3. $\forall n \in \mathbb{Z}^+$ $\displaystyle\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$

$$\frac{\sigma(n)}{n} = \frac{1}{n}\sum_{d|n} d = \sum_{d|n}\frac{d}{n} = \sum_{d|n}\frac{1}{n/d} = \sum_{d|n}\frac{1}{d} \quad \#$$

Lemma:

- Because $d|n$ the value n/d gives another factor, d' such that $d \cdot d' = n$.
- $d'|n$ as well, so it would also be encountered in the iteration of $\sum_{d|n}$ and when it is encountered, n/d' would give d. Therefore in the summation of factors d of n it is safe to replace n/d with d as it always gives related values of d in the set and the order of addition does not matter as addition is commutative.
- Edit: I could have used the Möbius inversion formula here but think argument holds.

4. Because the factors of n must include n and 1, the largest possible value of n would be 17.

Brute-Forcing on the limited range...

$\sigma(1) = 1$

$\sigma(2) = 3$

$\sigma(3) = 4$

$\sigma(4) = 1+2+4 = 7$

$\sigma(5) = 1+5 = 6$

$\sigma(6) = 1+2+3+6 = 12$

$\sigma(7) = 1+7 = 8$

$\sigma(8) = 1+2+4+8 = 15$

$\sigma(9) = 1+3+9 = 13$

$\sigma(10) = 1+2+5+10 = 18$ ✓ $\therefore n = 10$

$\sigma(11) = 1+11 = 12$

$\sigma(12) = 1+2+3+4+6+12 = 28$

$\sigma(13) = 1+13 = 14$

$\sigma(14) = 1+2+7+14 = \cancel{24}$

$\sigma(15) = 1+3+5+15 = X$

$\sigma(16) = 1+2+4+8+16 = X$

$\sigma(17) = 1+17 = 18$ ✓ $\therefore n = 17$

The only possible values of n are 10 and 17 as it would be impossible for a number >17 to make $\sigma(n) = 18$

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The only possible values of n are 10 and 17 as it would be impossible for a number > 17 to make $\sigma(n) = 18$

5. $\forall\, k \geq 2$;

a. $\sigma(2^{(k-1)}) = 2 \cdot 2^{k-1} - 1 = 2^k - 1$ $\qquad n = 2^{k-1}$

the prime factorization for a number of the form 2^{k-1} consists of $k-1$ 2's

via theorem 6.2:

$$\sigma(2^{(k-1)}) = \frac{2^{(k-1)+1} - 1}{2 - 1} = \frac{2 \cdot 2^{k-1} - 1}{1} = 2 \cdot 2^{k-1} - 1$$

$$\underline{\sigma(n) = 2n - 1}$$

b. if $2^k - 1$ is prime then $n = 2^{k-1}(2^k - 1)$ satisfies $\sigma(n) = 2n$

let $p = 2^k - 1$ in $n = 2^{k-1} p$

Because $2^k - 1$ is prime, the prime factorization of $2^{k-1}(2^k - 1)$ consists of $k-1$, 2's and one $(2^k - 1)$. Therefore we can apply theorem 6.2:

$$\sigma(n) = \sigma(2^{k-1}(2^k - 1)) = \frac{2^{(k-1)+1} - 1}{2-1} \cdot \frac{(2^k - 1)^{1+1} - 1}{(2^k - 1) - 1} = (2^k - 1) \cdot \frac{(2^k - 1)^2 - 1}{2^k - 2}$$

$$= (2^k - 1) \cdot \frac{(2^{2k} - 2 \cdot 2^k + 1) - 1}{2^k - 2} = \frac{(2^k - 1)(2^{2k} - 2 \cdot 2^k)}{2^k - 2}$$

$(2^k - 1)(2^{2k} - 2 \cdot 2^k)$

$2^{3k} - 2 \cdot 2^{2k} - 2^{2k} + 2 \cdot 2^k$

$2^{3k} - 3 \cdot 2^{2k} + 2 \cdot 2^k$

$2^k(2^{2k} - 3 \cdot 2^k + 2)$

$2^k(2^k - 2)(2^k - 1)$

$$= \frac{2^k \cancel{(2^k - 2)}(2^k - 1)}{\cancel{2^k - 2}} = 2^k(2^k - 1)$$

$$= 2(2^{k-1}(2^k - 1))$$

$$\underline{\sigma(n) = 2n}$$

c. Like with before we can apply thrm. 6.2

$$\sigma(n) = \sigma(2^{k-1}(2^2 - 3)) = \frac{2^{(k-1)+1} - 1}{2 - 1} \cdot \frac{(2^k - 3)^{1+1} - 1}{(2^k - 3) - 1} = (2^k - 1) \cdot \frac{(2^k - 3)^2 - 1}{2^k - 4}$$

$$= (2^k - 1)\frac{(2^{2k} - 6 \cdot 2^k + 9) - 1}{2^k - 4} = \frac{(2^k - 1)\cancel{(2^k - 4)}(2^k - 2)}{\cancel{2^k - 4}} = (2^k - 1)(2^k - 2)$$

$$= 2^{2k} - 3 \cdot 2^k + 2 = 2^k(2^k - 3) + 2$$

$$\sigma(n) = 2 \cdot 2^{k-1}(2^k - 3) + 2 = \underline{2n + 2}$$

6. a. Given two relatively prime integers m and n, they must have no prime factors in common (By FTA).

$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ $\quad n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$ $\quad$ where $p_i \neq q_i \; \forall i \in [0, r]$

$\omega(m) = r$ $\quad \omega(n) = s$

Therefore, the number of prime factors for $m \cdot n$ $(\omega(m \cdot n))$ must be the sum of each ~~products~~ m + n's number of prime factors $(\omega(m) + \omega(n))$

$\omega(m) = r \quad \omega(n) = s$

$$\omega(m \cdot n) = \omega(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \cdot q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}) = r + s = \omega(m) + \omega(n)$$

using exponent rules and this result we can show that $f(n) = 2^{\omega(n)}$ is multiplicative.

$$g(m \cdot n) \overset{?}{=} g(m) \cdot g(n)$$

$$2^{\omega(m \cdot n)} \overset{?}{=} 2^{\omega(m)} \cdot 2^{\omega(n)}$$

$$2^{r+s} \overset{?}{=} 2^r \cdot 2^s$$

$$2^{r+s} \overset{\checkmark}{=} 2^{r+s} \quad \#$$

b. Because $f(n) = 2^{\omega(n)}$ is a multiplicative function we can use theorem 6.4 to show that $g(n) = \sum_{d|n} 2^{\omega(d)}$ is also multiplicative. Additionally $h(n) = \tau(n^2)$ is also multiplicative as exponentiation is distributive and τ is multiplicative. As shown by

$$h(mn) = \tau((mn)^2) = \tau(m^2 n^2) = \tau(m^2)\tau(n^2) = \underline{h(m)h(n)}$$

Thus proving the relation for each term in the prime factorization allows us to prove for the entire problem. As the terms are all relatively prime and thus can be isolated via the multiplicative property

$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$n^2 = p_1^{2k_1} p_2^{2k_2} \cdots p_r^{2k_r}$ $\quad$ Isolate terms using multiplicative property

$\tau(p^{2k}) = \underline{2k+1}$ $\quad$ via theorem 6.2 as p is prime

$$\sum_{d|p^k} 2^{\omega(d)} = 2^{\omega(p^k)} + 2^{\omega(p^{k-1})} + \cdots + 2^{\omega(p^0)} = 2^1 + 2^1 + \cdots + 2^0 = \underline{2k+1}$$

$$\therefore \tau(p^{2k}) = \sum_{d|p^k} 2^{\omega(d)}$$

Because the proposition holds for the terms in the prime factorization of any $n \in \mathbb{Z}$ and both sides of the formula are multiplicative it must hold for all possible values.

7. a. $\mu(n)\mu(n+1)\mu(n+3) = 0 \quad \forall n \in \mathbb{Z}^+$

In order for the proposition to hold at least one of the terms $\mu(a)$ must equal zero where $n \leq a \leq n+3$.

According to the definition of $\mu(n)$ this happens when $p^2 | n$ for some prime p.

Fortunately, every 4 integers starting at $n = 1+3$ are divisible by $4 = 2^2$ a prime square, And because the starting first 4 values includes 4 it must hold for all $n \in \mathbb{Z}^+$.

b. $\sum_{k=1}^{n} \mu(k!) = 1 \quad \forall n \geq 3$

– Because μ is a multiplicative function $\mu(k!) = \mu(k)\mu(k-1)\ldots\mu(1)$

Because $4 = 2^2$ is the first prime square all $\mu(k!)$ where $k \geq 4$ will equal zero

Thus $\sum_{k=1}^{n} \mu(k!)$ will have the same value for all $n \geq 3$

That value is:

$= \mu(1!) + \mu(2!) + \mu(3!) = \mu(1) + \mu(2) + \mu(3 \cdot 2) = 1 - 1 + 1 = \underline{1}$

8. $\Lambda(n) = \begin{cases} \log(p) & \text{in } \Lambda(p^k) \text{ for any prime } p \text{ w/ } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$

a. for $\forall n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$

Because $\Lambda(n)$ is only non-zero for nonzero powers of primes the following holds

$$\sum_{d|n} \Lambda(d) = k_1 \log(p_1) + k_2 \log(p_2) + \ldots + k_r \log(p_r)$$

$$= \log(p_1^{k_1}) + \log(p_2^{k_2}) + \ldots + \log(p_r^{k_r})$$

$$= \log(p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r})$$

$$= \log(n) \quad \#$$

8. B, $\Lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\log d = -\sum_{d|n}\mu(d)\log d$

$n = p^k$ where p is prime & $k \geq 1$

$\Lambda(p^k) \overset{?}{=} \sum_{d|p^k} \mu\left(\frac{p^k}{d}\right)\log(d) = \cancel{\mu\left(\frac{p^k}{p^k}\right)}^{=1}\log(p^k) + \mu\left(\frac{p^k}{p^{k-1}}\right)\log(p^{k-1}) + \mu\left(\frac{p^k}{p^{k-2}}\right)\log(p^{k-2}) + \dots + \mu\left(\frac{p^k}{1}\right)\log(1)$

$= \log p^k + \mu(p^1)\log(p^{k-1}) + \cancel{\mu(p^2)\log(p^{k-2}) + \dots + \mu(p^k)\log 1}$

$\llcorner\ \mu(p^k) = 0$ for $k \geq 2$

$= \log p^k - \log p^{k-1} = \log\left(\frac{p^k}{p^{k-1}}\right) = \underline{\log p}$ ✓

$\Lambda(p^k) \overset{?}{=} -\sum_{d|p^k}\mu(d)\log(d) = -(\cancel{\mu(1)\log(1)} + \mu(p)\log(p) + \cancel{\mu(p^2)\log(p^2) + \dots + \mu(p^k)\log(p^k)})$

$\llcorner\ \mu(p^k) = 0$ for $k \geq 2$

$= -(-\log p) = \underline{\log p}$ ✓

$n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$:

- as with before μ eliminates square-divisible factors and 1
- the remaining terms follow the following patterns

$\sum_{d|n} \mu\left(\frac{n}{d}\right)\log d = -\sum_{d|n}\mu(d)\log d = \dots$

when $r = 2$: $= \log p_1 + \log p_2 - (\log p_1 p_2) = \log\left(\frac{p_1 p_2}{p_1 p_2}\right) = \emptyset$ ✓

when $r = 3$: $= \log p_1 + \log p_2 + \log p_3 - (\log p_1 p_2 + \log p_2 p_3 + \log p_1 p_3) + \log p_1 p_2 p_3$

$= \log\left(\frac{p_1^2 p_2^2 p_3^2}{p_1^2 p_2^2 p_3^2}\right) = \emptyset$ ✓

$\dots = (r-1)\log p_1 - (r-1)\log p_1 + (r-1)\log p_2 - (r-1)\log p_2 + \dots + (r-1)\log p_r - (r-1)\log p_r$

$= \emptyset$ ✓

This could likely be put into a proof involving binomial coefficients to make a lemma but I'm almost out of time and I feel like there's an easier way that's not coming to me.

9. $s(n)$ = # square free divisors of n

$$s(n) = \sum_{d|n} |\mu(d)| = 2^{\omega(n)}$$

a. $\sum_{d|n} |\mu(d)|$ — any integer divisible by a square prime is also divisible by square as a square of a composite # will be divisible by square of it's prime factors as well. $(pq)^2 = p^2 q^2$ w/ p, q prime

— $\mu(d) = 0$ for any d s.t. $p^2 | d$ thus we can eliminate any square terms

because $1 = |\mu(d)|$ for any square-free term it gives desired value

b. $2^{\omega(n)}$: Because $2^{\omega(n)}$ is a multiplicative function, we can break it into its individual terms of its prime factorization

$$2^{\omega(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r})} = 2^{\omega(p_1^{k_1})} 2^{\omega(p_2^{k_2})} \cdots 2^{\omega(p_r^{k_r})}$$

$$= 2^1 \cdot 2^1 \cdots 2^1$$

$$= 2^r$$

✓

lemma: $2^r = \sum_{i=0}^{r} \binom{r}{i}$ which is the ~~actual~~ # square free factors