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A description of Gorenstein projective modules over the tensor products of algebras

Dawei Shen

School of Mathematics and Statistics, Henan University Kaifeng, Henan, P. R. China

ABSTRACT

Let A be a two-sided coherent algebra over a field k and B be a finite-dimensional algebra. If B is Gorenstein, we give a description of finitely presented Gorenstein projective modules over the tensor product of A and B in terms of their underlying modules over A and B. If B has finite global dimension, we also give another description.

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1. Introduction

Finitely generated Gorenstein projective modules over two-sided noetherian rings are introduced in [1] under the name "modules of G-dimension zero". This notion for arbitrary modules over arbitrary rings are defined and investigated in [13,14]. By the pioneering work of [6], the stable categories of Gorenstein projective modules are closely related to singularity categories of rings.

One of the most important tasks in Gorenstein homological algebra is to describe Gorenstein projective modules. In this article, we consider the description of Gorenstein projective modules over tensor product of two algebras under certain conditions.

Another motivation of our work is from monomorphism categories. The work in [5] initiates this area by the study of monomorphisms between abelian groups. General submodule categories are investigated in [25,26]. Submodule categories are generalized to monomorphism categories of type $\mathbb{A}$ in [29,30]. Monomorphism categories over arbitrary finite acyclic quivers are studied in [18].

Let k be a field and A be an algebra over k. Let Q be a finite acyclic quiver. The monomorphism category $\mathrm{Mon}(Q, A)$ in [18] is in fact a full subcategory of the category of all $A\otimes_k kQ$-modules. They use monomorphism category $\mathrm{Mon}(Q, A)$ to describe Gorenstein projective $A\otimes_k kQ$-modules. They also use monomorphism category to describe Gorenstein projective modules when kQ is replaced by bound quiver algebras in [17,19].

Given an arbitrary finite-dimensional algebra B over k, how to give a description of Gorenstein projective modules over $A\otimes_k B$? If B is hereditary, the work in [18] gives an answer. If

CONTACT Dawei Shen dwshen@math.ecnu.edu.cn School of Mathematics and Statistics, Henan University Kaifeng, Henan 475004, P. R. China.

B is selfinjective, the result in [9] indicates that a finitely presented $A\otimes_k B$-module X is Gorenstein projective if and only if it is Gorenstein projective over A; compare [28].

We unify their results to some extent. If B is a Gorenstein algebra, we give a description of Gorenstein projective $A\otimes_k B$-modules by their underlying A-modules and underlying B-modules.

The following is our main result. Denote by D the k-dual functor.

Theorem I. *Let A be a two-sided coherent algebra and B be a finite-dimensional algebra over a field k. Let X be a finitely presented $A\otimes_k B$-module. Assume that B is Gorenstein. Then X is Gorenstein projective if and only if $D(B)\otimes_B X$ is Gorenstein projective over A, and X is Gorenstein projective over B.*

In general, it is not easy to check whether the A-module $D(B)\otimes_B X$ is Gorenstein projective or not. However, if the algebra B has finite global dimension, we can replace $D(B)\otimes_B X$ by the qotient module $X/\mathrm{rad}_B X$ in the previous theorem. For a B-module U, the radical $\mathrm{rad}_B U$ is the intersection of all maximal submodules.

We have the following main result.

Proposition II. *Keep the notation in Theorem 1. Assume furthur that B has finite global dimension. Then X is Gorenstein projective if and only if $X/\mathrm{rad}_B X$ is Gorenstein projective over A, and X is projective over B.*

Recently W. Hu, X.-H. Luo, B.-L. Xiong and G. Zhou also study independently Gorenstein projective modules over the tensor product of two algebras in [16]. They also obtain Proposition 2 with different method. Their paper has some overlaps with our article.

This article is organized as follows. In Section 2, we recall some facts about tensor products of algebras and Gorenstein projective modules. In Section 3, we state and prove some lemmas. The proofs of Theorem 1 and Proposition 2 are given in Section 4. We provide some examples and applications of our results in Section 5.

2. Preliminaries

Throughout this article, k is a fixed field. Denote by $\otimes$ the tensor product over k. Denote by D the k-dual functor $\mathrm{Hom}_k(-,k)$. From now on, "algebra" will always mean k-algebra, and "module" will always mean left module unless stated otherwise.

Given an arbitrary ring A, denote by A^{op} the opposite ring of A. Recall that a right module over a ring is exactly a left module over its opposite. Denote by A-Mod the abelain category of all (left) A-modules, and by A-Proj the full subcategory of projective A-modules.

Recall that a ring A is said to be *left coherent* if every finitely generated left ideal of A is finitely presented. Given a left coherent ring A, denote by A-mod the abelian category of finitely presented A-modules, and by A-proj the full subcategory of finitely generated projective A-modules.

We first recall Gorenstein projective modules.

Let A be an arbitrary ring. A complex $P^\bullet$ of projective A-modules is said to be *totally acyclic* [13, Definition 5.1] if $P^\bullet$ is acyclic and $\mathrm{Hom}_A(P^\bullet, T)$ is acyclic for every projective A-module T.

An A-module M is said to be *Gorenstein projective* if there exists a totally acyclic complex $P^\bullet$ of projective A-modules such that M is isomorphic to the cokernel of $d^{-1}: P^{-1}\to P^0$; this $P^\bullet$ is said to be a *complete (projective) resolution* of M.

Denote by A-GProj the full subcategory of A-Mod consisting of all Gorenstein projective A-modules. We observe that A-Proj is a full subcategory of A-GProj.

Let A be a two-sided coherent ring. Here, "two-sided coherent" means that both A and A^{op} are left coherent. Denote by A-Gproj the full subcategory of A-mod formed by Gorenstein projective modules. That is, A-Gproj is the intersection of A-GProj and A-mod.

Denote by $(-)^*$ the A-dual functors. For an A-module M, there in an *evaluation map* $\mathrm{ev}_M^A : M \to M^{**}$, where $\mathrm{ev}_M^A(m)(f) = f(m)$ for all $m \in M, f \in M^*$.

Recall from [14, Chapter X, Proposition 10.2.6] the following characterization of Gorenstein projective modules; see also [1].

Lemma 2.1. *Let M be in A-mod. Then M is Gorenstein projective if and only if M satisfies the following conditions:*

1. $Ext_A^n(M, A) = 0$ *for each* $n \geq 1$;
2. $Ext_{A^{op}}^n(M^*, A) = 0$ *for each* $n \geq 1$;
3. $ev_M^A : M \to M^{**}$ *is an isomorphism.*

Recall that an exact category $\mathcal{E}$ is called a *Frobenius category* [15, I.2] if $\mathcal{E}$ has enough projective objects and enough injective objects, and the projective objects coincide with the injective objects. The stable category $\underline{\mathcal{E}}$ of a Frobenius category $\mathcal{E}$ is triangulated [15, I.2]; it is obtained from $\mathcal{E}$ by factoring out the ideal of all maps which factor through projective objects in $\mathcal{E}$.

We will need the following facts; see [6].

1. $A-$Gproj is an extension closed subcategory of A-mod.
2. $A-$Gproj is closed under kernels of epimorphisms and direct summands.
3. $A-$Gproj is a Frobenius category, the projective objects are just $A-$proj.
4. The A-dual functors induce exact dualities of $A-$Gproj and $A^{\mathrm{op}}-$Gproj.

We will need the following well known result.

Lemma 2.2. *Let $M^\bullet : 0 \to M^{-m} \to \cdots \to M^{-1} \xrightarrow{d^{-1}} M^0 \to 0$ be a complex in $A-$Gproj. Then the following statements are equivalent.*

1. $H^n(Hom_A(M^\bullet, A)) = 0$ *for all* $n \geq 1$;
2. $H^0(M^\bullet) \in A-Gproj$ *and* $H^n(M^\bullet) = 0$ *for all* $n \leq -1$.

Proof. "(1) ⇒ (2)" Recall that $A^{\mathrm{op}}-$Gproj is closed under kernels of epimorphisms. We infer that $\mathrm{Ker}(d^{-1})^*$ is in $A^{\mathrm{op}}-$Gproj. Since the A-dual functors induce exact dualities of $A-$Gproj and $A^{\mathrm{op}}-$Gproj, $M^\bullet \simeq (M^\bullet)^{**}$ and $H^0(M^\bullet) \simeq (\mathrm{Ker}(d^{-1})^*)^*$. Then $H^0(M^\bullet)$ is in $A-$Gproj and $H^n(M^\bullet) = 0$ for all $n \leq -1$.

"(2) ⇒ (1)" Since the A-dual functors induce exact dualities of $A^{\mathrm{op}}-$Gproj and $A-$Gproj, then $H^n(\mathrm{Hom}_A(M^\bullet, A)) = 0$ for all $n \geq 1$. □

Next we recall tensor product of algebras and their modules.

Let A and B be two algebras. Denote by $A \otimes B$ the algebra of tensor product of A and B. We consider the abelian category $A \otimes B-$Mod of all modules over $A \otimes B$.

There are two forgetful functors which forget the action of B and A respectively

$$
\begin{aligned}
{}_A(-) &: A \otimes B-\mathrm{Mod} \to A-\mathrm{Mod}, \\
{}_B(-) &: A \otimes B-\mathrm{Mod} \to B-\mathrm{Mod}.
\end{aligned}
$$

We need the following facts.

1. Let X be in $A \otimes B-$Mod, M be in $A-$Mod and P be in $B^{\mathrm{op}}-$Proj. Then for each $n \geq 0$ there is an isomorphism
$$\mathrm{Ext}^n_{A\otimes B}(X, \mathrm{Hom}_k(P, M)) \simeq \mathrm{Ext}^n_A(X \otimes_B P, M). \tag{2.1}$$

Moreover, if P is finite dimensional, for each $n \geq 0$ there is an isomorphism

$$\mathrm{Ext}^n_{A\otimes B}\big(X, M\otimes D(P)\big) \simeq \mathrm{Ext}^n_A(X\otimes_B P, M). \tag{2.2}$$

See [7, Chapter IX, Theorem 2.8a].

2. Let M_1, M_2 be in $A-$Mod and U_1, U_2 be in $B-$Mod. Suppose that M_1, U_1 are finitely generated, then there is an isomorphism

$$\mathrm{Hom}_{A\otimes B}(M_1\otimes U_1, M_2\otimes U_2) \simeq \mathrm{Hom}_A(M_1, M_2)\otimes \mathrm{Hom}_B(U_1, U_2). \tag{2.3}$$

See [7, Chapter XI, Theorem 3.1].

3. Comparing Gorenstein projective modules

Let A be a two-sided coherent algebra and B be a finite-dimensional algebra. The tensor product $A\otimes B$ is a two-sided coherent algebra. Recall that $A\otimes B$-mod denotes the abelian category of finitely presented $A\otimes B$-modules.

For every X in $A\otimes B$-mod, we note that ${}_AX$ is finitely presented. In general, the B-module ${}_BX$ is not finitely presented; it is finitely presented if A is also a finite-dimensional algebra.

Let X be in $A\otimes B$-mod. Denote the $A\otimes B$-dual functors by $(-)^*$. Following Lemma 2.1 we know that X is Gorenstein projective if and only if X satisfies the following conditions:

(G1) $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes B) = 0$ for all $n \geq 1$;
(G2) $\mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}(X^*, A\otimes B) = 0$ for all $n \geq 1$;
(G3) $\mathrm{ev}^{A\otimes B}_X : X \to X^{**}$ is an isomorphism.

Let U be in B-mod. Since B is finite dimensional, U is also finite dimensional. The k-dual functors D induce exact dualities of B-mod and B^{op}-mod.

Recall that the tensor product $A\otimes D(B)$ is a bimodule over $A\otimes B$. Denote by $(-)^\vee$ the $A\otimes D(B)$-dual functors.

For every $A\otimes B$-module X, there is an evaluation map $\mathrm{ev}^{A\otimes D(B)}_X : X \to X^{\vee\vee}$. Given a finitely presented B-module U, we have $(A\otimes U)^\vee \simeq A\otimes D(U)$ and $(A\otimes D(U))^\vee \simeq A\otimes U$ by (2.3).

By Lemma 2.1 and (2.2) the A-module ${}_AX$ is Gorenstein projective if and only if X satisfies the following conditions:

(G1′) $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes D(B)) = 0$ for all $n \geq 1$;
(G2′) $\mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}(X^\vee, A\otimes D(B)) = 0$ for all $n \geq 1$;

(G3′) $\mathrm{ev}^{A\otimes D(B)}_X : X \to X^{\vee\vee}$ is an isomorphism.

By comparing (G1–G3) with (G1′–G3′) under certain circumstances, we will discuss the relation between Gorenstein projective $A\otimes B$-modules and Gorenstein projective A-modules.

Before doing this in the next section, we state and prove some lemmas.

Let U be in B-mod. Recall that a (finite-dimensional) injective resolution of U is a complex $I^\bullet_U$ of finite-dimensional injective B-modules

$$I^\bullet_U : 0 \to I^0 \xrightarrow{d^0} I^1 \to \cdots \to I^i \xrightarrow{d^i} I^{i+1} \to \cdots$$

such that $H^n(I^\bullet_U) = 0$ for all $n \geq 1$ and $H^0(I^\bullet_U) \simeq U$.

Lemma 3.1. *Let X be in $A\otimes B-$Mod satisfying (G1′). Let U be in B-mod and $I^\bullet_U$ be an injective resolution of U. Then the following statements hold true.*

1. *For each $n \geq 0$ there is an isomorphism*

$$\mathrm{Ext}^n_{A\otimes B}(X, A\otimes U) \simeq H^n\mathrm{Hom}_{A\otimes B}\big(X, A\otimes I_U^\bullet\big);$$

2. *For each $n \geq 0$ there is an isomorphism*

$$\mathrm{Ext}^n_{A\otimes B}(X, A\otimes U) \simeq \mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}\big((A\otimes U)^\vee, X^\vee\big).$$

Proof. (1) Let F be the functor $\mathrm{Hom}_{A\otimes B}(X, -)$. Since X satisfies (G1′), we infer that $A\otimes I_U^\bullet$ is an F-acyclic resolution of $A\otimes U$. Then the conclusion follows.

(2) We see from (2.3) that $(A\otimes U)^{\vee\vee} \simeq A\otimes U$. Then $\mathrm{Hom}_{A\otimes B}(X, A\otimes U)$ and $\mathrm{Hom}_{(A\otimes B)^{\mathrm{op}}}((A\otimes U)^\vee, X^\vee)$ are naturally isomorphic.

For each $n \geq 1$, we have following commutative diagram.

$$\begin{tikzcd}
\mathrm{Ext}^n_{A\otimes B}(X, A\otimes U) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}((A\otimes U)^\vee, X^\vee) \arrow[d, "\simeq"] \\
H^n\,\mathrm{Hom}_{A\otimes B}(X, A\otimes I_U^\bullet) \arrow[r, "\simeq"] & H^n\,\mathrm{Hom}_{(A\otimes B)^{\mathrm{op}}}((A\otimes I_U^\bullet)^\vee, X^\vee)
\end{tikzcd}$$

Therefore, $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes U) \simeq \mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}((A\otimes U)^\vee, X^\vee)$ for each $n \geq 1$. □

We will need the following key lemma.

Lemma 3.2. *Let X be in $A\otimes B$-mod with ${}_AX \in A$−Gproj. Let U be in B-mod and $I_U^\bullet$ be an injective resolution of U. Suppose that U has finite injective dimension. Then the following statements are equivalent.*

1. *$\mathrm{Ext}^n_{A\otimes B}(X, A\otimes U) = 0$ for all $n \geq 1$;*
2. *$H^n(\mathrm{Hom}_{A\otimes B}(X, A\otimes I_U^\bullet)) = 0$ for all $n \geq 1$;*
3. *$H^n(\mathrm{Hom}_A(D(I_U^\bullet)\otimes_B X, A)) = 0$ for all $n \geq 1$;*
4. *$D(U)\otimes_B X \in A$−Gproj and $\mathrm{Tor}^B_n(D(U), X) = 0$ for all $n \geq 1$.*

Proof. "(1) $\iff$ (2)" Since ${}_AX$ is in A−Gproj, it follows that X satisfies (G1′). By Lemma 3.1(1) we know that (1) and (2) are equivalent.

"(2) $\iff$ (3)" It follows from (2.2) that $\mathrm{Hom}_{A\otimes B}(X, A\otimes I_U^\bullet)$ is isomorphic to $\mathrm{Hom}_A(D(I_U^\bullet)\otimes_B X, A)$. Then (2) and (3) are equivalent.

"(3) $\iff$ (4)" Since $I_U^\bullet$ is an injective resolution of U, we know that $D(I_U^\bullet)$ is a projective resolution of $D(U)$. Then $D(I_U^\bullet)\otimes_B X$ is a complex in A−Gproj.

By Lemma 2.2, $H^n(\mathrm{Hom}_A(D(I_U^\bullet)\otimes_B X, A)) = 0$ for all $n \geq 1$ if and only if $H^0(D(I_U^\bullet)\otimes_B X) \in A$−Gproj and $H^n(D(I_U^\bullet)\otimes_B X)$ for all $n \leq -1$.

Since $H^{-n}(D(I_U^\bullet)\otimes_B X)$ and $\mathrm{Tor}^B_n(D(U), X)$ are isomorphic for all $n \geq 0$, it follows that (3) and (4) are equivalent. □

Let X be in $A\otimes B$-mod with ${}_AX \in A$−Gproj. We have the following corollaries.

Corollary 3.3. *If B has finite injective dimension, then X satisfies (G1) if and only if $D(B)\otimes_B X \in A$−Gproj and $\mathrm{Tor}^B_n(D(B), X) = 0$ for all $n \geq 1$.*

Proof. Take $U = B$ in Lemma 3.2. □

Corollary 3.4. *If B has finite global dimension, then X satisfies (G1) if and only if $X/\mathrm{rad}_B X \in A$−Gproj and X is projective over B.*

Proof. Since B has finite global dimension, the semisimple module $D(B/\mathrm{rad}B)$ has finite injective dimension. By Lemma 3.6 we have that X satisfies (G1) if and only if $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes D(B/\mathrm{rad}B)) = 0$ for all $n \geq 1$.

Following Lemma 3.2 for all $n \geq 1, \mathrm{Ext}^n_{A\otimes B}(X, A\otimes B/\mathrm{rad}B) = 0$ if and only if $B/\mathrm{rad}B\otimes_B X \in A{-}\mathrm{Gproj}$ and $\mathrm{Tor}^B_n(B/\mathrm{rad}B, X) = 0$ for all $n \geq 1$. Since B is a finite-dimensional algebra, $\mathrm{Tor}^B_n(B/\mathrm{rad}B, X) = 0$ for all $n \geq 1$ if and only if X is flat or, equivalently, projective over B. □

To give the proof of Lemma 3.6, we need right perpendicular categories.

Let A be a two-sided coherent algebra and B be a finite-dimensional algebra. Let X be in $A\otimes B$-mod. Let us denote by $\mathcal{X}$ the full subcategory of B-mod formed by object U such that $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes U) = 0$ for all $n \geq 1$.

Lemma 3.5. *The full subcategory $\mathcal{X}$ is closed under direct summands, extensions and cokernels of monomorphisms in B-mod.*

Proof. Since extension functors are additive, $\mathcal{X}$ is closed under direct summands. It remains to show that $\mathcal{X}$ is closed under extensions and cokernels of monomorphisms.

Let M be in $\mathcal{X}$ and take an exact sequence

$$\epsilon : 0 \to M \to N \to L \to 0$$

in B-mod. Since A is projective over k, the sequence

$$\epsilon' : 0 \to A\otimes M \to A\otimes N \to A\otimes L \to 0$$

in $A\otimes B$-mod is also exact. We apply $\mathrm{Hom}_{A\otimes B}(X, -)$ to the sequence ϵ'.

Since $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes M) = 0$ for all $n \geq 1$, it follows that $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes N)$ and $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes L)$ are isomorphic. Then N belongs to $\mathcal{X}$ if and only if L belongs to $\mathcal{X}$. Therefore, $\mathcal{X}$ is closed under extension and cokernels of monomorphisms. □

Lemma 3.6. *Suppose that B has finite global dimension. The following statements are equivalent.*

1. $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes B) = 0$ *for all* $n \geq 1$;
2. $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes S) = 0$ *for all* $n \geq 1$ *and all simple B-module S;*
3. $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes D(B/\mathrm{rad}B)) = 0$ *for all* $n \geq 1$.

Proof. "(1) ⇒ (2)" Since B has finite global dimension, each simple B-module S has finite projective dimension. By Lemma 3.5, the full subcategory $\mathcal{X}$ is closed under direct summands and cokernels of monomorphisms, then $B \in \mathcal{X}$ implies that each simple module $S \in \mathcal{X}$.

"(2) ⇒ (1)" By Lemma 3.5, the full subcategory $\mathcal{X}$ is closed under extensions. Since each simple B-module S belongs to $\mathcal{X}$, it follows that B belongs to $\mathcal{X}$.

"(2) ⟺ (3)" Since $D(B/\mathrm{rad}B)$ is a direct sum of simple B-modules and each simple B-module occurs as a direct summand of $D(B/\mathrm{rad}B)$, we know that (2) and (3) are equivalent. □

Given a two-sided coherent algebra A, we recall the *right perpendicular category* $A{-}\mathrm{Gproj}^{\perp}$ of $A{-}\mathrm{Gproj}$. It is the full subcategory of A-mod formed by object M such that $\mathrm{Ext}^n_A(G, M) = 0$ for all $n \geq 1$ and all $G \in A{-}\mathrm{Gproj}$.

Observe that $A{-}\mathrm{Gproj}^{\perp}$ contains every finitely generated projective A-module and it is closed under cokernels of monomorphisms; see Lemma 3.5. Then $A{-}\mathrm{Gproj}^{\perp}$ contains every finitely presented A-module of finite projective dimension.

Let M be in $A-$Gproj. Recall that M has a complete resolution $P^\bullet$ in $A-$proj. This means that $P^\bullet$ is a totally acyclic complex of finitely generated projective A-modules and M is isomorphic the cokernel of $d^{-1}: P^{-1} \to P_0$.

We have the following; see also [16].

Proposition 3.7. *Let X be in $A\otimes B-$Gproj and $P^\bullet$ be a complete resolution of X in $A\otimes B-$proj. Suppose that $B^{\rm op}$ has finite injective dimension.*

1. *$Ext^n_{A\otimes B}(X, A\otimes D(B)) = 0$ for all $n \geq 1$;*
2. *$Hom_{A\otimes B}(P^\bullet, A\otimes D(B))$ is acyclic;*
3. *${}_AX \in A-Gproj$.*

Proof. (1) Since the injective dimension of $B^{\rm op}$ is finite, the projective dimension of the $A\otimes B$-module $A\otimes D(B)$ is also finite. Then $A\otimes D(B)$ lies in $A\otimes B-{\rm Gproj}^\perp$. Therefore, we have $Ext^n_{A\otimes B}(X, A\otimes D(B)) = 0$ for all $n \geq 1$.

(2) We mention that every coboundary of $P^\bullet$ is in $A\otimes B-$Gproj. It follows from (1) that $\mathrm{Hom}_{A\otimes B}(P^\bullet, A\otimes D(B))$ is acyclic.

(3) By (2.2) $\mathrm{Hom}_{A\otimes B}(P^\bullet, A\otimes D(B))$ is isomorphic to $\mathrm{Hom}_A(P^\bullet, A)$, they are acyclic by (2). Since $P^\bullet$ is an acyclic complex in $A\otimes B-$proj, then ${}_AP^\bullet$ is an acyclic complex in $A-$proj. Then ${}_AP^\bullet$ is totally acyclic, it is a complete resolution of ${}_AX$. Therefore, the A-module ${}_AX$ is Gorenstein projective. □

4. The proofs of main results

Let A be a two-sided coherent algebra and B be a finite-dimensional algebra. Recall the two-sided coherent algebra $A\otimes B$ and the abelian category $A\otimes B$-mod of finitely presented $A\otimes B$-modules.

Let X be in $A\otimes B$-mod. Denote by $(-)^*$ the $A\otimes B$-dual functors and by $(-)^\vee$ the $A\otimes D(B)$-dual functors. Recall that X is Gorenstein projective if only if

(G1) $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes B) = 0$ for each $n \geq 1$;
(G2) $\mathrm{Ext}^n_{(A\otimes B)^{\rm op}}(X^*, A\otimes B) = 0$ for each $n \geq 1$;

(G3) $\mathrm{ev}_X^{A\otimes B}: X \to X^{**}$ is an isomorphism.

Recall that ${}_AX$ is Gorenstein projective if and only if

(G1′) $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes D(B)) = 0$ for all $n \geq 1$;
(G2′) $\mathrm{Ext}^n_{(A\otimes B)^{\rm op}}(X^\vee, A\otimes D(B)) = 0$ for all $n \geq 1$;

(G3′) $\mathrm{ev}_X^{A\otimes D(B)}: X \to X^{\vee\vee}$ is an isomorphism.

A finite-dimensional algebra B is said to be *Gorenstein* if the injective dimensions of B and $B^{\rm op}$ are both finite. In particular, if B is injective then B is called a *self-injective* algebra.

We now give the proof of Theorem 1. In fact, we are going to show that the following conditions (1–4) are equivalent under the assumptions of Theorem 1.

1. $X \in A\otimes B-$Gproj.
2. ${}_AX \in A-$Gproj and $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes B) = 0$ for all $n \geq 1$.
3. $D(B)\otimes_B X \in A-$Gproj and $\mathrm{Tor}^B_n(D(B), X) = 0$ for all $n \geq 1$.
4. $D(B)\otimes_B X \in A-$Gproj and ${}_BX \in B-$GProj.

Proof of Theorem 1. "(1) $\Rightarrow$ (2)" Since X is in $A\otimes B-$Gproj, it follows from (G1) that $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes B) = 0$ for all $n \geq 1$. Since B is a Gorenstein algebra, the injective dimension B^{op} is finite. By Proposition 3.7 ${}_AX$ lies in $A-$Gproj.

"(2) $\Rightarrow$ (1)" Recall that ${}_AX \in A-$Gproj implies X satisfying (G1$'$). Then by (G1) and Lemma 3.1 we have $\mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}((A\otimes B)^\vee, X^\vee) = 0$.

Since B is a Gorenstein algebra, $(A\otimes B)^\vee = A\otimes D(B)$ is a tilting module over $(A\otimes B)^{\mathrm{op}}$ in the sense of [10,20]. Following [20, Theorem 1.16 and Proposition 1.20], for each $n \geq 0$ we have

$$\begin{aligned}&\mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}\left(X^\vee, (A\otimes B)^\vee\right)\\ &\simeq \mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}\left(\mathrm{Hom}_{(A\otimes B)^{\mathrm{op}}}\left((A\otimes B)^\vee, X^\vee\right), \mathrm{Hom}_{(A\otimes B)^{\mathrm{op}}}\left((A\otimes B)^\vee, (A\otimes B)^\vee\right)\right).\end{aligned}$$

For each $n \geq 1$, we have

$$\begin{aligned}&\mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}(X^*, A\otimes B)\\ &\simeq \mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}(\mathrm{Hom}_{A\otimes B}(X, A\otimes B), \mathrm{Hom}_{A\otimes B}(A\otimes B, A\otimes B))\\ &\simeq \mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}\left(\mathrm{Hom}_{(A\otimes B)^{\mathrm{op}}}\left((A\otimes B)^\vee, X^\vee\right), \mathrm{Hom}_{(A\otimes B)^{\mathrm{op}}}\left((A\otimes B)^\vee, (A\otimes B)^\vee\right)\right)\\ &\simeq \mathrm{Ext}^n_{(A\otimes B)^{\mathrm{op}}}\left(X^\vee, (A\otimes B)^\vee\right).\end{aligned}$$

Since ${}_AX$ lies in $A-$Gproj, we know that X satisfies (G2$'$). Then X satisfies (G2). Similarly, we can prove that X satisfies (G3).

"(2) $\Rightarrow$ (3)" Since B has finite injective dimension, by Corollary 3.3 we have $D(B)\otimes_B X$ is in $A-$Gproj and $\mathrm{Tor}^B_n(D(B), X) = 0$ for $n \geq 1$.

"(3) $\Rightarrow$ (2)" Recall that $A-$Gproj is closed under kernels of epimorphisms. Since B^{op} has finite injective dimension, $D(B)\otimes_B X \in A-$Gproj implies ${}_AX \in A-$Gproj.

Since the B has finite injective dimension, it follows from Corollary 3.3 that $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes B) = 0$ for all $n \geq 1$.

"(3) $\Longleftrightarrow$ (4)" A module U over the Gorenstein algebra B is Gorenstein projective if and only if $\mathrm{Tor}^B_n(D(B), U) = 0$ for all $n \geq 1$; see [4, Proposition 3.10]. □

In particular, if B is self-injective, we have the following. We mention that it is a special case of [9, Theorem 3.6]; compare [28].

Corollary 4.1. *Let A be a two-sided coherent algebra and B be a finite-dimensional self-injective algebra. Suppose that X is a finitely presented $A\otimes B$-module. Then X is Gorenstein projective if and only if X is Gorenstein projective over A.*

Next we prove Proposition II.

Proof of Proposition II. "$\Rightarrow$" Since X is a Gorenstein projective module, following the proof of Theorem 1 we know that ${}_AX$ is in $A-$Gproj and $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes B) = 0$ for all $n \geq 1$.

Since the algebra B has finite global dimension, it follows that $X/\mathrm{rad}_B X$ lies in $A-$Gproj and X is projective over B by Corollary 3.4.

"$\Leftarrow$" Observe that $B\otimes_B X$ is a finite extension of direct summands of copies of $(B/\mathrm{rad}B)\otimes_B X$. Then ${}_AX$ is in $A-$Gproj.

Following Corollary 3.4 we have $\mathrm{Ext}^n_{A\otimes B}(X, A\otimes B) = 0$ for all $n \geq 1$. Then X is Gorenstein projective by the proof of Theorem 1. □

5. Examples and applications

In this section, we give some examples and applications of our results.

First we discuss representations of bound quiver algebras over fields.

Let $Q = (Q_0, Q_1)$ be a finite quiver; see [3, Chapter III, §1]. Here, Q_0 is the set of vertices and Q_1 is the set of arrows. Recall that $s, t : Q_1 \to Q_0$ are the starting map and the terminating map respectively.

Denote by e_i the trivial path at each $i \in Q_0$. A nontrivial path p is a sequence $\alpha_l \cdots \alpha_2\alpha_1$ of arrows such that $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \le i \le l-1$.

Let kQ be the path algebra of Q. It is the vector space having all the (trivial and nontrivial) paths in Q as basis. The multiplication is given by concatenation of paths, if two paths cannot be concatenated, their product is zero. Recall that $e_i^2 = e_i$ for each $i \in Q_0$, and $\alpha e_{s(\alpha)} = e_{t(\alpha)}\alpha = \alpha$ for each $\alpha \in Q_1$.

Denote by J the ideal of kQ generated by all arrows in Q. Let I be an ideal of kQ satisfying $J^k \subseteq I \subseteq J^2$ for some $k \ge 2$. The bound quiver algebra $B = kQ/I$ is finite dimensional. In particular, if $I = J^2$, we obtain the radical square zero algebra kQ/J^2 of Q.

For each $i \in Q_0$, denote by $S(i)$ the simple B-module at i, $I(i)$ the indecomposable injective B-module at i, and $P(i)$ the indecomposable projective B-module at i respectively.

Example 5.1. Let A be a two-sided coherent algebra and $B = kQ/I$ be a bound quiver algebra. Let X be in $A \otimes B$-mod.

For each $i \in Q_0$, we have $X_i = e_iX \in A$-mod. For each $\alpha \in Q_1$, there is an A-module map $X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}$ induced by multiplication of α.

For each $i \in Q_0$, there is an exact sequence

$$\coprod_\alpha X_{s(\alpha)} \xrightarrow{f_i} X_i \to \operatorname{Coker} f_i \to 0,$$

where $f_i = (X_\alpha)$ and α runs through all arrows with $t(\alpha) = i$.

Take a minimal injective copresentation of $S(i)$

$$0 \to S(i) \to I(i) \xrightarrow{g_i} \coprod_\alpha I(s(\alpha)),$$

where $g_i = (\alpha)$ and α runs through all arrows with $t(\alpha) = i$.

Recall the k-dual functor D. Applying $D(-)\otimes_B X$ to the previous sequence, we obtain an isomorphism $DS(i)\otimes_B X \simeq \operatorname{Coker} f_i$.

Suppose that the bound quiver algebra $B = kQ/I$ has finite global dimension. It follows from Proposition 2 that X is Gorenstein projective if and only if the B-module ${}_BX$ is projective and each A-module $\operatorname{Coker} f_i$ is Gorenstein projective for $i \in Q_0$; compare [18,19].

If B has infinite global dimension, the description may be more complicated. The following is an example.

Example 5.2. Let Q be the following quiver.

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "\alpha"] & \overset{2}{\bullet} \arrow[loop right, "\beta"] \end{tikzcd}$$

Let $B = kQ/\langle \beta^2 \rangle$. Then B is a Gorenstein algebra of infinite global dimension.

Take minimal injective resolutions of $P(1)$ and $P(2)$ respectively

$$0 \to P(1) \to I(2) \xrightarrow{\alpha} I(1) \to 0,$$
$$0 \to P(2) \to I(2) \xrightarrow{\binom{\alpha}{\beta\alpha}} I(1) \oplus I(1) \to 0.$$

Let A be a two-sided coherent algebra. Let X be in $A \otimes B$-mod.

We have $DP(1) \otimes_B X \simeq \mathrm{Coker}\, X_\alpha$ and $DP(2) \otimes_B X \simeq \mathrm{Coker}\, f$, where

$$f = (X_\alpha, X_{\beta\alpha}) : X_1 \bigoplus X_1 \to X_2.$$

By Theorem 1, the $A \otimes B$-module X is Gorenstein projective if and only if ${}_BX$ belongs to $B-\mathrm{GProj}$ and $\mathrm{Coker}\, X_\alpha, \mathrm{Coker}\, f$ belong to $A-\mathrm{Gproj}$.

Note that ${}_BX \in B-\mathrm{GProj}$ if and only if $\mathrm{Tor}_1^B(D(B), X) = 0$ or, equivalently, f is injective. The injectivity of f implies $\mathrm{Coker}\, X_\alpha \simeq \mathrm{Coker}\, f \coprod X_1$.

In summary, $X \in A \otimes B$-mod is Gorenstein projective if and only if

1. f is injective;
2. $\mathrm{Coker}\, f, X_1 \in A-\mathrm{Gproj}$.

We mention that this description can also be obtained by [31, Theorem 1.4].

Example 5.3. Keep the notation in Example 5.2.

Assume further that A is *CM-free*, that is, $A-\mathrm{Gproj} = A-\mathrm{proj}$. Let $B' = e_2 B e_2$. Then B' is the algebra of dual numbers; it is a self-injective algebra.

Let X be in $A \otimes B-\mathrm{Gproj}$. By Example 5.2 f is injective and $\mathrm{Coker}\, f, X_1$ are projective. Then $\mathrm{Coker}\, f$ lies in $A \otimes B'-\mathrm{Gproj}$ and $\mathrm{Im} f$ lies in $A \otimes B'-\mathrm{proj}$.

The exact sequence $0 \to \mathrm{Im} f \to X_2 \to \mathrm{Coker}\, f \to 0$ in $A \otimes B'-\mathrm{Gproj}$ splits. There is an isomorphism $X \simeq (X_1 \otimes P(1)) \coprod \mathrm{Coker}\, f$.

We note that $X_1 \otimes P(1)$ is a projective $A \otimes B$-module. However, the action of β on $\mathrm{Coker}\, f$ is rather complicated. The algebra $A \otimes B'$ is not CM-free in general; see [22,27] for more examples.

Next we consider periodic complexes.

Let $n \geq 1$ be an arbitrary positive integer. Denote by $[n]$ the set $\{1, 2, \ldots, n\}$ of positive integers $\leq n$.

Let A be an arbitrary ring. Recall that an *n-periodic complex* X of A-modules is a collection $(X^i, d_X^i)_{i \in [n]}$, where each X^i is an A-module and each $d_X^i : X^i \to X^{i+1}$ is an A-module map such that $d_X^{i+1} d_X^i = 0$ for all $i \in [n]$. Here, we identify $n+1$ with 1.

Given two n-periodic complexes X and Y of A-modules, an *n-periodic cochain map* $f : X \to Y$ is a collection $(f^i)_{i \in [n]}$, where $f^i : X^i \to Y^i$ is an A-module map such that $f^{i+1} d_X^i = d_Y^i f^i$ for $i \in [n]$. An n-periodic cochain map $f : X \to Y$ is said to be *null-homotopic* if there exist A-module maps $s^i : X^i \to Y^{i-1}$ for $i \in [n]$, such that $f^i = d_Y^{i-1} s^i + s^{i+1} d_X^i$. An n-periodic complex X of A-modules is said to be *contractible* if the identity map of X is null-homotopic.

Let $\mathbb{Z}_n$ be the quiver with n vertices and n arrows which forms an oriented cycle. The vertex set of $\mathbb{Z}_n$ is $[n] = \{1, 2, \ldots, n\}$; and there is a unique arrow α_i from i to $i+1$ for each $i \in [n]$. Here, we also identify $n+1$ with 1.

Now we suppose that A is a two-sided coherent algebra. Denote by $\mathbf{C}_n(A\text{-mod})$ the abelian category of n-periodic complexes of finitely presented A-modules.

Let $B_n = k\mathbb{Z}_n / J^2$ be the radical square zero algebra of $\mathbb{Z}_n$. We note that B_n is finite dimensional and self-injective. Then $A \otimes B_n$ is a two-sided coherent algebra.

There exists an equivalence $R : A \otimes B_n\text{-mod} \to \mathbf{C}_n(A\text{-mod})$ of abelian categories; see [18, Lemma 2.1]. Given an object X in $A \otimes B_n$-mod, recall that $R(X)$ is given by $(X^i, d_X^i)_{i \in [n]}$, where $X^i = e_i X$ and d_X^i is induced by multiplication of α_i.

Denote by $\mathbf{C}_n(A-\mathrm{proj})$ the category of n-periodic complexes of finitely generated projective A-modules; it is a Frobenius category [21, Proposition 7.1]. An object C in $\mathbf{C}_n(A-\mathrm{proj})$ is projective if only if C is contractible. Denote by $\mathbf{K}_n(A-\mathrm{proj})$ the associated homotopy category; it is a triangulated category.

Denotes by $A \otimes B_n-\mathrm{Gproj}$ the category of finitely presented Gorenstein projective modules over $A \otimes B_n$; it is a Frobenius category. The projective objects are exactly $A \otimes B_n-\mathrm{proj}$. Denote by $A \otimes B_n-\underline{\mathrm{Gproj}}$ the stable category of $A \otimes B_n-\mathrm{Gproj}$; this is a triangulated category [15, I.2]. Recall that $\underline{A \otimes B_n-\mathrm{Gproj}}$ is obtained from $A \otimes B_n-\mathrm{Gproj}$ by factoring out the ideal of all maps which factor through projective $A \otimes B_n$-modules.

Recall that a two-sided coherent ring A is said to be CM-free if the categories $A-\mathrm{Gproj}$ and $A-\mathrm{proj}$ coincide. If A has finite weak dimension, then it is CM-free. See [8] for more examples.

Lemma 5.4. *Let A be a two-sided coherent CM-free algebra. Then R induces*

1. *an exact equivalence $A \otimes B_n-\mathrm{Gproj} \simeq \mathbf{C}_n(A-\mathrm{proj})$; and*
2. *a triangle equivalence $A \otimes B_n-\underline{\mathrm{Gproj}} \simeq \mathbf{K}_n(A-\mathrm{proj})$.*

Proof. (1) Let X be in $A \otimes B_n$-mod. By Corollary 4.1 X belongs to $A \otimes B_n-\mathrm{Gproj}$ if and only if $R(X)$ belongs to $\mathbf{C}_n(A-\mathrm{Gproj})$. Since A is CM-free, $A-\mathrm{Gproj}$ is equal to $A-\mathrm{proj}$. Since R is exact, it preserves admissible sequences. Then R induces an exact equivalence of $A \otimes B_n-\mathrm{Gproj}$ and $\mathbf{C}_n(A-\mathrm{proj})$.

(2) Let X be in $A \otimes B_n-\mathrm{Gproj}$. Note that X is a projective module if and only if $R(X)$ is a contractible complex. Then R induces a triangle equivalence of stable categories by (1). □

Before giving an application of Lemma 5.4, we recall some facts about singularity categories and Ding-Chen rings.

Given a left coherent ring A, we denote by $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ the bounded derived category of finitely presented A-modules. Two left coherent rings A, A' are said to be *derived equivalent* if $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ and $\mathbf{D}^{\mathrm{b}}(A'\text{-mod})$ are triangle equivalent.

Denote by $\mathbf{K}^{\mathrm{b}}(A-\mathrm{proj})$ the bounded homotopy category of finitely generated projective A-modules; it is a thick subcategory of the derived category $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$. Following [6] the Verdier quotient

$$\mathbf{D}_{\mathrm{sg}}(A) := \mathbf{D}^{\mathrm{b}}(A\text{-mod})/\mathbf{K}^{\mathrm{b}}(A-\mathrm{proj})$$

is called the *singularity category* of A.

Let A be an arbitrary ring. Recall that an A-module I is said to be *FP-injective* if $\mathrm{Ext}^1_A(F, I) = 0$ for every finitely presented A-module F, and A is said to be a *Ding-Chen ring* if A is two-sided coherent and the FP-injective dimensions of A and A^{op} are both finite [11]. Two-sided coherent rings of finite weak dimension are automatically Ding-Chen rings.

For a two-sided coherent ring A, recall that there exists a natural fully faithful triangle functor $\imath : A-\underline{\mathrm{Gproj}} \to \mathbf{D}_{\mathrm{sg}}(A)$. It is dense if A is a Ding-Chen ring; see [12, Theorem 6.19].

We give a new proof of the main theorem in [32].

Proposition 5.5. *Let A, A' be two-sided coherent algebras of finite weak dimension. If A and A' are derived equivalent, then $\mathbf{K}_n(A-\mathrm{proj})$ and $\mathbf{K}_n(A'-\mathrm{proj})$ are triangle equivalent.*

Proof. Since A and A' are derived equivalent, it follows from [24, Theorem 2.1] that $A \otimes B_n$ and $A' \otimes B_n$ are also derived equivalent. Then by [23, Theorem 1.1] the singularity categories $\mathbf{D}_{\mathrm{sg}}(A \otimes B_n)$ and $\mathbf{D}_{\mathrm{sg}}(A' \otimes B_n)$ are triangle equivalent.

Since A and A' have finite weak dimension and B_n is self-injective, we know that $A \otimes B_n$ and $A' \otimes B$ are Ding-Chen algebras. By Lemma 5.4(2), we infer that the horizontal functors in the following diagram are all triangle equivalences.

$$\begin{tikzcd}
\mathbf{K}_n(A\text{-proj}) & A \otimes B_n\text{-}\underline{\mathrm{Gproj}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, dashed] & \mathbf{D}_{\mathrm{sg}}(A \otimes B_n) \arrow[d, "\simeq"] \\
\mathbf{K}_n(A'\text{-proj}) & A' \otimes B_n\text{-}\underline{\mathrm{Gproj}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathbf{D}_{\mathrm{sg}}(A' \otimes B_n)
\arrow[from=1-1, to=2-1, dashed]
\end{tikzcd}$$

Therefore, $\mathbf{K}_n(A-\text{proj})$ and $\mathbf{K}_n(A'-\text{proj})$ are triangle equivalent. $\square$

We end this section by an application of Theorem 1.

Let A be a finite-dimensional algebra. Denote by $A^e = A \otimes A^{op}$ the *enveloping algebra* of A. Recall that an A^e-module is exactly an A-A-bimodule. In particular, A itself is an A^e-module. Recall the k-dual functor D.

Proposition 5.6. *Let A be a finite-dimensional Gorenstein algebra. Then A is a Gorenstein projective A^e-module if and only if A is a self-injective algebra.*

Proof. "$\Rightarrow$" Since A is a Gorenstein projective A^e-module, by Theorem 1 we know that $D(A)$ is a Gorenstein projective A-module. Since A is Gorenstein, $D(A)$ has finite projective dimension. Then we infer that $D(A)$ is a projective A-module. Therefore, A is a self-injective algebra.

"$\Leftarrow$" Since A is a self-injective algebra, A^e is also a self-injective algebra by [2, Proposition 2.2]. Then every A^e-modules is Gorenstein projective. In particular, A is a Gorenstein projective A^e-module. $\square$

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