# Chapter 1

# FEM for Poisson problem in 1D

Consider the one dimensional Poisson problem

$$-u''(x) = f(x) \qquad \text{in } \Omega = (x_\ell, x_r) \tag{1.1a}$$
$$u(x_\ell) = u(x_r) = 0. \tag{1.1b}$$

In order to solve the problem (1.1), we multiply both sides of (1.1a) by the test function $v(x)$ and integrate over the domain,

$$-\int_\Omega u''(x)v(x)\ dx = \int_\Omega f(x)v(x)\ dx. \tag{1.2}$$

Then the weak formulation of the problem (1.1) is as follows: find $u(x) \in H_0^1(\Omega)$ such that

$$\int_\Omega u'(x)v'(x)\ dx = \int_\Omega f(x)v(x)\ dx, \tag{1.3}$$

for all $v(x) \in H_0^1(\Omega)$. The weak formulation (1.3) is obtained from (1.2) by using the boundary condition (1.1b) and integration by parts.

For the finite element approach, we need a conforming finite element space

$$V_h^k = \{v_h \in H_0^1(\Omega) \mid v_h|_I \in P_k(I),\ \forall I \in \mathcal{T}_h\},$$

where $P_k(I)$ is the polynomial function space of degrees $\leq k$. Then the variational formulation of the problem (1.1) is as follows: find $u_h \in V_h^k$ such that

$$\int_\Omega u_h'v_h'\ dx = \int_\Omega fv_h\ dx, \tag{1.4}$$

for all $v_h \in V_h^k$.

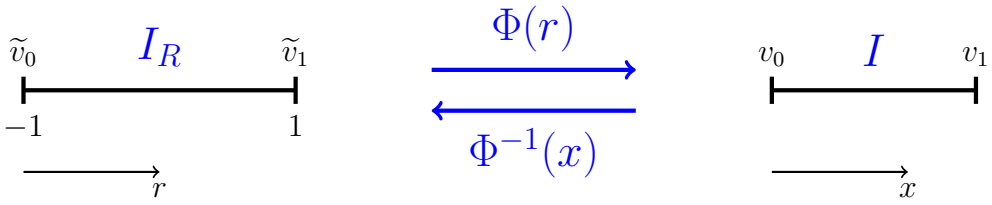



Figure 1.1: An affine mapping from the reference interval $I_R$ to an interval $I$.

## 1.1 Affine mapping

Let us define the reference interval $I_R$ as

$$I_R = \{r \mid -1 \le r \le 1\}.$$

Barycentric coordinates $(\widetilde{\lambda}_0,\ \widetilde{\lambda}_1)$ have the properties

$$\begin{cases} 0 \le \widetilde{\lambda}_i(r) \le 1, & i = 0,\ 1, \\ \widetilde{\lambda}_0(r) + \widetilde{\lambda}_1(r) = 1. \end{cases} \tag{1.5}$$

Let us consider an interval $I$,

$$I = \{x \mid v_0 \le x \le v_1\}.$$

Then we have an affine mapping $\Phi$ such that

$$\Phi(r) = v_0\widetilde{\lambda}_0(r) + v_1\widetilde{\lambda}_1(r) = x, \tag{1.6}$$

where $x \in I$. Then (1.5), $v_0 = -1$ and $v_1 = 1$ imply

$$\begin{aligned} r &= -\widetilde{\lambda}_0(r) + \widetilde{\lambda}_1(r) = -\widetilde{\lambda}_0(r) + (1 - \widetilde{\lambda}_0(r)) \\ &= 1 - 2\widetilde{\lambda}_0(r). \end{aligned}$$

Thus, barycentric coordinates can be written as

$$\widetilde{\lambda}_0(r) = \frac{1-r}{2}, \qquad \widetilde{\lambda}_1(r) = \frac{1+r}{2}. \tag{1.7}$$

From (1.6) and (1.7),

$$\frac{dx}{dr} = \frac{v_1 - v_0}{2}.$$

A simple calculation yields that

$$I = \frac{dx}{dr}\frac{dr}{dx} = x_r r_x.$$

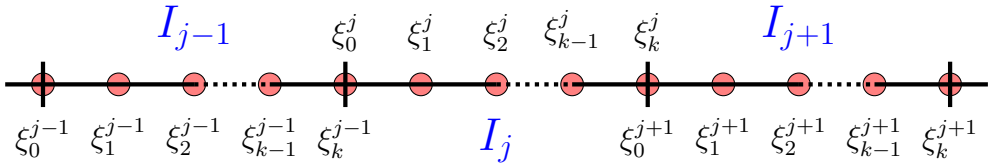



Figure 1.2: Nodes in $\mathcal{T}_h$ with $k$.

Therefore, we have

$$r_x = \frac{1}{J},$$

where $J$ is the Jacobian, $J = (v_1 - v_0)/2$.

Let $\lambda_0(x)$ and $\lambda_1(x)$ be the barycentric coordinates on $I$ such that

$$\lambda_0(x) = \frac{v_1 - x}{v_1 - v_0}, \qquad \lambda_1(x) = \frac{x - v_0}{v_1 - v_0}. \tag{1.8}$$

Since $\lambda_i(x)$ and $\widetilde{\lambda}_i(r)$ $(i = 0, 1)$ are linear functions and $\Phi$ is a linear mapping, we have

$$\lambda_i(x) = \widetilde{\lambda}_i(\Phi^{-1}(x)), \tag{1.9a}$$
$$\widetilde{\lambda}_i(r) = \lambda_i(\Phi(r)). \tag{1.9b}$$

Also we can obtain the following relations by simple calculation.

$$\frac{d}{dx}\lambda_i(x) = \frac{dr}{dx}\frac{d}{dr}\widetilde{\lambda}_i(\Phi^{-1}(x)) = r_x \frac{d}{dr}\widetilde{\lambda}_i(r). \tag{1.10}$$

## 1.2 Triangulation

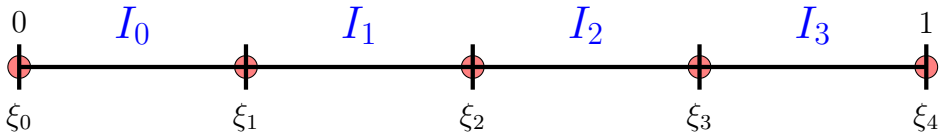
Consider the domain $\Omega = [0, 1]$. The number of nodes in the triangulation $\mathcal{T}_h$ is determined by the number of elements and the polynomial order (see, Figure 1.2). Let $N$ be the number of nodes, $M$ be the number of elements, $I_j$ be the $j$-th element in $\mathcal{T}_h$, $\xi_i$ be the $i$-th node in $\mathcal{T}_h$, and $\xi_i^j$ be the $i$-th node in $I_j$. Then the following relations hold

$$\begin{aligned} N &= kM + 1, \\ \xi_i^j &= \xi_{jk+i}, \\ \xi_k^j &= \xi_0^{j+1}. \end{aligned}$$

Here the indices $i$ and $j$ satisfy $0 \le j \le M - 1$ and $0 \le i \le k$, respectively.

The data for a given triangulation $\mathcal{T}_h$ are described in $N$ dimensional vector **c4n**, $M \times 2$ matrix **n4e**, 2 dimensional vector **n4db** and $M \times (k+1)$ matrix **ind4e**. The vector c4n means coordinates for nodes and its $i$-th row represents the coordinate for $i$-th node $\xi_i$. The matrix n4e means nodes for elements and $j$-th row of n4e shows the left-end and right-end nodes in $I_j$. The matrix ind4e means index for elements and its $j$-th row represents all indices of nodes in $I_j$. The vector n4db means nodes for Dirichlet boundary and n4db $= [\xi_1,\ \xi_N]^t$.

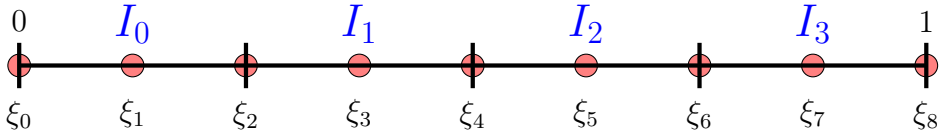
For example if $\Omega = (0,\ 1)$, $M = 4$ and $k = 1$, data are stored as follows.




$$
\begin{aligned}
\text{c4n} &= [0,\ 1/4,\ 1/2,\ 3/4,\ 1], \\
\text{n4e} &= [[0,1],\ [1,2],\ [2,3],\ [3,4]], \\
\text{ind4e} &= [[0,1],\ [1,2],\ [2,3],\ [3,4]], \\
\text{n4db} &= [0,4]
\end{aligned}
$$

In this case, each element has two nodes which are left-end and right-end nodes. Note that n4e and ind4e are exactly the same for $k = 1$.

If $\Omega = (0,\ 1)$, $M = 4$ and $k = 2$, data are stored as follows.




$$
\begin{aligned}
\text{c4n} &= [0,\ 1/8,\ 1/4,\ 3/8,\ 1/2,\ 5/8,\ 3/4,\ 7/8,\ 1], \\
\text{n4e} &= [[0,2],\ [2,4],\ [4,6],\ [6,8]], \\
\text{ind4e} &= [[0,1,2],\ [2,3,4],\ [4,5,6],\ [6,7,8]], \\
\text{n4db} &= [0,8]
\end{aligned}
$$

In this case, each element has an extra point except left-end and right-end nodes. Thus ind4e is different from n4e. Here the first column of ind4e is the same as the first column of n4e and the third column of ind4e is the same as the second column of n4e. In general, for a fixed $k$, the first column of ind4e is the same as the first column of n4e and the last column of ind4e is the same as the second column of n4e.

**mesh_fem_1d** The following python code generates an uniform mesh on the domain $\Omega = (a, b)$ in $\mathbb{R}$ with mesh size $h = 1/M$. Also this code returns an index matrix for continuous $k$-th order polynomial approximations.

```python
def mesh_fem_1d(a,b,M,k):
    nrNodes = k*M + 1
    c4n = np.linspace(a, b, nrNodes)
    n4e = np.array([[i*k, (i+1)*k] for i in range(M)])
    n4db = np.array([0, nrNodes-1])
    ind4e = np.array([list(range(i*k, (i+1)*k+1)) for i in range(M)])
    return (c4n,n4e,n4db,ind4e)
```

## 1.3 Basis functions of $V_h^k$ and Numerical solution

Basis functions of $V_h^k$ are piecewise polynomial and they have Kronecker delta property,

$$\phi_i(\xi_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{1.11}$$

$$\sum_{i=1}^{N} \phi_i(x) = 1 \qquad \forall x \in \Omega, \tag{1.12}$$

where $\phi_i$ is the $i$-th basis function of $V_h^k$ and $\xi_j$ is the $j$-th node point in $\mathcal{T}_h$. These basis functions can be locally expressed similar to the node points. The local basis function $\phi_i^j(x)$ is the $i$-th basis function in the $j$-th element. Then the following relations hold

$$\begin{aligned} \phi_i^j(x) &= \phi_{jk+i}(x)\Big|_{I_j} \\ \phi_i^j(x) &= 0 \quad \text{if } x \in \Omega \setminus I_j \\ \sum_{i=0}^{k} \phi_i^j(x) &= 1 \quad \forall x \in I_j. \end{aligned}$$

Figure 1.3 and 1.4 show global and local basis functions, respectively.

Using these basis functions, the interpolate function of $f(x)$ can be written as follows

$$\mathcal{I}f(x) = \sum_{i=0}^{N-1} f_i \phi_i(x) \tag{1.13}$$

where $f_i = f(\xi_i)$. Clearly, $f(\xi_i) = \mathcal{I}f(\xi_i)$ for all $0 \leq i \leq N-1$ by (1.11). If $f(x) \in V_h^k$, the interpolate function is the same as $f(x)$, i.e., $f(x) = \mathcal{I}f(x)$. Thus, the numerical solution can be written as follows

$$u_h(x) = \sum_{i=0}^{N-1} u_i \phi_i(x) \tag{1.14}$$

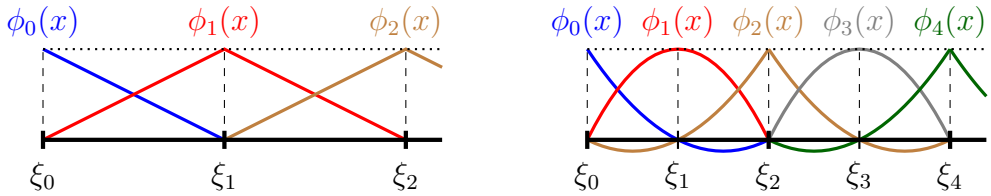



Figure 1.3: Global basis functions of $V_h^1$ (left) and $V_h^2$ (right) on an interval.

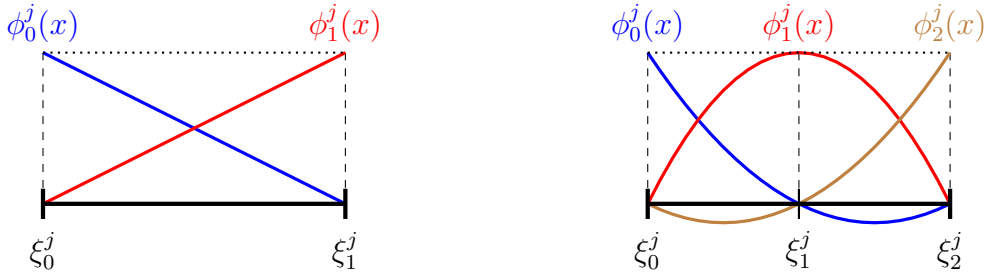



Figure 1.4: Local basis functions of $V_h^1$ (left) and $V_h^2$ (right) on the $j$-th element.

where $u_i = u_h(\xi_i)$. Especially, the numerical solution can be written locally because the solution is piecewise polynomial function:

$$u_h(x)\Big|_{I_j} = \sum_{i=0}^{k} u_i^j \phi_i^j(x) \tag{1.15}$$

where $u_i^j = u_h(\xi_i^j)$. Then the derivative of the solution is easily obtained from (1.14) and (1.15).

$$\frac{d}{dx}u_h(x) = \sum_{i=0}^{N-1} u_i \frac{d}{dx}\phi_i(x), \tag{1.16}$$

$$\frac{d}{dx}u_h(x)\Big|_{I_j} = \sum_{i=0}^{k} u_i^j \frac{d}{dx}\phi_i^j(x). \tag{1.17}$$

## 1.4 Mass matrix and Stiffness matrix

For a test function $\phi_i(x) \in V_h^k$, the variational formulation (1.4) can be rewritten as

$$\sum_{j=0}^{N-1} u_j \int_\Omega \phi_i'(x)\phi_j'(x)\ dx = \int_\Omega f(x)\phi_i(x)\ dx \tag{1.18}$$

by (1.14) and (1.15). Then, for all basis functions in $V_h^k$, we have the following finite element system

$$A\boldsymbol{u} = \boldsymbol{b} \tag{1.19}$$

where

$$(A)_{ij} = \int_\Omega \phi'_{i-1}(x)\phi'_{j-1}\ dx \tag{1.20a}$$

$$(\boldsymbol{b})_i = \int_\Omega f(x)\phi_{i-1}(x)\ dx \tag{1.20b}$$

$$(\boldsymbol{u})_j = u_{j-1}. \tag{1.20c}$$

Here, the matrix $A$ is called the global stiffness matrix and the right-hand side $\boldsymbol{b}$ is called the load vector. In order to compute the load vector, $f$ is replaced by the interpolate function $\mathcal{I}f(x)$ in general. Thus, FE system (1.19) can be rewritten as

$$A\boldsymbol{u} = M\boldsymbol{f} \tag{1.21}$$

where

$$(M)_{ij} = \int_\Omega \phi_{i-1}(x)\phi_{j-1}(x)\ dx \tag{1.22a}$$

$$(\boldsymbol{f})_j = f_{j-1}. \tag{1.22b}$$

Here, the matrix $M$ is called the global mass matrix. For the accurate computation, $f$ can be replaced by the $L^2$-orthogonal projection $\pi f(x)$.

As in the previous section, a basis function $\phi_i(x)$ is zero except at most two elements and $\phi'_i(x)$ is also zero in elements where $\phi_i(x) = 0$. Thus, the global stiffness matrix and the global mass matrix can be assembled by using local basis functions,

$$A = \sum_{\ell=1}^{M} A_{I_\ell}, \tag{1.23}$$

$$M = \sum_{\ell=1}^{M} M_{I_\ell}. \tag{1.24}$$

Here, $(k+1)$-by-$(k+1)$ matrices $A_{I_\ell}$ and $M_{I_\ell}$ are defined as

$$(A_{I_\ell})_{ij} = \int_{I_\ell} \frac{d}{dx}\phi^\ell_{i-1}(x)\frac{d}{dx}\phi^\ell_{j-1}(x)\ dx \tag{1.25}$$

$$(M_{I_\ell})_{ij} = \int_{I_\ell} \phi^\ell_{i-1}(x)\phi^\ell_{j-1}(x)\ dx \tag{1.26}$$

for $1 \le i,\ j \le k+1$. The matrices $A_{I_\ell}$ and $M_{I_\ell}$ are called the local stiffness matrix and the local mass matrix, respectively.

The following lemma is helpful to compute the local matrices.

**Lemma 1.1.** *For a given interval $I = [v_0, v_1]$ with barycentric coordinates $\lambda_0$ and $\lambda_1$, it holds for $a,\ b \in \mathbb{N}_0$ that*

$$\int_I \lambda_0^a \lambda_1^b \, dx = |I| \frac{a!\ b!}{(a+b+1)!} \tag{1.27}$$

*where $|I| = v_1 - v_0$.*

*Proof.* From the definition of barycentric coordinates (1.8),

$$\begin{aligned}\int_I \lambda_0^a \lambda_1^b \, dx &= \int_I \left(\frac{v_1 - x}{|I|}\right)^a \left(\frac{x - v_0}{|I|}\right)^b \, dx \\ &= \frac{1}{|I|^{a+b}} \int_I (v_1 - x)^a (x - v_0)^b \, dx.\end{aligned}$$

Then repeated integration by parts yield that

$$\begin{aligned}\int_I \lambda_0^a \lambda_1^b \, dx &= \frac{1}{|I|^{a+b}} \frac{b}{a+1} \int_{v_0}^{v_1} (v_1 - x)^{a+1} (x - v_0)^{b-1} \, dx \\ &= \frac{1}{|I|^{a+b}} \frac{b(b-1)}{(a+1)(a+2)} \int_{v_0}^{v_1} (v_1 - x)^{a+2} (x - v_0)^{b-2} \, dx \\ &\quad \vdots \\ &= \frac{1}{|I|^{a+b}} \frac{b!}{(a+1)(a+2)\cdots(a+b)} \int_{v_0}^{v_1} (v_1 - x)^{a+b} \, dx \\ &= |I| \frac{a!\ b!}{(a+b+1)!}.\end{aligned}$$

□

Since the measure of the reference interval $I_R$ is 2, i.e. $|I_R| = 2$, we have

$$\int_{I_R} \widetilde{\lambda}_0^a \widetilde{\lambda}_1^b \, dr = 2 \frac{a!\ b!}{(a+b+1)!}$$

and

$$\int_I \lambda_0^a \lambda_1^b \, dx = \frac{v_1 - v_0}{2}\ 2\ \frac{a!\ b!}{(a+b+1)!} = J \int_{I_R} \widetilde{\lambda}_1^a \widetilde{\lambda}_2^b \, dr.$$

Therefore, we can compute the integration (1.27) on an arbitrary interval $I$ by using the Jacobian and the integration over the reference interval $I_R$.

In order to compute derivatives, we now introduce a differentiation matrix $D$ such that

$$(D)_{ij} = \phi'_{j-1}(\xi_{i-1}). \tag{1.28}$$

Clearly, $\phi'_i(x) \in V_h^{k-1} \subset V_h^k$ because $\phi_i(x) \in V_h^k$. Thus, $\phi'_i(x)$ satisfies

$$\phi'_{i-1}(x) = \sum_{j=0}^{N-1} \phi'_{i-1}(\xi_j)\phi_j(x) = (D^t)_i \boldsymbol{\phi}$$

where $(D^t)_i$ is the $i$-th row of the matrix $D^t$, i.e. $(D^t)_i = [\phi'_{i-1}(\xi_1) \ \cdots \ \phi'_{i-1}(\xi_N)]$, and $\boldsymbol{\phi} = [\phi_0(x) \ \cdots \ \phi_{N-1}(x)]^t$. Similar to the stiffness and mass matrices, $D$ can be assembled by the local differentiation matrix

$$D = \sum_{\ell=0}^{M-1} D_{I_\ell}, \qquad (D_{I_\ell})_{ij} = \frac{d\phi^\ell_{j-1}}{dx}(\xi^\ell_{i-1}).$$

Thus, the derivative for the local basis function $\phi^\ell_i(x)$ is written as

$$\frac{d}{dx}\phi^\ell_{i-1}(x) = \sum_{j=0}^{k} \frac{d\phi^\ell_{i-1}}{dx}(\xi^\ell_j)\phi^\ell_j(x) = (D^t_{I_\ell})_i \boldsymbol{\phi}^\ell \tag{1.29}$$

where $(D^t_{I_\ell})_i$ is the $i$-th row of the matrix $D^t_{I_\ell}$ and $\boldsymbol{\phi}^\ell = [\phi^\ell_0(x) \ \cdots \ \phi^\ell_k(x)]^t$. Then, we have

$$u'_h(\xi_{m-1}) = \sum_{i=0}^{N-1} u_i\phi'_i(\xi_{m-1}) = \sum_{i=0}^{N-1} u_i \sum_{j=0}^{N-1} \phi'_i(\xi_j)\phi_j(\xi_{m-1}) = \sum_{i=0}^{N-1} u_i\phi'_i(\xi_{m-1}) = (D)_m \boldsymbol{u}$$

$$u'_h(\xi^\ell_{n-1}) = \sum_{i=0}^{k} u^\ell_i \frac{d\phi^\ell_i}{dx}(\xi^\ell_{n-1}) = \sum_{i=0}^{k} u^\ell_i \sum_{j=0}^{k} \frac{d\phi^\ell_i}{dx}(\xi_j)\phi_j(\xi^\ell_{n-1}) = \sum_{i=0}^{k} u^\ell_i \frac{d\phi^\ell_i}{dx}(\xi^\ell_{n-1}) = (D_{I_\ell})_m \boldsymbol{u}^\ell,$$

where $1 \le m \le N$, $1 \le n \le k+1$, $\boldsymbol{u} = [u_0, \ ..., \ u_{N-1}]$, and $\boldsymbol{u}^\ell = [u^\ell_0, \ ..., \ u^\ell_k]$.

**Lemma 1.2.** *For a given interval $I = [v_0, v_1]$ with barycentric coordinates $\lambda_0$ and $\lambda_1$, it holds for $a, \ b \in \mathbb{N}_0$ that*

$$\frac{d}{dx}(\lambda^a_0(x)\lambda^b_1(x)) = r_x \frac{d}{dr}(\widetilde{\lambda}^a_0(r)\widetilde{\lambda}^b_1(r)) \tag{1.30}$$

*where $r = \Phi^{-1}(x)$, $r_x = 1/J$ and $J = (v_1 - v_0)/2$.*

*Proof.* By the product rule and (1.10),

$$\begin{aligned}\frac{d}{dx}\left(\lambda_0^a(x)\lambda_1^b(x)\right) &= \lambda_0^{a-1}(x)\lambda_1^b(x)\frac{d}{dx}\lambda_0(x) + \lambda_0^a(x)\lambda_1^{b-1}(x)\frac{d}{dx}\lambda_1(x)\\ &= \widetilde{\lambda}_0^{a-1}(r)\widetilde{\lambda}_1^b(r)r_x\frac{d}{dr}\widetilde{\lambda}_0(r) + \widetilde{\lambda}_0^a(r)\widetilde{\lambda}_1^{b-1}(r)r_x\frac{d}{dr}\widetilde{\lambda}_1(r)\\ &= r_x\left(\widetilde{\lambda}_0^{a-1}(r)\widetilde{\lambda}_1^b(r)\frac{d}{dr}\widetilde{\lambda}_0(r) + \widetilde{\lambda}_0^a(r)\widetilde{\lambda}_1^{b-1}(r)\frac{d}{dr}\widetilde{\lambda}_1(r)\right)\\ &= r_x\frac{d}{dr}\left(\widetilde{\lambda}_0^a(r)\widetilde{\lambda}_1^b(r)\right).\end{aligned}$$

□

Now we are ready to compute the local stiffness and mass matrices. For convenience, we will use the notations $M$ and $S$ instead of $M_{I_\ell}$ and $A_{I_\ell}$ for a fixed element $I$, respectively. Also, $\phi$ and $\widetilde{\phi}$ are basis functions on the interval $I$ and the reference interval $I_R$, respectively. By the affine mapping, we have

$$(M)_{ij} = \int_I \phi_{i-1}(x)\phi_{j-1}(x)\,dx = J\int_{I_R}\widetilde{\phi}_{i-1}(r)\widetilde{\phi}_{j-1}(r)\,dr = J(M_R)_{ij} \tag{1.31}$$

where $M_R$ is the mass matrix on $I_R$. By (1.29) and (1.30),

$$\begin{aligned}(S)_{ij} &= \int_I \phi'_{i-1}(x)\phi'_{j-1}(x)\,dx\\ &= \int_I \Big(\sum_{\ell=0}^{k}\frac{d\phi_{i-1}}{dx}(\xi_\ell)\phi_\ell(x)\Big)\Big(\sum_{m=0}^{k}\frac{d\phi_{j-1}}{dx}(\xi_m)\phi_m(x)\Big)\,dx\\ &= J\int_{I_R}\Big(\sum_{\ell=0}^{k} r_x\frac{d\widetilde{\phi}_{i-1}}{dr}(\Phi^{-1}(\xi_\ell))\widetilde{\phi}_\ell(r)\Big)\Big(\sum_{m=0}^{k} r_x\frac{d\widetilde{\phi}_{j-1}}{dr}(\Phi^{-1}(\xi_m))\widetilde{\phi}_m(r)\Big)\,dr\\ &= r_x^2 J(S_R)_{ij} = \frac{1}{J}(S_R)_{ij}\end{aligned} \tag{1.32}$$

and

$$\begin{aligned}(S_R)_{ij} &= \int_{I_R}\Big((D_R^t)_i\widetilde{\boldsymbol{\phi}}\Big)\Big((D_R^t)_j\widetilde{\boldsymbol{\phi}}\Big)\,dr\\ &= (D_R^t)_i M_R (D_R)_j\\ &= (D_R^t M_R D_R)_{ij}\end{aligned} \tag{1.33}$$

where $S_R$ is the local stiffness matrix on $I_R$, $D_R$ is the differentiation matrix on $I_R$ and $\widetilde{\boldsymbol{\phi}} = [\widetilde{\phi}_0(r)\ \cdots\widetilde{\phi}_k(r)]^t$. Thus, the local stiffness and mass matrices are obtained from the reference mass and stiffness matrices by using affine mapping.

$$M = JM_R, \qquad S = \frac{1}{J}S_R \tag{1.34}$$

Here, we compute the reference matrices for the linear ($k = 1$) and quadratic ($k = 2$) approximations. For the $k$-th order approximation, the matrices are obtained similarly.

**$P_1$ matrices** The basis functions of $P_1(I_R)$ are the same as the barycentric coordinates. Thus the basis functions and their derivatives are

$$\begin{aligned} \widetilde{\phi}_0(r) &= \widetilde{\lambda}_0(r), \qquad & \widetilde{\phi}_0'(r) &= -\frac{1}{2}, \\ \widetilde{\phi}_1(r) &= \widetilde{\lambda}_1(r), \qquad & \widetilde{\phi}_1'(r) &= \frac{1}{2}. \end{aligned}$$

Then, (1.26) and (1.27) yield

$$M_R = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{1.35}$$

$$\because \begin{cases} (M_R)_{11} = \displaystyle\int_{I_R} \widetilde{\lambda}_0^2(r)\ dr = \frac{2!}{3!}|I_R| = \frac{2}{3} \\ (M_R)_{12} = \displaystyle\int_{I_R} \widetilde{\lambda}_0(r)\widetilde{\lambda}_1(r)\ dr = \frac{1}{3!}|I_R| = \frac{1}{3} \\ (M_R)_{22} = \displaystyle\int_{I_R} \widetilde{\lambda}_1^2(r)\ dr = \frac{2}{3} \end{cases}$$

The differentiation matrix $D_R$ is obtained from (1.28)

$$D_R = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{1.36}$$

Therefore, the local stiffness matrix is obtained from (1.33)

$$S_R = D_R^t M_R D_R = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{1.37}$$

**$P_2$ matrices** The basis functions in $P_2(I_R)$ are obtained from the barycentric coordinates such that

$$\begin{aligned} \widetilde{\phi}_0(r) &= \widetilde{\lambda}_0(r)(2\widetilde{\lambda}_0(r) - 1) \\ \widetilde{\phi}_1(r) &= 4\widetilde{\lambda}_0(r)\widetilde{\lambda}_1(r) \\ \widetilde{\phi}_2(r) &= \widetilde{\lambda}_1(r)(2\widetilde{\lambda}_1(r) - 1). \end{aligned}$$

Then, their derivatives are obtained by using the chain rule

$$\begin{aligned} \widetilde{\phi}_0'(r) &= -2\widetilde{\lambda}_0(r) + \frac{1}{2} \\ \widetilde{\phi}_1'(r) &= 2\widetilde{\lambda}_0(r) - 2\widetilde{\lambda}_1(r) \\ \widetilde{\phi}_2'(r) &= 2\widetilde{\lambda}_1(r) - \frac{1}{2}. \end{aligned}$$

Similar to the $P_1$ matrices, we have the mass matrix

$$M_R = \frac{1}{15}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}, \tag{1.38}$$

$$\because \begin{cases} (M_R)_{11} = \int_{I_R} 4\widetilde{\lambda}_0^4 - 4\widetilde{\lambda}_0^3 + \widetilde{\lambda}_0^2 dr = 2\left(4\frac{4!}{5!} - 4\frac{3!}{4!} + \frac{2!}{3!}\right) = \frac{4}{15} \\ (M_R)_{12} = \int_{I_R} 8\widetilde{\lambda}_0^3\widetilde{\lambda}_1 - 4\widetilde{\lambda}_0^2\widetilde{\lambda}_1 \; dr = 2\left(8\frac{3!}{5!} - 4\frac{2!}{4!}\right) = \frac{2}{15} \\ (M_R)_{13} = \int_{I_R} 4\widetilde{\lambda}_0^2\widetilde{\lambda}_1^2 - 2\widetilde{\lambda}_0^2\widetilde{\lambda}_1 - 2\widetilde{\lambda}_0\widetilde{\lambda}_1^2 + \widetilde{\lambda}_0\widetilde{\lambda}_1 \; dr = 2\left(4\frac{2!2!}{5!} - 2\frac{2!}{4!} - 2\frac{2!}{4!} + \frac{1}{3!}\right) = \frac{-1}{15} \\ (M_R)_{22} = \int_{I_R} 16\widetilde{\lambda}_0^2\widetilde{\lambda}_1^2 \; dr = 32\frac{2!2!}{5!} = \frac{16}{15} \\ (M_R)_{23} = \int_{I_R} 8\widetilde{\lambda}_0\widetilde{\lambda}_1^3 - 4\widetilde{\lambda}_0\widetilde{\lambda}_1^2 \; dr = 2\left(8\frac{3!}{5!} - 4\frac{2!}{4!}\right) = \frac{2}{15} \\ (M_R)_{33} = \int_{I_R} 4\widetilde{\lambda}_1^4 - 4\widetilde{\lambda}_1^3 + \widetilde{\lambda}_1^2 dr = 2\left(4\frac{4!}{5!} - 4\frac{3!}{4!} + \frac{2!}{3!}\right) = \frac{4}{15} \end{cases}$$

the differentiation matrix

$$D_R = \frac{1}{2}\begin{pmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{pmatrix}, \tag{1.39}$$

and the stiffness matrix

$$S_R = D_R^t M_R D_R = \frac{1}{6}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}. \tag{1.40}$$

**get_matrices_1d** The following python code generates the mass matrix $M_R$, the stiffness matrix $S_R$ and the differentiation matrix $D_R$ for continuous $k$-th order polynomial approximations on the reference interval $I_R$.

```
def get_matrices_1d(k=1):
    if k == 1:
        M_R = np.array([[2, 1],[1, 2]], dtype=np.float64) / 3.
        S_R = np.array([[1, -1],[-1, 1]], dtype=np.float64) / 2.
        D_R = np.array([[-1, 1],[-1, 1]], dtype=np.float64) / 2.
    elif k == 2:
        M_R = np.array([[4, 2, -1],[2, 16, 2],[-1, 2, 4]], dtype=np.float64) / 15.
        S_R = np.array([[7, -8, 1],[-8, 16, -8],[1, -8, 7]], dtype=np.float64) / 6.
        D_R = np.array([[-3, 4, -1],[-1, 0, 1],[1, -4, 3]], dtype=np.float64) / 2.
    elif: ...
    return (M_R, S_R, D_R)
```

## 1.5 Matlab codes in 1D

Now we are ready to assemble the global stiffness matrix $A$ and the global load vector $\boldsymbol{b}$ in (1.19). In the matlab code, $A$ and $\boldsymbol{b}$ are assembled by using the local stiffness matrix (1.32) and the local mass matrix (1.31).

**fem_for_poisson_1d** The following python code solves the Poisson problem. In order to use this code, mesh information (c4n, n4e, n4db, ind4e), matrices $(M_R, S_R)$, the source $f$, and the boundary condition u_D are necessary. Then, the results of this code are the numerical solution $u$, the global stiffness matrix $A$, the global load vector $\boldsymbol{b}$ and the freenodes.

```
def fem_for_poisson_1d(c4n,n4e,n4db,ind4e,k,M_R,S_R,f,u_D):
    number_of_nodes = len(c4n)
    number_of_elements = len(n4e)
    b = np.zeros(number_of_nodes)
    u = np.zeros(number_of_nodes)
    J = np.array([(c4n[n4e[i,1]]-c4n[n4e[i,0]])/2  ...
                  for i in range(number_of_elements)])
    Aloc = np.array([S_R.flatten()/J[i] for i in range(number_of_elements)])
    for i in range(number_of_elements):
        b[ind4e[i]] += J[i]*np.matmul(M_R, f(c4n[ind4e[i]]))
    row_ind = np.tile(ind4e.flatten(),(k+1,1)).T.flatten()
    col_ind = np.tile(ind4e,(1,k+1)).flatten()
    A_COO = coo_matrix((Aloc.flatten(), (row_ind, col_ind)),
                       shape=(number_of_nodes, number_of_nodes))
    A = A_COO.tocsr()
    dof = np.setdiff1d(range(0,number_of_nodes), n4db)
    u[dof] = spsolve(A[dof, :].tocsc()[:, dof].tocsr(), b[dof])
    return u
```

**compute_error_fem_1d** The following python code computes the semi H1 error between the exact solution and the numerical solution.

```
def compute_error_fem_1d(c4n,ind4e,M_R,D_R,u,Du):
    error = 0
    number_of_elements = len(ind4e)
    J = np.array([(c4n[n4e[i,1]]-c4n[n4e[i,0]])/2  ...
                  for i in range(number_of_elements)])
    for i in range(number_of_elements):
        u_x = np.matmul(D_R, u[ind4e[i]]) / J[i]
```

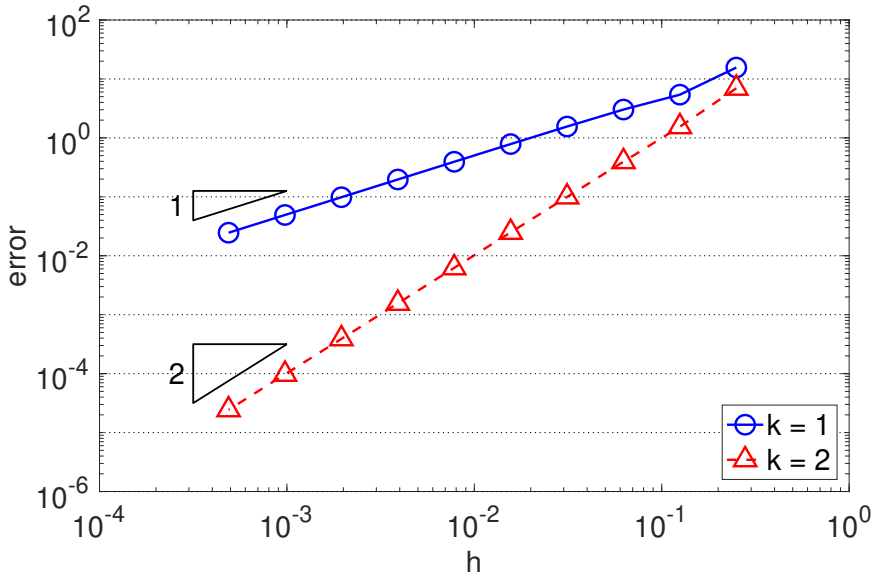



Figure 1.5: Convergence history for Example 1.1

```
        D_e = Du(c4n[ind4e[i]]) - u_x
        error += J[i] * np.matmul(D_e,np.matmul(M_R,D_e))
    return np.sqrt(error)
```

The following numerical example is introduced to verify above matlab codes.

**Example 1.1.** *Consider the domain* $\Omega = [0, 1]$. *The source term* $f$ *is chosen such that*

$$u = \sin(5\pi x) \tag{1.41}$$

*is the analytical solution to* (1.1).

The results of this example are displayed in Figure 1.5. Here the optimal rates of convergence are obtained. The matlab code for this example is as follows.

**main_1d** The following python code solves the Poisson problem by using several matlab codes such as mesh_fem_1d.m, get_matrices_1d.m, fem_for_poisson_1d.m and compute_error_fem_1d.m.

```
iter = 10
a = 0
b = 1
```

```
k = 1
M = 2 ** np.arange(2,iter+2)
f = lambda x: 25 * np.pi**2 * np.sin(5 * np.pi * x)
u_D = lambda x: 0 * x
Du = lambda x: 5 * np.pi * np.cos(5 * np.pi * x)

error = np.zeros(iter)
h = 1 / M

M_R, S_R, D_R = get_matrices_1d(k)
for i in range(iter):
    c4n, n4e, n4db, ind4e = mesh_fem_1d(a,b,M[i],k)
    u = fem_for_poisson_1d(c4n,n4e,n4db,ind4e,k,M_R,S_R,f,u_D)
    error[i] = compute_error_fem_1d(c4n,ind4e,M_R,D_R,u,Du)

rate = (np.log(error[1:])-np.log(error[:-1]))/(np.log(h[1:])-np.log(h[:-1]))
print(rate)
```

# Exercises

1. Add the matrices for the cubic approximations ($k = 3$) in **get_matrices_1d** and check the convergence rate.

2. Modify **fem_for_poisson_1d_ex2** to solve the Poisson problem with non-homogeneous Dirichlet boundary condition,

$$\begin{aligned} -u''(x) &= f(x) && \text{in } \Omega \\ u(x) &= u_D(x) && \text{on } \partial\Omega. \end{aligned}$$

3. Modify **fem_for_poisson_1d_ex3** to solve the Poisson problem with mixed boundary condition,

$$\begin{aligned} -u''(x) &= f(x) && \text{in } \Omega \\ u(x) &= u_D(x) && \text{on } \Gamma_D \\ u'(x)\boldsymbol{n} &= u_N(x) && \text{on } \Gamma_N, \end{aligned}$$

where $\Gamma_D$ denotes the Dirichlet boundary, $\Gamma_N$ denotes the Neumann boundary, and $\boldsymbol{n}$ is the outward unit normal vector.