1. $g = 17k_1 + 3 \Rightarrow x \equiv 3 \pmod{17}$
$g = 16k_2 + 10 \Rightarrow x \equiv 10 \pmod{16}$
$g = 15k_3 + 0 \Rightarrow x \equiv 0 \pmod{15}$

$\gcd(17,16)=1$
$\gcd(15,16)=1$
$\gcd(15,17)=1$
$\therefore$ all naturally relatively prime so we can apply chinese remainder theorem

$n = 17 \cdot 16 \cdot 15 = 4080$

$N_1 = \frac{n}{17} = 240 \qquad N_2 = \frac{n}{16} = 255 \qquad N_3 = \frac{n}{15} = 272$

$240x_1 \equiv 1 \pmod{17} \qquad 255x_2 \equiv 1 \pmod{16} \qquad 272x_3 \equiv 1 \pmod{15}$

$x_1 \equiv 9 \qquad x_2 \equiv 15 \qquad x_3 = 8$

$x \equiv a_1N_1x_1 + \dots + a_rN_rx_r \pmod{n}$
$x \equiv 3 \cdot 240 \cdot 9 + 10 \cdot 255 \cdot 15 + \cancel{0 \cdot 272 \cdot 8} \pmod{4080}$
$x \equiv 44730 \equiv 3930 \pmod{4080}$

there were at least 3930 coins

solutions of form $x = 4080k + 3930$

$44730 \bmod 17 = 3$
$44730 \bmod 16 = 10$ ✓
$44730 \bmod 15 = 0$

2. $2x \equiv 1 \pmod 5 \Rightarrow 2x \equiv 6 \pmod 5 \Rightarrow x = 3 \pmod 5$
$3x \equiv 9 \pmod 6 \Rightarrow 3x \equiv 3 \pmod 6 \Rightarrow x \equiv 1 \pmod 2$
$4x \equiv 1 \pmod 7 \Rightarrow 4x \equiv 8 \pmod 7 \Rightarrow x \equiv 2 \pmod 7$
$5x \equiv 9 \pmod{11} \Rightarrow 45x \equiv 81 \pmod{11} \Rightarrow x \equiv 4 \pmod{11}$

2, 5, 7 and 11 are all prime and ~~thus~~ their gcd's should be 1 with eachother allowing us to use the chinese remainder theorem

$n = 5 \cdot 2 \cdot 7 \cdot 11 = 770$

$N_1 = \frac{n}{5} = 2 \cdot 7 \cdot 11 = 154 \qquad N_2 = \frac{n}{2} = 5 \cdot 7 \cdot 11 = 385 \qquad N_3 = \frac{n}{7} = 5 \cdot 2 \cdot 11 = 110 \qquad N_4 = \frac{n}{11} = 5 \cdot 2 \cdot 7 = 70$

$154x_1 \equiv 1 \pmod 5 \qquad 385x_2 \equiv 1 \pmod 2 \qquad 110x_3 \equiv 1 \pmod 7 \qquad 70x_4 \equiv 1 \pmod{11}$

$x_1 \equiv 4 \qquad x_2 \equiv 1 \qquad x_3 \equiv 3 \qquad x_4 = 3$

$\bar{x} \equiv 3 \cdot 154 \cdot 4 + 1 \cdot 385 \cdot 1 + 2 \cdot 110 \cdot 3 + 4 \cdot 70 \cdot 3 \pmod{770}$
$\bar{x} \equiv 3733 \equiv 653 \pmod{770}$

therefore solutions have the form $x = 770k + 653$

3. $17x \equiv 3 \pmod{2\cdot3\cdot5\cdot7} \Rightarrow \begin{cases} 17x \equiv 3 \pmod 2 \Rightarrow x \equiv 1 \pmod 2 \\ 17x \equiv 3 \pmod 3 \Rightarrow 17x \equiv 0 \pmod 3 \Rightarrow x \equiv 0 \pmod 3 \\ 17x \equiv 3 \pmod 5 \Rightarrow 51x \equiv 9 \pmod{15} \Rightarrow x \equiv 4 \pmod 5 \\ 17x \equiv 3 \pmod 7 \Rightarrow 17x \equiv 17 \pmod 7 \Rightarrow x \equiv 1 \pmod 7 \end{cases}$

2, 3, 5 and 7 are all prime they also relatively prime so we can apply the chinese remainder theorem.

$n = 2\cdot3\cdot5\cdot7 = 210$

$N_1 = \frac{n}{2} = 105 \quad N_2 = \frac{n}{3} = 70 \quad N_3 = \frac{n}{5} = 42 \quad N_4 = \frac{n}{7} = 30$

$105x_1 \equiv 1 \pmod 2 \quad 70x_2 \equiv 1 \pmod 3 \quad 42x_3 \equiv 1 \pmod 5 \quad 30x_4 \equiv 1 \pmod 7$

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 3 \quad x_4 = 4$

$\bar{x} \equiv 1\cdot105\cdot1 + \cancel{0\cdot70\cdot1} + 4\cdot42\cdot3 + 1\cdot30\cdot4 \pmod{210}$

$\equiv 729 \equiv \underline{99 \pmod{210}}$

therefore solutions have the form $\underline{x = 210k + 99}$

4. $\begin{cases} 2^2 \mid x \\ 3^2 \mid x+1 \\ 5^2 \mid x+2 \end{cases} \Rightarrow \begin{cases} x \equiv 0 \pmod{2^2} \Rightarrow x \equiv 0 \pmod 4 \\ x \equiv -1 \pmod{3^2} \Rightarrow x \equiv 8 \pmod 9 \\ x \equiv -2 \pmod{5^2} \Rightarrow x \equiv 23 \pmod{25} \end{cases}$

Because there is no overlap in prime factorizations 4, 9 and 25 are relatively prime and thus we can apply CRT.

$n = 4\cdot9\cdot25 = 900$

N_1 doesn't matter as $\equiv 0 \quad N_2 = \frac{n}{9} = 100 \quad N_3 = \frac{n}{25} = 36$

$100x_2 \equiv 1 \pmod 9 \quad 36x_3 \equiv 1 \pmod{25}$

$x_2 \equiv 1 \quad x_3 = 16$

$x \equiv 0 + 8\cdot100\cdot1 + 23\cdot36\cdot16 \pmod{900}$

$\equiv 14048 \equiv 548 \pmod{900}$

$\therefore$ solutions of form $x = 900k + 548$

using $k=0$, $x = 548$

548, 549, 550

$2^2 \mid 548$ ✓

$3^2 \mid 549$ ✓ 99 18 9

$5^2 \mid \underline{550}$ ✓

5. $a^2 \mid x$ $\quad b^3 \mid x+1$ $\quad c^4 \mid x+2$ $\Rightarrow \begin{cases} x \equiv 0 \pmod{5^2} \\ x \equiv -1 \pmod{3^3} \\ x \equiv -2 \pmod{2^4} \end{cases} \Rightarrow \begin{array}{l} x \equiv 0 \pmod{25} \\ x \equiv 26 \pmod{27} \\ x \equiv 14 \pmod{16} \end{array}$

Because there is no overlap in the prime factorizations of 25, 27 and 8 they must be relatively prime and they can apply CRT

$n = 25 \cdot 27 \cdot 16 = 10800$

N_1 doesn't matter b/c coeff 0 $\quad N_2 = \frac{n}{27} = 400 \quad N_3 = \frac{n}{16} = 675$

$400x_2 \equiv 1 \pmod{27}$ $\quad 675x_3 \equiv 1 \pmod{16}$

$x_2 = 16$ $\quad x_3 = 11$

$x \equiv 0 + 26 \cdot 400 \cdot 16 + 14 \cdot 675 \cdot 11 \pmod{10800}$

$x \equiv 270350 \equiv \underline{350 \pmod{10800}}$ $\therefore x = 10800k + 350$

using $k=0$, $x = 350$

350, 351, 352

$5^2 \mid 350$ ends w/ 50

$3^3 \mid 351$ 9 digits sum to 9

$2^4 \mid 352$

6. $\begin{cases} x \equiv 5 \pmod 6 \\ x \equiv 7 \pmod{15} \end{cases}$ $\quad \gcd(6, 15) = 3 \neq 1$ $\therefore$ cannot use CRT

according to the theorem proved in question 9 of homework 6, a congruence system $x \equiv a \pmod n$, $x \equiv b \pmod m$ has a solution iff $\gcd(n,m) \mid a-b$. In this problem because

$\gcd(6, 15) \nmid 5-7$
$3 \nmid 2$

the theorem ensures that there does not exist a simultaneous solution

7. $3x + 4y \equiv 5 \pmod 8$

$3x - 5 \equiv -4y \pmod 8$

$6x - 10 \equiv \cancel{-8y} \pmod 8$

$6x - 10 \equiv 0 \pmod 8$

$6x \equiv 2 \pmod 8$

$3x \equiv 1 \pmod 4$ $\quad \gcd(3,4) = 1$

$x_0 = 3$

$x = 3 + \frac{4}{\gcd(3,4)}k \rightarrow x \equiv 3 \pmod 4$ by theorem 4-7

$x \equiv \{3, 7\} \pmod 8$

Theorem 4.7: $ax \equiv b \pmod n$

solutions: $x = x_0 + \frac{n}{\gcd(a,n)}t$

iff $\gcd(a,n) \mid b$

exist $\gcd(a,n)$ incong solns.

$x=3$: $3(3) + 4y \equiv 5 \pmod 8$

$4y \equiv -4 \pmod 8$

$y \equiv -1 \equiv 1 \pmod 2$

$x=7$: $3(7) + 4y \equiv 5 \pmod 8$

$4y \equiv -16 \pmod 8$

$y \equiv -4 \pmod 2$

$y \equiv 0 \pmod 2$

($x=3$	$y=1$)	($x=7$	$y=0$)
($x=3$	$y=3$)	($x=7$	$y=2$)
($x=3$	$y=5$)	($x=7$	$y=4$)
($x=3$	$y=7$)	($x=7$	$y=6$)

} incongruent solutions

8. $\begin{cases} 11x + 5y \equiv 7 \pmod{20} \\ 6x + 3y \equiv 8 \pmod{20} \end{cases}$

Theorem 4-9: $ax + by \equiv r \pmod n$
$cx + dy \equiv s \pmod n$
unique sln. when $\gcd(ad - bc, n) = 1$

$\gcd(11\cdot 3 - 5\cdot 6, 20) = \gcd(3, 20) = 1$ $\therefore$ has solution

$(11\cdot 3 - 5\cdot 6)x \equiv 3\cdot 7 - 5\cdot 8 \pmod{20}$

$3x \equiv -19 \pmod{20}$

$3x \equiv 1 \pmod{20}$

$x \equiv 7 \pmod{20}$

$d(ax + by) \equiv rd \Rightarrow dax + dby \equiv dr$

$b(cx + dy) \equiv sb \quad bcx + bdy \equiv bs$

$dax + \cancel{dby} - bcx + \cancel{bdy} \equiv dr - bs$

$\therefore (da - bc)x \equiv dr - bs \pmod n$

$6(7) + 3y \equiv 8 \pmod{20}$

$3y \equiv -34 \equiv 6 \pmod{20}$

$y \equiv 2 \pmod{20}$

$x \equiv 7 \pmod{20}$

$y \equiv 2 \pmod{20}$