Arch. Math. 108 (2017), 251–261

0003-889X/17/030251-11
published online January 27, 2017
DOI 10.1007/s00013-016-1016-x

Archiv der Mathematik

A note on homological properties of Nakayama algebras

Dawei Shen

Abstract. Using the resolution quiver for a connected Nakayama algebra, a fast algorithm is given to decide whether its global dimension is finite or not and whether it is Gorenstein or not. The latter strengthens a result of Ringel.

Mathematics Subject Classification. Primary 16G20; Secondary 13E10.

Keywords. Nakayama algebra, Resolution quiver, Cartan matrix.

1. Introduction. Let A be a connected Nakayama algebra. Following [4,7,8], its *resolution quiver* is defined as follows: the vertex set is the set of non-isomorphic simple A-modules; there is a unique arrow from each simple A-module S to $\gamma(S) = \tilde{\tau}\operatorname{soc} P(S)$. Here, $P(S)$ is the projective cover of S and 'soc' is the socle of a module. If A has a simple projective module, denote by S_{inj} the unique simple injective A-module up to isomorphism. Then

$$\tilde{\tau}(S) = \begin{cases} \tau(S) & \text{if } S \text{ is not projective} \\ S_{\text{inj}} & \text{otherwise} \end{cases}$$

for each simple A-module S, where τ is the Auslander–Reiten translation [1].

Let A be a connected Nakayama algebra. Denote by $R(A)$ its resolution quiver. It is known that each connected component of $R(A)$ has a unique cycle. For a cycle C in $R(A)$ with vertices $S_1, S_2, \ldots, S_m$, the *weight* $w(C)$ of C is $\sum_{k=1}^{m} \frac{c_k}{n}$. Here, n is the number of non-isomorphic simple A-modules and c_k is the composition length of $P(S_k)$. It turns out that $w(C)$ is an integer and all cycles in $R(A)$ have the same weight [8]. The weight $w(C)$ is called the weight of the algebra A.

The resolution quiver is a very efficient tool for investigating the homological properties of Nakayama algebras. The Gorenstein projective modules for Nakayama algebras are described by resolution quivers [7]. Resolution quivers are also used to study the singularity categories of Nakayama algebras [9].

The resolution quiver of a connected Nakayama algebra gives a fast algorithm to decide whether it is Gorenstein or not and whether it is CM-free or not [7]. In this paper, we show that the resolution quiver of a connected Nakayama algebra also gives a fast algorithm to decide whether its global dimension is finite or not.

More precisely, we have the following.

Proposition 1.1. *Let A be a connected Nakayama algebra. Then A has finite global dimension if and only if its resolution quiver is connected and its weight is* 1.

As a consequence of Proposition 1.1, the resolution quiver is connected for a connected Nakayama algebra with finite global dimension.

For a connected Nakayama algebra, recall from [7] that a cycle in its resolution quiver is called *black* provided that the projective dimension of each simple module on this cycle is not equal to 1.

The following result strengthens [7, Proposition 5(a)].

Proposition 1.2. *Let A be a connected Nakayama algebra with infinite global dimension. Then A is a Gorenstein algebra if and only if all cycles in its resolution quiver are black.*

Let A be a connected Nakayama algebra. Take a complete set $\{S_1, S_2, \ldots, S_n\}$ of pairwise non-isomorphic simple A-modules. The *Cartan matrix* $\mathbf{C}_A$ of A is an $n \times n$ matrix (c_{ij}), where c_{ij} is the number of copies of S_i appearing in a composition series for the projective cover of S_j. Denote by $\mathbf{C}_A^T$ the transpose of $\mathbf{C}_A$.

Denote by c the number of cycles and by b the number of black cycles in the resolution quiver of A.

The following result gives a connection between Cartan matrices and resolution quivers.

Proposition 1.3. *Let A be a connected Nakayama algebra.*

(1) *The rank of $\mathbf{C}_A$ is $n + 1 - c$.*
(2) *If b is nonzero, then the rank of $(\mathbf{C}_A, \mathbf{C}_A^T)$ is $n + 1 - b$.*

The paper is organised as follows. The proofs of Proposition 1.1 and Proposition 1.2 are given in Sections 2 and 3, respectively. In Section 4, we study the connection between Cartan matrices and resolution quivers for Nakayama algebras and prove Proposition 1.3.

2. Retractions and resolution quivers. Let A be a connected Nakayama algebra. Denote by $n = n(A)$ the number of non-isomorphic simple A-modules. Take a sequence $(S_1, S_2, \ldots, S_n)$ of pairwise non-isomorphic simple A-modules such that the radical of P_i is a factor module of P_{i+1} for $1 \leq i \leq n-1$ and the radical of P_n is a factor module of P_1. Here, P_i is the projective cover of S_i. Denote by c_i the composition length of P_i. The *admissible sequence* for A is given by $\mathbf{c}(A) = (c_1, c_2, \ldots, c_n)$; see [1, Chapter IV. 2].

Following [4], there exists a map $f_A \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that n divides $f_A(i) - (c_i + i)$. For $1 \leq i \leq n$, we have $\gamma(S_i) = S_{f_A(i)}$. Then for

$1 \leq i, j \leq n$, there is an arrow $S_i \to S_j$ in the resolution quiver of A if and only if $f_A(i) = j$.

Suppose now that A is not selfinjective. If A has no simple projective modules, after possible cyclic permutations, we may assume that its admissible sequence is *normalized* [3], that is, $p(A) = c_1 = c_n - 1$. Here, $p(A)$ is the minimal integer among c_i. For convenience, if A has a simple projective module, its admissible sequence is always normalized.

Following [3], there exists a *left retraction* $\eta\colon A \to L(A)$, where $L(A)$ is a connected Nakayama algebra with admissible sequence $\mathbf{c}(L(A)) = (c'_1, c'_2, \ldots, c'_{n-1})$ such that $c'_i = c_i - \left[\frac{c_i+i-1}{n}\right]$ for $1 \leq i \leq n-1$. In particular, $n(L(A)) = n(A) - 1$. Here, $[x]$ is the largest integer not greater than a real number x. The corresponding sequence of simple $L(A)$-modules is denoted by $(S'_1, S'_2, \ldots, S'_{n-1})$.

We need the map $\pi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n-1\}$ such that $\pi(i) = i$ for $1 \leq i \leq n-1$ and $\pi(n) = 1$.

Recall from [8, Lemma 2.1] the following result. For the convenience of the reader, we give a proof here.

Lemma 2.1. *Let A be a connected Nakayama algebra which is not selfinjective. Then $\pi f_A(i) = f_{L(A)}(i)$ for $1 \leq i \leq n-1$.*

Proof. Let $f_A(i) = j$ and $c_i + i = kn + j$ with $k \in \mathbb{N}$. For $1 \leq i \leq n-1$, we have

$$c'_i + i = c_i + i - \left[\frac{c_i + i - 1}{n}\right] = kn + j - \left[\frac{kn + j - 1}{n}\right] = k(n-1) + j. \quad (2.1)$$

It follows that $\pi f_A(i) = \pi(j) = f_{L(A)}(i)$. $\square$

Denote by γ' the map $\tilde{\tau}\,\mathrm{soc}\,P(-)$ for $L(A)$. It follows from Lemma 2.1 that $\gamma'(S'_i) = S'_{\pi f_A(i)}$ for $1 \leq i \leq n-1$. Hence the resolution quiver of $L(A)$ can be obtained from the resolution quiver of A just by "merging" the vertices S_1 and S_n.

Observe that $\gamma(S_n) = \gamma(S_1)$ if A has no simple projective modules and $\gamma(S_n) = S_1$ if A has a simple projective module. In particular, the vertices S_1 and S_n lie on the same connected component in the resolution quiver $R(A)$ of A. Then $R(A)$ and $R(L(A))$ have the same number of connected components. It follows that they have the same number of cycles since each connected component of a resolution quiver has a unique cycle.

Let C be a cycle with vertices $S_{x_1}, S_{x_2}, \ldots, S_{x_s}$ in $R(A)$. The weight $w(C)$ of C is $\sum_{k=1}^{s} \frac{c_{x_k}}{n(A)}$ and the size of C is the number of vertices on C.

The following result strengthens [8, Lemma 2.2], where the connected Nakayama algebra is required to have no simple projective modules.

Lemma 2.2. *Let A be a connected Nakayama algebra which is not selfinjective. Then there exists a weight preserving bijection between the set of cycles in $R(A)$ and the set of cycles in $R(L(A))$. Moreover, if A has no simple projective modules or the simple projective A-module does not lie on a cycle in $R(A)$, then the bijection also preserves the size.*

Proof. If A has no simple projective modules, then the bijection follows from [8, Lemma 2.2]. We may assume that A has a simple projective module S_n.

Let C be a cycle with vertices $S_{x_1}, S_{x_2}, \dots, S_{x_s}$ in $R(A)$. Assume $x_{i+1} = f_A(x_i)$. Here, we identify x_{s+1} with x_1. Let $c_{x_i} + x_i = k_i n + x_{i+1}$ with $k_i \in \mathbb{N}$. Then

$$w(C) = \frac{\sum_{i=1}^{s} c_{x_i}}{n} = \sum_{i=1}^{s} k_i.$$

It follows from (2.1) that for $x_i < n$, we have $c'_{x_i} + x_i = k_i(n-1) + x_{i+1}$.

There exist two cases:

Case 1. S_n does not lie on C. Then $S'_{x_1}, S'_{x_2}, \dots, S'_{x_s}$ form a cycle C' in $R(L(A))$.

Observe that $c'_{x_i} + x_i = k_i(n-1) + x_{i+1}$ for $1 \leq i \leq s$. Then

$$\sum_{i=1}^{s} c'_{x_i} = (n-1) \sum_{i=1}^{s} k_i.$$

Therefore, $w(C) = w(C')$.

Case 2. S_n lies on C. Since the admissible sequence of A is normalized, we have $x_s = n$ and $x_1 = 1$. It follows that $S'_{x_1}, S'_{x_2}, \dots, S'_{x_{s-1}}$ form a cycle C'' in $R(L(A))$.

Observe that $k_s = 1$ and $c'_{x_i} + x_i = k_i(n-1) + x_{i+1}$ for $1 \leq i \leq s-1$. Then

$$\sum_{i=1}^{s-1} c'_{x_i} = (n-1) \sum_{i=1}^{s-1} k_i + n - 1 = (n-1) \sum_{i=1}^{s} k_i.$$

Therefore, $w(C) = w(C'')$.

We have shown that there exists an injective weight preserving map from the set of cycles in $R(A)$ to the set of cycles in $R(L(A))$. Since $R(L(A))$ and $R(A)$ have the same number of cycles, this map is also surjective. This finishes our proof. $\square$

Recall from [3, Theorem 3.8] that there exists a sequence of left retractions

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \longrightarrow \cdots \longrightarrow A_{r-1} \xrightarrow{\eta_{r-1}} A_r \tag{2.2}$$

such that each A_i is a connected Nakayama algebra, each $\eta_i \colon A_i \to A_{i+1}$ is a left retraction, and A_r is selfinjective; the global dimension of A is finite if and only if A_r is simple. Note that a similar argument has already appeared in [5, end of Section 3].

For a connected Nakayama A, denote by $c(A)$ the number of cycles and by $w(A)$ the weight of a cycle in $R(A)$. We mention that $w(A)$ is an integer and all cycles in $R(A)$ have the same weight; see [8].

We now prove Proposition 1.1.

Proof of Proposition 1.1. Applying Lemma 2.2 to (2.2) repeatedly, we obtain $c(A) = c(A_r)$ and $w(A) = w(A_r)$.

Now A_r is a connected selfinjective Nakayama algebra. Denote by m its radical length. Then the admissible sequence of A_r is $(m, m, \dots, m)$. It is routine to show that the resolution quiver of A_r consists of $\gcd(m, n)$ cycles of the same weight $\frac{m}{\gcd(m,n)}$. Here, 'gcd' is the greatest common divisor.

Since A_r is a connected Nakayama algebra, we infer that A_r is simple if and only if $m = c(A_r)w(A_r) = 1$. It follows that A_r is simple if and only if $c(A_r) = 1$ and $w(A_r) = 1$. Then the global dimension of A is finite if and only if $c(A) = 1$ and $w(A) = 1$. Each connected component of a resolution quiver has a unique cycle. Then $c(A) = 1$ if and only if the resolution quiver of A is connected. $\square$

3. Two maps on simple modules. Let A be a connected Nakayama algebra. For an A-module M, denote by $\operatorname{pd} M$ its projective dimension and by $\operatorname{id} M$ its injective dimension. Recall that the *syzygy* $\Omega(M)$ of M is the kernel of its projective cover $p\colon P(M) \longrightarrow M$. Dually, the *cosyzygy* $\Omega^{-1}(M)$ of M is the cokernel of its injective envelope $i\colon M \longrightarrow I(M)$.

It is known that A has a simple projective module if and only if it has a simple injective module. In this case, denote by S_{inj} the unique simple injective A-module up to isomorphism. Then there exists a map $\tilde{\tau}$ defined by

$$\tilde{\tau}(S) = \begin{cases} \tau(S) & \text{if } S \text{ is not projective} \\ S_{\text{inj}} & \text{otherwise} \end{cases}$$

for each simple A-module S, where τ is the Auslander-Reiten translation [1].

Take a complete set $\mathcal{S}$ of pairwise non-isomorphic simple A-modules. Recall from [4,7] that there exist two maps $\gamma, \psi\colon \mathcal{S} \to \mathcal{S}$ given by

$$\gamma(S) = \tilde{\tau} \operatorname{soc} P(S) \text{ and } \psi(S) = \tilde{\tau}^{-1} \operatorname{top} I(S),$$

for each S in $\mathcal{S}$. Here, 'soc' is the socle and 'top' is the top of a module.

The map γ determines the resolution quiver for A. Denote by A^{op} the opposite algebra of A and by γ^{op} the map $\tilde{\tau} \operatorname{soc} P(-)$ for A^{op}. Then $D\psi(S) = \gamma^{\text{op}}(DS)$ for each S in $\mathcal{S}$, where D is the usual dual for finitely generated A-modules. Hence the resolution quiver of A^{op} is isomorphic to the quiver determined by the map ψ.

The following terminology is taken from [7]. A simple A-module S is called *γ-black* provided that $\operatorname{pd} S$ is not equal to 1; it is called *γ-cyclic* provided that $\gamma^m(S) = S$ for some integer $m > 0$. Dually, one can define ψ-black and ψ-cyclic simple A-modules.

We need the following lemma.

Lemma 3.1. *Let S and T be simple A-modules.*

(1) *If S is not projective, then $\psi(T) = S$ if and only if T is a composition factor in $\Omega^2(S)$.*
(2) *S is ψ-cyclic if and only if* $\operatorname{pd} S$ *is not odd.*
(3) *A has infinite global dimension if and only if the set of ψ-cyclic simple A-modules is exactly the set of simple A-modules of infinite projective dimension.*

Proof. If A has no simple projective modules, then the arguments follow from [6, Section 3]. We mention that they are valid for any connected Nakayama algebra. $\square$

We need the following result from [7, Lemma 2]; see also [4].

Lemma 3.2. *Let M be an indecomposable A-module. Then either* $\operatorname{pd} M \leq 1$ *or* $\operatorname{top} \Omega^2(M) = \gamma(\operatorname{top} M)$.

The following lemma provides a connection between the maps γ and ψ.

Lemma 3.3. *A simple A-module S is γ-black if and only if $\psi\gamma(S) = S$.*

Proof. Suppose that S is a γ-black simple A-module. If S is projective, then by definition $\psi\gamma(S) = S$. If $\operatorname{pd} S \geq 2$, then by Lemma 3.2 we have $\gamma(S) = \operatorname{top} \Omega^2(S)$. It follows from Lemma 3.1(1) that $\psi\gamma(S) = S$.

Suppose $\psi\gamma(S) = S$. If S is projective, then S is γ-black. If S is not projective, then it follows from Lemma 3.1(1) that $\gamma(S)$ is a composition factor in $\Omega^2(S)$. In particular, $\Omega^2(S)$ is nonzero and thus $\operatorname{pd} S \geq 2$. □

Recall that a cycle in the resolution quiver of A is called black provided that each vertex on this cycle is γ-black.

We have the following observation.

Proposition 3.4. *Let C be a cycle in the resolution quiver of A. Then the following statements are equivalent.*

(1) *The vertices of C form a ψ-cycle.*
(2) *Each vertex on C is ψ-cyclic.*
(3) *C is a black cycle.*

Proof. (1) $\Longrightarrow$ (2) This is obvious.

(2) $\Longrightarrow$ (3) By Lemma 3.1(2) each ψ-cyclic simple A-module is γ-black.

(3) $\Longrightarrow$ (1) Assume that the vertices of C are $S_1, S_2, \ldots, S_m$ with $S_{i+1} = \gamma(S_i)$ for $i \geq 1$. Here, we identify S_{m+i} with S_i.

Since C is a black cycle, each S_i is γ-black for $i \geq 1$. It follows from Lemma 3.3 that $\psi(S_{i+1}) = \psi\gamma(S_i) = S_i$. Then $S_m, S_{m-1}, \ldots, S_1$ form a ψ-cycle. □

Suppose that the global dimension of A is infinite. In particular, there is no simple projective A-module. The following lemma describes projective A-modules of finite injective dimension and injective A-modules of finite projective dimension.

Lemma 3.5. *Let A be a connected Nakayama algebra with infinite global dimension, and let P be an indecomposable projective A-module and I be an indecomposable injective A-module.*

(1) *The injective dimension of P is infinite if and only if P is a nontrivial submodule of $P(S)$ with S a γ-cyclic simple A-module.*
(2) *The projective dimension of I is infinite if and only if I is a nontrivial factor module of $I(S)$ with S a ψ-cyclic simple A-module.*

Proof. (1) "$\Longrightarrow$" If the injective dimension of P is infinite, then the injective dimension of its cosyzygy $\Omega^{-1}(P)$ is also infinite. It follows that $\Omega^{-1}(P)$ contains at least one composition factor with infinite injective dimension.

We claim that $\Omega^{-1}(P)$ contains at most one composition factor which is γ-cyclic. It follows from Lemma 3.1(3) that $\Omega^{-1}(P)$ contains precisely one composition factor which is γ-cyclic. Since P is a nontrivial submodule of $P(T)$ for each composition factor T in $\Omega^{-1}(P)$, we infer that P is a nontrivial submodule of $P(S)$ with S a γ-cyclic simple A-module.

For the claim, observe that $\operatorname{soc} P(T) = \operatorname{soc} P$ and $\gamma(T) = \gamma(\operatorname{top} P)$ for each composition factor T in $\Omega^{-1}(P)$. Since the composition factors in $\Omega^{-1}(P)$ have the same image under the map γ, at most one of them is γ-cyclic.

"$\Longleftarrow$" Suppose that P is a nontrivial submodule of $P(S)$ with S a γ-cyclic simple A-module. By the previous claim, one can show that there exists only one composition factor in $\Omega^{-1}(P)$ which is γ-cyclic, namely S. The injective dimension of S is infinite and the injective dimension of the other composition factor of $\Omega^{-1}(P)$ is finite by Lemma 3.1(3). Since $\Omega^{-1}(P)$ has precisely one composition factor of infinite injective dimension, the injective dimension of $\Omega^{-1}(P)$ is infinite. Then the injective dimension of P is also infinite.

(2) This is dual to (1). □

Recall that an Artin algebra A is called a *Gorenstein algebra* if both $\operatorname{id} A$ and $\operatorname{pd} DA$ are finite. Here, D is the usual dual for finitely generated A-modules.

We are now ready to prove Proposition 1.2. Indeed, using the maps γ and ψ as above, several characterizations are given to decide whether a connected Nakayama algebra with infinite global dimension is Gorenstein or not. It strengthens [7, Proposition 5(a)].

Proposition 3.6. *Let A be a connected Nakayama algebra with infinite global dimension. Then the following statements are equivalent.*

(1) *A is a Gorenstein algebra.*
(2) *Each γ-cyclic simple A-module is γ-black.*
(3) *Each ψ-cyclic simple A-module is ψ-black.*
(4) *The set of γ-cyclic simple A-modules is exactly the set of ψ-cyclic simple A-modules.*

Proof. (1) $\Longrightarrow$ (2). Let S be a γ-cyclic simple A-module. By Lemma 3.5(1) the projective cover of S has no nontrivial projective submodules. Then the projective dimension of S greater than 1 and thus S is γ-black. Similarly, we have (1) $\Longrightarrow$ (3).

(2) $\Longrightarrow$ (4). Following Proposition 3.4 the set of γ-cyclic simple A-modules is contained in the set of ψ-cyclic simple A-modules. By [8, Corollary 2.3] the two sets have the same finite number of modules. Then they must coincide.

(4) $\Longrightarrow$ (2). Since each ψ-cyclic simple A-module is γ-black, it follows that each γ-cyclic simple A-module is γ-black.

Similarly, one can prove that (3) and (4) are equivalent.

(2) + (3) $\Longrightarrow$ (1). Observe that a simple A-module S is γ-black if and only if its projective cover $P(S)$ has no nontrivial projective submodule. By Lemma 3.5(1) there is no indecomposable projective A-module of infinite injective dimension, that is, each indecomposable projective A-module has finite injective dimension. Similarly, by Lemma 3.5(2) each indecomposable injective

A-module has finite projective dimension. It follows that A is a Gorenstein algebra. □

We mention that the global dimension condition in Proposition 3.6 cannot be omitted; see [7, Example 2].

4. Cartan matrices and resolution quivers. In this section, we study the connection between the Cartan matrix and the resolution quiver for a fixed connected Nakayama algebra A.

Denote by $n = n(A)$ the number of non-isomorphic simple A-modules. Take a complete set $\{S_1, S_2, \ldots, S_n\}$ of pairwise non-isomorphic simple A-modules. The Cartan matrix $\mathbf{C}_A = (c_{ij})$ of A is an $n \times n$ matrix, where c_{ij} is the number of copies of S_i appearing in a composition series for the projective cover of S_j.

Recall that two $n \times n$ integer matrices $\mathbf{X}$ and $\mathbf{Y}$ are *$\mathbb{Z}$-equivalent* provided that there exist invertible integer matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{PXQ} = \mathbf{Y}$. For an $n \times n$ integer matrix, its *Smith normal form* is the $n \times n$ diagonal integer matrix

$$\operatorname{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0)$$

where $d_1, d_2, \ldots, d_r \in \mathbb{N}^+$ and d_i divides d_{i+1} for $1 \le i \le r-1$. The Smith normal form always exists and is unique; it is $\mathbb{Z}$-equivalent to the original matrix.

Denote by $c(A)$ the number of cycles and by $w(A)$ the weight of a cycle in the resolution quiver of A.

The following result provides a connection between the Cartan matrix and the resolution quiver for A.

Proposition 4.1. *Let A be a connected Nakayama algebra. Then the Smith normal form of its Cartan matrix $\mathbf{C}_A$ is the diagonal matrix*

$$\operatorname{diag}(1, \ldots, 1, w(A), 0, \ldots, 0)$$

with $c(A) - 1$ zeros on the diagonal. In particular, the rank of $\mathbf{C}_A$ is $n(A) + 1 - c(A)$.

Proof. Recall (2.2) in Section 2. For $1 \le i \le r-1$, it is easy to show that the Cartan matrix $\mathbf{C}_{A_i}$ of A_i and the block diagonal matrix $\operatorname{diag}(1, \mathbf{C}_{A_{i+1}})$ are $\mathbb{Z}$-equivalent.

By Lemma 2.2 we have $c(A_i) = c(A_{i+1})$ and $w(A_i) = w(A_{i+1})$. Then it suffices to prove the assertion for the selfinjective algebra A_r.

Assume that A is a connected selfinjective Nakayama algebra. Denote by m its radical length. Using the same argument as in the proof of Proposition 1.1, we infer that the resolution quiver of A consists of $\gcd(m, n)$ cycles of the same weight $w = \frac{m}{\gcd(m,n)}$. Here, 'gcd' is the greatest common divisor.

Let $m = kn + r$ with $k \in \mathbb{N}$ and $1 \le r \le n$. After possible permutations on simple A-modules, the Cartan matrix $\mathbf{C}_A$ is a circulant matrix given by

$$c_{ij} = \begin{cases} k+1 & \text{if } 0 \le i-j < r \text{ or } j-i > n-r \\ k & \text{otherwise.} \end{cases}$$

It follows from [10, Theorem 3.1] that the Smith normal form of the Cartan matrix $\mathbf{C}_A$ is the diagonal matrix $\mathrm{diag}(1, \ldots, 1, w, 0, \ldots, 0)$ with $\gcd(m, n) - 1$ zeros on the diagonal. Then the rank of $\mathbf{C}_A$ is $n + 1 - \gcd(m, n)$. This finishes our proof. $\square$

Remark 4.2. Use the notation in the Proposition 4.1.

(1) Following [2, Theorem 6], the global dimension of A is finite if and only if the determinant of the Cartan matrix $\mathbf{C}_A$ is 1. In fact, by [3, Proposition 2.2(5)] left retractions also preserve the determinants of Cartan matrices. Then the determinant of the Cartan matrix $\mathbf{C}_A$ is $w(A)$ if the resolution quiver of A is connected; see also [2, Lemma 2]. This provides another proof of Proposition 1.1.

(2) Denote by A^{op} the opposite algebra of A. Then the Cartan matrix of A^{op} is the transpose of $\mathbf{C}_A$. Since $\mathbf{C}_A$ and its transpose $\mathbf{C}_A^T$ have the same Smith normal form, it follows that $c(A) = c(A^{\mathrm{op}})$ and $w(A) = w(A^{\mathrm{op}})$; see also [9, Proposition 5.2].

Let X be a subset of $\{S_1, S_2, \ldots, S_n\}$. There exists a $n \times 1$ vector ξ_X associated with X, where the ith entry of ξ_X is 1 if S_i is in X and the ith entry of ξ_X is 0 if S_i is not in X for $1 \leq i \leq n$. Denote by $\mathbf{1}$ the $n \times 1$ vector with all entries being 1.

We have the following observation.

Proposition 4.3. *Let A be a connected Nakayama algebra. Denote by Γ the set of cycles and by $B\Gamma$ the set of black cycles in the resolution quiver of A.*

(1) The vectors $\{\xi_C\}_{C\in\Gamma}$ are maximal linearly independent solutions to the linear system $\mathbf{C}_A\xi = w(A)\mathbf{1}$.

(2) The vectors $\{\xi_C\}_{C\in\Gamma}$ are the entire nonnegative integer solutions to the linear system $\mathbf{C}_A\xi = w(A)\mathbf{1}$.

(3) The vectors $\{\xi_E\}_{E\in B\Gamma}$ are maximal linearly independent solutions to the linear system $\mathbf{C}_A^T\xi = \mathbf{C}_A\xi = w(A)\mathbf{1}$.

Proof. (1) Since C is a cycle in the resolution quiver, for $1 \leq i \leq n$ the simple A-module S_i appears exactly $w(A)$ times in the direct sum $\oplus_{S\in C} P(S)$. It follows that $\mathbf{C}_A\xi_C = w(A)\mathbf{1}$. Since the vectors $\{\xi_C\}_{C\in\Gamma}$ have disjoint support, they are linearly independent. Then we obtain $c(A)$ linearly independent solutions to the linear system $\mathbf{C}_A\xi = w(A)\mathbf{1}$. By Proposition 4.1 the number of these solutions is $n + 1 - \operatorname{rank} \mathbf{C}_A$. Therefore, the solutions $\{\xi_C\}_{C\in\Gamma}$ are maximal.

(2) This follows from (1).

(3) Let E be a cycle in the resolution quiver of A. By Proposition 3.4 the cycle E is black if and only if the vertices of E form a ψ-cycle. By (2) the vertices of E form a ψ-cycle if and only if $\mathbf{C}_A^T\xi_E = w(A)\mathbf{1}$. Then the vectors $\{\xi_E\}_{E\in B\Gamma}$ are linearly independent solutions to the linear system $\mathbf{C}_A^T\xi = \mathbf{C}_A\xi = w(A)\mathbf{1}$.

Denote by Ψ is the set of ψ-cycles for A. We claim that the vectors $\{\xi_C\}_{C\in\Gamma\cup\Psi}$ are linearly independent. For a solution ξ to the desired linear system, by (1) we have $\xi = \sum_{C\in\Gamma} a_C\xi_C = \sum_{D\in\Psi} b_D\xi_D$. Observe that the intersection $\Gamma \cap \Psi$ is the set of black cycles. Since the vectors $\{\xi_C\}_{C\in\Gamma\cup\Psi}$ are linearly independent, we have $\xi = \sum_{C\in B\Gamma} a_C\xi_C$. This proves (3).

For the claim, let $\sum_{C\in\Gamma\cup\Psi} a_C\xi_C = 0$. For a non-black cycle $C \in \Gamma$, it follows from Proposition 3.4 that there exists some S_i on C which is not ψ-cyclic. Observe that the ith entry of $\sum_{C\in\Gamma\cup\Psi} a_C\xi_D$ is a_C. It follows that $a_C = 0$ for each non-black cycle $C \in \Gamma$. Since the vectors $\{\xi_C\}_{C\in\Psi}$ have disjoint support, $a_C = 0$ for each ψ-cycle C. Therefore, the vectors $\{\xi_C\}_{C\in\Gamma\cup\Psi}$ are linearly independent. $\square$

Denote by $b(A)$ be the number of black cycles in the resolution quiver of A. Recall that $\mathbf{C}_A$ is the Cartan matrix of A and $\mathbf{1}$ is the $n \times 1$ vector with all entries being 1.

We have the following.

Corollary 4.4. *Let A be a connected Nakayama algebra.*

(1) *$b(A)$ is nonzero if and only if* $\operatorname{rank}\begin{pmatrix}\mathbf{C}_A^T\\ \mathbf{C}_A\end{pmatrix} = \operatorname{rank}\begin{pmatrix}\mathbf{C}_A^T & \mathbf{1}\\ \mathbf{C}_A & \mathbf{1}\end{pmatrix}$.

(2) *If $b(A)$ is nonzero, then $b(A) = n(A) + 1 - \operatorname{rank}(\mathbf{C}_A, \mathbf{C}_A^T)$.*

Proof. By Proposition 4.3(3) we know $b(A) \neq 0$ if and only if the linear system

$$\begin{pmatrix}\mathbf{C}_A^T\\ \mathbf{C}_A\end{pmatrix}\xi = \begin{pmatrix}\mathbf{1}\\ \mathbf{1}\end{pmatrix}$$

is solvable. However, it is solvable if and only if $\operatorname{rank}\begin{pmatrix}\mathbf{C}_A^T\\ \mathbf{C}_A\end{pmatrix} = \operatorname{rank}\begin{pmatrix}\mathbf{C}_A^T & \mathbf{1}\\ \mathbf{C}_A & \mathbf{1}\end{pmatrix}$.

If it is solvable, then by Proposition 4.3(3) the number of maximal linearly independent solutions $n(A) + 1 - \operatorname{rank}(\mathbf{C}_A, \mathbf{C}_A^T)$ is equal to $b(A)$. $\square$

Acknowledgements. The author is very grateful to the referee for helpful comments, which improve the exposition of this paper. He thanks Professor Xiao-Wu Chen and Professor Guodong Zhou for numerous inspiring discussions. This work is supported by China Postdoctoral Science Foundation (No. 2015M581563) and Science and Technology Commission of Shanghai Municipality (No. 13DZ2260400).

References

[1] M. Auslander, I. Reiten, and S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Adv. Math. **36**, Cambridge Univ. Press, Cambridge, 1995.

[2] W.D. Burgess, K.R. Fuller, E.R. Voss, and B. Zimmermann-Huisgen, The Cartan matrix as an indicator of finite global dimension for Artinian rings, Proc. Amer. Math. Soc. **95** (1985), 157–165.

[3] X.-W. Chen and Y. Ye, Retractions and Gorenstein homological properties, Algebr. Represent. Theory **17** (2014), 713–733.

[4] W.H. Gustafson, Global dimension in serial rings, J. Algebra **97** (1985), 14–16.

[5] K. Igusa and D. Zacharia, On the cyclic homology of monomial relation algebras, J. Algebra **151** (1992), 502–521.

[6] D. Madsen, Projective dimensions and Nakayama algebras, In: Representations of algebras and related topics, Fields Inst. Commun. **45**, Amer. Math. Soc., Providence, RI, 2005, 247–265.

[7] C.M. Ringel, The Gorenstein projective modules for the Nakayama algebras. I, J. Algebra **385** (2013), 241–261.

[8] D. Shen, A note on resolution quivers, J. Algebra Appl. **13** (2014), 1350120.

[9] D. Shen, The singularity category of a Nakayama algebra, J. Algebra **429** (2015), 1–18.

[10] G. Williams, Smith forms for adjacency matrices of circulant graphs, Linear Algebra Appl. **443** (2014), 21–33.

Dawei Shen
Department of Mathematics,
Shanghai Key Laboratory of PMMP,
East China Normal University,
Shanghai 200241,
People's Republic of China
e-mail: dwshen@math.ecnu.edu.cn

Received: 11 June 2016