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A NOTE ON SINGULAR EQUIVALENCES AND IDEMPOTENTS

DAWEI SHEN

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ABSTRACT. Let Λ be an Artin algebra and let e be an idempotent in Λ. We study certain functors which preserve the singularity categories. Suppose $\operatorname{pd} \Lambda e_{e\Lambda e} < \infty$ and $\operatorname{id}_{\Lambda} \frac{\Lambda/\langle e\rangle}{\operatorname{rad}\Lambda/\langle e\rangle} < \infty$, we show that there is a singular equivalence between $e\Lambda e$ and Λ.

1. INTRODUCTION

Let R be a left noetherian ring. Let $\operatorname{mod} R$ be the category of finitely generated left R-modules and $\operatorname{proj} R$ be the full subcategory of projective left R-modules. A complex in $\operatorname{mod} R$ is called *perfect* if it is quasi-isomorphic to a bounded complex in $\operatorname{proj} R$. The *singularity category* of R is the quotient triangulated category of the bounded derived category of $\operatorname{mod} R$ by the thick subcategory of perfect complexes [2, 10]. If there is a triangle equivalence between the singularity categories of two rings R and S, then such an equivalence is called a *singular equivalence* [4].

Let e be an idempotent in R. The functor i_λ from $\operatorname{mod} eRe$ to $\operatorname{mod} R$ takes any $N \in \operatorname{mod} eRe$ to $Re \otimes_{eRe} N$; it admits an exact right adjoint i_e. Suppose

(1) $\operatorname{pd}_{eRe} eR < \infty$, and
(2) $\operatorname{pd}_R M < \infty$ for every $M \in \operatorname{mod} R/\langle e\rangle$.

Then i_e induces a singular equivalence between R and eRe; see [3].

In the present paper, we investigate the singular equivalence induced by the functor i_λ.

We have the following:

Theorem I. *Let R be a left noetherian ring and e be an idempotent in R. Suppose*
(1) $\operatorname{fd} Re_{eRe} < \infty$, *and*
(2) *every $M \in \operatorname{mod} R$ admits a projective resolution P such that P^{-i} belongs to $\operatorname{add} Re$ for every sufficiently large i.*
Then i_λ induces a singular equivalence between eRe and R.

Let Λ be an artin algebra and e be an idempotent in Λ. The following conditions are considered in [11].

$$(\alpha)\ \operatorname{id}_{\Lambda} \frac{\Lambda/\langle e\rangle}{\operatorname{rad}\Lambda/\langle e\rangle} < \infty, \qquad (\beta)\ \operatorname{pd}_{e\Lambda e} e\Lambda < \infty,$$
$$(\gamma)\ \operatorname{pd}_{\Lambda} \frac{\Lambda/\langle e\rangle}{\operatorname{rad}\Lambda/\langle e\rangle} < \infty, \qquad (\delta)\ \operatorname{pd} \Lambda e_{e\Lambda e} < \infty.$$

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It turns out that

(1) (α) and (β) hold if and only if (γ) and (δ) hold;

(2) i_e induces a singular equivalence if and only if (β) and (γ) hold.

As a complement, we have the following Theorem II.

Theorem II. *Let Λ be an artin algebra and e be an idempotent in Λ. Then i_λ induces a singular equivalence if and only if (α) and (δ) hold.*

2. Singularity categories

Given a left noetherian ring R, denote by $\mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} R)$ the homotopy category of bounded above complexes in $\operatorname{proj} R$ with bounded cohomologies. It is a triangulated category whose translation functor Σ is the shift of complexes [12]. An object P in $\mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} R)$ is perfect if there is an integer ℓ such that the i-th coboundary of P is projective for any $i \leq \ell$. Let $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} R)$ be the full subcategory of perfect complexes.

The singularity category of R is the quotient triangulated category

$$\mathbf{D}_{\mathrm{sg}}(R) \coloneqq \mathbf{K}^{-,\mathrm{b}}(\operatorname{proj} R)/\mathbf{K}^{\mathrm{b}}(\operatorname{proj} R).$$

Let $\underline{\operatorname{mod}}\, R$ be the projectively stable category of $\operatorname{mod} R$. For any M in $\operatorname{mod} R$, take a projective precover $\pi\colon P \to M$; the syzygy ΩM of M is the kernel of π. There is functor

$$\Omega\colon\ \underline{\operatorname{mod}}\, R \to \underline{\operatorname{mod}}\, R$$

sending a module M to the syzygy ΩM. For any M in $\operatorname{mod} R$, take a projective resolution pM of M. There is a functor

$$p\colon\ \underline{\operatorname{mod}}\, R \to \mathbf{D}_{\mathrm{sg}}(R)$$

sending a module M to the projective resolution pM. For any integer $\ell \geq 0$, there is a natural isomorphism

$$p\Omega^\ell M \cong \Sigma^{-\ell} pM.$$

Lemma 2.1 ([4, Lemma 2.1])**.** *For any X in $\mathbf{D}_{\mathrm{sg}}(R)$, there is an integer $\ell \geq 0$ and M in $\underline{\operatorname{mod}}\, R$ such that*

$$\Sigma^{-\ell} X \cong pM.$$

Lemma 2.2 ([9, Example 2.3])**.** *For any M, N in $\underline{\operatorname{mod}}\, R$, there is a natural isomorphism*

$$\varinjlim_{\ell \geq 0} \underline{\operatorname{Hom}}_R(\Omega^\ell M, \Omega^\ell N) \cong \operatorname{Hom}_{\mathbf{D}_{\mathrm{sg}}(R)}(pM, pN).$$

Let e be an idempotent in R. Recall the functor

$$i_\lambda\colon\ \operatorname{mod} eRe \to \operatorname{mod} R$$

where $i_\lambda(N) = Re \otimes_{eRe} N$ for any N in $\operatorname{mod} eRe$; it is a full faithful functor which preserves projectives. The restriction of i_λ has a factorization

$$i_\lambda\colon\ \operatorname{proj} eRe \xrightarrow{\sim} \operatorname{add} Re \xrightarrow{\subsetneq} \operatorname{proj} R.$$

Here, we denote by $\operatorname{add} Re$ the full subcategory of $\operatorname{mod} R$ whose objects are summands of finite direct sums of copies of Re.

We have the following strong form of Theorem I.

Theorem 2.3. *Suppose* $\operatorname{fd} Re_{eRe} < \infty$, *then* i_λ *induces a full faithful triangle functor*

$$\mathbf{D}_{\rm sg}(i_\lambda)\colon \mathbf{D}_{\rm sg}(eRe) \to \mathbf{D}_{\rm sg}(R).$$

Moreover, $\mathbf{D}_{\rm sg}(i_\lambda)$ *is a triangle equivalence if and only if for every* M *in* $\operatorname{mod} R$, *there is an integer* n *and a projective resolution* P *of* M *such that* $P^{-i} \in \operatorname{add} Re$ *for every* $i > n$.

Proof. Let (P, d) be in $\mathbf{K}^{-,\rm b}(\operatorname{proj} eRe)$ which is exact at degree $\leq \ell$. For any $i \leq \ell - \operatorname{fd} Re_{eRe}$ the cohomology group

$$H^i(i_\lambda P) = H^i(Re \otimes_{eRe} P) \cong \operatorname{Tor}^{eRe}_{-i+\ell+1}(Re, \operatorname{Coker} d^\ell) = 0.$$

Then $i_\lambda P$ belongs to $\mathbf{K}^{-,\rm b}(\operatorname{proj} R)$ and thus i_λ induces a triangle functor

$$\mathbf{K}^{-,\rm b}(i_\lambda)\colon \mathbf{K}^{-,\rm b}(\operatorname{proj} eRe) \to \mathbf{K}^{-,\rm b}(\operatorname{proj} R).$$

Since i_λ is full faithful, $\mathbf{K}^{-,\rm b}(i_\lambda)$ is full faithful. Since i_λ preserves perfect complexes, it induces a triangle functor

$$\mathbf{D}_{\rm sg}(i_\lambda)\colon \mathbf{D}_{\rm sg}(eRe) \xrightarrow{\sim} \mathbf{K}^{-,\rm b}(\operatorname{add} Re)/\mathbf{K}^{\rm b}(\operatorname{add} Re) \to \mathbf{D}_{\rm sg}(R).$$

Let $f\colon X \to Y$ be a morphism in $\mathbf{K}^{-,\rm b}(\operatorname{proj} R)$, where X is perfect and Y is in $\mathbf{K}^{-,\rm b}(\operatorname{add} Re)$. Suppose $X^i = 0$ for any $i < m$, and let $\sigma_{\geq m} Y$ be the stupid truncation of Y at degree $\geq m$. Since f factors through an object $\sigma_{\geq m} Y$ from $\mathbf{K}^{\rm b}(\operatorname{add} Re)$, by [8, Proposition 10.2.6] the functor $\mathbf{D}_{\rm sg}(i_\lambda)$ is full faithful.

By Lemma 2.1 and 2.2, the singularity category of R is equivalent to the stabilization of $\underline{\operatorname{mod}} R$. Following [5, Corollary 2.3] the functor $\mathbf{D}_{\rm sg}(i_\lambda)$ is dense if and only if for any $M \in \operatorname{mod} R$, there is an integer n and a projective resolution P such that if $i > n$, then $P^{-i} \in \operatorname{add} Re$. Then the desired statement holds. ☐

3. Triangular matrix rings

Let T and S be two rings and M be an S-T-bimodule. Let

$$R = \begin{pmatrix} T & 0 \\ M & S \end{pmatrix}$$

be the triangular matrix ring. Following [1, III.2] a left R-module is

$$(X, Y, \phi) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x \in X, y \in Y \right\},$$

where X is a left T-module, Y is a left S-module and $\phi\colon M \otimes_T X \to Y$ is a left S-module map. The action is given by

$$\begin{pmatrix} t & 0 \\ m & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} tx \\ \phi(m \otimes x) + sy \end{pmatrix}$$

for every $t \in T$, $s \in S$ and $m \in M$.

Let $e = \operatorname{diag}(0, 1)$ be an idempotent in R.

Lemma 3.1. *Let* X *be a left* T*-module and* Y *be a left* S*-module.*

(1) $\operatorname{add} Re = \{(0, Q, 0) \mid Q \in \operatorname{proj} S\}$;
(2) $\operatorname{pd}_R (0, Y, 0) = \operatorname{pd}_S Y$;
(3) ${}_R(X, M \otimes_T X, 1)$ *is projective if and only if* ${}_T X$ *is projective.*

We have Proposition 3.2; compare [3, Theorem 4.1].

Proposition 3.2. *Let T and S be left noetherian rings, and let M be an S-T-bimodule such that ${}_SM$ is finitely generated. Assume that T has finite left global dimension, then there is a triangle equivalence*

$$\mathbf{D}_{\mathrm{sg}}(\begin{pmatrix} T & 0 \\ M & S \end{pmatrix}) \simeq \mathbf{D}_{\mathrm{sg}}(S).$$

Proof. One checks that the triangular matrix ring R is left noetherian. Let (X, Y, ϕ) be in $\operatorname{mod} R$. Since $\operatorname{pd}_T X$ is finite, by Lemma 3.1 there is a projective resolution P of (X, Y, ϕ) such that $P^{-i} \in \operatorname{add} Re$ for any $i > \operatorname{pd}_T X$. Note that eRe is isomorphic to S and Re is projective over eRe. By Theorem I there is a singular equivalence between S and R. □

Example 3.3 (see [4, Proposition 4.1])**.** Let k be a field, Λ be a finite dimensional k-algebra and M be a finite dimensional left Λ-module. Then M is a Λ-k-bimodule. The *one-point extension* of Λ by M is the triangular matrix algebra

$$\Lambda[M] = \begin{pmatrix} k & 0 \\ M & \Lambda \end{pmatrix}.$$

By Proposition 3.2 there a singular equivalence between Λ and $\Lambda[M]$.

4. Artin algebras

We need the following well known:

Lemma 4.1. *Let Λ be an artin algebra and e be an idempotent in Λ.*

(1) *Let M be in $\operatorname{mod}\Lambda$ and P_M be a minimal projective resolution of M. For any semi-simple Λ-module S and $i \geq 0$ there is an isomorphism*

$$\operatorname{Ext}^i_\Lambda(M, S) \cong \operatorname{Hom}_\Lambda(P_M^{-i}, S).$$

(2) *Let P be in $\operatorname{proj}\Lambda$. Then $\operatorname{Hom}_\Lambda(P, \frac{\Lambda/\langle e\rangle}{\operatorname{rad}\Lambda/\langle e\rangle}) = 0$ if and only if $P \in \operatorname{add}\Lambda e$.*

Theorem 4.2. *Let Λ be an artin algebra and e be an idempotent in Λ. Suppose $\operatorname{pd}\Lambda e_{e\Lambda e} < \infty$, then $\mathbf{D}_{\mathrm{sg}}(i_\lambda)$ is a triangle equivalence if and only if $\operatorname{id}_\Lambda \frac{\Lambda/\langle e\rangle}{\operatorname{rad}\Lambda/\langle e\rangle} < \infty$.*

Proof. “ $\Longrightarrow$” Let $P_{\Lambda/\operatorname{rad}\Lambda}$ be a minimal projective resolution of $\Lambda/\operatorname{rad}\Lambda$. Since $\mathbf{D}_{\mathrm{sg}}(i_\lambda)$ is a triangle equivalence, by Theorem 2.3 there is an integer n such that $P^{-i}_{\Lambda/\operatorname{rad}\Lambda} \in \operatorname{add}\Lambda e$ for every $i > n$.

By Lemma 4.1 $\operatorname{Ext}^{n+1}_\Lambda(\Lambda/\operatorname{rad}\Lambda, \frac{\Lambda/\langle e\rangle}{\operatorname{rad}\Lambda/\langle e\rangle}) = 0$. Then $\operatorname{id}\frac{\Lambda/\langle e\rangle}{\operatorname{rad}\Lambda/\langle e\rangle}$ is finite.

“ $\Longleftarrow$” Let M be in $\operatorname{mod}\Lambda$ and P_M be a minimal projective resolution of M. Since $n = \operatorname{id}_\Lambda \frac{\Lambda/\langle e\rangle}{\operatorname{rad}\Lambda/\langle e\rangle}$ is finite, by Lemma 4.1 $P_M^{-i} \in \operatorname{add}\Lambda e$ for any $i > n$. Then $\mathbf{D}_{\mathrm{sg}}(i_\lambda)$ is a triangle equivalence by Theorem I. □

Note that $i_\lambda P_{\Lambda/\operatorname{rad}\Lambda}$ having bounded cohomologies implies $\operatorname{pd}\Lambda e_{e\Lambda e}$ is finite. Then Theorem II holds.

Let Λ be an artin algebra and S be a semisimple left Λ-module with $\operatorname{id}_\Lambda S \leq 1$. Denote the perpendicular subcategory by

$${}^\perp S = \{M \in \operatorname{mod}\Lambda \mid \operatorname{Hom}_\Lambda(M, S) = \operatorname{Ext}^1_\Lambda(M, S) = 0\}.$$

For an object M in $\mathrm{mod}\,\Lambda$, we know from Lemma 4.1 that M belongs to ${}^{\perp}S$ if and only if M admits a projective presentation

$$P^{-1} \to P^{0} \to M \to 0$$

such that both P^{-1} and P^{0} belong to ${}^{\perp}S$.

Let e be an idempotent in Λ such that $\mathrm{proj}\,{}^{\perp}S$ equals $\mathrm{add}\,\Lambda e$. Recall the functor

$$i_{\lambda}\colon\ \mathrm{mod}\ e\Lambda e \overset{\sim}{\to} {}^{\perp}S \overset{\subseteq}{\to} \mathrm{mod}\Lambda.$$

We have Proposition 4.3; compare [6, Proposition 2.13].

Proposition 4.3. *Keep the notation as previous.*

(1) $\mathrm{gl.dim}\, e\Lambda e \leq \mathrm{gl.dim}\,\Lambda \leq \mathrm{gl.dim}\, e\Lambda e + 2$*;*

(2) *there is a singular equivalence between* $e\Lambda e$ *and* Λ.

Proof. (1) Let $\Omega^2 M$ be the minimal second syzygy for M in $\mathrm{mod}\Lambda$. Since $\mathrm{id}_{\Lambda} S \leq 1$, $\mathrm{Ext}^i_{\Lambda}(M, S) = 0$ for every $i \geq 2$. By Lemma 4.1 $\Omega^2 M \in {}^{\perp}S$. Since i_{λ} preserves minimal projective resolutions, for any N in $\mathrm{mod} e\Lambda e$ we have

$$\mathrm{pd}_{e\Lambda e} N = \mathrm{pd}_{\Lambda} i_{\lambda} N.$$

Then

$$\mathrm{pd}_{\Lambda} M \leq \mathrm{pd}_{\Lambda} \Omega^2 M + 2 = \mathrm{pd}_{e\Lambda e}\, i(\Omega^2 M) + 2.$$

(2) By [7, Proposition 1.1] ${}^{\perp}S$ is an exact subcategory of $\mathrm{mod}\Lambda$. Then $\Lambda e_{e\Lambda e}$ is projective. By Theorem II there is a singular equivalence between $e\Lambda e$ and Λ. $\square$

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School of Mathematics and Statistics, Henan University, 475004, Kaifeng, Henan, People's Republic of China

Email address: sdw12345@mail.ustc.edu.cn