1. a. $1 + 3 + 5 + \dots + (2n-1) = n^2$ for all $n \geq 1$

Using induction

Basis: $n = 1 \Rightarrow$

$$n^2 = 1$$
$$1^2 = 1$$
$$1 \overset{\checkmark}{=} 1$$

Inductive:

$$(n+1)^2 = n^2 + (2(n+1) - 1)$$
$$n^2 + 2n + 1 \overset{\checkmark}{=} n^2 + 2n + 1$$

#

b. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$

Using induction

Basis: $n = 1 \Rightarrow$

$$1 \cdot 2 \overset{?}{=} \frac{n(n+1)(n+2)}{3}$$
$$1 \cdot 2 = \frac{1(1+1)(1+2)}{\cancel{3}}$$
$$2 \overset{\checkmark}{=} 2$$

$n^3 + 2n^2 + n^2 + 2n$

$n+1$

Inductive:

$2n + n$

$(n+1)(n+2)(n+3)$

$(n^2+3n+2)(n+3)$

$n^3+3n^2+3n^2+9n+2n+6$

$6n^2$

$11n$

$$\frac{(n+1)((n+1)+1)((n+1)+2)}{3} = \frac{n(n+1)(n+2)}{3} + (n+1)((n+1)+1)$$
$$\frac{n^3 + 6n^2 + 11n + 6}{3} = \frac{n^3 + 3n^2 + 2n}{3} + n^2 + 3n + 2$$
$$\cancel{\frac{n^3}{3}} + \cancel{2n^2} + \cancel{\frac{11}{3}n} + \cancel{2} = \cancel{\frac{n^3}{3}} + \cancel{n^2} + \cancel{\frac{2}{3}n} + \cancel{n^2} + \cancel{3n} + \cancel{2}$$
$$0 \overset{\checkmark}{=} 0 \quad \text{true for all } n$$

2.

2. $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \dots + a + 1)$ for all $n \geq 1$

Basis Step:

$n=1$: $a^1 - 1 = (a-1)(1)$
$a - 1 \overset{\checkmark}{=} a - 1$

$n=2$: $a^2 - 1 = (a-1)(a+1)$
$a^2 - 1 \overset{\checkmark}{=} a^2 - 1$

Inductive

$$\frac{a^n - 1}{a-1} = a^{n-1} + a^{n-2} + \dots + a + 1$$

next step

$$(a-1)\ \frac{a^n - 1}{a-1} + a^n = \frac{a^{n+1} - 1}{a-1}\ (a-1)$$

don't think I need this

$$a^{n+1} - 1 = a^{n+1} - a + a^n - 1 - a^n + a$$
$$= (a+1)(a^n - 1) - a(a^{n-1} - 1)$$

$$a^n - 1 + a^n(a-1) = a^{n+1} - 1$$
$$\cancel{a^n} - 1 + a^{n+1} - \cancel{a^n} = a^{n+1} - 1$$
$$a^{n+1} - 1 \overset{\checkmark}{=} a^{n+1} - 1$$

confirming induction hypothesis

3. a.

$n = 4$: $4! + 1 = 5^2$

$n = 5$: $5! + 1 = 11^2$

$n = 7$: $7! + 1 = 71^2$

b. No, counterexample: $m=2$ $n=2$

$2!\,2! \overset{?}{=} (2 \cdot 2)!$
$2 \cdot 2 \overset{?}{=} 4!$
$4 \neq 24$

$2! + 2! \overset{?}{=} (2+2)!$
$2 + 2 = 4!$
$4 \neq 24$

4.

Basis Step : $n=1$: $\frac{1}{1} \leq 2 - \frac{1}{1}$

$1 \leq 1$ ✓

Inductive Step :

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$$

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} - \frac{1}{n} \leq 0$$

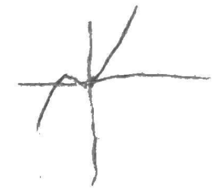
$$\frac{n(n+1) + n - (n+1)^2}{n(n+1)^2} \leq 0$$

$$\frac{-1}{n(n+1)^2} \leq 0$$

$$\frac{1}{n(n+1)^2} \geq 0$$

$n(n+1)^2 = 0$

$n = 0, -1$



Because $n(n+1)^2$ is positive for $n > 0$ the inequality should be true for all relevant values of n.

5. $\begin{cases} a_n = n & , n \in [1,3] \\ a_n = a_{n-1} + a_{n-2} + a_{n-3} & , n > 3 \end{cases}$ $a_n < 2^n$

Basis Step :

$n=1$: $1 < 2^1$ ✓ $n=2$: $2 < 2^2$, $2 < 4$ ✓ $n=3$: $3 < 2^3$, $3 < 8$

Inductive Step :

$$2^{n-1} + 2^{n-2} + 2^{n-3} < 2^n$$

$$\frac{2^n}{2} + \frac{2^n}{4} + \frac{2^n}{8} < 2^n$$

$$\frac{4 \cdot 2^n + 2 \cdot 2^n + 2^n}{8} < 2^n$$

$$\frac{7}{8} 2^n < 2^n \checkmark$$

$$\frac{7}{8} < 1 \checkmark$$

Not proofs a "justify"

4. a. using pascals triangle the sums of the elements on each row totals to a power of two following the pattern

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

This likely stems from the fact that each value is the sum of the two above it.

b. ~~For~~ For odd values of n: because pascals triangle has horizontal symmetry the equivalent values will have opposite signs and thus cancel eachother out.

For even values of n: the values come from the sum of the items in the odd n row above it and thus the values cancel out

c. $$\sum_{k=0}^{n} 9^{(n-k)} \binom{n}{k} = 10^n$$

probably has to ~~do~~ do w/ polynomial terms formula

7. $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \binom{2n+1}{3} = \frac{(2n+1)!}{(2n-2)!\,6}$

Basis: $n = 1$: $(2(1)-1)^2 = \binom{2(1)+1}{3}$

$1 = \binom{3}{3}$

$1 = 1$ ✓

Inductive: $\binom{2n+1}{3} + (2(n+1)-1)^2 = \binom{2(n+1)+1}{3}$

$$\frac{(2n+1)!}{(2n-2)!\,6} + (2n+1)^2 = \frac{(2n+3)!}{(2n)!\,6}$$

$$\frac{(2n+1)(2n)(2n-1)\cancel{(2n-2)!}}{6\cancel{(2n-2)!}} + 4n^2 + 4n + 1 = \frac{(2n+3)(2n+2)(2n+1)\cancel{(2n)!}}{6\cancel{(2n)!}}$$

$$\frac{8n^3 - 2n}{6} + 4n^2 + 4n + 1 = \frac{8n^3 + 24n^2 + 22n + 6}{6}$$

$$\frac{8n^3 - 2n + 24n^2 + 24n + 6}{6} = \text{''}$$

$$\frac{8n^3 + 24n^2 + 22n + 6}{6} \overset{\checkmark}{=} \frac{8n^3 + 24n^2 + 22n + 6}{6}$$

8. The square of any integer is either $3k$ or $3k+1$

using the "division algorithm" theorem we can classify all integers as multiples of $b=3$ where $a = 3q + r$ and $0 \le r < 3$

thus all integers come in one of the following forms:

$3q$, $3q+1$, $3q+2$.

$(3q)^2 = 9q^2 = 3(3q^2) \in 3k$

$(3q+1)^2 = 3q^2 + 6q + 1 = 3(q^2 + 2q) + 1 \in 3k+1$

$(3q+2)^2 = 3q^2 + 12q + 4 = 3(q^2 + 4q + 1) + 1 \in 3k+1.$

~~wh~~ which are all specific cases of the more generalized patterns

$3k$ ~~and~~ $3k+1$.

9. $3a^2 - 1$ need $0 \le r < 3$

$= 3(a^2 - 1) + 2$

$\in 3k + 2$

Because values resulting from this expression have a remainder ~~of~~ 2 they cannot be perfect squares as proof 8 shows that perfect squares can only have a remainder of 1 or 0 with 3.

10. No integer in the sequence 11, 111, 1111, 11111, ... is a perfect square

the remainder of any term in the sequence mod 4 is 3

however as we showed in class (and here again) perfect squares must have a remainder of 0 or 1 with 4

All integers can be described as

$2k$ $2k+1$ where $k \in \mathbb{Z}$

thus squares can be described as such

$(2k)^2 = 4k^2 \in 4k$

$(2k+1)^2 = 4k^2+4k+1 = 4(k^2+k)+1 \in 4k+1$

Thus the remainder for perfect squares with 4 must be either 1 or 0 and because items in the series have a remainder of 3 they cannot be perfect squares.