3. $15 \mid 2^{4n} - 1 \quad \forall \; n \geq 1$

Basis: $\quad 2^{4(1)} - 1 = 16 - 1 = 15(1)$ ✓ divisible by 15 ✓

Inductive:

Assume holds for n

$2^{4n} - 1 = 15q$

$2^{4n} = 15q + 1$

$2^{4(n+1)} - 1$

$= 2^{4n+4} - 1$

$= 2^{4n} 2^{4} - 1$

$= (15q+1)16 - 1$

$= 16 \cdot 15q + 16 - 1$

$= 15 \cdot 16q + 15$

$= 15(16q+1) \quad \in 15k$ ✓

4. $5 \mid 3^{3n+1} + 2^{n+1} \quad \forall \; n \geq 1$

Basis: $\quad 3^{3(1)+1} + 2^{(1)+1} = 3^{4} + 2^{2} = 81 + 4 = 85 = \underline{5(17)}$ ✓ divisible by 5

Induction: $\quad$ Assume holds for $n \quad 3^{3n+1} + 2^{n+1} = 5q$

$3 \cdot 27^n + 2 \cdot 2^n = 5q$

for $n+1$ $\quad\quad 2 \cdot 2^n = 5q - 3 \cdot 27^n$

$3^{3(n+1)+1} + 2^{(n+1)+1}$

$= 3^{3n+4} + 2^{n+2}$

$= 3^4 \cdot 27^n + 4 \cdot 2^n$

$= 3^4 \cdot 27^n + 2(5q - 3 \cdot 27^n)$

$= 3^4 \cdot 27^n + 10q - 2 \cdot 3 \cdot 27^n$

$= 3(3^3 - 2)27^n + 10q$

$= 3(25)27^n + 10q$

$= 3 \cdot 5 \cdot 5 \cdot 27^n + 10q$

$= 5(3 \cdot 5 \cdot 27^n + 2q) \quad \in 5k$ ✓

5. $3 \mid n(2n^2+7) \quad \forall\ n \in \mathbb{Z}$

Basis $(n=1)$: $(1)(2(1)^2+7) = 2+7 = 9 = 3(3)$ ✓

Inductive: assume holds for n

$n(2n^2+7) = 3q$

$2n^3+7n = 3q$

$2n^3 = 3q-7n$

$(n+1)(2(n+1)^2+7)$

$= (n+1)(2(n^2+2n+1)+7)$

$= (n+1)(2n^2+4n+9)$

$= 2n^3+2n^2+4n^2+4n+9n+9$

$= 2n^3+6n^2+13n+9$

$= (3q-7n)+6n^2+13n+9$

$= 6n^2+6n+3q+9$

$= 3(2n^2+2n+q+3) \in 3k$ ✓

6. $\gcd(a, a+n) \mid n \quad \forall\ a \in \mathbb{Z},\ n \in \mathbb{Z}^+$

Basis $(n=1)$: $\gcd(a, a+1) \mid 1$

$\gcd(a, a+1) = d(1)$

$ax + (a+1)y = d(1)$

$ax + ay + y = d$

$a(x+y) + y = d$

let $x=-y$

$\cancel{a(-y+y)} + y = d$

$y = d$ ∴ $\gcd(a, a+1) = 1$ ✓

Because y is a free-variable and d must be on $\mathbb{Z}^+$ the gcd must be 1 as it's the lowest value in the set

Inductive:

$\gcd(a, a+n) \mid n$

$ax + (a+n)y = qn$

$ax = qn-(a+n)y$

$ax = qn-ay-ny$

$\gcd(a, a+(n+1)) \mid (n+1)$

$ax + (a+n+1)y = (n+1)q$

$ax + ay + ny + y = qn + q$

$(qn-ay-ny) + ay + ny + y = qn + q$

$y = q$ ✓ this prooves that there exists solutions

$ax + (a+n)q = qn$

$ax + aq = 0$

$a(x+q) = 0$

$x = -q$ ✓ this is in line w/ basis case

$= -y$