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The singularity category of a Nakayama algebra

Dawei Shen

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China

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ABSTRACT

Let A be a Nakayama algebra. We give a description of the singularity category of A inside its stable module category, which provides a new approach to the singularity category of a Nakayama algebra. We prove that there is a duality between the singularity category of A and the singularity category of its opposite algebra. As a consequence, the resolution quiver of A and the resolution quiver of its opposite algebra have the same number of cycles and the same number of cyclic vertices.

1. Introduction

Let A be an Artin algebra. Denote by A-mod the category of finitely generated left A-modules, and by $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ the bounded derived category of A-mod. Recall that a complex in $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex of finitely generated projective A-modules. Following [4,12,17], the *singularity category* $\mathbf{D}_{\mathrm{sg}}(A)$ of A is the quotient triangulated category of $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ with respect to the

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URL: http://home.ustc.edu.cn/~sdw12345/.

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full subcategory consisting of perfect complexes. Recently, the singularity category of a Nakayama algebra was described in [8].

Let A be a connected Nakayama algebra without simple projective modules. Following [20], the *resolution quiver* $R(A)$ of A is defined as follows: the vertex set is the set of isomorphism classes of simple A-modules, and there is an arrow from S to $\tau \operatorname{soc} P(S)$ for each simple A-module S; see also [10]. Here, $P(S)$ is the projective cover of S, 'soc' denotes the socle of a module, and $\tau = D\operatorname{Tr}$ is the Auslander–Reiten translation [2]. A simple A-module is called *cyclic* provided that it lies in a cycle of $R(A)$.

The following consideration is inspired by [20]. We emphasize that the treatment here is different from [20]; compare [20, Example in the introduction] with Example 3.13. Let A be a connected Nakayama algebra of infinite global dimension. Let $\mathcal{S}_c$ be a complete set of pairwise non-isomorphic cyclic simple A-modules. Let $\mathcal{X}_c$ be the set formed by indecomposable A-modules X such that $\operatorname{top} X$ and $\tau \operatorname{soc} X$ both belong to $\mathcal{S}_c$. Here, 'top' denotes the top of a module. Denote by $\mathcal{F}$ the full subcategory of A-mod whose objects are finite direct sums of objects in $\mathcal{X}_c$. It turns out that $\mathcal{F}$ is a Frobenius abelian category, and it is equivalent to A'-mod with A' a connected selfinjective Nakayama algebra. Denote by $\underline{\mathcal{F}}$ the stable category of $\mathcal{F}$ modulo projective objects; it is a triangulated category by [11]. We emphasize that the stable category $\underline{\mathcal{F}}$ is a full subcategory of the stable module category A-$\underline{\mathrm{mod}}$ of A.

The well-known result of [4,12] describes the singularity category of a Gorenstein algebra A via the subcategory of A-$\underline{\mathrm{mod}}$ formed by Gorenstein projective modules. Here, we recall that an Artin algebra is *Gorenstein* if the injective dimension of the regular module is finite on both sides. In general, a Nakayama algebra is not Gorenstein [20,8]. The following result describes the singularity category of a Nakayama algebra via the subcategory $\underline{\mathcal{F}}$ of A-$\underline{\mathrm{mod}}$. For a Gorenstein Nakayama algebra, these two descriptions coincide; compare [20].

Theorem 1.1. *Let A be a connected Nakayama algebra of infinite global dimension. Then the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ and the stable category $\underline{\mathcal{F}}$ are triangle equivalent.*

Denote by A-inj the category of finitely generated injective A-modules, and by $\mathbf{K}^{\mathrm{b}}(A\text{-inj})$ the bounded homotopy category of A-inj. We view $\mathbf{K}^{\mathrm{b}}(A\text{-inj})$ as a thick subcategory of $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ via the canonical functor. Then the quotient triangulated category $\mathbf{D}^{\mathrm{b}}(A\text{-mod})/\mathbf{K}^{\mathrm{b}}(A\text{-inj})$ is triangle equivalent to the opposite category of the singularity category $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ of A^{op}. Here, A^{op} is the opposite algebra of A. It is well known that for a Gorenstein algebra A, the singularity categories $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ are triangle dual. In general, it seems that for an arbitrary Artin algebra A, there is no obvious relation between $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$. Indeed, there are examples of algebras A of radical square zero such that $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ are neither triangle equivalent nor triangle dual; see Example 4.4. However, we have the following result for Nakayama algebras.

Proposition 1.2. *Let A be a Nakayama algebra. Then the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is triangle equivalent to $\mathbf{D}^{\mathrm{b}}(A\text{-mod})/\mathbf{K}^{\mathrm{b}}(A\text{-inj})$. Equivalently, there is a triangle duality between $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$.*

Let A be a connected Nakayama algebra of infinite global dimension. Recall from [21] that the resolution quivers $R(A)$ and $R(A^{\mathrm{op}})$ have the same number of cyclic vertices. The following result strengthens the previous one by a different method.

Proposition 1.3. *Let A be a connected Nakayama algebra of infinite global dimension. Then the resolution quivers $R(A)$ and $R(A^{\mathrm{op}})$ have the same number of cycles and the same number of cyclic vertices.*

The paper is organized as follows. In Section 2, we recall some facts on singularity categories of Artin algebras and the simplification in the sense of [19]. In Section 3, we introduce the Frobenius subcategory $\mathcal{F}$ and prove Theorem 1.1. The proofs of Propositions 1.2 and 1.3 are given in Sections 4 and 5, respectively.

2. Preliminaries

We first recall some facts on the singularity category of an Artin algebra.

Let A be an Artin algebra over a commutative artinian ring R. Recall that A-mod denotes the category of finitely generated left A-modules. Let A-proj denote the full subcategory consisting of projective A-modules, and A-inj the full subcategory consisting of injective A-modules. Denote by A-$\underline{\mathrm{mod}}$ the projectively stable category of finitely generated A-modules; it is obtained from A-mod by factoring out the ideal of all maps which factor through projective A-modules; see [2, IV.1].

Recall that for an A-module M, its *syzygy* $\Omega(M)$ is the kernel of its projective cover $P(M) \to M$. This gives rise to the *syzygy functor* $\Omega : A\text{-}\underline{\mathrm{mod}} \to A\text{-}\underline{\mathrm{mod}}$. Let $\Omega^0(M) = M$ and $\Omega^{i+1}(M) = \Omega(\Omega^i(M))$ for $i \geq 0$. Denote by $\Omega^i(A\text{-mod})$ the full subcategory of A-mod formed by modules M such that there is an exact sequence $0 \to M \to P_{i-1} \to \cdots \to P_1 \to P_0$ with each P_j projective. We also denote by $\Omega_0^i(A\text{-mod})$ the full subcategory of $\Omega^i(A\text{-mod})$ formed by modules without indecomposable projective direct summands.

Recall that $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ denotes the bounded derived category of A-mod, whose translation functor is denoted by [1]. For each integer n, let $[n]$ denote the n-th power of [1]. The category A-mod is viewed as a full subcategory of $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ by identifying an A-module with the corresponding stalk complex concentrated at degree zero. Recall that a complex in $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex of finitely generated projective A-modules. Perfect complexes form a thick subcategory of $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$, which is denoted by $\mathbf{perf}(A)$. Here, we recall that a triangulated subcategory is *thick* if it is closed under direct summands.

Following [4,12,17], the quotient triangulated category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^{\mathrm{b}}(A\text{-mod})/\mathbf{perf}(A)$$

is called the *singularity category* of A. Denote by $q : \mathbf{D}^{\mathrm{b}}(A\text{-mod}) \to \mathbf{D}_{\mathrm{sg}}(A)$ the quotient functor. We recall that the objects in $\mathbf{D}_{\mathrm{sg}}(A)$ are bounded complexes of finitely generated A-modules. The translation functor of $\mathbf{D}_{\mathrm{sg}}(A)$ is also denoted by [1].

The following results are well known.

Lemma 2.1. *(See [6, Lemma 2.1].) Let X be a complex in $\mathbf{D}_{\mathrm{sg}}(A)$ and $n_0 > 0$ be an arbitrary natural number. Then for any n sufficiently large, there exists a module M in $\Omega^{n_0}(A\text{-mod})$ such that $X \simeq q(M)[n]$.*

Lemma 2.2. *(See [6, Lemma 2.2].) Let $0 \to N \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be an exact sequence in A-mod with each P_i projective. Then there is an isomorphism $q(M) \simeq q(N)[n]$ in $\mathbf{D}_{\mathrm{sg}}(A)$. Moreover, if $N = \Omega^n(M)$, then there is a natural isomorphism $\theta^n_M : q(M) \simeq q(\Omega^n(M))[n]$ for any M in A-mod and $n \geq 0$.*

Observe that the composition $A\text{-mod} \to \mathbf{D}^{\mathrm{b}}(A\text{-mod}) \xrightarrow{q} \mathbf{D}_{\mathrm{sg}}(A)$ vanishes on projective modules. Then it induces a unique functor $q' : A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$. It follows from Lemma 2.2 that for each $n \geq 0$, the following diagram of functors

$$\begin{tikzcd} A\text{-}\underline{\mathrm{mod}} \arrow[r, "\Omega^n"] \arrow[d, "q'"'] & A\text{-}\underline{\mathrm{mod}} \arrow[d, "q'"] \\ \mathbf{D}_{\mathrm{sg}}(A) \arrow[r, "{[-n]}"] & \mathbf{D}_{\mathrm{sg}}(A) \end{tikzcd}$$

is commutative. Let M and N be in A-mod and $n \geq 0$. Lemma 2.2 yields a natural map

$$\Phi^n : \underline{\mathrm{Hom}}_A(\Omega^n(M), \Omega^n(N)) \longrightarrow \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(M), q(N)).$$

Here, Φ^0 is induced by q' and $\Phi^n(f) = (\theta^n_N)^{-1} \circ (q'(f)[n]) \circ \theta^n_M$ for $n \geq 1$.

Consider the following chain of maps $\{G^{n,n+1}\}_{n\geq 0}$ such that

$$G^{n,n+1} : \underline{\mathrm{Hom}}_A(\Omega^n(M), \Omega^n(N)) \longrightarrow \underline{\mathrm{Hom}}_A(\Omega^{n+1}(M), \Omega^{n+1}(N))$$

is induced by the syzygy functor Ω. The sequence $\{\Phi^n\}_{n\geq 0}$ is compatible with $\{G^{n,n+1}\}_{n\geq 0}$, that is, $\Phi^{n+1} \circ G^{n,n+1} = \Phi^n$ for each $n \geq 0$. Then we obtain an induced map

$$\Phi : \varinjlim_{n\geq 0} \underline{\mathrm{Hom}}_A(\Omega^n(M), \Omega^n(N)) \longrightarrow \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(M), q(N)).$$

The following lemma is contained in [3, Corollary 3.9(1)]. Indeed, we use the isomorphism between Hom-spaces in the *stabilization* category of A-$\underline{\mathrm{mod}}$ and in $\mathbf{D}_{\mathrm{sg}}(A)$, which is a consequence of the triangle equivalence in [3, Corollary 3.9(1)].

Lemma 2.3. *(See [14, Exemple 2.3].) Let M and N be in A-mod. Then there is a natural isomorphism*

$$\Phi : \varinjlim_{n\geq 0} \underline{\mathrm{Hom}}_A(\Omega^n(M), \Omega^n(N)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(q(M), q(N)).$$

Next we recall the *simplification* in the sense of [19].

Let $\mathcal{A}$ be an abelian category. Recall that an object X in $\mathcal{A}$ is a *brick* if $\mathrm{End}_{\mathcal{A}}(X)$ is a division ring. Two objects X and Y are *orthogonal* if $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$ and $\mathrm{Hom}_{\mathcal{A}}(Y, X) = 0$. A full subcategory $\mathcal{W}$ of $\mathcal{A}$ is called a *wide subcategory* if it is closed under kernels, cokernels and extensions. In particular, $\mathcal{W}$ is an abelian category and the inclusion functor is exact. Recall that an abelian category $\mathcal{A}$ is called a *length category* provided that each object in $\mathcal{A}$ has a composition series.

Let $\mathcal{E}$ be a set of objects in an abelian category $\mathcal{A}$. For an object C in $\mathcal{A}$, an *$\mathcal{E}$-filtration* of C is given by a sequence of subobjects

$$0 = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_m = C,$$

such that each factor C_i/C_{i-1} belongs to $\mathcal{E}$ for $1 \leq i \leq m$. Denote by $\mathcal{F}(\mathcal{E})$ the full subcategory of $\mathcal{A}$ formed by objects in $\mathcal{A}$ with an $\mathcal{E}$-filtration.

Lemma 2.4. *(See [19, Theorem 1.2].) Let $\mathcal{E}$ be a set of pairwise orthogonal bricks in $\mathcal{A}$. Then $\mathcal{F}(\mathcal{E})$ is a wide subcategory of $\mathcal{A}$; moreover, $\mathcal{F}(\mathcal{E})$ is a length category and $\mathcal{E}$ is a complete set of pairwise non-isomorphic simple objects in $\mathcal{F}(\mathcal{E})$.*

Let A be a connected Nakayama algebra without simple projective modules. Recall that the vertex set of the *resolution quiver* $R(A)$ of A is the set of isomorphism classes of simple A-modules, and there is an arrow from S to $\gamma(S) = \tau\,\mathrm{soc}\,P(S)$ for each simple A-module S. Since each vertex in $R(A)$ is the start of a unique arrow, each connected component of $R(A)$ contains precisely one cycle. A simple A-module is called *cyclic* provided that it lies in a cycle of $R(A)$.

Let A be a connected Nakayama algebra of infinite global dimension. In particular, there are no simple projective A-modules. Let $\mathcal{S}$ be a complete set of pairwise non-isomorphic simple A-modules. Denote by $\mathcal{S}_c$ the subset of all cyclic simple A-modules, and by $\mathcal{S}_{nc}$ the subset of all *noncyclic* simple A-modules.

Lemma 2.5. *Let A be a connected Nakayama algebra of infinite global dimension. Then $\mathcal{S}_c$ is a complete set of pairwise non-isomorphic simple A-modules of infinite injective dimension, and $\mathcal{S}_{nc}$ is a complete set of pairwise non-isomorphic simple A-modules of finite injective dimension.*

Proof. This is dual to [15, Corollary 3.6]. $\square$

We will need the following fact. Recall that 'top' denotes the top a module.

Lemma 2.6. *(See [20, Corollary to Lemma 2].) Let A be a connected Nakayama algebra without simple projective modules. Assume that M is an indecomposable A-module and $m \geq 0$. Then either $\Omega^{2m}(M) = 0$ or else* $\operatorname{top}\Omega^{2m}(M) = \gamma^m(\operatorname{top} M)$.

3. A Frobenius subcategory

In this section, we introduce a Frobenius subcategory in the module category of a Nakayama algebra, whose stable category is triangle equivalent to the singularity category of the given algebra.

Throughout this section, A is a connected Nakayama algebra of infinite global dimension. Denote by $n(A)$ the number of the isomorphism classes of simple A-modules. Denote by $l(M)$ the composition length of an A-module M. Recall that $\mathcal{S}$ denotes a complete set of pairwise non-isomorphic simple A-modules, $\mathcal{S}_c$ the subset of all cyclic simple A-modules and $\mathcal{S}_{nc}$ the subset of all noncyclic simple A-modules. Observe that the map γ restricts to a permutation on $\mathcal{S}_c$. Let $\mathcal{X}_c$ be the set formed by indecomposable A-modules X such that both $\operatorname{top} X$ and $\tau \operatorname{soc} X$ belong to $\mathcal{S}_c$.

We recall some well-known facts on indecomposable modules over the Nakayama algebra A; see [2, IV.3 and VI.2]. Each indecomposable A-module Z is uniserial, and it is uniquely determined by its top and its composition length. Its composition factors from the top are $S, \tau S, \cdots, \tau^{l-1} S$, where $S = \operatorname{top} Z$ and $l = l(Z)$. In particular, the projective cover $P(Z)$ of Z is indecomposable.

Lemma 3.1. *Let M be an indecomposable A-module which contains a nonzero projective submodule P. Then M is projective.*

Proof. Recall that the projective cover $P(M)$ of M is uniserial, in particular, each nonzero submodule of $P(M)$ is indecomposable.

Suppose that, on the contrary, M is nonprojective. Then there is a proper surjective map $\pi : P(M) \to M$. We have a proper surjective map $\pi^{-1}(P) \to P$; it splits, since P is projective. This is impossible, since $\pi^{-1}(P)$, as a submodule of $P(M)$, is indecomposable. $\square$

Recall that each indecomposable A-module Z is uniserial. Moreover, for any nonzero proper submodule W of Z, we have $\operatorname{top} Z/W = \operatorname{top} Z$, $\operatorname{soc} W = \operatorname{soc} Z$ and $\tau \operatorname{soc} Z/W = \operatorname{top} W$.

The following two lemmas are parallel to [20, Lemmas 6 and 7].

Lemma 3.2. *Let $f : X \to Y$ be a morphism in $\mathcal{X}_c$. Then* $\operatorname{Ker} f$, $\operatorname{Coker} f$ *and* $\operatorname{Im} f$ *belong to* $\mathcal{X}_c \cup \{0\}$.

Proof. We may assume that f is nonzero. Then we have $\operatorname{top}(\operatorname{Im} f) = \operatorname{top} X$ and $\tau \operatorname{soc}(\operatorname{Im} f) = \tau \operatorname{soc} Y$, both of which belong to $\mathcal{S}_c$. Thus, $\operatorname{Im} f$ belongs to $\mathcal{X}_c$.

If f is not a monomorphism, then $\operatorname{top}(\operatorname{Ker} f) = \tau \operatorname{soc}(\operatorname{Im} f) = \tau \operatorname{soc} Y$ and $\tau \operatorname{soc}(\operatorname{Ker} f) = \tau \operatorname{soc} X$. Thus, $\operatorname{Ker} f$ belongs to $\mathcal{X}_c$.

If f is not an epimorphism, then $\operatorname{top}(\operatorname{Coker} f) = \operatorname{top} Y$ and $\tau \operatorname{soc}(\operatorname{Coker} f) = \operatorname{top}(\operatorname{Im} f) = \operatorname{top} X$. Thus, $\operatorname{Coker} f$ belongs to $\mathcal{X}_c$. □

Lemma 3.3. *Let X be an object in $\mathcal{X}_c$. If $0 \subsetneq X'' \subsetneq X' \subsetneq X$ are subobjects of X such that X'/X'' belongs to $\mathcal{X}_c$, then X'' and X/X' belong to $\mathcal{X}_c$.*

Proof. Since both $\operatorname{top} X'' = \tau \operatorname{soc} X'/X''$ and $\tau \operatorname{soc} X'' = \tau \operatorname{soc} X$ belong to $\mathcal{S}_c$, it follows from the definition that X'' belongs to $\mathcal{X}_c$. Similarly, since both $\operatorname{top} X/X' = \operatorname{top} X$ and $\tau \operatorname{soc} X/X' = \operatorname{top} X'/X''$ belong to $\mathcal{S}_c$, it follows from the definition that X/X' belongs to $\mathcal{X}_c$. □

Remark 3.4. Under the same assumption as in Lemma 3.3, the same argument proves that X' and X/X'' belong to $\mathcal{X}_c$.

Denote by $\mathcal{P}_c$ a complete set of projective covers of modules in $\mathcal{S}_c$. We claim that $\mathcal{P}_c$ is a subset of $\mathcal{X}_c$. Indeed, we have $\operatorname{top} P(S) = S$ and $\tau \operatorname{soc} P(S) = \gamma(S)$, both of which belong to $\mathcal{S}_c$. It follows that $\mathcal{X}_c$ is closed under projective covers.

For each S in $\mathcal{S}_c$, let $E(S)$ denote the indecomposable A-module of the least composition length among those objects X in $\mathcal{X}_c$ with $\operatorname{top} X = S$. Inspired by [20, Section 4], we call $E(S)$ the *elementary module* associated to S. Denote by $\mathcal{E}_c$ the set of elementary modules. Recall that $\mathcal{F}(\mathcal{E}_c)$ is the full subcategory of A-mod formed by A-modules with an $\mathcal{E}_c$-filtration.

The *support* of an A-module M, denoted by $\operatorname{supp} M$, is the subset of $\mathcal{S}$ consisting of those simple A-modules appearing as a composition factor of M. For a set $\mathcal{X}$ of A-modules, we denote by $\operatorname{add} \mathcal{X}$ the full subcategory of A-mod whose objects are direct summands of finite direct sums of objects in $\mathcal{X}$.

The following result is in spirit close to [20, Proposition 2]. In particular, we prove that each elementary module E is a brick and thus $l(E) \leq n(A)$. Here, we use the well-known fact that a brick Z over the Nakayama algebra A satisfies that $l(Z) \leq n(A)$.

Proposition 3.5. *Let A be a connected Nakayama algebra of infinite global dimension. Then the following statements hold.*

(1) *The set $\mathcal{E}_c$ of elementary modules is a set of pairwise orthogonal bricks, and thus $\mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of A-mod.*
(2) *$\mathcal{F}(\mathcal{E}_c) = \operatorname{add} \mathcal{X}_c$, which is closed under projective covers.*
(3) *Let E and E' be elementary modules. Then $E = E'$ if and only if $\operatorname{supp} E \cap \operatorname{supp} E' \neq \emptyset$.*

Proof. (1) Let $f : E \to E'$ be a nonzero map between elementary modules. By Lemma 3.2 $\operatorname{Im} f$ belongs to $\mathcal{X}_c$. However, $\operatorname{Im} f$ is a factor module of E. By the definition of the elementary module E we have $E = \operatorname{Im} f$. Then f is an injective map. Similarly, f is a surjective map and thus an isomorphism. Therefore $\mathcal{E}_c$ is a set of pairwise orthogonal bricks.

By Lemma 2.4 $\mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of A-mod. In particular, it is closed under direct sums and direct summands.

(2) We prove that any module X in $\mathcal{X}_c$ belongs to $\mathcal{F}(\mathcal{E}_c)$, and thus $\operatorname{add} \mathcal{X}_c \subseteq \mathcal{F}(\mathcal{E}_c)$. We use induction on $l(X)$. Set $S = \operatorname{top} X \in \mathcal{S}_c$. If $X = E(S) \in \mathcal{E}_c$, we are done. Otherwise, there is a proper surjective map $\pi : X \to E(S)$. By Lemma 3.2 we have $\operatorname{Ker} \pi \in \mathcal{X}_c$. Then by induction $\operatorname{Ker} \pi \in \mathcal{F}(\mathcal{E}_c)$. Therefore $X \in \mathcal{F}(\mathcal{E}_c)$.

Recall that each elementary module E satisfies that $\operatorname{top} E \in \mathcal{S}_c$ and $\tau \operatorname{soc} E \in \mathcal{S}_c$. It follows from its $\mathcal{E}_c$-filtration that each indecomposable object X in $\mathcal{F}(\mathcal{E}_c)$ satisfies that $\operatorname{top} X \in \mathcal{S}_c$ and $\tau \operatorname{soc} X \in \mathcal{S}_c$. Then by definition X belongs to $\mathcal{X}_c$. Therefore $\operatorname{add} \mathcal{X}_c \supseteq \mathcal{F}(\mathcal{E}_c)$, and thus $\mathcal{F}(\mathcal{E}_c) = \operatorname{add} \mathcal{X}_c$. Since $\mathcal{X}_c$ is closed under projective covers, we infer that $\operatorname{add} \mathcal{X}_c$ is closed under projective covers.

(3) Suppose that $E \neq E'$ and they have a common composition factor. Assume that $\operatorname{soc} E \in \operatorname{supp} E'$. Recall that E and E' are orthogonal bricks. We infer that $\operatorname{top} E \in \operatorname{supp} E'$, otherwise there is a nonzero map from E' to E. For the same reason, we have $\operatorname{soc} E \neq \operatorname{soc} E'$ and $\operatorname{top} E \neq \operatorname{top} E'$. Then there exists a chain $0 \subsetneq E_1 \subsetneq E_2 \subsetneq E'$ of A-modules such that $E_2/E_1 = E$. By Lemma 3.3 we know that E'/E_2 belongs to $\mathcal{X}_c$. This contradicts to the definition of the elementary module E'. $\square$

The second statement of the following lemma is parallel to [20, Lemma 8].

Lemma 3.6. *Let S be a cyclic simple A-module in $\mathcal{S}_c$. Then the following statements hold.*

(1) *The injective dimension of $E(S)$ is infinite, and the injective dimension of $P(S)$ is finite.*
(2) *There is a unique cyclic simple A-module S' in $\mathcal{S}_c$ such that $\operatorname{top} E(S') = \tau \operatorname{soc} E(S)$ and $\operatorname{Ext}^1_A(E(S), E(S')) \neq 0$.*

Proof. (1) We recall from Lemma 2.5 that $\mathcal{S}_c$ is a complete set of pairwise non-isomorphic simple A-modules of infinite injective dimension. Since the elementary modules have pairwise disjoint supports, for each S in $\mathcal{S}_c$, $\operatorname{supp} E(S)$ contains precisely one cyclic simple A-module, that is, S. In other words, each composition factor of $E(S)$ different from S is a noncyclic simple A-module, and thus has finite injective dimension. It follows that $E(S)$ has infinite injective dimension.

Let $h : P(S) \to I$ be an injective envelope of the A-module $P(S)$. We claim that each composition factor S' of $\operatorname{Coker} h$ is a noncyclic simple A-module, and thus has finite injective dimension. Consequently, the injective dimension of $\operatorname{Coker} h$ is finite. Therefore the injective dimension of $P(S)$ is finite.

For the claim, we observe by Lemma 3.1 that $P(S) \subsetneq P(S') \subseteq I$. Then we have $\gamma(S') = \gamma(S)$. Recall that the restriction of γ on cyclic simple A-modules is injective. Therefore S' is a noncyclic simple A-module, since S is a cyclic simple A-module and $S' \neq S$.

(2) Let $E = E(S)$. Recall that $P(S)$ lies in $\mathcal{X}_c$ and thus in $\mathcal{F}(\mathcal{E}_c)$. Consider the $\mathcal{E}_c$-filtration of $P(S)$, say

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{t-1} \subsetneq M_t = P(S),$$

such that M_i/M_{i-1} is elementary for $1 \leq i \leq t$. We observe that $M_t/M_{t-1} = E$ and $t \geq 2$, since by (1) we have $E(S) \neq P(S)$. Set $E' = M_{t-1}/M_{t-2}$. Note that $E' = E(S')$ for some cyclic simple A-module S'. Then

$$\operatorname{top} E' = \operatorname{top}(M_{t-1}/M_{t-2}) = \tau \operatorname{soc}(M_t/M_{t-1}) = \tau \operatorname{soc} E.$$

Since $M_t/M_{t-2} = P(S)/M_{t-2}$ is indecomposable, the exact sequence

$$0 \to M_{t-1}/M_{t-2} \to M_t/M_{t-2} \to M_t/M_{t-1} \to 0$$

does not split. Then we have $\operatorname{Ext}^1_A(E, E') \neq 0$. The uniqueness of S' is obvious, since $S' = \tau \operatorname{soc} E(S)$. $\square$

Recall that by definition $\tau \operatorname{soc} E$ lies in $\mathcal{S}_c$ for each elementary module E. We have a map $\delta : \mathcal{S}_c \to \mathcal{S}_c$, which sends a cyclic simple A-module S to $\delta(S) = \tau \operatorname{soc} E(S)$. We claim that δ is injective and thus bijective. Indeed, if $\delta(S) = \delta(\bar{S})$, then $\operatorname{soc} E(S) = \operatorname{soc} E(\bar{S})$. It follows from Proposition 3.5(3) that $S = \bar{S}$.

Corollary 3.7. *Let S be a cyclic simple A-module in $\mathcal{S}_c$ and t the minimal positive integer such that $\delta^t(S) = S$. Then $\mathcal{S}_c = \{S, \delta(S), \cdots, \delta^{t-1}(S)\}$ and $\mathcal{S}$ is the disjoint union of the supports of all elementary modules.*

Proof. Since A is a connected Nakayama algebra without simple projective modules, any nonempty subset of $\mathcal{S}$ which is closed under τ must be $\mathcal{S}$. We claim that the union $\cup_{i=0}^{t-1} \operatorname{supp} E(\delta^i(S))$ is closed under τ, therefore this union equals $\mathcal{S}$.

For the claim, let $S' \in \cup_{i=0}^{t-1} \operatorname{supp} E(\delta^i(S))$. Assume that $S' \in \operatorname{supp} E(\delta^i S)$. If $S' \neq \operatorname{soc} E(\delta^i S)$, then $\tau S' \in \operatorname{supp} E(\delta^i S)$. If $S' = \operatorname{soc} E(\delta^i S)$, then by the definition of δ, we have $\tau S' = \operatorname{top} E(\delta^{i+1} S) \in \operatorname{supp} E(\delta^{i+1} S)$.

Let S' be a cyclic simple A-module in $\mathcal{S}_c$. Then there exists an integer $0 \leq i \leq t-1$ such that $\operatorname{supp} E(\delta^i(S)) \cap \operatorname{supp} E(S') \neq \emptyset$. It follows from Proposition 3.5(3) that $S' = \delta^i(S)$. $\square$

The following result is close to [20, Proposition 1].

Proposition 3.8. *Let A be a connected Nakayama algebra of infinite global dimension. Then $\mathcal{F}(\mathcal{E}_c)$ is equivalent to A'-mod, where A' is a connected selfinjective Nakayama algebra.*

Proof. Let $P = \oplus_{S \in \mathcal{S}_c} P(S)$ and $A' = \mathrm{End}_A(P)^{\mathrm{op}}$. Then P is a projective object in $\mathcal{F}(\mathcal{E}_c)$, since $\mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of A-mod. The natural projection $P(S) \to E(S)$ is a projective cover in the category $\mathcal{F}(\mathcal{E}_c)$. Recall from Lemma 2.4 that $\mathcal{F}(\mathcal{E}_c)$ is a length category with $\mathcal{E}_c = \{E(S) \mid S \in \mathcal{S}_c\}$ a complete set of pairwise non-isomorphic simple objects. We infer that for each object X in $\mathcal{F}(\mathcal{E}_c)$, there is an epimorphism $P' \to X$ with P' in $\mathrm{add}\, P$. Then P is a projective generator for $\mathcal{F}(\mathcal{E}_c)$. We have an equivalence $\mathcal{F}(\mathcal{E}_c) \simeq A'$-mod; compare [16, Chapter IV, Theorem 5.3].

Since each indecomposable object in $\mathcal{F}(\mathcal{E}_c)$ is uniserial, we infer that A' is a Nakayama algebra. Denote by τ' the Auslander–Reiten translation of A'. Then we have $\tau' E(S) = E(\delta(S))$ by Lemma 3.6(2). It follows from Corollary 3.7 that all simple A'-modules are in the same τ'-orbit. Therefore, the Nakayama algebra A' is connected.

It remains to show that A' is selfinjective. Since γ restricts to a permutation on $\mathcal{S}_c$, the modules in $\mathcal{P}_c$ have pairwise distinct socles. Therefore, we have $\mathcal{S}_c = \{\tau \,\mathrm{soc}\, P \mid P \in \mathcal{P}_c\}$. Let E be an elementary module. Since $\tau \,\mathrm{soc}\, E$ lies in $\mathcal{S}_c$, there exists P in $\mathcal{P}_c$ with $\mathrm{soc}\, P = \mathrm{soc}\, E$ and $\mathrm{soc}\, P \neq \mathrm{soc}\, E'$ for any elementary module $E' \neq E$. It follows that the socle of P in the category $\mathcal{F}(\mathcal{E}_c)$ is E. We have proven that every simple A'-module S' can embed into an indecomposable projective A'-module P', and thus the injective envelope of S' contains P' as a submodule. It follows from Lemma 3.1 that every injective A'-module is projective. Therefore, A' is selfinjective. $\square$

The following result is analogous to [20, Proposition 4].

Lemma 3.9. *The following statements are equivalent for an indecomposable nonprojective A-module M.*

1. *M belongs to $\mathcal{F}(\mathcal{E}_c)$.*
2. *There is an exact sequence $0 \to M \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ for some $n \geq 1$ such that each P_i belongs to $\mathcal{P}_c$.*
3. *There is an exact sequence $P_1 \to P_0 \to M \to 0$ such that P_i belongs to $\mathcal{P}_c$ for $i = 0, 1$.*

Proof. "(1) $\Rightarrow$ (2)" Let A' be a connected selfinjective Nakayama algebra. Recall that the syzygy functor Ω induces a bijective map on the finite set of isomorphism classes of indecomposable nonprojective A'-modules; compare [2, IV, Proposition 3.5]. Then for any indecomposable nonprojective A'-module M', there exists an exact sequence $0 \to M' \to P'_n \to \cdots \to P'_1 \to P'_0 \to M' \to 0$ of A'-modules for some $n \geq 1$ such that each P'_i is indecomposable projective. Then (2) follows from Proposition 3.8.

"(2) $\Rightarrow$ (3)" This is obvious.

"(3) $\Rightarrow$ (1)" Observe that $\operatorname{top} M = \operatorname{top} P_0$ and $\tau \operatorname{soc} M = \operatorname{top} \Omega(M) = \operatorname{top} P_1$, both of which belong to $\mathcal{S}_c$. Then by definition M belongs to $\mathcal{X}_c$. □

Recall that each component of the resolution quiver $R(A)$ has a unique cycle. For each noncyclic vertex S in $R(A)$, there exists a unique path of minimal length starting with S and ending in a cycle. We call the length of this path the distance between S and the cycle. Let $d = d(A)$ be the maximal distance between noncyclic vertices and cycles. Observe that $\gamma^d(S)$ is cyclic for each simple A-module S.

Lemma 3.10. *Let $d = d(A)$ be the maximal distance between noncyclic vertices and cycles in $R(A)$. Then the following statements hold.*

(1) *$\Omega^{2d}(M)$ belongs to $\mathcal{F}(\mathcal{E}_c)$ for any M in A-mod.*
(2) *$\Omega_0^{2d}(A\text{-mod}) \subseteq \mathcal{F}(\mathcal{E}_c) \subseteq \Omega^{2d}(A\text{-mod})$.*

Proof. (1) We may assume that M is indecomposable. It follows from Lemma 2.6 that either $\Omega^{2d}(M)$ is zero or $\operatorname{top} \Omega^{2d}(M) = \gamma^d(\operatorname{top} M)$. If $\Omega^{2d}(M)$ is indecomposable projective, then $\Omega^{2d}(M)$ belongs to $\mathcal{P}_c$. If $\Omega^{2d}(M)$ is indecomposable nonprojective, then $\operatorname{top} \Omega^{2d}(M) = \gamma^d(\operatorname{top} M)$ and $\tau \operatorname{soc} \Omega^{2d}(M) = \operatorname{top} \Omega^{2d+1}(M) = \gamma^d(\operatorname{top} \Omega(M))$. Then by definition M belongs to $\mathcal{X}_c$.

(2) The first inclusion follows from (1), and the second one is a direct consequence of Lemma 3.9. □

By Proposition 3.8 $\mathcal{F}(\mathcal{E}_c)$ is a Frobenius category whose projective objects are precisely $\operatorname{add} \mathcal{P}_c$. Denote by $\underline{\mathcal{F}}(\mathcal{E}_c)$ the stable category of $\mathcal{F}(\mathcal{E}_c)$ modulo projective objects. It is a triangulated category; see [11].

Recall from Proposition 3.5 that $\mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of A-mod which is closed under projective covers. Consider the inclusion functor $i : \mathcal{F}(\mathcal{E}_c) \to A\text{-mod}$. It induces uniquely a fully-faithful functor $i' : \underline{\mathcal{F}}(\mathcal{E}_c) \to A\text{-}\underline{\mathrm{mod}}$. We recall the induced functor $q' : A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$ in Section 2.

The following is the main result of this section, which describes the singularity category of A as a subcategory of the stable module category of A.

Theorem 3.11. *Let A be a connected Nakayama algebra of infinite global dimension. Then the composite functor $q' \circ i' : \underline{\mathcal{F}}(\mathcal{E}_c) \to A\text{-}\underline{\mathrm{mod}} \to \mathbf{D}_{\mathrm{sg}}(A)$ is a triangle equivalence.*

Proof. Observe that the composite functor $\mathcal{F}(\mathcal{E}_c) \xrightarrow{i} A\text{-mod} \to \mathbf{D}^{\mathrm{b}}(A\text{-mod}) \xrightarrow{q} \mathbf{D}_{\mathrm{sg}}(A)$ is a ∂-functor in the sense of [13, Section 1]; compare [5, Lemma 2.4]. Then the functor $q' \circ i'$ is a triangle functor; see [5, Lemma 2.5].

Recall that the subcategory $\mathcal{F}(\mathcal{E}_c)$ of A-mod is wide and closed under projective covers; moreover, $\mathcal{F}(\mathcal{E}_c)$ is a Frobenius category. Then the restriction of the syzygy functor

$\Omega : A\text{-}\underline{\mathrm{mod}} \to A\text{-}\underline{\mathrm{mod}}$ on $\underline{\mathcal{F}}(\mathcal{E}_c)$ is an autoequivalence, in particular, it is fully faithful. It follows from Lemma 2.3 that the canonical map $\underline{\mathrm{Hom}}_A(M, N) \to \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A)}(M, N)$ is an isomorphism for any M and N in $\mathcal{F}(\mathcal{E}_c)$. Observe that the canonical map is induced by the functor $q' \circ i'$. Then we infer that the functor $q' \circ i'$ is fully faithful.

It remains to show that the functor $q' \circ i'$ is also dense. Let X be an object in $\mathbf{D}_{\mathrm{sg}}(A)$. It follows from Lemmas 2.1 and 3.10(1) that there exists a module M in $\mathcal{F}(\mathcal{E}_c)$ and n sufficiently large such that $X \simeq q(M)[n]$ in $\mathbf{D}_{\mathrm{sg}}(A)$. By above, the image $\mathrm{Im}(q' \circ i')$ is a triangulated subcategory of $\mathbf{D}_{\mathrm{sg}}(A)$, in particular, it is closed under $[m]$ for all $m \in \mathbb{Z}$. It follows from $X \simeq q(M)[n]$ that X lies in $\mathrm{Im}(q' \circ i')$. This finishes our proof. $\square$

We observe the following immediate consequence of Proposition 3.8 and Theorem 3.11.

Corollary 3.12. *(Compare [8, Corollary 3.11].) Let A be a connected Nakayama algebra of infinite global dimension. Then there is a triangle equivalence between $\mathbf{D}_{\mathrm{sg}}(A)$ and $A'\text{-}\underline{\mathrm{mod}}$ for a connected selfinjective Nakayama algebra A'.*

Let A be a connected Nakayama algebra of infinite global dimension. The notion of *Gorenstein core* $\mathcal{C}(A)$ is introduced in [20]. It is a wide subcategory of A-mod, which is Frobenius. The stable category $\underline{\mathcal{C}}(A)$ is triangle equivalent to the stable category of finitely generated Gorenstein projective A-modules. The simple objects of $\mathcal{C}(A)$ are called *elementary Gorenstein projective modules*. It follows from [20, Proposition 3] that each elementary Gorenstein projective module belongs to $\mathcal{X}_c$, therefore $\mathcal{C}(A) \subseteq \mathcal{F}(\mathcal{E}_c)$. Moreover, if A is Gorenstein, then by [20, Proposition 5(a)] the elementary modules defined here coincide with the elementary Gorenstein projective modules, therefore $\mathcal{C}(A) = \mathcal{F}(\mathcal{E}_c)$.

The following example shows that the Gorenstein core $\mathcal{C}(A)$ and the Frobenius category $\mathcal{F}(\mathcal{E}_c)$ may not be equal in general.

Example 3.13. (Compare [20, Example in the introduction].) Let A be a connected Nakayama algebra with admissible sequence $(13, 13, 12, 12, 12)$. Let $\{S_1, S_2, S_3, S_4, S_5\}$ be a complete set of pairwise non-isomorphic simple A-modules such that $\tau S_i = S_{i+1}$ for $1 \le i \le 4$ and $\tau S_5 = S_1$. Then we have $l(P_1) = l(P_2) = 13$ and $l(P_3) = l(P_4) = l(P_5) = 12$. There is an arrow from S_i to S_j in $R(A)$ if and only if 5 divides $i - j + l(P_i)$. The resolution quiver $R(A)$ looks like:

$$\begin{tikzcd} S_1 \arrow[r, bend left] & S_4 \arrow[l, bend left] & S_3 \arrow[r] & S_5 \arrow[r, bend left] & S_2 \arrow[l, bend left] \end{tikzcd} .$$

There are four elementary modules: $E(S_1) = S_1$, $E(S_2)$ with composition factors given by S_2 and S_3, $E(S_4) = S_4$ and $E(S_5) = S_5$. Observe that there are only two elementary Gorenstein projective modules $E(1)$ and $E(4)$; see [20, Example in the introduction]. They are given such that $E(1)$ has composition factors S_1, S_2, S_3, and $E(4)$ has composition factors S_4, S_5. Therefore $\mathcal{C}(A) \ne \mathcal{F}(\mathcal{E}_c)$.

4. A duality between singularity categories

In this section, we prove that there is a triangle duality between the singularity category of a Nakayama algebra and the singularity category of its opposite algebra. The proof uses the Frobenius subcategory in the previous section.

Let A be a connected Nakayama algebra of infinite global dimension. We recall from Propositions 3.5 and 3.8 that the category $\mathcal{F} = \mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of A-mod closed under projective covers; it is equivalent to A'-mod for a connected selfinjective Nakayama algebra A'.

Consider the inclusion functor $i : \mathcal{F} \to A$-mod. We claim that it admits an exact right adjoint $i_\rho : A\text{-mod} \to \mathcal{F}$.

For the claim, recall from the proof of Proposition 3.8 that $A' = \mathrm{End}_A(P)^{\mathrm{op}}$ with $P = \oplus_{S \in \mathcal{S}_c} P(S)$. We identify $\mathcal{F}$ with A'-mod. Then the inclusion i is identified with $P \otimes_{A'} -$. The right adjoint is given by $i_\rho = \mathrm{Hom}_A(P, -)$. It is exact since ${}_A P$ is projective.

The adjoint pair (i, i_ρ) induces an adjoint pair (i^*, i_ρ^*) of triangle functors between bounded derived categories. Here, for an exact functor F between abelian categories, F^* denotes its extension on bounded derived categories.

Recall that $\mathbf{K}^{\mathrm{b}}(A\text{-inj})$ denotes the bounded homotopy category of A-inj. We view $\mathbf{K}^{\mathrm{b}}(A\text{-inj})$ as a thick subcategory of $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ via the canonical functor. We mention that by the usual duality on module categories, the quotient triangulated category $\mathbf{D}^{\mathrm{b}}(A\text{-mod})/\mathbf{K}^{\mathrm{b}}(A\text{-inj})$ is triangle equivalent to the opposite category of the singularity category $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ of A^{op}. Here, A^{op} is the opposite algebra of A.

The proof of the following result is similar to [8, Proposition 2.13]. Recall that $\mathcal{P}_c$ is a complete set of pairwise non-isomorphic indecomposable projective objects in $\mathcal{F} = \mathcal{F}(\mathcal{E}_c)$.

Lemma 4.1. *Let A be a connected Nakayama algebra of infinite global dimension. Then the above functors i_ρ^* and i^* induce mutually inverse triangle equivalences between $\mathbf{D}^{\mathrm{b}}(A\text{-mod})/\mathbf{K}^{\mathrm{b}}(A\text{-inj})$ and $\mathbf{D}^{\mathrm{b}}(\mathcal{F})/\mathbf{K}^{\mathrm{b}}(\mathrm{add}\,\mathcal{P}_c)$.*

Proof. Observe by [7, Lemma 3.3.1] that $i^* : \mathbf{D}^{\mathrm{b}}(\mathcal{F}) \to \mathbf{D}^{\mathrm{b}}(A\text{-mod})$ is fully faithful. It follows that its right adjoint i_ρ^* induces a triangle equivalence

$$\overline{i_\rho^*} : \mathbf{D}^{\mathrm{b}}(A\text{-mod})/\operatorname{Ker} i_\rho^* \simeq \mathbf{D}^{\mathrm{b}}(\mathcal{F});$$

see [9, Chapter I, Section 1, 1.3 Proposition]. Here, $\operatorname{Ker} F$ denotes the essential kernel of an additive functor F.

We claim that $\operatorname{Ker} i_\rho^* = \mathrm{thick}\langle \mathcal{S}_{nc} \rangle$, the smallest thick subcategory of $\mathbf{D}^{\mathrm{b}}(A\text{-mod})$ containing $\mathcal{S}_{nc}$. Here, we recall that $\mathcal{S}_{nc}$ denotes the set of noncyclic simple A-modules. By Lemma 2.5 each noncyclic simple A-module has finite injective dimension. It follows from the claim that $\operatorname{Ker} i_\rho^* \subseteq \mathbf{K}^{b}(A\text{-inj})$.

For the claim, we observe that $\operatorname{Ker} i_\rho = \mathcal{F}(\mathcal{S}_{nc})$, the full subcategory of A-mod formed by A-modules with an $\mathcal{S}_{nc}$-filtration. The claim follows from the fact that a complex X is in $\operatorname{Ker} i_\rho^*$ if and only if each cohomology $H^i(X)$ is in $\operatorname{Ker} i_\rho$.

We observe that i_ρ preserves injective objects since it has an exact left adjoint. It follows that $i_\rho^*(\mathbf{K}^{\mathrm{b}}(A\text{-inj})) \subseteq \mathbf{K}^{\mathrm{b}}(\operatorname{add}\mathcal{P}_c)$. By Lemma 3.6(1) each module Q in $\mathcal{P}_c$ has finite injective dimension. Note that $i_\rho^* Q = Q$. Therefore $i_\rho^*(\mathbf{K}^{\mathrm{b}}(A\text{-inj})) \supseteq \mathbf{K}^{\mathrm{b}}(\operatorname{add}\mathcal{P}_c)$, and thus $i_\rho^*(\mathbf{K}^{\mathrm{b}}(A\text{-inj})) = \mathbf{K}^{\mathrm{b}}(\operatorname{add}\mathcal{P}_c)$. From this equality, the triangle equivalence $\overline{i_\rho^*}$ restricts to a triangle equivalence

$$\mathbf{K}^{\mathrm{b}}(A\text{-inj})/\operatorname{Ker} i_\rho^* \simeq \mathbf{K}^{\mathrm{b}}(\operatorname{add}\mathcal{P}_c).$$

The desired equivalence follows from [22, Chapitre I, §2, 4-3 Corollaire]. $\square$

Proposition 4.2. *Let A be a Nakayama algebra. Then the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is triangle equivalent to $\mathbf{D}^{\mathrm{b}}(A\text{-mod})/\mathbf{K}^{\mathrm{b}}(A\text{-inj})$. Equivalently, there is a triangle duality between $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$.*

Proof. Recall from [1, Corollary 5] that A has finite global dimension if and only if A^{op} does. In this case, all the categories $\mathbf{D}_{\mathrm{sg}}(A)$, $\mathbf{D}^{\mathrm{b}}(A\text{-mod})/\mathbf{K}^{\mathrm{b}}(A\text{-inj})$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ are trivial.

Assume that A is a connected Nakayama algebra of infinite global dimension. Then the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is triangle equivalent to the stable category $\underline{\mathcal{F}}$ by Theorem 3.11.

Observe that the proof in [18, Theorem 2.1] is valid for any Frobenius abelian category. Since $\mathcal{F}$ is a Frobenius abelian category, it follows that the stable category $\underline{\mathcal{F}}$ is triangle equivalent to $\mathbf{D}^{\mathrm{b}}(\mathcal{F})/\mathbf{K}^{\mathrm{b}}(\operatorname{add}\mathcal{P}_c)$; see also [4,12,14]. Then the conclusion follows from Lemma 4.1. $\square$

Remark 4.3. Let A be a connected Nakayama algebra of infinite global dimension. Applying Corollary 3.12 to A and A^{op}, there are triangle equivalences $\mathbf{D}_{\mathrm{sg}}(A) \simeq A'\text{-}\underline{\mathrm{mod}}$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}}) \simeq (A'')^{\mathrm{op}}\text{-}\underline{\mathrm{mod}}$, where A', A'' are connected selfinjective Nakayama algebras; compare [8, Corollary 3.11]. However, we do not know a direct relation between A' and A''. On the other hand, by Proposition 4.2, they are stably equivalent.

The following example shows that there are algebras A such that the singularity categories $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ are neither triangle equivalent nor triangle dual. We recall the functor $q : \mathbf{D}^{\mathrm{b}}(A\text{-mod}) \to \mathbf{D}_{\mathrm{sg}}(A)$ in Section 2.

Example 4.4. Consider the following quiver Q and it opposite quiver Q^{op}.

$$Q: \begin{tikzcd} \underset{1}{\bullet} \arrow[loop left] & \underset{2}{\bullet} \arrow[l] \arrow[loop above] \arrow[loop below] \end{tikzcd}, \qquad Q^{\mathrm{op}}: \begin{tikzcd} \underset{1'}{\bullet} \arrow[loop left] \arrow[r] & \underset{2'}{\bullet} \arrow[loop above] \arrow[loop below] \end{tikzcd}.$$

Let k be a field. Let A be the finite dimensional algebra over k which is of radical square zero given by Q. Denote S_i the simple A-module corresponding to the vertex i. It follows from [6, Proposition 2.5] that $\mathbf{D}_{\mathrm{sg}}(A) = \mathrm{add}\, q(S_1 \oplus S_2)$. We observe that $\mathbf{D}_{\mathrm{sg}}(A)$ has a unique nontrivial thick triangulated subcategory $\mathrm{add}\, q(S_1)$ which is Hom-finite. Here, we recall that a k-linear category $\mathcal{C}$ is *Hom-finite* if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a finite-dimensional k-vector space for any $X, Y \in \mathcal{C}$.

The opposite algebra A^{op} of A is of radical square zero given by Q^{op}. We have $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}}) = \mathrm{add}\, q(S_{1'} \oplus S_{2'})$ and that $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ has a unique nontrivial thick triangulated subcategory $\mathrm{add}\, q(S_{2'})$. We observe that the category $\mathrm{add}\, q(S_{2'})$ is not Hom-finite.

Therefore, we conclude that the singularity categories $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ are neither triangle equivalent nor triangle dual.

5. The resolution quivers

Let A be a connected Nakayama algebra of infinite global dimension. Recall that $n(A)$ denotes the number of isomorphism classes of simple A-modules. By Corollary 3.12 the Auslander–Reiten quiver of the singularity category $\mathbf{D}_{\mathrm{sg}}(A)$ is isomorphic to a truncated tube $\mathbb{Z}\mathbb{A}_m/\langle \tau^t \rangle$, where $m = m(A)$ denotes its height and $t = t(A)$ denotes its rank. Here, we use the fact that the Auslander–Reiten quiver of the stable module category of a connected selfinjective Nakayama algebra is a truncated tube; compare [2, VI.2].

Recall that $R(A)$ denotes the resolution quiver of A. We denote by $c(A)$ the number of cycles in $R(A)$. Let C be a cycle in $R(A)$. Then the *size* $s(C)$ of C is the number of vertices in C, and the *weight* $w(C)$ of C is $\frac{\sum_S l(P(S))}{n(A)}$, where S runs though all vertices of C. Here, $l(P(S))$ is the composition length of the projective cover $P(S)$ of a simple A-module S. Recall from [21] that all cycles in the resolution quiver $R(A)$ have the same size and the same weight. We denote $s(A) = s(C)$ and $w(A) = w(C)$ for an arbitrary cycle C in $R(A)$.

For two positive integers a and b, we denote their greatest common divisor by (a, b).

Lemma 5.1. *Let $m = m(A)$ and $t = t(A)$ be as above. Then $c(A) = (m + 1, t)$, $s(A) = \frac{t}{(m+1,t)}$ and $w(A) = \frac{m+1}{(m+1,t)}$.*

Proof. Recall from [8, Theorem 3.8] that there exists a sequence of algebra homomorphisms

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \to \cdots \to A_{r-1} \xrightarrow{\eta_{r-1}} A_r$$

such that each A_i is a connected Nakayama algebra and A_r is selfinjective; moreover, each η_i induces a triangle equivalence between $\mathbf{D}_{\mathrm{sg}}(A_i)$ and $\mathbf{D}_{\mathrm{sg}}(A_{i+1})$. Following [21, Lemma 2.2], each η_i induces a bijection between the set of cycles in $R(A_i)$ and the set of cycles in $R(A_{i+1})$, which preserves sizes and weights. Then we have $m(A_i) = m(A_{i+1})$, $t(A_i) = t(A_{i+1})$, $c(A_i) = c(A_{i+1})$, $s(A_i) = s(A_{i+1})$ and $w(A_i) = w(A_{i+1})$ for $0 \leq i \leq r - 1$. Therefore it is enough to prove the equations for selfinjective Nakayama algebras.

Let A be a connected selfinjective Nakayama algebra. Then t equals the number of isomorphism classes of simple A-modules, and $m + 1$ equals the radical length of A. We claim that $s(A) = \frac{t}{(m+1,t)}$. Therefore, we have $c(A) = \frac{t}{s(A)} = (m+1, t)$ and $w(A) = \frac{(m+1)s(A)}{t} = \frac{m+1}{(m+1,t)}$.

For the claim, let $\{S_1, \cdots, S_t\}$ be a complete set of pairwise non-isomorphic simple A-modules such that $\tau S_i = S_{i+1}$ for $1 \leq i \leq t$. Here, we let $S_{t+j} = S_j$ for each $j > 0$. Then we have $\gamma(S_i) = S_{i+m+1}$. Therefore $\gamma^d(S_i) = S_i$ if and only if t divides $d(m+1)$. Since t divides $d(m+1)$ if and only if d is a multiple of $\frac{t}{(m+1,t)}$, it follows that $R(A)$ consists of cycles of size $\frac{t}{(m+1,t)}$. □

The following result establishes the relationship between the resolution quiver of a Nakayama algebra and the resolution quiver of its opposite algebra.

Proposition 5.2. *Let A be a connected Nakayama algebra of infinite global dimension. Then the following statements hold.*

(1) *The resolution quivers $R(A)$ and $R(A^{\mathrm{op}})$ have the same number of cycles and the same number of cyclic vertices.*
(2) *All cycles in $R(A)$ and $R(A^{\mathrm{op}})$ have the same weight.*

Proof. By Proposition 4.2 there is a triangle duality between $\mathbf{D}_{\mathrm{sg}}(A)$ and $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$. Then the Auslander–Reiten quiver of $\mathbf{D}_{\mathrm{sg}}(A^{\mathrm{op}})$ is isomorphic to the opposite quiver of the Auslander–Reiten quiver of $\mathbf{D}_{\mathrm{sg}}(A)$. Therefore, we have $m(A) = m(A^{\mathrm{op}})$ and $t(A) = t(A^{\mathrm{op}})$. It follows from Lemma 5.1 that $c(A) = c(A^{\mathrm{op}})$, $s(A) = s(A^{\mathrm{op}})$ and $w(A) = w(A^{\mathrm{op}})$. □

Remark 5.3. The proof of Proposition 5.2 uses the singularity categories. In fact, we already know, without using the singularity categories, that the resolution quivers $R(A)$ and $R(A^{\mathrm{op}})$ have the same number of cyclic vertices; compare [21]. However, we do not know a direct proof of the fact that they have the same number of cycles.

The following example shows that these two resolution quivers $R(A)$ and $R(A^{\mathrm{op}})$ may not be isomorphic in general.

Example 5.4. Let B be a connected Nakayama algebra with admissible sequence $(7, 6, 6, 5)$. Let $\{S_1, S_2, S_3, S_4\}$ be a complete set of pairwise non-isomorphic simple B-modules such that $\tau S_i = S_{i+1}$ for $1 \leq i \leq 3$ and $\tau S_4 = S_1$. Then we have $l(P_1) = 7$, $l(P_2) = 6$, $l(P_3) = 6$ and $l(P_4) = 5$. There is an arrow from S_i to S_j in $R(A)$ if and only if 4 divides $i - j + l(P_i)$.

Denote by D the usual duality, and by $(-)^*$ the duality on finitely generated projective modules. Then $\{DS_4, DS_3, DS_2, DS_1\}$ is a complete set of pairwise non-isomorphic simple B^{op}-modules such that $\tau' DS_i = DS_{i-1}$ for $2 \leq i \leq 4$ and $\tau' D(S_1) = D(S_4)$. Here, τ' is the Auslander–Reiten translation of B^{op}. We observe that $l(P_4^*) = 6$, $l(P_3^*) = 7$,

$l(P_2^*) = 6$ and $l(P_1^*) = 5$. Therefore the admissible sequence of B^{op} is $(6, 7, 6, 5)$. There is an arrow from DS_i to DS_j in $R(B^{\mathrm{op}})$ if and only if 4 divides $i - j - l(P_i^*)$.

The resolution quivers $R(B)$ and $R(B^{\mathrm{op}})$ are shown as follows.

$$\begin{tikzcd} S_3 \arrow[r] & S_1 \arrow[r, bend left] & S_4 \arrow[l, bend left] & S_2 \arrow[l] \end{tikzcd}, \qquad \begin{tikzcd} DS_1 \arrow[dr] & & \\ & DS_4 \arrow[r, bend left] & DS_2 \arrow[l, bend left] \\ DS_3 \arrow[ur] & & \end{tikzcd}.$$

We mention that, for the algebra A in Example 3.13, the resolution quivers $R(A)$ and $R(A)^{\mathrm{op}}$ are isomorphic.

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