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A NOTE ON RESOLUTION QUIVERS

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Recently, Ringel introduced the resolution quiver for a connected Nakayama algebra. It is known that each connected component of the resolution quiver has a unique cycle. We prove that all cycles in the resolution quiver are of the same size. We introduce the notion of weight for a cycle in the resolution quiver. It turns out that all cycles have the same weight.

Keywords: Resolution quiver; Nakayama algebra; left retraction.

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1. Introduction

Let A be a connected Nakayama algebra without simple projective modules. All modules are left modules of finite length. We denote the number of simple A-modules by $n(A)$. Following [5], let $\gamma(S) = \tau \mathrm{soc} P(S)$ for a simple A-module S, where $P(S)$ is the projective cover of S and $\tau = \mathrm{DTr}$ is the Auslander–Reiten translation [1]. Ringel [5] defined the *resolution quiver* $R(A)$ of A as follows: the vertices correspond to simple A-modules and there is an arrow from S to $\gamma(S)$ for each simple A-module S. The resolution quiver gives a fast algorithm to decide whether A is a Gorenstein algebra or not, and whether it is CM-free or not; see [5].

Using the map f introduced in [3], the notion of resolution quiver applies to any connected Nakayama algebra. It is known that each connected component of $R(A)$ has a unique cycle.

Let A be a connected Nakayama algebra and C be a cycle in $R(A)$. Assume that the vertices of C are $S_1, S_2, \ldots, S_m$. We define the *weight* of C to be $\frac{\sum_{k=1}^{m} c_k}{n(A)}$, where c_k is the length of the projective cover of S_k. The aim of this note is to prove the following result.

Proposition 1.1. *Let A be a connected Nakayama algebra. Then all cycles in its resolution quiver are of the same size and of the same weight.*

As a consequence of Proposition 1.1, if the resolution quiver has a loop, then all cycles are loops; this result is obtained by Ringel [5, 6]. The proof of Proposition 1.1 uses *left retractions* of Nakayama algebras studied in [2].

2. The Proof of Proposition 1.1

Let A be a connected Nakayama algebra. Recall that $n = n(A)$ is the number of simple A-modules. Let $S_1, S_2, \ldots, S_n$ be a complete set of pairwise non-isomorphic simple A-modules and P_i be the projective cover of S_i. We require that $\mathrm{rad}P_i$ is a factor module of P_{i+1}. Here, we identify $n+1$ with 1.

Recall that $\mathbf{c}(A) = (c_1, c_2, \ldots, c_n)$ is an *admissible sequence* for A, where c_i is the length of P_i; see [1, Chap. IV. 2]. We denote $p(A) = \min\{c_1, c_2, \ldots, c_n\}$. The algebra A is called a *line algebra* if $c_n = 1$ or, equivalently, the valued quiver of A is a line; otherwise, A is called a *cycle algebra* or, equivalently, the valued quiver of A is a cycle. Then A is a cycle algebra if and only if A has no simple projective modules.

Following [3], we introduce a map $f_A : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that n divides $f_A(i) - (c_i + i)$ for $1 \leq i \leq n$. The *resolution quiver* $R(A)$ of A is defined as follows: its vertices are $1, 2, \ldots, n$ and there is an arrow from i to $f_A(i)$. Observe that for a cycle algebra A we have $\gamma(S_i) = S_{f_A(i)}$. Then by identifying i with S_i, the resolution quiver $R(A)$ coincides with that in [5].

Assume that A is a cycle algebra which is not self-injective. After possible cyclic permutations, we may assume that its admissible sequence $\mathbf{c}(A) = (c_1, c_2, \ldots, c_n)$ is *normalized* [2], that is, $p(A) = c_1 = c_n - 1$. Recall from [2] that there is an algebra homomorphism $\eta : A \to L(A)$ with $L(A)$ a connected Nakayama algebra such that its admissible sequence $\mathbf{c}(L(A)) = (c'_1, c'_2, \ldots, c'_{n-1})$ is given by $c'_i = c_i - [\frac{c_i+i-1}{n}]$ for $1 \leq i \leq n-1$; in particular, $n(L(A)) = n(A) - 1$. Here, for a real number x, $[x]$ denotes the largest integer not greater than x. The algebra homomorphism η is called the *left retraction* [2] of A with respect to S_n.

We introduce a map $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n-1\}$ such that $\pi(i) = i$ for $i < n$ and $\pi(n) = 1$. The following result is contained in the proof of [2, Lemma 3.7].

Lemma 2.1. *Let A be a cycle algebra which is not self-injective. Then $\pi f_A(i) = f_{L(A)}\pi(i)$ for $1 \leq i \leq n$.*

Proof. Let $c_i + i = kn + j$ with $k \in \mathbb{N}$ and $1 \leq j \leq n$. In particular, $f_A(i) = j$.

For $i < n$, we have

$$c'_{\pi(i)} + i = c_i + i - \left[\frac{c_i + i - 1}{n}\right] = kn + j - \left[\frac{kn + j - 1}{n}\right] = k(n-1) + j. \qquad (1)$$

Then $\pi f_A(i) = \pi(j)$ and $f_{L(A)}\pi(i) = f_{L(A)}(i) = \pi(j)$.

For $i = n$, we have

$$\begin{aligned} c'_{\pi(n)} + n &= c_n - 1 + n - \left[\frac{c_n - 1}{n}\right] \\ &= kn + j - 1 - \left[\frac{kn + j - n - 1}{n}\right] = k(n-1) + j. \end{aligned} \tag{2}$$

Then $\pi f_A(n) = \pi(j)$ and $f_{L(A)}\pi(n) = f_{L(A)}(1) = \pi(j)$. □

The previous lemma gives rise to a unique morphism of resolution quivers

$$\tilde{\pi} : R(A) \to R(L(A))$$

such that $\tilde{\pi}(i) = \pi(i)$. Then $\tilde{\pi}$ sends the unique arrow from i to $f_A(i)$ to the unique arrow in $R(L(A))$ from $\pi(i)$ to $f_{L(A)}\pi(i) = \pi f_A(i)$. The morphism $\tilde{\pi}$ identifies the vertices 1 and n as well as the arrows starting from 1 and n. Because 1 and n are in the same connected component of $R(A)$, we infer that $R(A)$ and $R(L(A))$ have the same number of connected components.

Let A be a connected Nakayama algebra and C be a cycle in $R(A)$. The *size* of C is the number of vertices in C. We recall that the *weight* of C is given by $w(C) = \frac{\sum_k c_k}{n(A)}$, where k runs over all vertices in C. We mention that $w(C)$ is an integer; see (3). A vertex in $R(A)$ is said to be *cyclic* provided that it belongs to a cycle.

Lemma 2.2. *Let A be a cycle algebra which is not self-injective. Then $\tilde{\pi}$ induces a bijection between the set of cycles in $R(A)$ and the set of cycles in $R(L(A))$, which preserves sizes and weights.*

Proof. We observe that for two vertices x and y in $R(A)$, $\pi(x) = \pi(y)$ if and only if $x = y$ or $\{x, y\} = \{1, n\}$. Note that $f_A(1) = f_A(n)$. So the vertices 1 and n are in the same connected component of $R(A)$ and they are not cyclic at the same time.

Let C be a cycle in $R(A)$ with vertices $x_1, x_2, \ldots, x_s$ such that $x_{i+1} = f_A(x_i)$. Here, we identify $s+1$ with 1. Since the vertices 1 and n are not cyclic at the same time, we have that $\pi(x_1), \pi(x_2), \ldots, \pi(x_s)$ are pairwise distinct and $\tilde{\pi}(C)$ is a cycle in $R(L(A))$. Hence $\tilde{\pi}$ induces a map from the set of cycles in $R(A)$ to the set of cycles in $R(L(A))$. Obviously the map is injective. On the other hand, recall that $R(L(A))$ and $R(A)$ have the same number of connected components, thus they have the same number of cycles. Hence $\tilde{\pi}$ induces a bijection between the set of cycles in $R(A)$ and the set of cycles in $R(L(A))$ which preserves sizes.

It remains to prove that $w(C) = w(\tilde{\pi}(C))$. We assume that $c_{x_i} + x_i = k_i n + x_{i+1}$ with $k_i \in \mathbb{N}$. Then we have

$$w(C) = \frac{\sum_{i=1}^{s} c_{x_i}}{n} = \sum_{i=1}^{s} k_i. \tag{3}$$

Recall that $n(L(A)) = n - 1$. We note that $c'_{\pi(x_i)} + x_i = k_i(n-1) + x_{i+1}$; see (1) and (2). Hence $\sum_{i=1}^{s} c'_{\pi(x_i)} = (n-1)\sum_{i=1}^{s} k_i$ and the assertion follows. □

Recall from [2, Theorem 3.8] that there exists a sequence of algebra homomorphisms

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \to \cdots \to A_{r-1} \xrightarrow{\eta_{r-1}} A_r \tag{4}$$

such that each A_i is a connected Nakayama algebra, $\eta_i : A_i \to A_{i+1}$ is a left retraction and A_r is self-injective.

We now prove Proposition 1.1.

Proof of Proposition 1.1. Assume that A is a connected self-injective Nakayama algebra with $n(A) = n$ and admissible sequence $\mathbf{c}(A) = (c, c, \ldots, c)$. Then a direct calculation shows that $R(A)$ consists entirely of cycles and each cycle is of size $\frac{n}{(n,c)}$ and of weight $\frac{c}{(n,c)}$, where (n, c) is the greatest common divisor of n and c. In particular, all cycles in $R(A)$ are of the same size and of the same weight.

In general, let A be a connected Nakayama algebra whose admissible sequence is $\mathbf{c}(A) = (c_1, c_2, \ldots, c_n)$. Take A' to be a connected Nakayama algebra with admissible sequence $\mathbf{c}(A') = (c_1 + n, c_2 + n, \ldots, c_n + n)$. Then $R(A) = R(A')$ and for any cycle C in $R(A)$, the corresponding cycle C' in $R(A')$ satisfies $w(C') = w(C) + s(C)$, where $s(C)$ denotes the size of C. The statement for A holds if and only if it holds for A'.

We now assume that A is a connected Nakayama algebra with $p(A) > n(A)$. One proves by induction that each A_i in the sequence (4) satisfies $p(A_i) > n(A_i)$. In particular, each A_i is a cycle algebra. We can apply Lemma 2.2 repeatedly. Then the statement for A follows from the statement for the self-injective Nakayama algebra A_r, which is already proved above. □

We conclude this note with a consequence of the above proof.

Corollary 2.3. *Let A be a connected Nakayama algebra of infinite global dimension. Then we have the following statements*:

(1) *The number of cyclic vertices of the resolution quiver $R(A)$ equals the number of simple A-modules of infinite projective dimension.*
(2) *The number of simple A-modules of infinite projective dimension equals the number of simple A-modules of infinite injective dimension.*

Proof. (1) All the algebras A_i in the sequence (4) have infinite global dimension; see [2, Lemma 2.4]. In particular, they are cycle algebras. We apply Lemma 2.2 repeatedly and obtain a bijection between the set of cyclic vertices of $R(A)$ and the set of cyclic vertices of $R(A_r)$. Recall that all vertices of $R(A_r)$ are cyclic, and $n(A_r)$ equals $n(A)$ minus the number of simple A-modules of finite projective dimension; see [2, Theorem 3.8]. Then the statement follows immediately.

(2) Recall from [4, Corollary 3.6] that a simple A-module S is cyclic in $R(A)$ if and only if S has infinite injective dimension. Then (2) follows from (1). □

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References

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, Vol. 36 (Cambridge University Press, Cambridge, 1995).

[2] X.-W. Chen and Y. Ye, Retractions and Gorenstein homological properties, *Algebr. Represent. Theory*, doi: 10.1007/s10468-013-9415-1.

[3] W. H. Gustafson, Global dimension in serial rings, *J. Algebra* **97** (1985) 14–16.

[4] D. Madson, Projective dimensions and Nakayama algebras, in *Representations of Algebras and Related Topics*, Fields Institute Communications, Vol. 45 (American Mathematical Society, Providence, RI, 2005), pp. 247–265.

[5] C. M. Ringel, The Gorenstein projective modules for the Nakayama algebras. I, *J. Algebra* **385** (2013) 241–261.

[6] C. M. Ringel, The Gorenstein projective modules for the Nakayama algebras. II, in preparation.