Teorija zvezdanih spektara -4-

Osobine (matematička svojstva) integro-eksponencijalnih funkcija (eksponencijalnih integrala). Po definiciji imamo (vidi sliku 1):

$$E_n(x) = \int_1^\infty e^{-xy}\ \frac{dy}{y^n}, \qquad n = 0, 1, 2, ..., \qquad x \geq 0.$$

Lako se pokazuje da važi:

$$E_0(x) = \int_1^\infty e^{-xy}\ dy = \frac{1}{x}\int_1^\infty e^{-xy}\ d(xy) = -\frac{1}{x}e^{-xy}\Big|_1^\infty = \frac{e^{-x}}{x}.$$

Takođe, imamo:

$$E_n(0) = \int_1^\infty \frac{dy}{y^n} = \begin{cases} \infty, & n = 0, 1 \\ \frac{1}{n-1}, & n \geq 2 \end{cases}.$$

Za velike vrednosti argumenta, integro-eksponencijalna funkcija postaje:

$$E_n(x) = \frac{e^{-x}}{x}\left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} - \cdots\right] \approx \frac{1}{xe^x}, \qquad x >> 1.$$

Dakle, važi:

$$x \to \infty \Rightarrow E_n(x) \to 0.$$

1. Dokazati da važi:

$$\frac{d}{dx}E_n(x) = \begin{cases} -\frac{1+x}{x}\ E_0(x), & n = 0 \\ -E_{n-1}(x), & n \geq 1 \end{cases}.$$

Najpre razmatramo slučaj kada je $n = 0$:

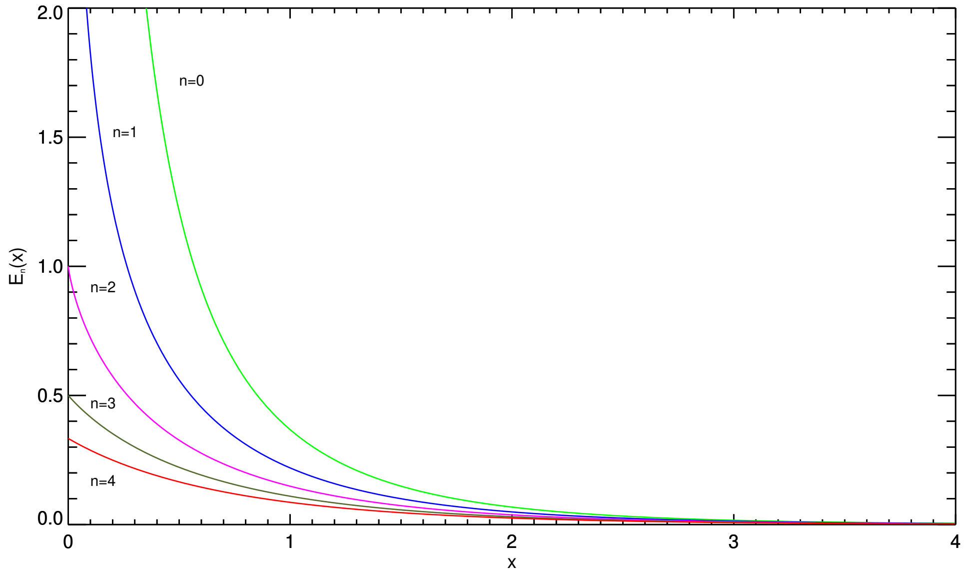
$$\frac{dE_0(x)}{dx} = \frac{d}{dx}\left(\frac{e^{-x}}{x}\right) = -\frac{1}{x^2}e^{-x} - \frac{1}{x}e^{-x} = -\frac{x+1}{x}\ E_0(x).$$

Sada nalazimo rešenje za $n \geq 1$:

$$\frac{dE_n(x)}{dx} = \frac{d}{dx}\int_1^\infty e^{-xy}\ \frac{dy}{y^n} = \int_1^\infty \frac{dy}{y^n}\left(\frac{d}{dx}e^{-xy}\right) = \int_1^\infty \frac{dy}{y^n}\left(-ye^{-xy}\right),$$

$$\frac{dE_n(x)}{dx} = -\int_1^\infty e^{-xy}\ \frac{dy}{y^{n-1}} = -E_{n-1}(x),$$

pri čemu važe uslovi ravnomerne konvergencije te smo mogli *podvući* operator diferenciranja pod integral.



Slika 1: Grafici $E_n(x)$, $n = 0, 1, 2, 3, 4$.

2. Pokazati da važi:

$$\int_x^\infty E_n(x)dx = E_{n+1}(x).$$

Imamo:

$$\int_x^\infty E_n(x)dx = \int_x^\infty dx \int_1^\infty e^{-xy}\ \frac{dy}{y^n} = \int_1^\infty \frac{dy}{y^n} \int_x^\infty e^{-xy}dx = \int_1^\infty e^{-xy}\ \frac{dy}{y^{n+1}} = E_{n+1}(x).$$

3. Pokazati da važi:

$$\int_x^\infty xE_n(x)dx = xE_{n+1}(x) + E_{n+2}(x).$$

Imamo:

$$\int_x^\infty xE_n(x)dx = \int_x^\infty xdx \int_1^\infty e^{-xy}\ \frac{dy}{y^n} = \int_1^\infty \frac{dy}{y^n} \int_x^\infty xe^{-xy}dx =$$

$$= \int_1^\infty \frac{dy}{y^n} \left(-\frac{x}{y}e^{-xy}\Big|_x^\infty + \frac{1}{y}\int_x^\infty e^{-xy}dx \right) = \int_1^\infty \frac{dy}{y^n} \left(\frac{x}{y}e^{-xy} - \frac{1}{y^2}e^{-xy}\Big|_x^\infty \right) =$$

$$= \int_1^\infty \frac{dy}{y^n} \left(\frac{x}{y}e^{-xy} + \frac{1}{y^2}e^{-xy} \right) = x\int_1^\infty e^{-xy}\frac{dy}{y^{n+1}} + \int_1^\infty e^{-xy}\frac{dy}{y^{n+2}} = xE_{n+1}(x) + E_{n+2}(x).$$

4. Pokazati da se integro-eksponencijalna funkcija može zapisati kao:

$$E_n(x) = x^{n-1} \int_x^\infty e^{-y}\, \frac{dy}{y^n}.$$

Imamo:

$$E_n(x) = \int_1^\infty e^{-xy}\, \frac{dy}{y^n}, \quad xy = z, \quad \Rightarrow$$

$$E_n(x) = x^{n-1} \int_x^\infty e^{-z} \frac{dz}{z^n} \equiv x^{n-1} \int_x^\infty e^{-y}\, \frac{dy}{y^n}.$$

5. Dokazati da važi:

$$E_n(x) = \frac{1}{n-1}\left[e^{-x} - xE_{n-1}(x)\right], \quad n > 1.$$

Imamo:

$$E_n(x) = \int_1^\infty e^{-xy}\, \frac{dy}{y^n} = \int_1^\infty e^{-xy}\, d\left(\frac{y^{1-n}}{1-n}\right) =$$

$$= e^{-xy}\, \frac{1}{1-n}\, y^{1-n}\Big|_1^\infty + \frac{1}{1-n}\int_1^\infty xy^{1-n}e^{-xy}dy =$$

$$= -e^{-x}\frac{1}{1-n} + \frac{x}{1-n}\int_1^\infty e^{-xy}\, \frac{dy}{y^{n-1}} = \frac{e^{-x}}{n-1} - \frac{x}{n-1}E_{n-1}(x).$$

Sada je jasno da važi:

$$E_n(x) = \frac{1}{n-1}\left[e^{-x} - xE_{n-1}(x)\right], \quad n > 1.$$

Za $E_1(x)$ inače važi:

$$E_1(x) = -\gamma - \ln x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k!}, \quad x > 0, \quad \gamma = 0.5572156...^1.$$

6. Izračunati: $\Lambda_\tau\{a + bt\}$, gde su a i b konstante.

Usled linearnosti Λ operatora imamo:

$$\Lambda_\tau\{a + bt\} = a\Lambda_\tau\{1\} + b\Lambda_\tau\{t\}.$$

Imamo:

$$\Lambda_\tau\{1\} = \frac{1}{2}\int_0^\infty E_1(|t-\tau|)dt = \frac{1}{2}\left(\int_0^\tau E_1(\tau - t)dt + \int_\tau^\infty E_1(t-\tau)dt\right),$$

$$\Lambda_\tau\{1\} = \frac{1}{2}\left(\int_0^\tau dt \int_1^\infty e^{-(\tau-t)y}\frac{dy}{y} + \int_\tau^\infty dt \int_1^\infty e^{-(t-\tau)y}\frac{dy}{y}\right),$$

[1] $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n}\frac{1}{k} - \ln n) = 0.5772156649015328606065120900824024310421593359399 2...$

$$\Lambda_\tau\{1\} = \frac{1}{2}\left(\int_1^\infty e^{-\tau y}\frac{dy}{y}\int_0^\tau e^{ty}dt + \int_1^\infty e^{\tau y}\frac{dy}{y}\int_\tau^\infty e^{-ty}dt\right),$$

$$\Lambda_\tau\{1\} = \frac{1}{2}\int_1^\infty \frac{dy}{y^2}\left(1 - e^{-\tau y}\right) + \frac{1}{2}\int_1^\infty \frac{dy}{y^2},$$

$$\Lambda_\tau\{1\} = 1 - \frac{1}{2}E_2(\tau).$$

Sa druge strane, imamo:

$$\Lambda_\tau\{t\} = \frac{1}{2}\int_0^\infty tE_1(|t-\tau|)dt = \frac{1}{2}\left(\int_0^\tau tE_1(\tau - t)dt + \int_\tau^\infty tE_1(t-\tau)dt\right),$$

$$\Lambda_\tau\{t\} = \frac{1}{2}\left(\int_0^\tau tdt\int_1^\infty e^{-(\tau-t)y}\frac{dy}{y} + \int_\tau^\infty tdt\int_1^\infty e^{-(t-\tau)y}\frac{dy}{y}\right),$$

$$\Lambda_\tau\{t\} = \frac{1}{2}\left(\int_1^\infty e^{-\tau y}\frac{dy}{y}\int_0^\tau te^{ty}dt + \int_1^\infty e^{\tau y}\frac{dy}{y}\int_\tau^\infty te^{-ty}dt\right),$$

$$\int_0^\tau te^{ty}dt = \frac{\tau}{y}e^{\tau y} - \frac{1}{y^2}e^{\tau y} + \frac{1}{y^2}, \quad u = t, \quad dv = e^{ty}dt,$$

$$\int_\tau^\infty te^{-ty}dt = \frac{\tau}{y}e^{-\tau y} + \frac{1}{y^2}e^{-\tau y}, \quad u = t, \quad dv = e^{-ty}dt,$$

$$\Lambda_\tau\{t\} = \tau\int_1^\infty \frac{dy}{y^2} + \frac{1}{2}\int_1^\infty e^{-\tau y}\frac{dy}{y^3},$$

$$\Lambda_\tau\{t\} = \tau + \frac{1}{2}E_3(\tau).$$

Konačno imamo:

$$\Lambda_\tau\{a + bt\} = a + b\tau + \frac{1}{2}\left[bE_3(\tau) - aE_2(\tau)\right].$$

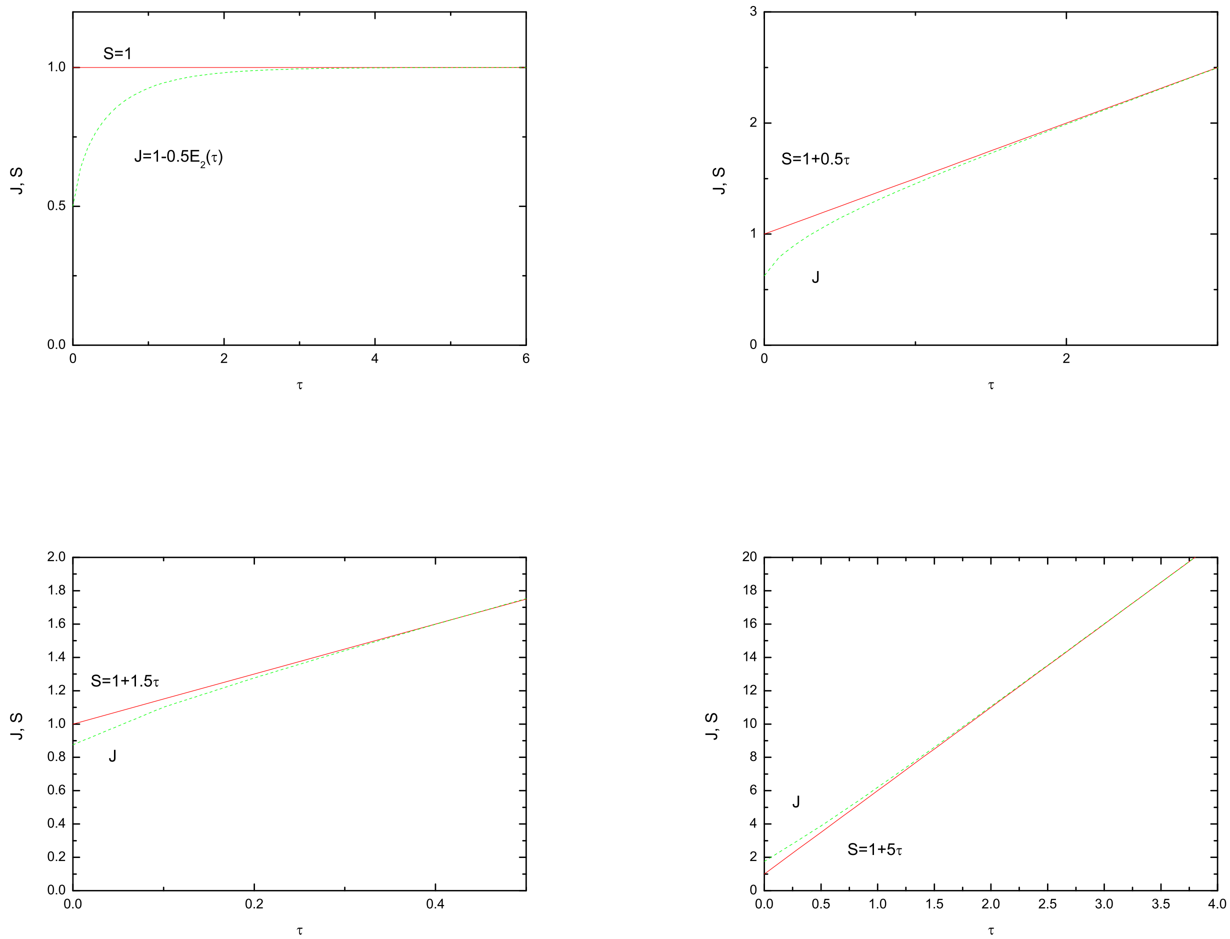
Znači, ukoliko aproskimiramo funkciju izvora linearnom funkcijom po optičkim dubinama $S(\tau) = a + b\tau$ dobijamo:

$$J(\tau) = \Lambda_\tau\{S(t)\} = S(\tau) + \frac{1}{2}\left[bE_3(\tau) - aE_2(\tau)\right].$$

Sada možemo nacrtati grafike[2] $J(\tau)$ za različite $S(\tau)$, odnosno, koeficijente u funkciji izvora (vidi sliku 2). Jasno se uočava da za $\tau \to \infty$ imamo da J teži S (tada $E_n(\tau) \to 0$). Kako je: $E_n(0) = \frac{1}{n-1}$ za $n \geq 2$ sledi da je na površini $J(0) = \frac{a}{2} + \frac{b}{2}$, odnosno, $J(0) = \frac{1}{2}S(\tau = \frac{1}{2})$. Ako je $S = a$ onda je $J(0) = \frac{a}{2} = \frac{S}{2}$. Samim tim sledi da kada je $S = 1$ (izotermalna atmosfera) imamo $J(0) = \frac{1}{2}$. Dakle, imamo naredne slučaje:

- $(a = 1, b = 0)$: $J(0) = 1/2$
- $(a = 1, b = 1/2)$: $J(0) = 5/8$
- $(a = 1, b = 3/2)$: $J(0) = 7/8$
- $(a = 1, b = 5)$: $J(0) = 5/4$

[2]Svi grafici su dobijeni upotrebom IDL funkcije `EXPINT`.



Slika 2: Grafici J za različite S.

7. Izračunati: $\Phi_\tau\{a+bt\}$, gde su a i b konstante.

Usled linearnost Λ operatora imamo:

$$\Phi_\tau\{a+bt\} = a\Phi_\tau\{1\} + b\Phi_\tau\{t\}.$$

Imamo:

$$\Phi_\tau\{1\} = 2\int_\tau^\infty E_2(t-\tau)dt - 2\int_0^\tau E_2(\tau-t)dt,$$

$$\Phi_\tau\{1\} = 2\int_\tau^\infty dt\int_1^\infty e^{-(t-\tau)y}\frac{dy}{y^2} - 2\int_0^\tau dt\int_1^\infty e^{-(\tau-t)y}\frac{dy}{y^2},$$

$$\Phi_\tau\{1\} = 2\int_1^\infty e^{\tau y}\frac{dy}{y^2}\int_\tau^\infty e^{-ty}dt - 2\int_1^\infty e^{-\tau y}\frac{dy}{y^2}\int_0^\tau e^{ty}dt,$$

$$\Phi_\tau\{1\} = 2\int_1^\infty \frac{dy}{y^3} - 2\int_1^\infty \left(1-e^{-\tau y}\right)\frac{dy}{y^3},$$

$$\Phi_\tau\{1\} = 2E_3(\tau).$$

Sa druge strane, imamo:

$$\Phi_\tau\{t\} = 2\int_\tau^\infty tdt \int_1^\infty e^{-(t-\tau)y}\frac{dy}{y^2} - 2\int_0^\tau tdt\int_1^\infty e^{-(\tau-t)y}\frac{dy}{y^2},$$

$$\Phi_\tau\{t\} = 2\int_1^\infty e^{\tau y}\frac{dy}{y^2}\int_\tau^\infty te^{-ty}dt - 2\int_1^\infty e^{-\tau y}\frac{dy}{y^2}\int_0^\tau te^{ty}dt,$$

$$\int_\tau^\infty te^{-ty}dt = \frac{\tau}{y}e^{-\tau y} + \frac{1}{y^2}e^{-\tau y}, \qquad u = t, \quad dv = e^{-ty}dt,$$

$$\int_0^\tau te^{ty}dt = \frac{\tau}{y}e^{\tau y} - \frac{1}{y^2}e^{\tau y} + \frac{1}{y^2}, \qquad u = t, \quad dv = e^{ty}dt,$$

$$\Phi_\tau\{t\} = 4\int_1^\infty \frac{dy}{y^4} - 2E_4(\tau),$$

$$\Phi_\tau\{t\} = \frac{4}{3} - 2E_4(\tau).$$

Konačno dobijamo:

$$\Phi_\tau\{a+bt\} = \frac{4}{3}b + 2\left[aE_3(\tau) - bE_4(\tau)\right].$$

Znači, ukoliko aproskimiramo funkciju izvora linearnom funkcijom $S(\tau) = a + b\tau$ dobijamo:

$$F(\tau) = \Phi_\tau\{S(t)\} = \frac{4}{3}b + 2\left[aE_3(\tau) - bE_4(\tau)\right].$$

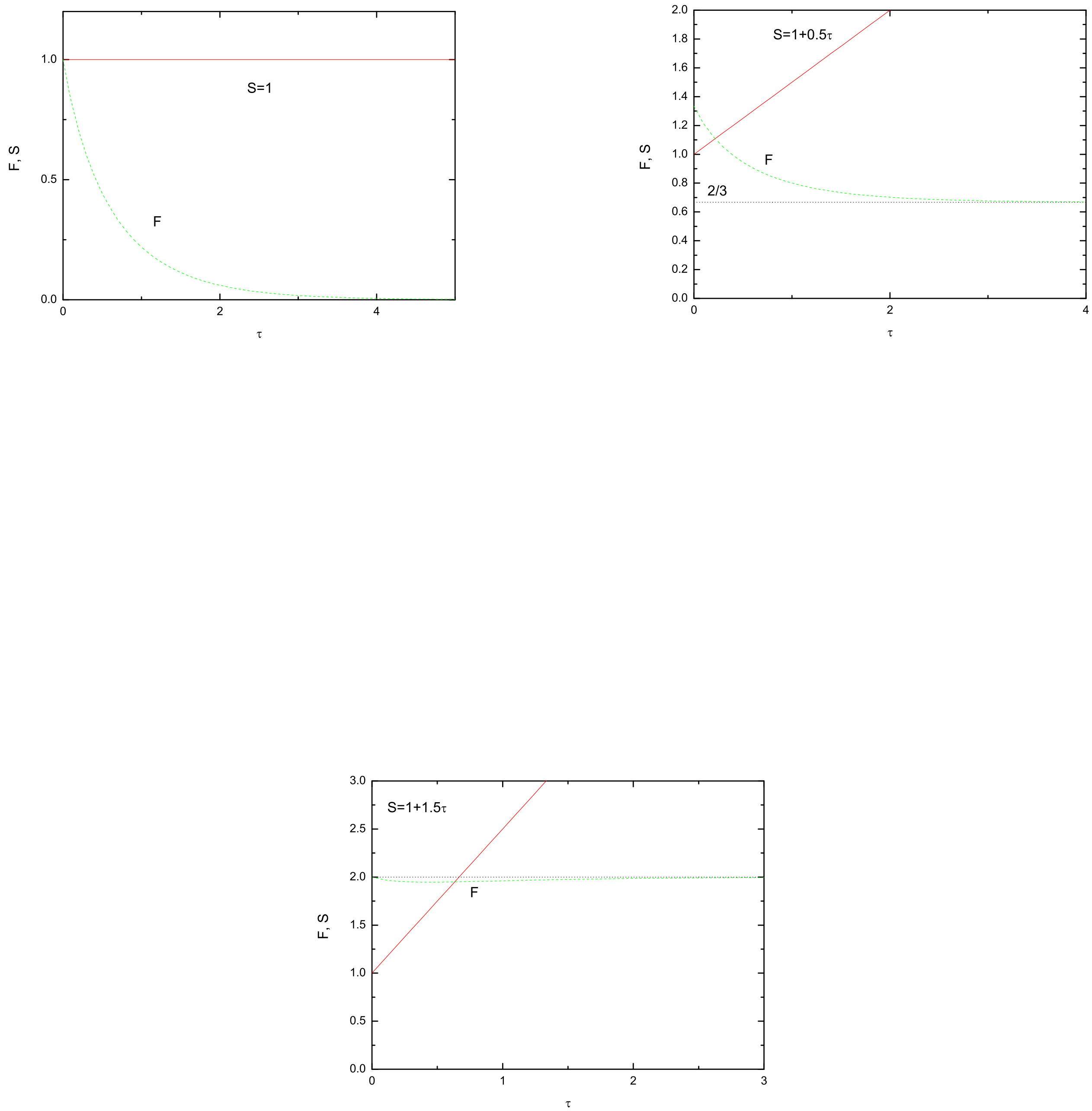
Prisetimo se i da je $H = F/4$. Sada možemo nacrtati grafike $F(\tau)$ za različite $S(\tau)$ (vidi sliku 3). Jasno se uočava da za $\tau \to \infty$ imamo da F teži $\frac{4}{3}b$ (tada $E_n(\tau) \to 0$). Drugim rečima, fluks je tada određen samo gradijentom funkcije izvora ($dS/d\tau = b$). Na površini imamo $F(0) = a + \frac{2}{3}b = S(\tau = \frac{2}{3})$. Ako je funkcija izvora konstantna ($b = 0$) sledi da je $F(0) = a$ pa je tada i $F(0) = 2J(0)$ kako je od ranije $J(0) = a/2$ (II Edingtonova aproksimacija). Za sledeće slučajeve imamo:

- $(a = 1, b = 0)$: $F(0) = 1$; $F(\tau \to \infty) \to 0$
- $(a = 1, b = 1/2)$: $F(0) = 4/3$; $F(\tau \to \infty) \to \frac{2}{3}$
- $(a = 1, b = 3/2)$: $F(0) = 2$; $F(\tau \to \infty) \to 2$

Primetimo da je $F(\tau) \approx \text{const}$ (uslovi RZ) za $(a = 1, b = 3/2)$.

Dodatni zadaci za vežbu.

1. Skicirati grafik $F(\tau)$ za $S(\tau) = 1 + 3\tau$.
2. Izračunati: $\mathcal{X}_\tau\{a + bt\}$.



Slika 3: Grafici F za različite S.