. Via Dirichlet's theorem, show there are infinitely many primes containing the block of digits 123456789 without ending with it

$a_k = 10^{11}k + 1234567891 \ \forall\, k \in \mathbb{Z}^+$ represents all integers

$\gcd(10^{11}, 1234567891) = 1$ via sagemath

∴ relatively prime

Because the formula for values in the set matches the form $a+bk$ and a & b are relatively prime as shown with their gcd of 1 Dirichlet's formula tells us that there are infinitely many primes in the set.

7. a. $53^{103} + 103^{53} \equiv 0 \pmod{39}$

$53^1 = 53 \equiv 14 \pmod{39}$

$53^2 \equiv 14^2 \equiv 196 \equiv 1 \pmod{39}$

$\therefore 53^{2k} \equiv 1 \pmod{39}$

$53^{103} = 53^{2\cdot 51+1} = 53^{2\cdot 51}\, 53 \equiv 53 \pmod{39} \equiv 14 \pmod{39}$

$103 \equiv 25 \pmod{39}$

$103^2 \equiv 25^2 \equiv 625 \equiv 1 \pmod{39}$

$\therefore 103^{2k} \equiv 1 \pmod{39}$

$103^{53} = 103^{2\cdot 26+1} = 103^{2\cdot 26}\, 103 \equiv 103 \pmod{39}$

$\equiv 25 \pmod{39}$

$14 + 25 \equiv 0 \pmod{39}$

$39 \equiv 0 \pmod{39}$ ✓

b. $111^{333} + 333^{111} \equiv 0 \pmod 7$

$111 \equiv 6 \pmod 7$

$111^2 \equiv 6^2 \equiv 36 \equiv 1 \pmod 7$

$\therefore 111^{2k} \equiv 1 \pmod 7$

$111^{333} = 111^{2\cdot 166}\, 111 \equiv 1 \cdot 6 \equiv 6 \pmod 7$

$333^1 \equiv 4 \pmod 7$

$333^2 \equiv 4^2 \equiv 16 \equiv 2 \pmod 7$

$333^3 \equiv 333^2 \cdot 333 \equiv 2 \cdot 4 \equiv 8 \equiv 1 \pmod 7$

$\therefore 333^{3k} \equiv 1 \pmod 7$

$333^{111} = 333^{37\cdot 3} \equiv 1 \pmod 7$

$6 + 1 \equiv 0 \pmod 7$

$7 \equiv 0 \pmod 7$

$0 \equiv 0 \pmod 7$ ✓

8c a. $89 \mid 2^{44} - 1$

$2^{44} - 1 \equiv 0 \pmod{89}$

$2^{44} \equiv 1 \pmod{89}$

$2^{11} = 2048 \equiv 1 \pmod{89}$

$\therefore 2^{11k} \equiv 1 \pmod{89}$

$2^{44} = 2^{4 \cdot 11} \equiv 1 \pmod{89}$

$1 \equiv 1 \pmod{89}$

b. $97 \mid 2^{48} - 1$

$2^{48} - 1 \equiv 0 \pmod{97}$

$2^{48} \equiv 1 \pmod{97}$

$2^{8} = 256 \equiv 62 \pmod{97}$

$2^{16} = 62^2 \equiv 3844 \equiv 61 \pmod{97}$

$2^{32} = 61^2 \equiv 3721 \equiv 35 \pmod{97}$

$2^{48} = 2^{32} \cdot 2^{16} \equiv 35 \cdot 61 \equiv 2135 \equiv 1 \pmod{97}$

$1 \equiv 1 \pmod{97}$

9.

~~for each a_i on the set of residuals let r_i denote the corresponding remainder modulo n of the [illegible]~~

for each $i \neq j$ where $i, j < n$

$aa_i \not\equiv aa_j \pmod{n}$ must be true as all items in the set must have unique residuals

using theorem 4-3 we can ~~simplify~~ remove the factors

$a_i \not\equiv a_j \pmod{n \mid \gcd(a, n)}$

$a_i \not\equiv a_j \pmod{n}$

Thus proving that all residuals remain unique, and the result will remain a complete set of residuals.