Finite Element Methods for PDEs

- Finite Difference Method and Finite Element Method -

Dongwook Shin

AK-SSAM

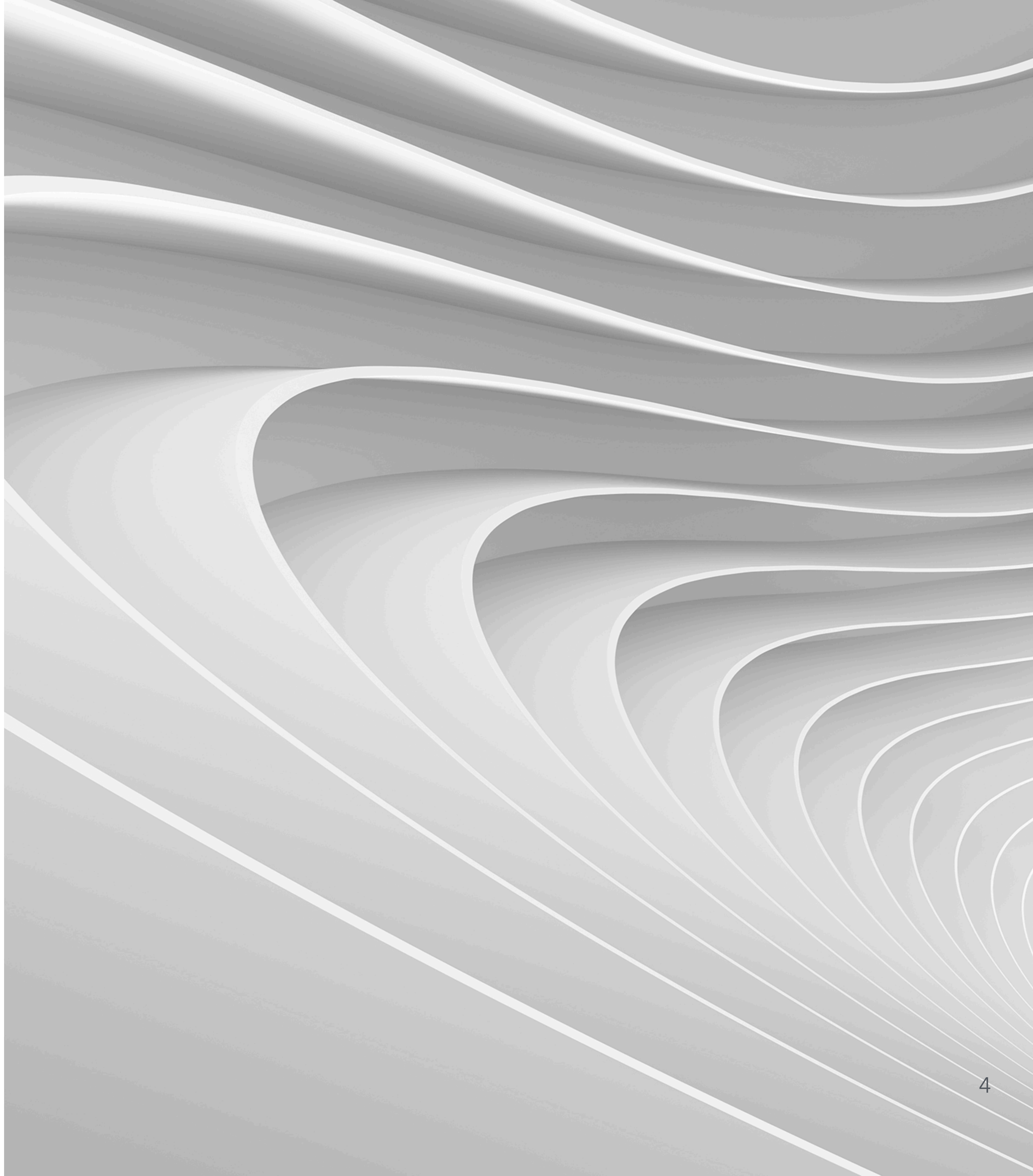
Online Lectures					
	7/1(Mon)	7/2(Tue)	7/3(Wed)	7/4(Thu)	7/5(Fri)
17:00-18:30		Lecture 1 Dongwook Shin	Lecture 2 Pierluigi Cesana	Lecture 3 Nguyen Dinh Hoa	

AJOU-KYUSHU SUMMER SCHOOL ON APPLIED MATHEMATICS					
	7/8(Mon)	7/9(Tue)	7/10(Wed)	7/11(Thu)	7/12(Fri)
08:40-10:10					Presentation 1
10:30-12:00		Lecture 6 Dongwook Shin	Lecture 8 Pierluigi Cesana	Project 1	Presentation 2
12:00-13:00		Lunch	Lunch	Lunch	Lunch
13:00-14:30	Opening	Lecture 7 Nguyen Dinh Hoa	Lecture 9 Nguyen Dinh Hoa	Project 2	
14:50-16:20	Lecture 4 Dongwook Shin			Project 3	
16:40-18:10	Lecture 5 Pierluigi Cesana		IMI Colloquium	Project 4	

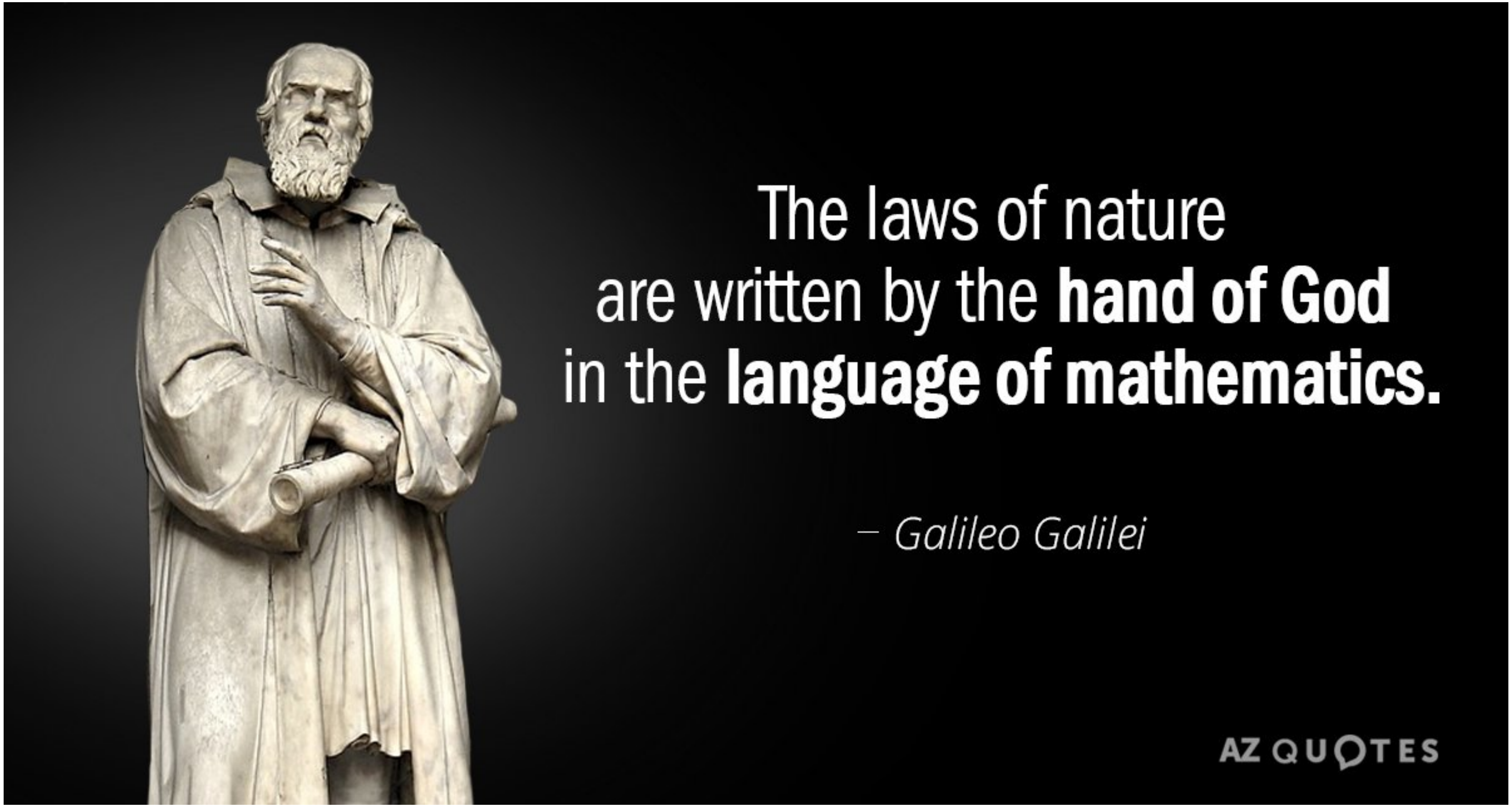
Outline

- Introduction
- Finite Difference Method (FDM) in 1D
- Finite Element Method (FEM) in 1D
- FDM and FEM in 2D
- A Posteriori Error Estimate
- Q&A

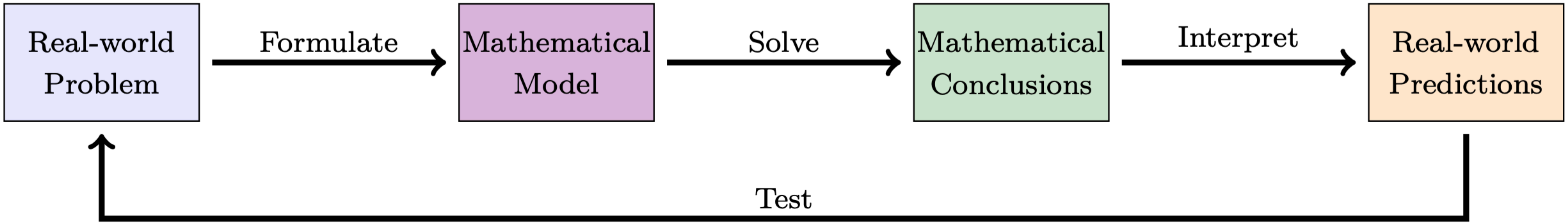
Introduction



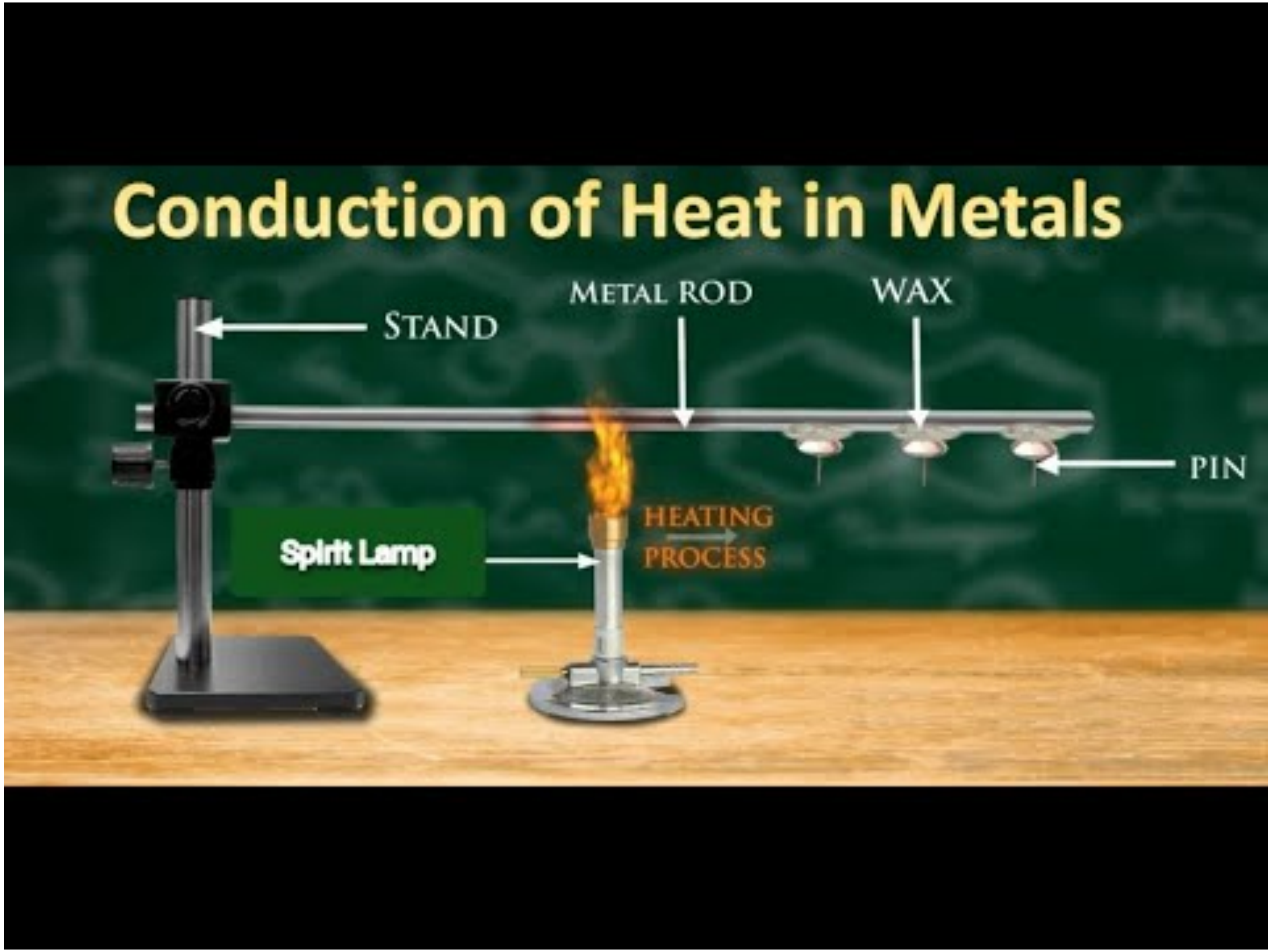
Mathematics



Modeling Process



Example (Real-World Problem)

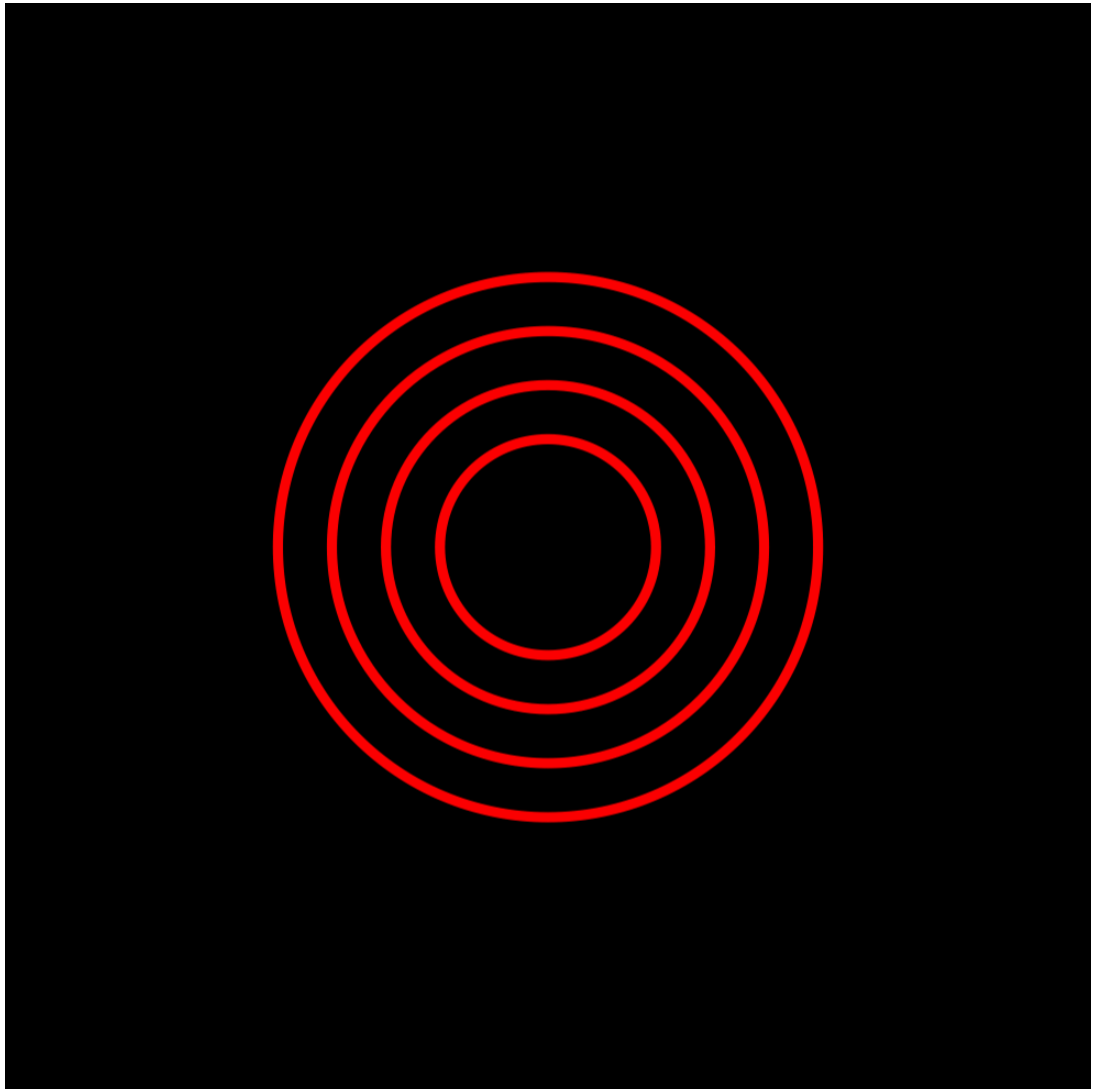


Example (Mathematical Modeling)

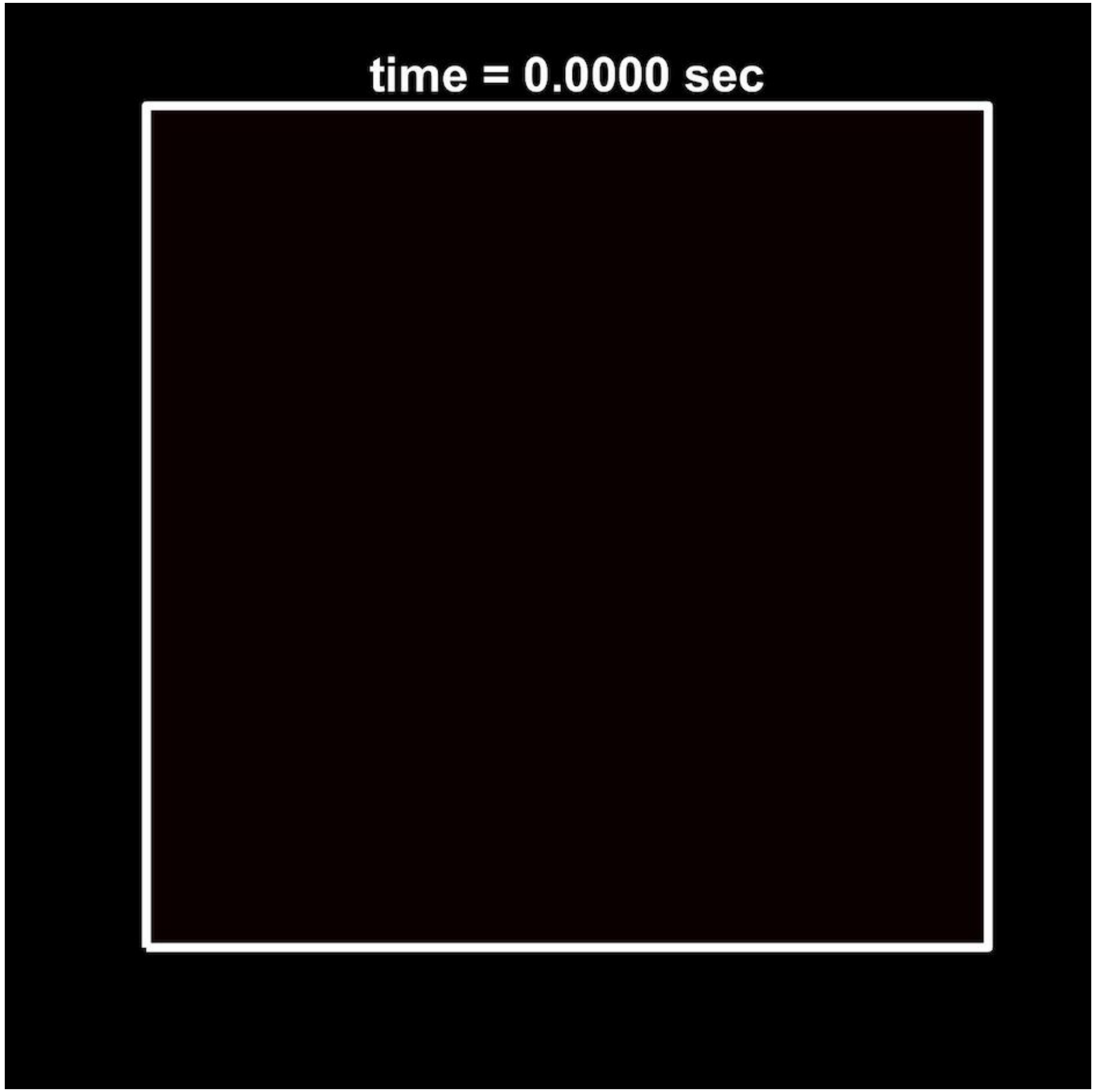
- Heat Equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) = f$$

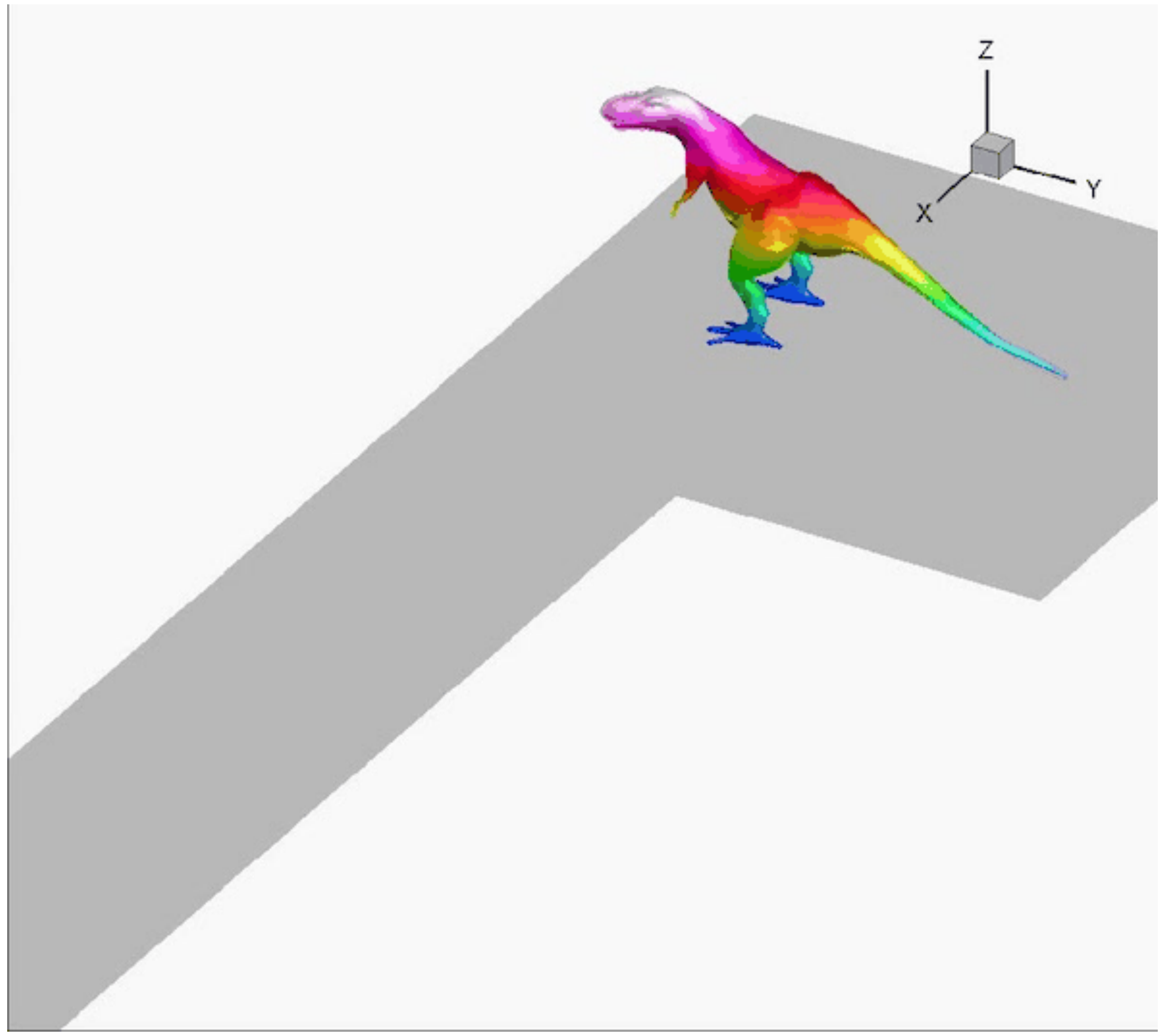
Example (Real-World Prediction)

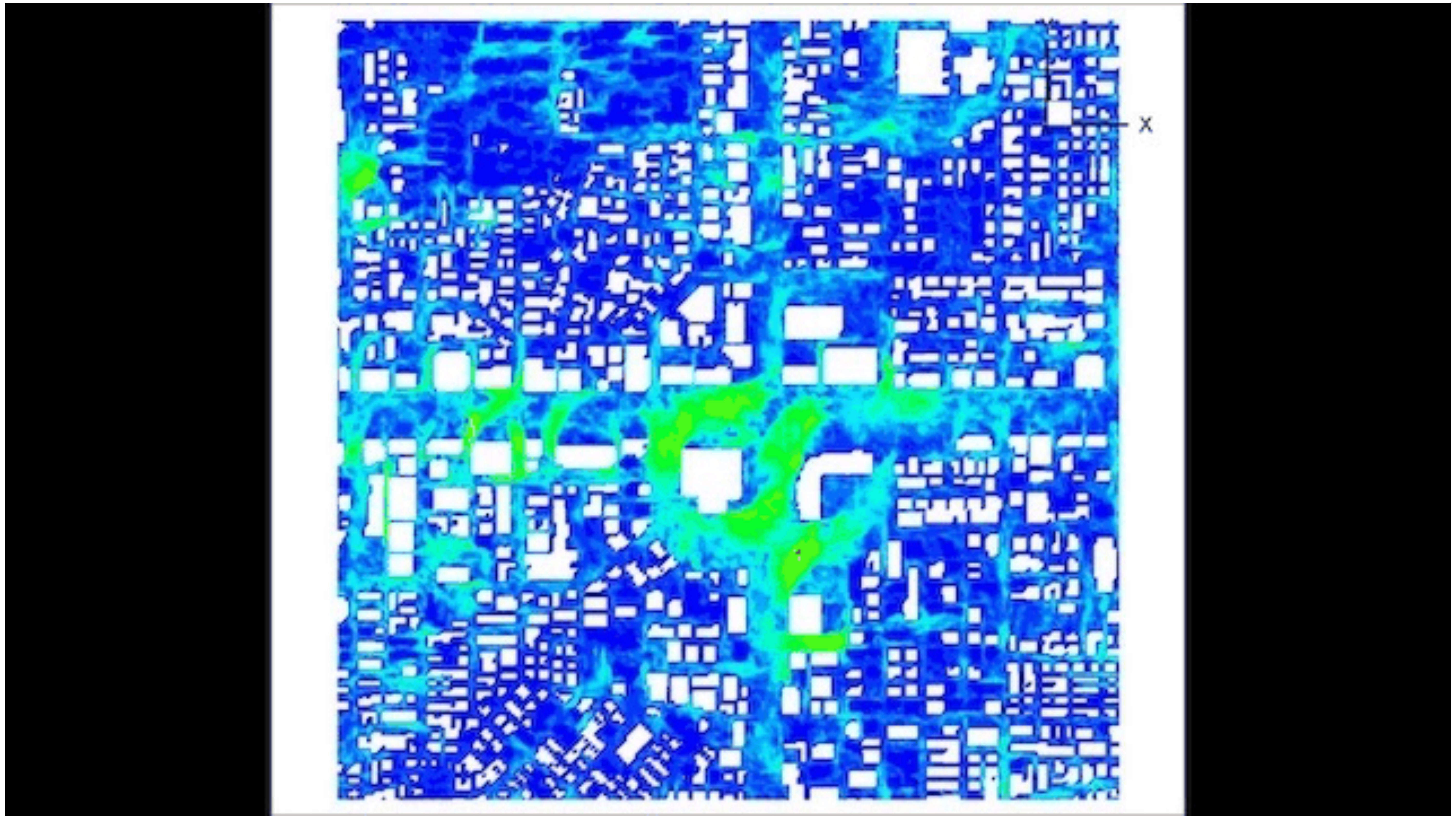


Example (Real-World Prediction)



Example (Real-World Prediction)




x

Poisson's equation

- Heat Equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) = f$$

- Steady-State: $\partial u / \partial t = 0$

$$-\nabla \cdot (k \nabla u) = f$$

- Poisson's equation

$$-\nabla \cdot (\nabla u) = f \qquad \text{or} \qquad -\Delta u = f$$

Poisson's equation

- Fourier's law

$$-q = k\nabla u$$

- Conservation of energy

$$\nabla \cdot q = f$$

- Poisson's equation ($k = 1$)

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

FDM in 1D

Model Problem in 1D

$$\begin{aligned} -u''(x) &= f(x) \qquad\qquad \text{in } \Omega = (0,1) \\ u(0) &= u(1) = 0. \end{aligned}$$

Finite Difference Approaches

- Derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Forward difference

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

- Backward difference

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}$$

- Central difference

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

Finite Difference Approaches

$$f''(a) \approx \frac{f'(a+h) - f'(a)}{h}$$

$$\approx \frac{\dfrac{f(a+h) - f(a)}{h} - \dfrac{f(a) - f(a-h)}{h}}{h}$$

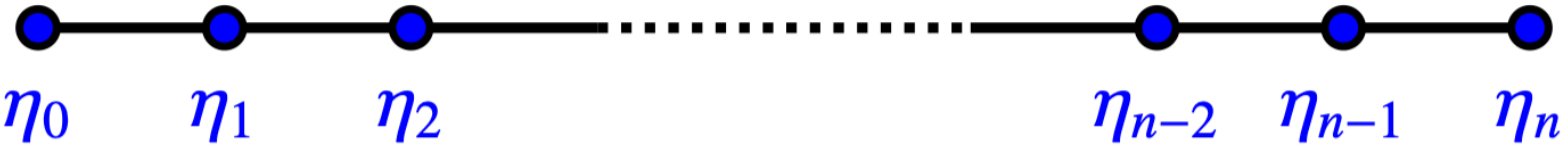
$$= \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Finite Difference Approaches

$$\frac{-u(\eta_{i-1}) + 2u(\eta_i) - u(\eta_{i+1})}{h^2} \approx -u''(\eta_i) = f(\eta_i)$$

$$\Rightarrow \qquad -u(\eta_{i-1}) + 2u(\eta_i) - u(\eta_{i+1}) = h^2 f(\eta_i)$$

Finite Difference Method



$$-u(\eta_{i-1}) + 2u(\eta_i) - u(\eta_{i+1}) = h^2 f(\eta_i)$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u(\eta_1) \\ u(\eta_2) \\ u(\eta_3) \\ \vdots \\ u(\eta_{n-2}) \\ u(\eta_{n-1}) \end{bmatrix} = h^2 \begin{bmatrix} f(\eta_1) \\ f(\eta_2) \\ f(\eta_3) \\ \vdots \\ f(\eta_{n-2}) \\ f(\eta_{n-1}) \end{bmatrix}$$

Taylor's Theorem

- Let $f(x)$ have $n+1$ continuous derivatives on $[a,b]$ for some $n \geq 0$, and let $x,\ x_0 \in [a,b]$. Then

$$f(x) = p_n(x) + R_{n+1}(x)$$
$$p_n(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$
$$R_{n+1}(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\ dt = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some ξ between x_0 and x.

Error Estimate

- Taylor's Theorem

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \frac{h^3}{6}f'''(a) + \frac{h^4}{24}f^{(4)}(\xi_1)$$
$$f(a-h) = f(a) - hf'(a) + \frac{h^2}{2}f''(a) - \frac{h^3}{6}f'''(a) + \frac{h^4}{24}f^{(4)}(\xi_2)$$

- 2nd order finite difference

$$f''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} + \mathcal{O}(h^2)$$

FEM in 1D

Finite Element Approach

- Model Problem

$$
\begin{aligned}
-u''(x) &= f(x) \qquad \text{in } \Omega = (0,1) \\
u(0) &= u(1) = 0.
\end{aligned}
$$

- Notations

  - $(v, w) = \int_\Omega v(x)w(x)\ dx$

  - $V = \{v :\ v$ is a continuous function on $[0,1]$,
    v' is piecewise continuous and bounded on $[0,1]$ and $v(0) = v(1) = 0\}$

Finite Element Method

- Variational Formulation

(V) Find $u \in V$ such that $(u', v') = (f, v) \quad \forall v \in V$

$$-\int_0^1 u''(x)v(x)\ dx = \int_0^1 f(x)v(x)\ dx$$

$$\Rightarrow \qquad \int_0^1 u'(x)v'(x)\ dx - \Big[u'(x)v(x)\Big]_0^1 = \int_0^1 f(x)v(x)\ dx$$

$$\Rightarrow \qquad \int_0^1 u'(x)v'(x)\ dx = \int_0^1 f(x)v(x)\ dx$$

Finite Element Method

- Linear Functional $F : V \to \mathbb{R}$

$$F(v) = \frac{1}{2}(v', v') - (f, v)$$

- Minimization problem

(M) Find $u \in V$ such that $F(u) \leq F(v) \quad \forall v \in V$

Equivalence

- (V) ⇒ (M): Let $v \in V$ and set $w = v - u$ so that $v = u + w$ and $w \in V$,

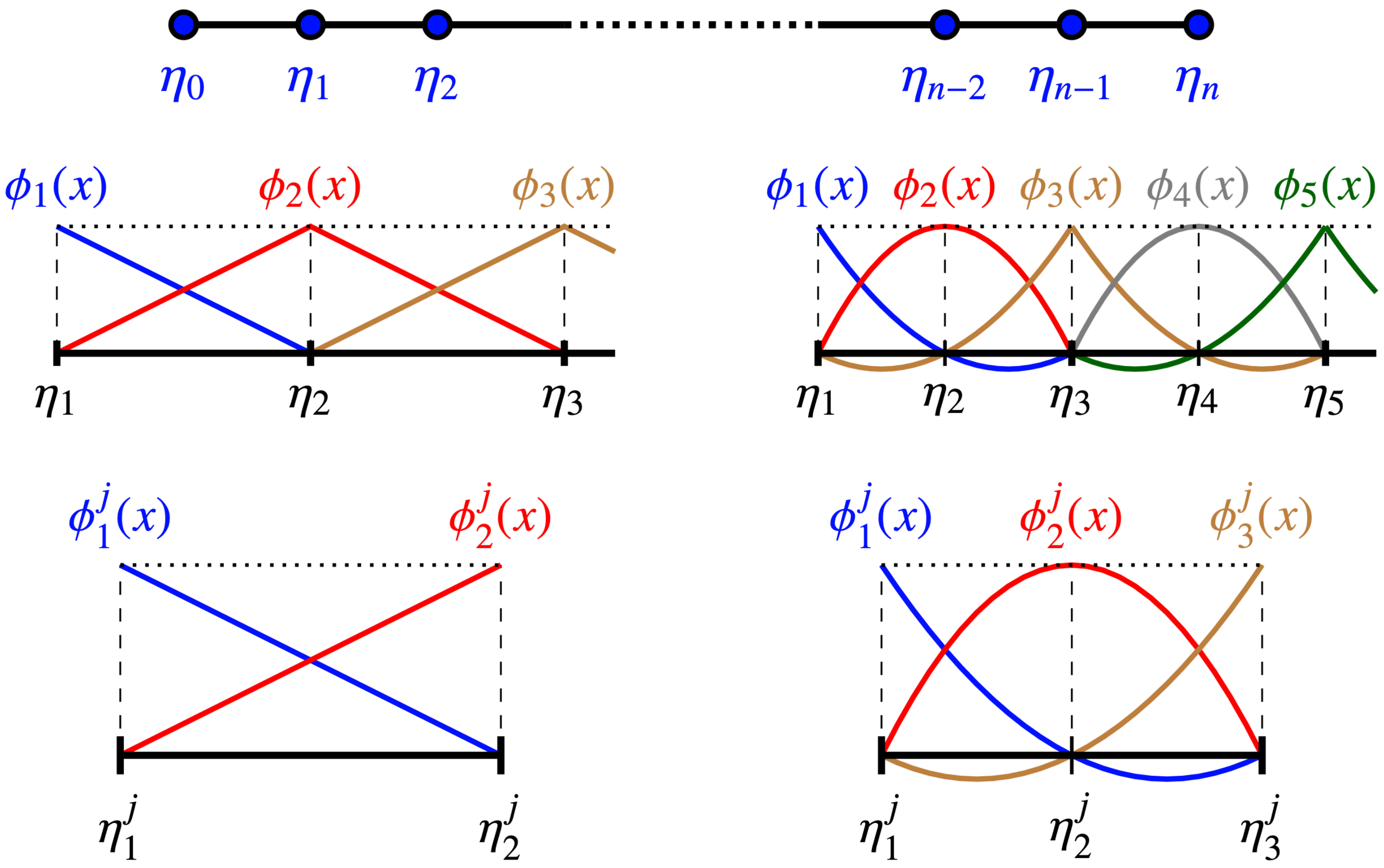
$$F(v) = F(u+w) = \frac{1}{2}(u' + w', u' + w') - (f, u + w)$$
$$= \frac{1}{2}(u', u') - (f, u) + \underbrace{(u', w') - (f, w)}_{=0} + \underbrace{\frac{1}{2}(w', w')}_{\geq 0} \geq F(u)$$

- (M) ⇒ (V): For any $v \in V$ and real number ϵ, $F(u) \leq F(u + \epsilon v)$ since $u + \epsilon v \in V$.

$$g(\epsilon) = F(u + \epsilon v) = \frac{1}{2}(u', u') + \epsilon(u', v') + \frac{\epsilon^2}{2}(v', v') - (f, u) - \epsilon(f, v)$$

$$g'(\epsilon) = (u', v') - (f, v) + \epsilon(v', v') \qquad \Rightarrow \qquad g'(0) = (u', v') - (f, v)$$

Finite Element Method



Finite Element Method

- Approximate solution

$$u_h(x) = \sum_{i=1}^{N} u_i \phi_i(x) \qquad \Rightarrow \qquad u_h' = \sum_{i=1}^{N} u_i \phi_i'(x)$$

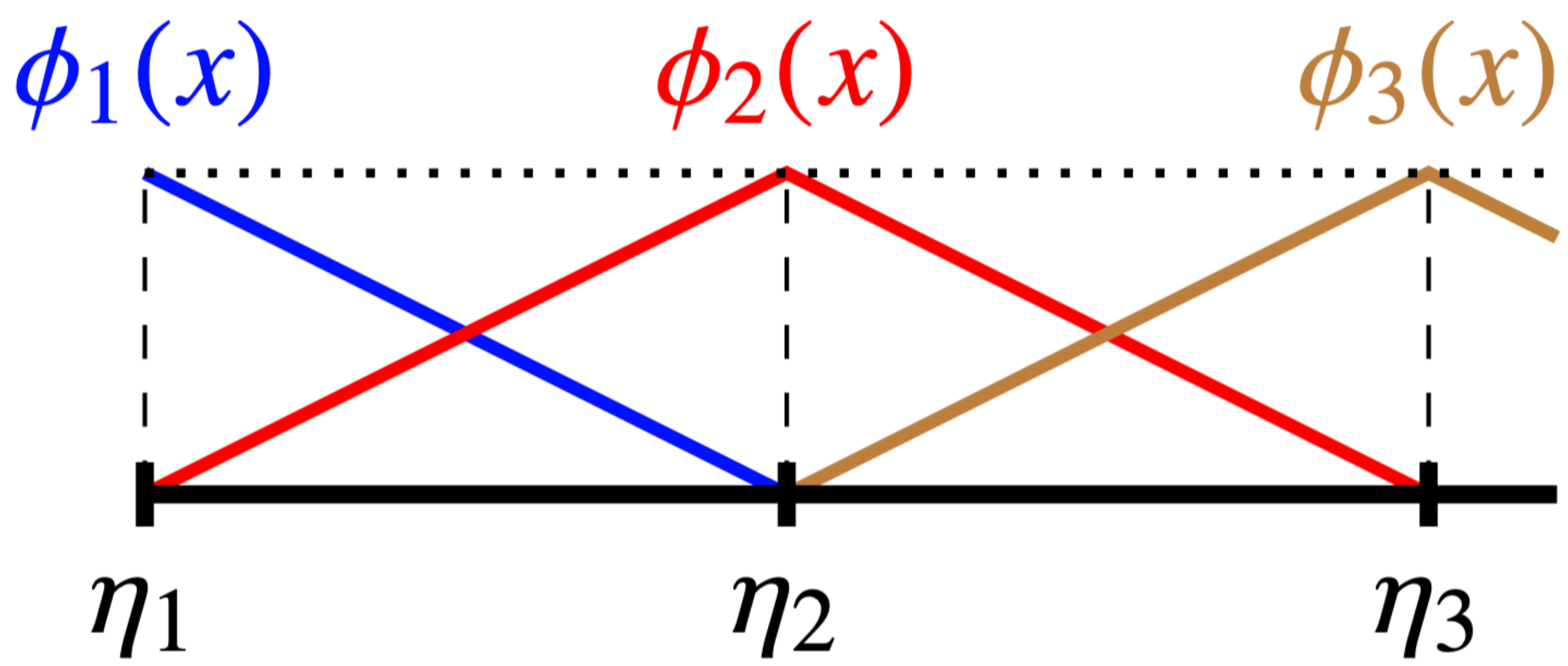
- For a test function $\phi_i(x) \in V_h^k$,

$$\int_\Omega u_h' v_h' \, dx = \int_\Omega f v_h \, dx \qquad \Rightarrow \qquad \sum_{j=1}^{N} u_j \int_\Omega \phi_i'(x)\phi_j'(x) \, dx = \int_\Omega f(x)\phi_i(x) \, dx$$

Finite Element Method

- Find $u_h \in V_h$ such that

$$\sum_{j=1}^{N} u_j \int_\Omega \phi_i'(x)\phi_j'(x)\ dx = \int_\Omega f(x)\phi_i(x)\ dx$$



$$\int_\Omega \phi_i'(x)\phi_j'(x)\ dx = \begin{cases} 0, \quad (|i-j|>1) \\ \displaystyle\int_{\eta_i}^{\eta_{i+1}} \phi_i'(x)\phi_{i+1}'(x)\ dx = -\frac{1}{h} \quad (j=i+1) \\ \displaystyle\int_{\eta_{i-1}}^{\eta_i} \phi_i'(x)\phi_{i-1}'(x)\ dx = -\frac{1}{h} \quad (j=i-1) \\ \displaystyle\int_{\eta_{i-1}}^{\eta_{i+1}} \left(\phi_i'(x)\right)^2\ dx = \frac{2}{h} \quad (j=i) \end{cases}$$

Finite Element Method

$$\sum_{j=1}^{N} u_j \int_\Omega \phi_i'(x)\phi_j'(x)\ dx = \int_\Omega f(x)\phi_i(x)\ dx$$

$$\Rightarrow \quad \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u(\eta_1) \\ u(\eta_2) \\ u(\eta_3) \\ \vdots \\ u(\eta_{n-2}) \\ u(\eta_{n-1}) \end{bmatrix} = h \begin{bmatrix} \int_\Omega f(x)\phi_1(x)\ dx \\ \int_\Omega f(x)\phi_2(x)\ dx \\ \int_\Omega f(x)\phi_3(x)\ dx \\ \vdots \\ \int_\Omega f(x)\phi_{n-2}(x)\ dx \\ \int_\Omega f(x)\phi_{n-1}(x)\ dx \end{bmatrix}$$

FDM vs FEM in 1D

- Finite Difference Method

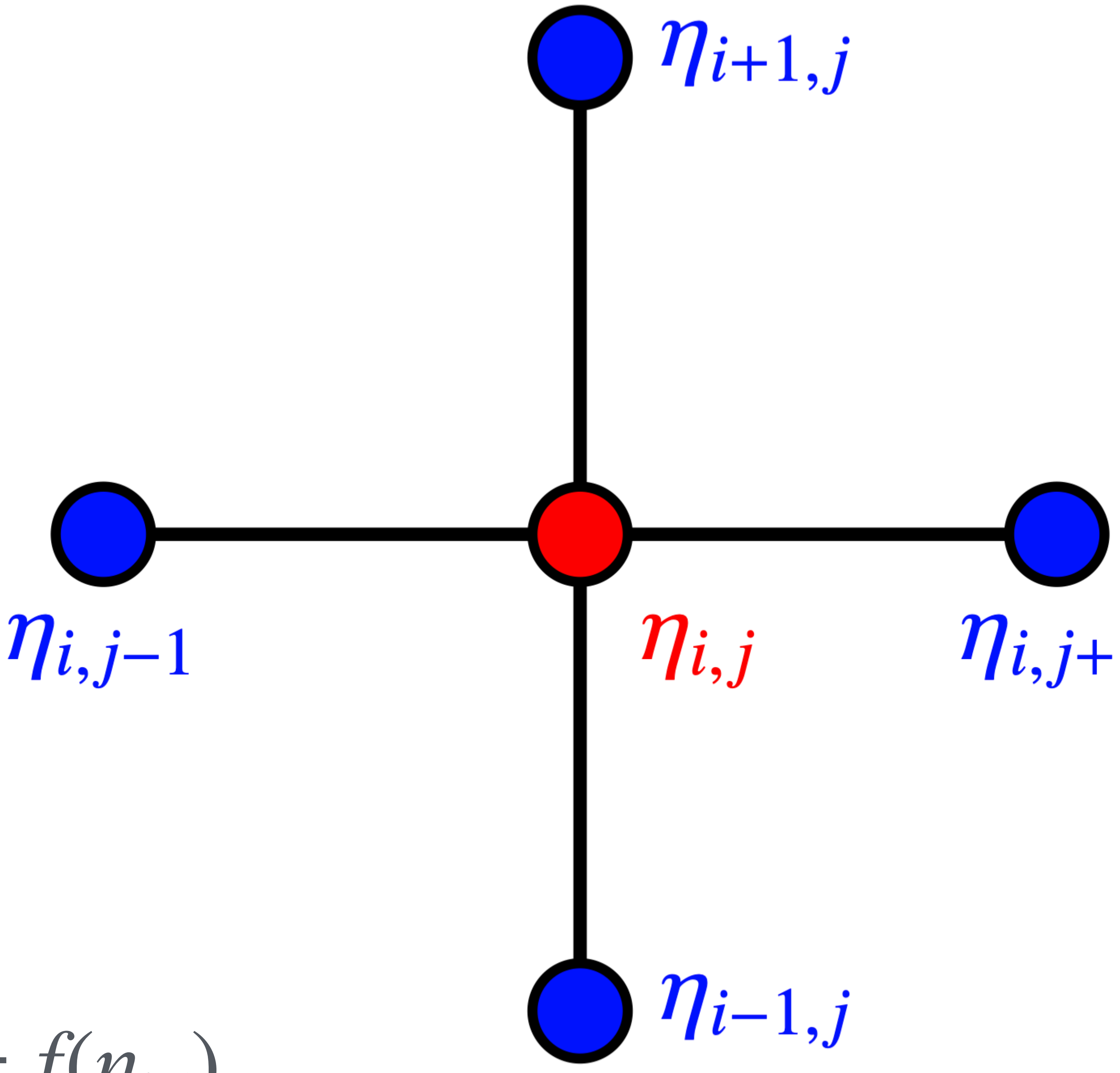
$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u(\eta_1) \\ u(\eta_2) \\ u(\eta_3) \\ \vdots \\ u(\eta_{n-2}) \\ u(\eta_{n-1}) \end{bmatrix} = h^2 \begin{bmatrix} f(\eta_1) \\ f(\eta_2) \\ f(\eta_3) \\ \vdots \\ f(\eta_{n-2}) \\ f(\eta_{n-1}) \end{bmatrix}$$

- Finite Element Method

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u(\eta_1) \\ u(\eta_2) \\ u(\eta_3) \\ \vdots \\ u(\eta_{n-2}) \\ u(\eta_{n-1}) \end{bmatrix} = h \begin{bmatrix} \int_\Omega f(x)\phi_1(x)\,dx \\ \int_\Omega f(x)\phi_2(x)\,dx \\ \int_\Omega f(x)\phi_3(x)\,dx \\ \vdots \\ \int_\Omega f(x)\phi_{n-2}(x)\,dx \\ \int_\Omega f(x)\phi_{n-1}(x)\,dx \end{bmatrix}$$

FDM and FEM in 2D

Finite Difference Approach



$$
\begin{aligned}
-\Delta u(\eta_{i,j}) &= -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \\
&\approx \frac{2u(\eta_{i,j}) - u(\eta_{i-1,j}) - u(\eta_{i+1,j})}{h^2} + \frac{2u(\eta_{i,j}) - u(\eta_{i,j-1}) - u(\eta_{i,j+1})}{h^2} = f(\eta_{i,j})
\end{aligned}
$$

$$
\Rightarrow \qquad -\Delta u(\eta_{i,j}) \approx \frac{4u(\eta_{i,j}) - u(\eta_{i-1,j}) - u(\eta_{i+1,j}) - u(\eta_{i,j-1}) - u(\eta_{i,j+1})}{h^2} = f(\eta_{i,j})
$$

Finite Element Approach

- (V) Find $u \in H_0^1(\Omega)$ such that

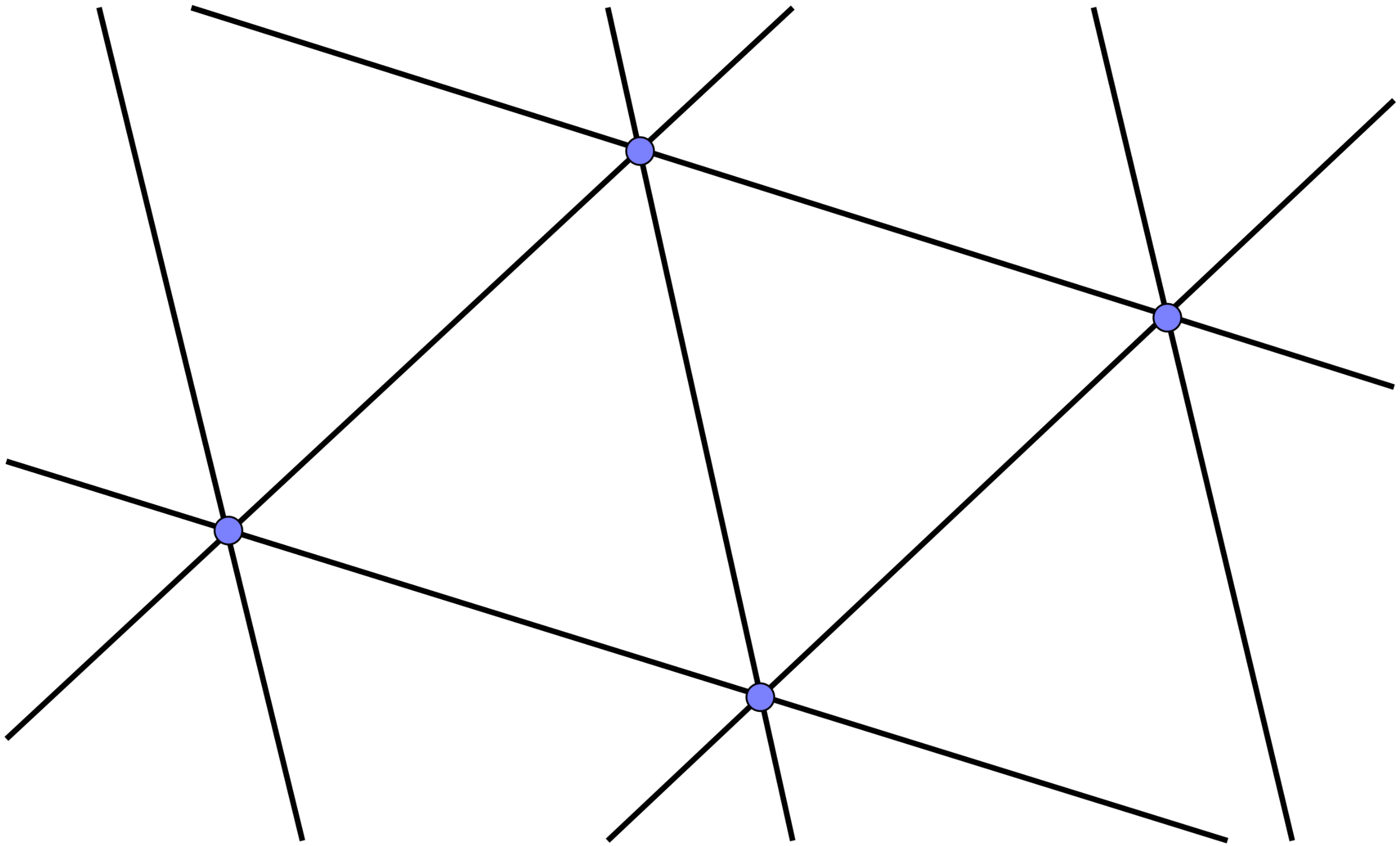
$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

  - $(-\Delta u, v) = -\int_\Omega \Delta u v \; dx = (f, v)$
  - $\int_\Omega \nabla u \cdot \nabla v \; dx - \underbrace{\int_{\partial\Omega} \nabla u \cdot \boldsymbol{n} v \; ds}_{=0} = (f, v)$

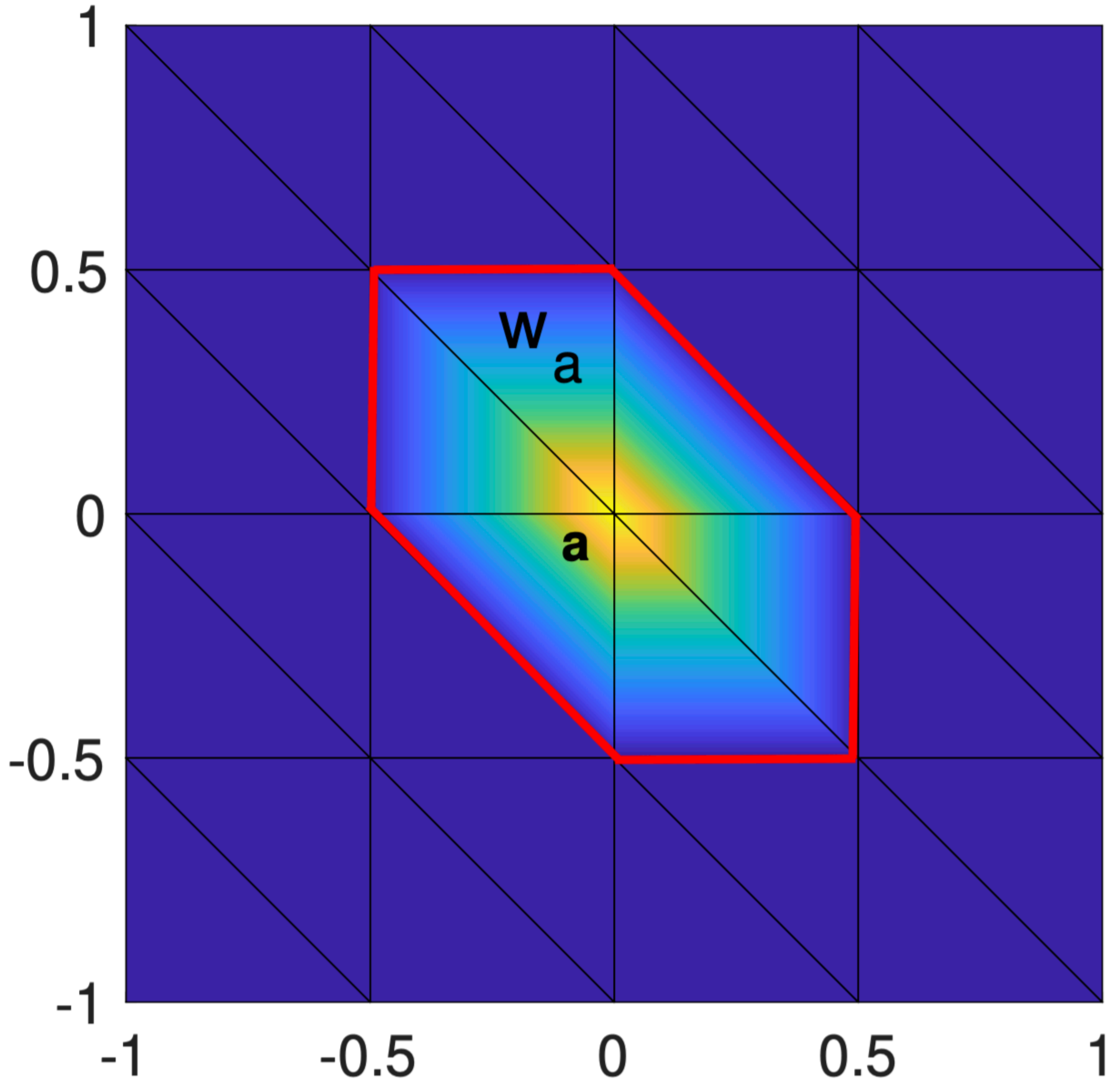
- (M) Find $u \in V$ such that $F(u) \leq F(v), \quad \forall v \in V,$ where $F(v) = \frac{1}{2}(\nabla v, \nabla v) - (f, v)$

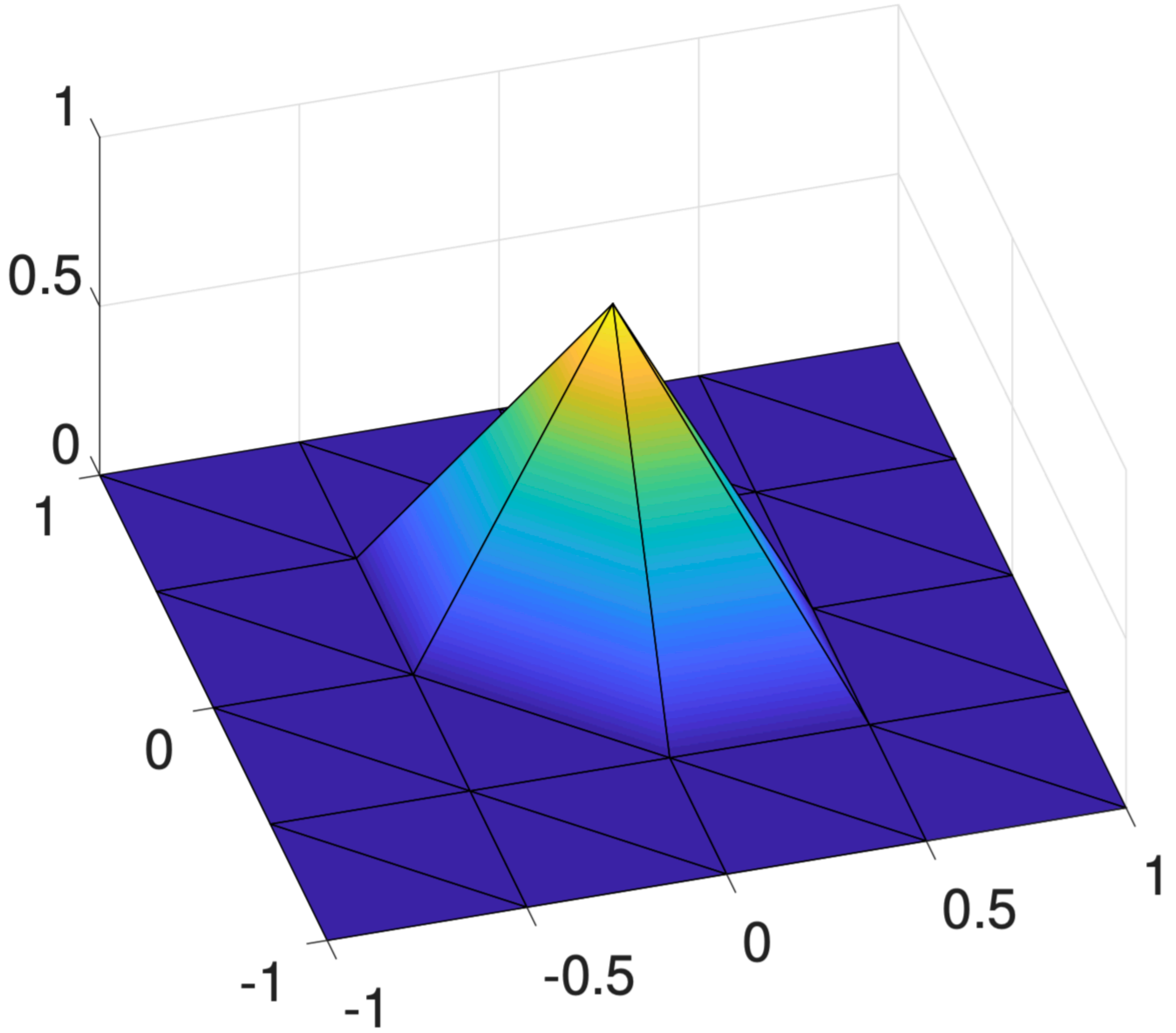
<HW> Show (V) and (M) are equivalent.

Elements for FEM



Elements for FEM





Finite Element Method

- Find $u_h \in V_h^k$ satisfying

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h^k$$

where

$$V_h^k := \left\{ v \in H_0^1(\Omega) : \ v|_T \in P_k(T),\ T \in \mathscr{T}_h \right\}$$

Analysis for FEM

- Weak Formulation: Find $u_h \in V_h^k$ satisfying

$$a_h(u_h, v_h) = b_h(v_h)$$

where $a_h(u_h, v_h) = (\nabla u_h, \nabla v_h), \quad b_h(v_h) = (f, v_h)$

- Consistency

$$a_h(u, v) = b_h(v) \quad \forall v \in H_0^1(\Omega)$$

- Galerkin orthogonality

$$a_h(u - u_h, v_h) = 0 \quad \forall v_h \in V_h^k$$

Analysis for FEM

- Weak Formulation: Find $u_h \in V_h^k$ satisfying

$$a_h(u_h, v_h) = b_h(v_h)$$

where $a_h(u_h, v_h) = (\nabla u_h, \nabla v_h), \quad b_h(v_h) = (f, v_h)$

- Coercivity

$$a_h(v_h, v_h) \geq \|\nabla v_h\|_{L^2(\Omega)}^2 \quad \forall v_h \in V_h^k$$

- Continuity

$$a_h(u_h, v_h) \leq \|\nabla u_h\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)}$$
$$b_h(v_h) \leq C(\Omega)\|f\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)}$$

Analysis for FEM

- Weak Formulation: Find $u_h \in V_h^k$ satisfying

$$a_h(u_h, v_h) = b_h(v_h)$$

where $a_h(u_h, v_h) = (\nabla u_h, \nabla v_h), \quad b_h(v_h) = (f, v_h)$

- A priori error estimate

If $u \in H^{k+1}(\Omega)$ solves the model problem, and u_h is the solution obtained from FEM, then

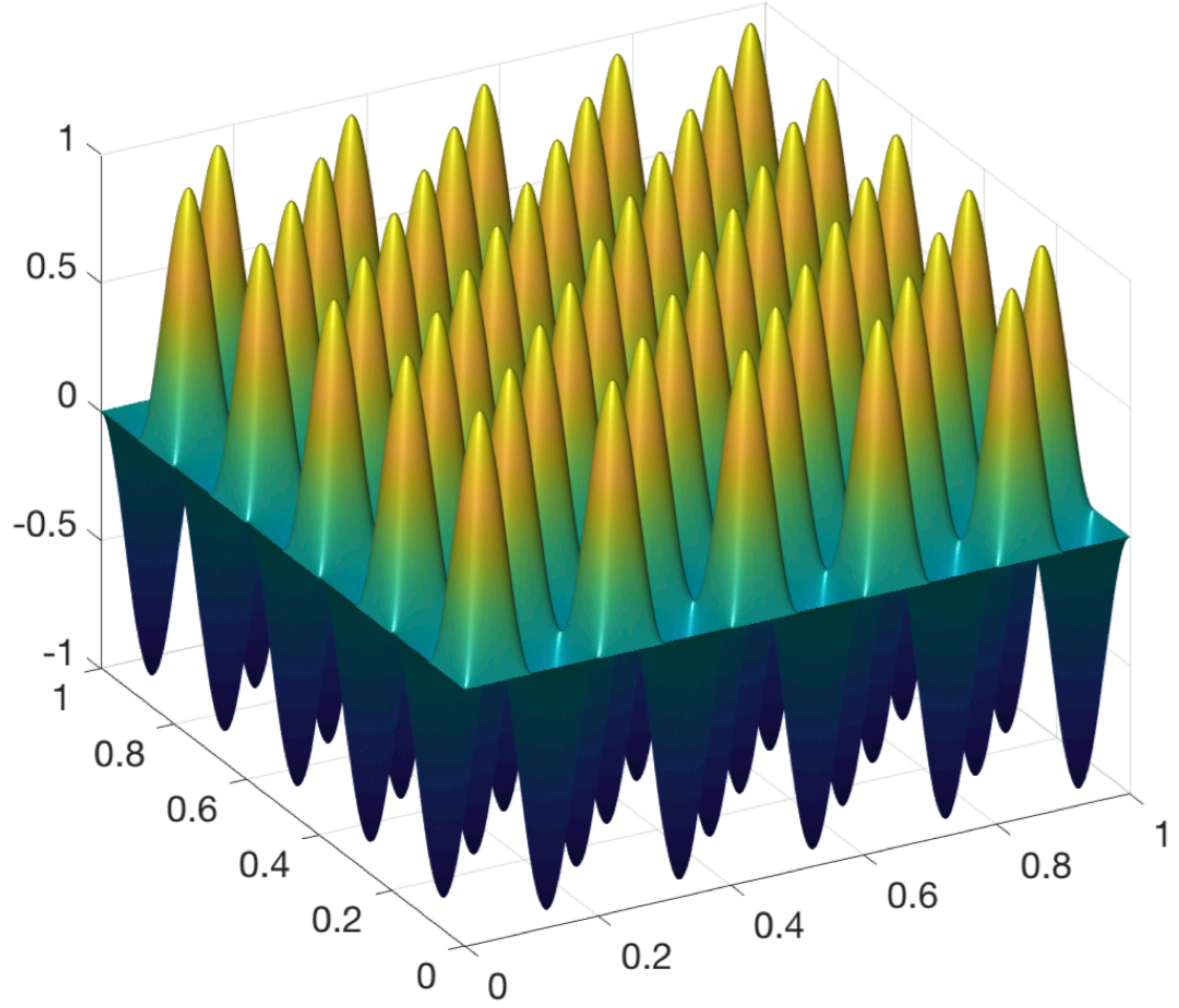
$$\|\nabla(u - u_h)\| \le Ch^k \|u\|_{k+1}$$

Numerical Results

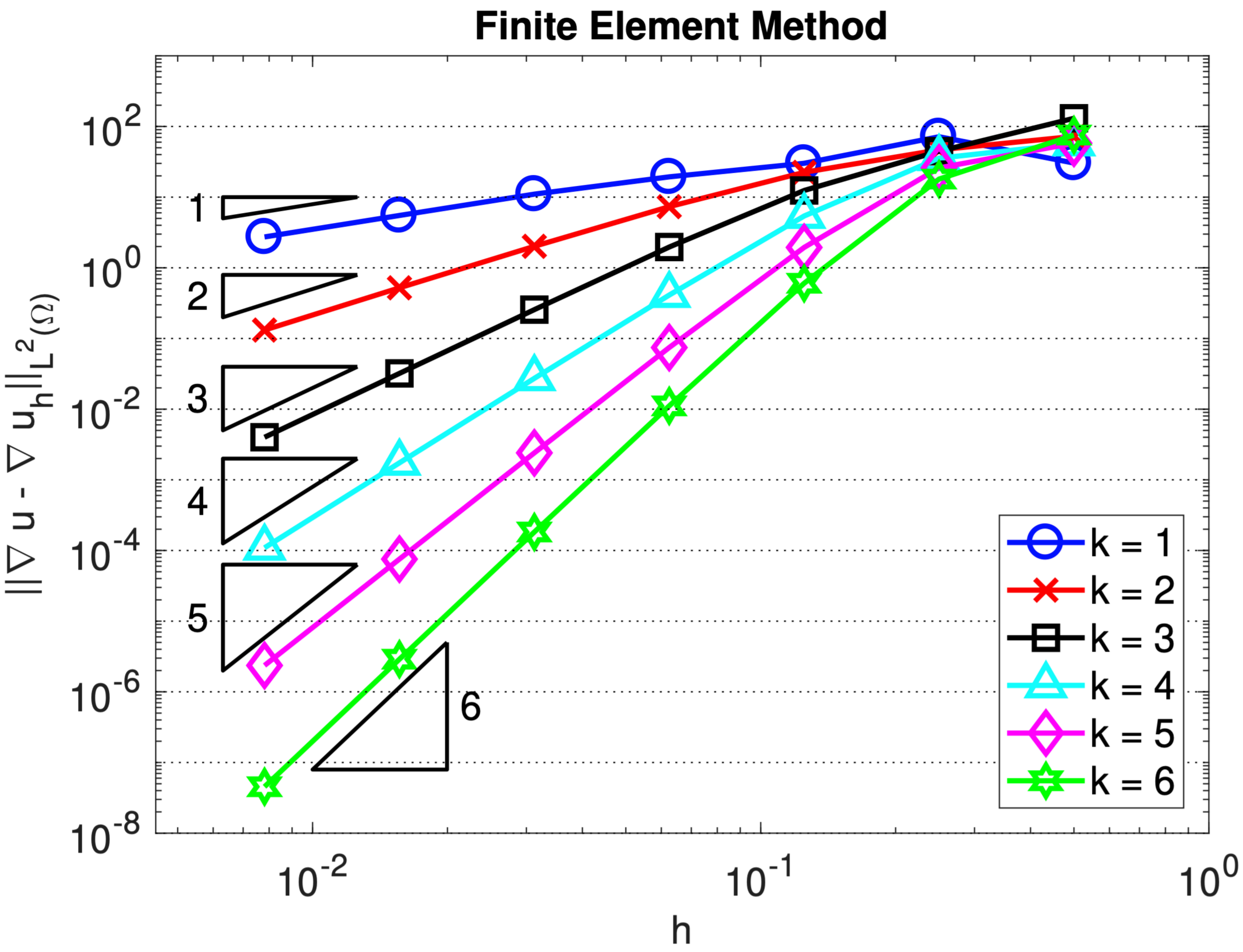
- Consider the domain $\Omega = (0,1) \times (0,1)$. Here $f(x, y)$ is chosen so that

$$u(x, y) = \sin(10\pi x)\sin(10\pi y)$$

is the exact solution.



Numerical Results



A Posteriori Error Estimate

A Priori Error Estimates

- For steady-state problems,

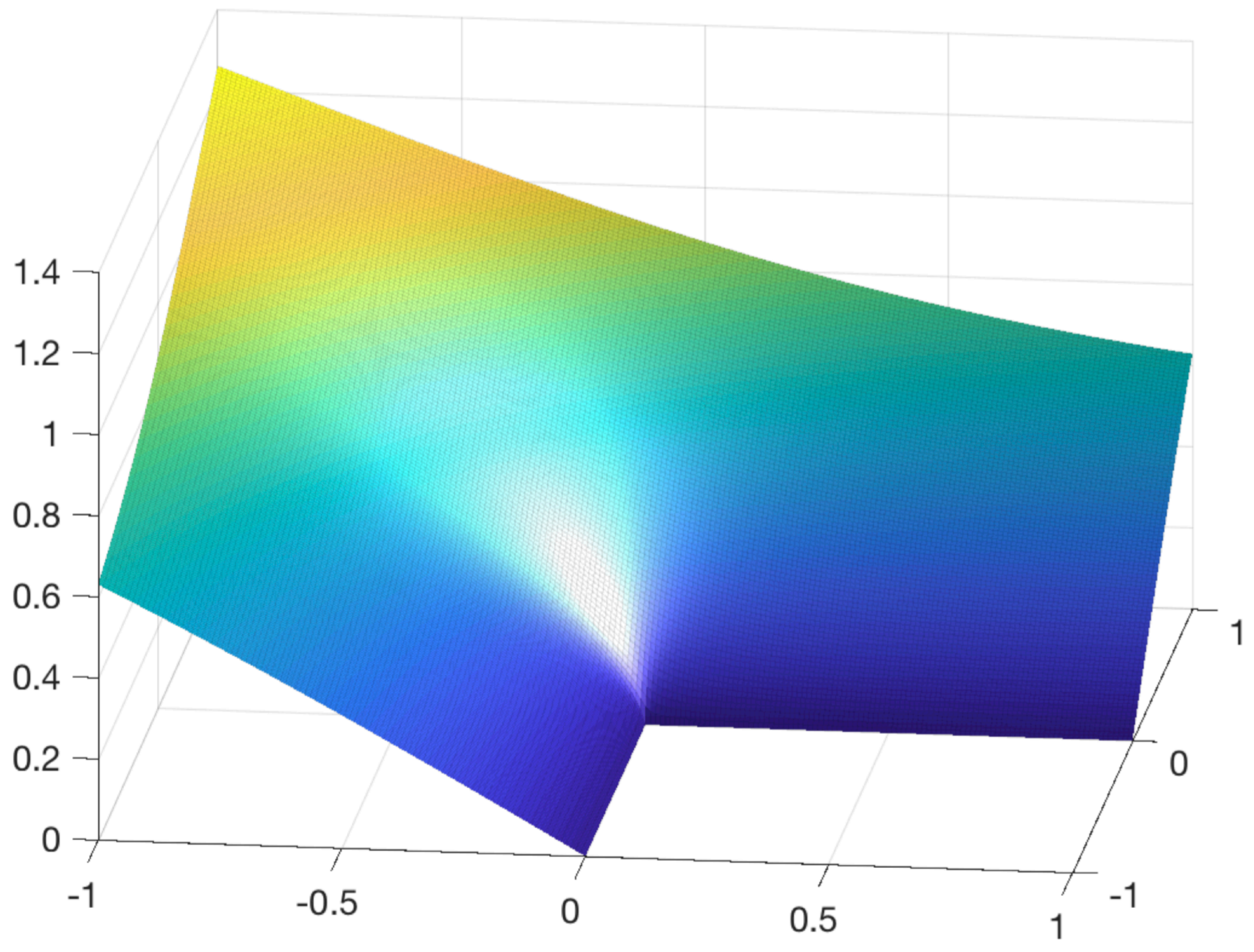
$$|||u - u_h||| \leq Ch^\alpha$$

where constants C and α are positive and $|||\cdot|||$ is a suitable norm.

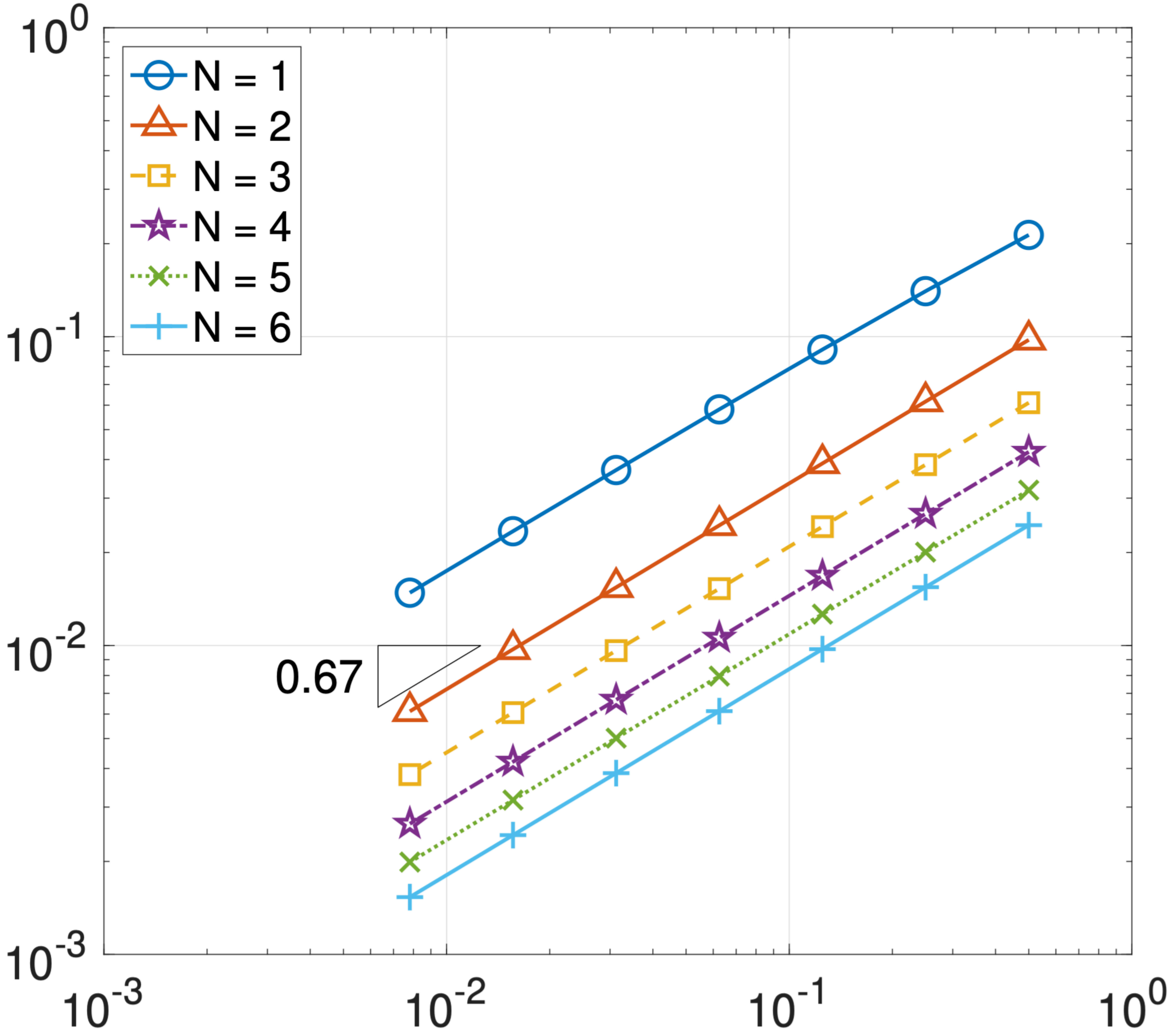
Non-smooth Solution

- Consider the domain $\Omega = (-1,1)^2 \backslash ([0,1] \times [-1,0])$. The source term $f = 0$ with the exact solution

$$u(r,\theta) = r^{2/3} \sin\left(\frac{2}{3}\theta\right).$$



Numerical Results



A Posteriori Error Estimates

- For steady-state problems,

$$|||u - u_h||| \leq \eta := \left(\sum_{T \in \mathcal{T}_h} \eta^2(T) \right)^{1/2}$$

where η is computed via known values such as the approximate solution u_h, the right hand side f, etc.

A Posteriori Error Estimates

1. Guaranteed upper bound

$$|||u - u_h||| \leq \eta := \left(\sum_{T \in \mathcal{T}_h} \eta^2(T) \right)^{1/2}$$

2. Local efficiency

$$\eta(T) \leq C|||u - u_h|||_{\mathfrak{T}(T)}, \quad \forall T \in \mathcal{T}_h$$

3. Asymptotic exactness

$$I_{\text{eff}} := \frac{\eta}{|||u - u_h|||}$$

4. Robustness

5. Small computational cost

Adaptive Algorithm

- Input

  initial triangulation $\mathscr{T}_0$

- Loop

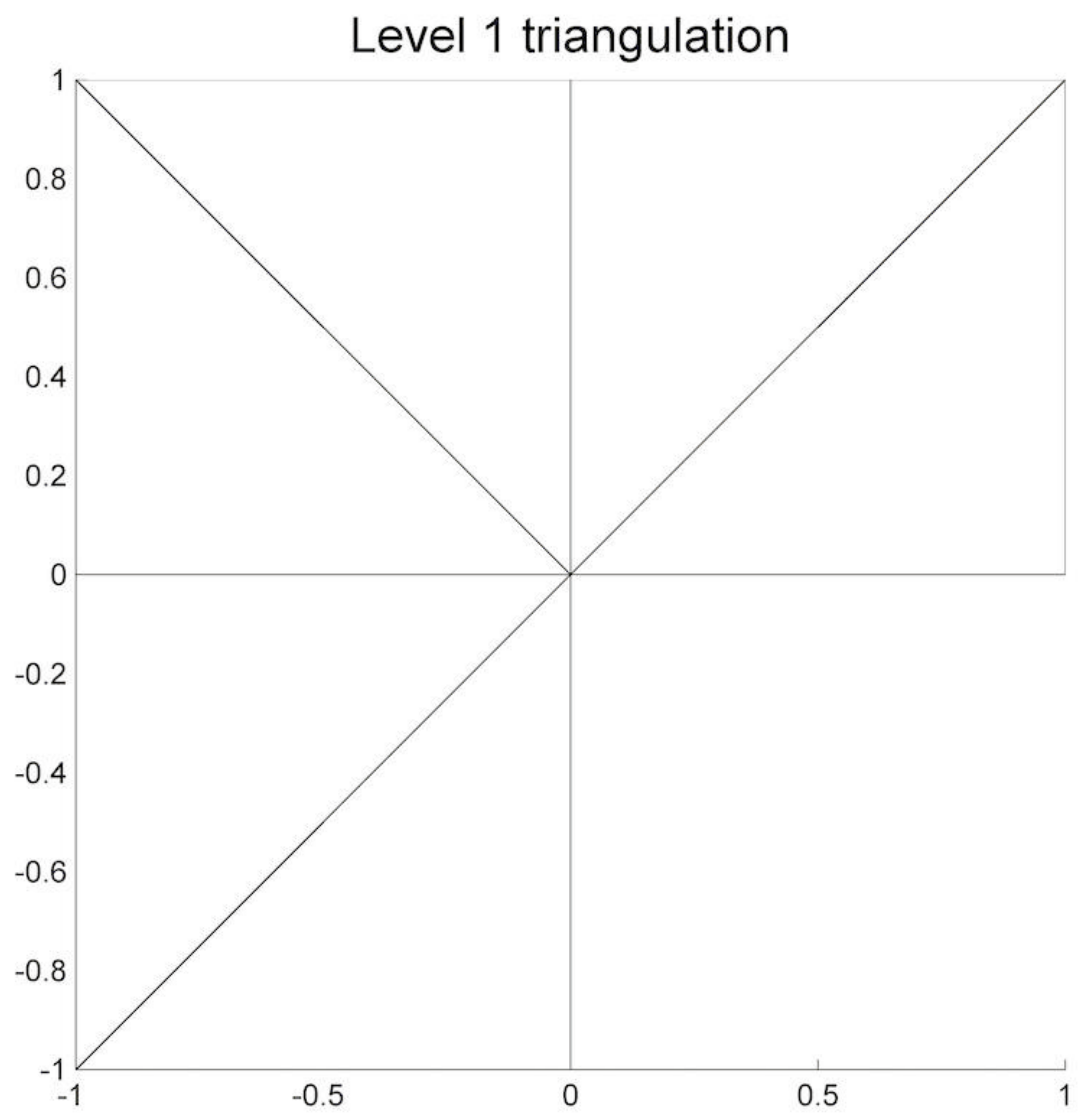
$$
\begin{tikzcd}
\boxed{\text{Solve}} \arrow[r, "\longrightarrow"] & \boxed{\text{Estimate}} \arrow[r] & \boxed{\text{Mark}} \arrow[r] & \boxed{\text{Refine}} \arrow[lll, bend left]
\end{tikzcd}
$$

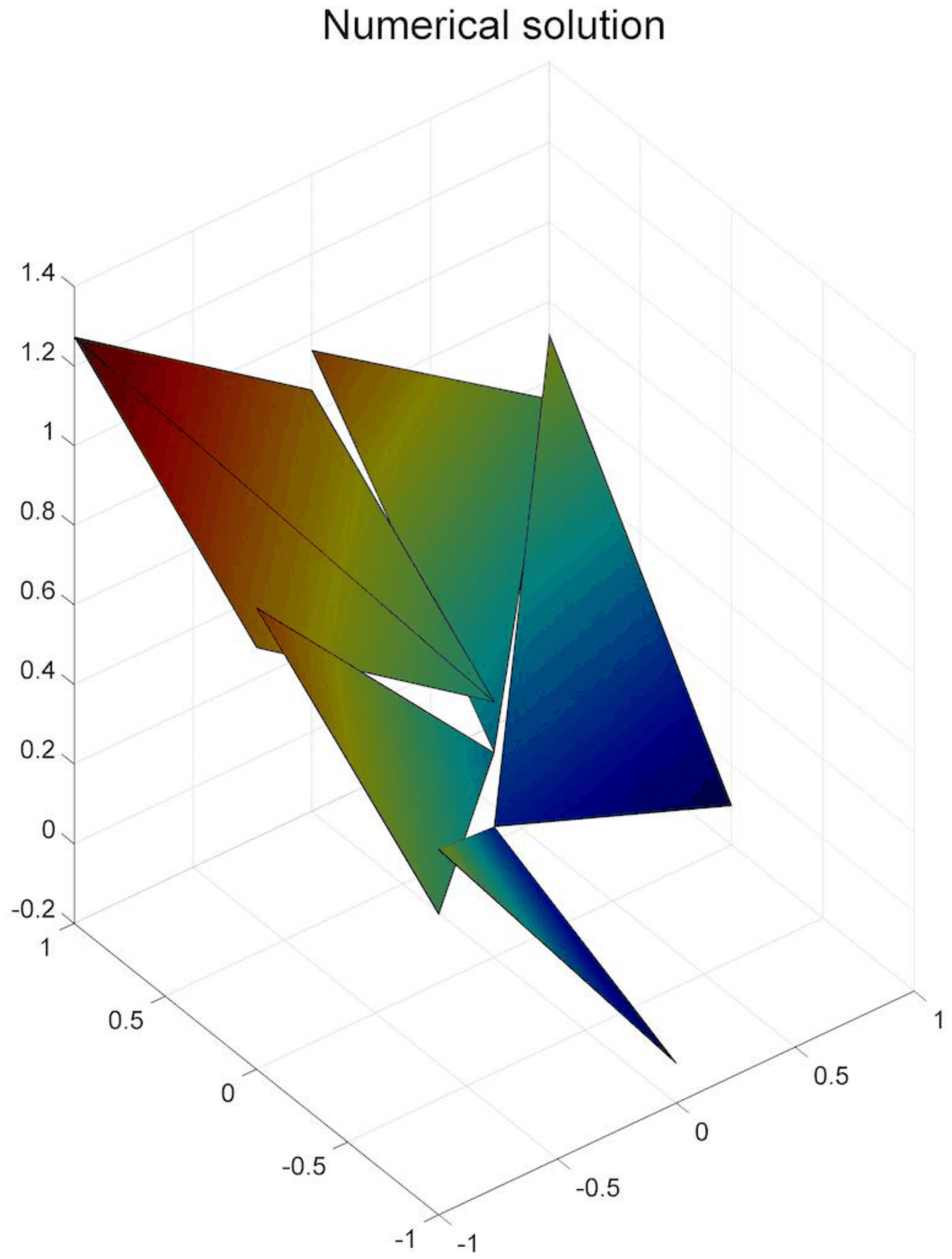
- Output

  Sequence of meshes $(\mathscr{T}_\ell)_{\ell\in\mathbb{N}_0}$, approximations $(u_\ell)_{\ell\in\mathbb{N}_0}$

Adaptive Algorithm

SOLVE

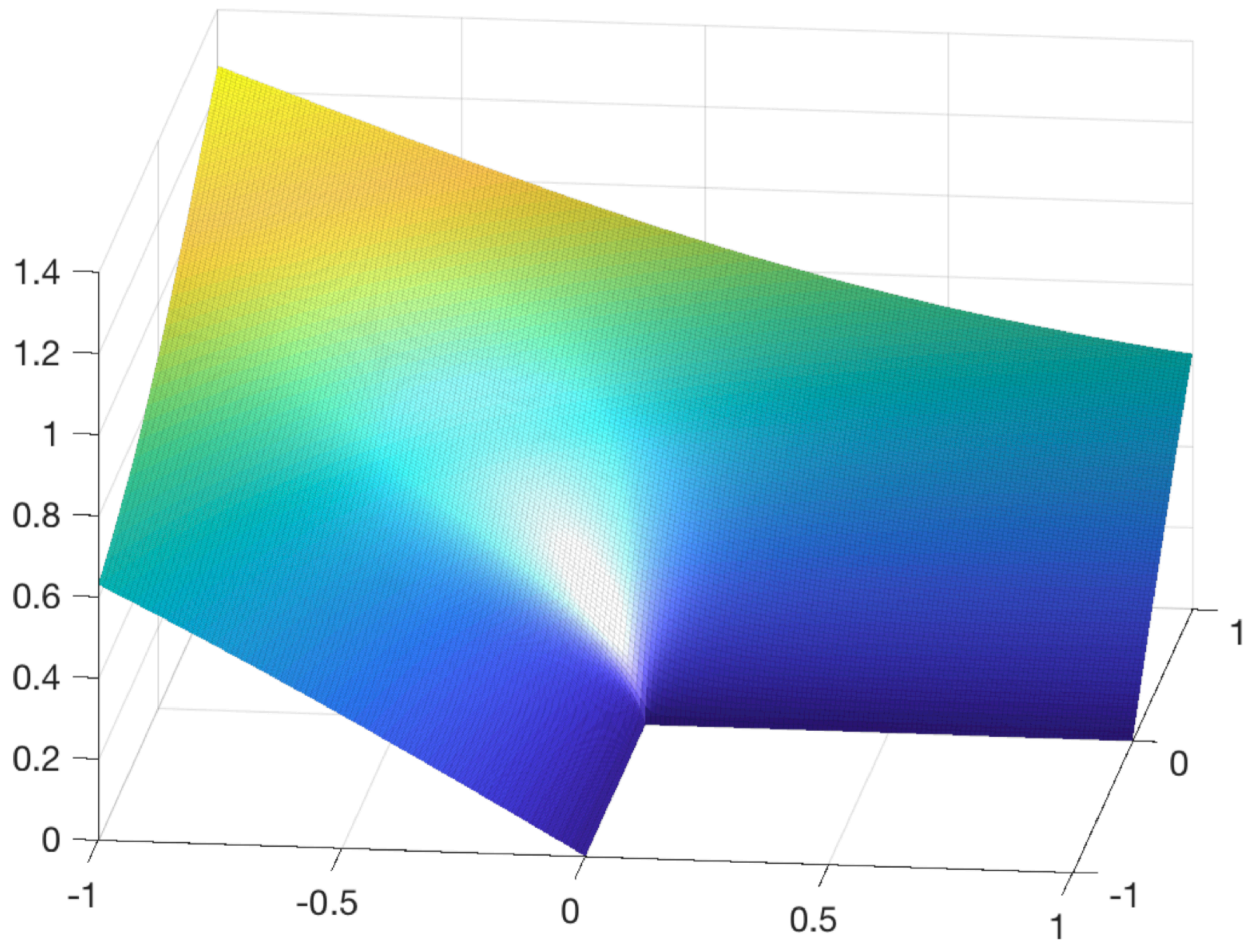




Example

- Consider the domain $\Omega = (-1,1)^2 \backslash ([0,1] \times [-1,0])$. The source term $f = 0$ with the exact solution

$$u(r, \theta) = r^{2/3} \sin\left(\frac{2}{3}\theta\right).$$



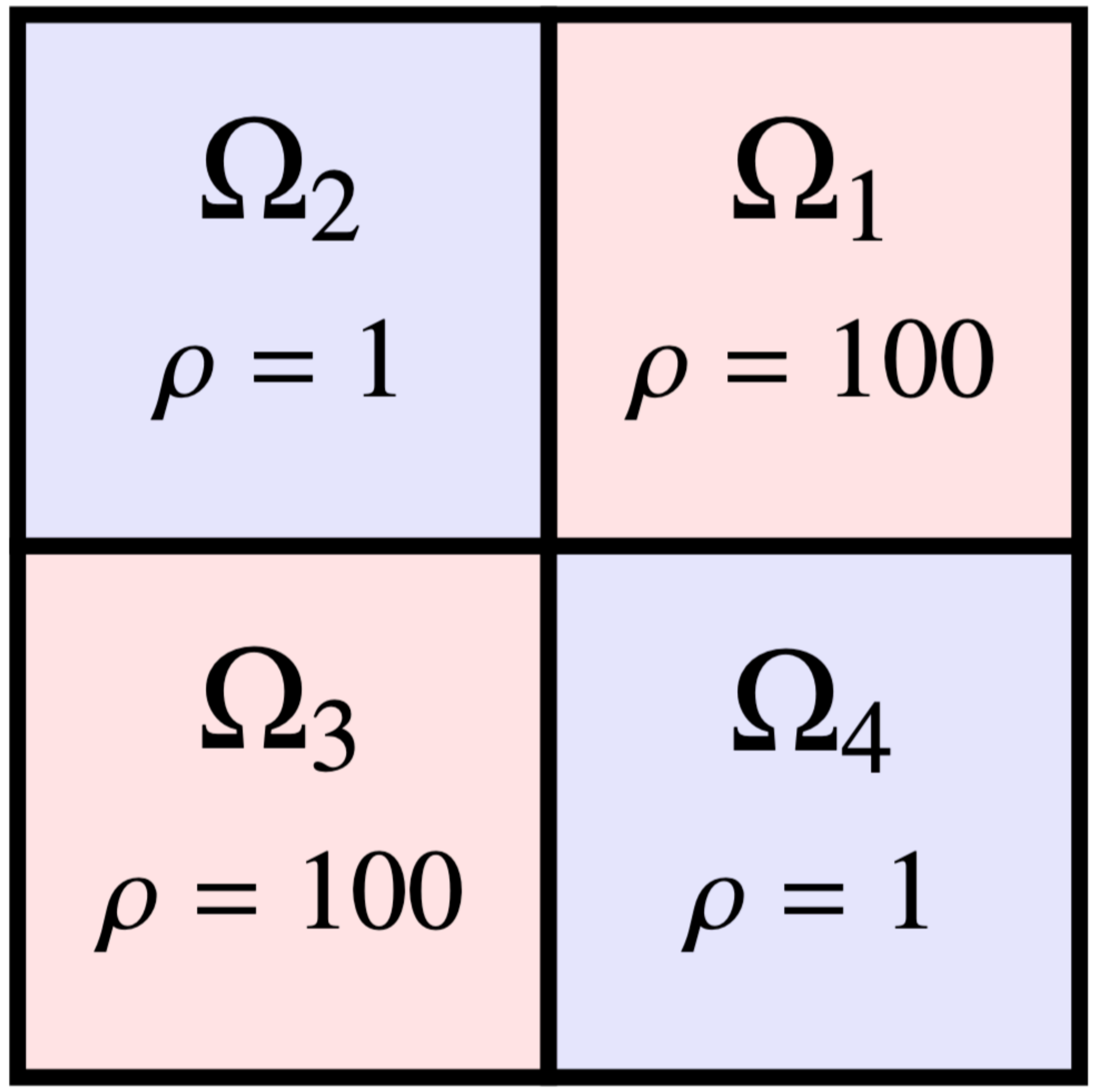
Example

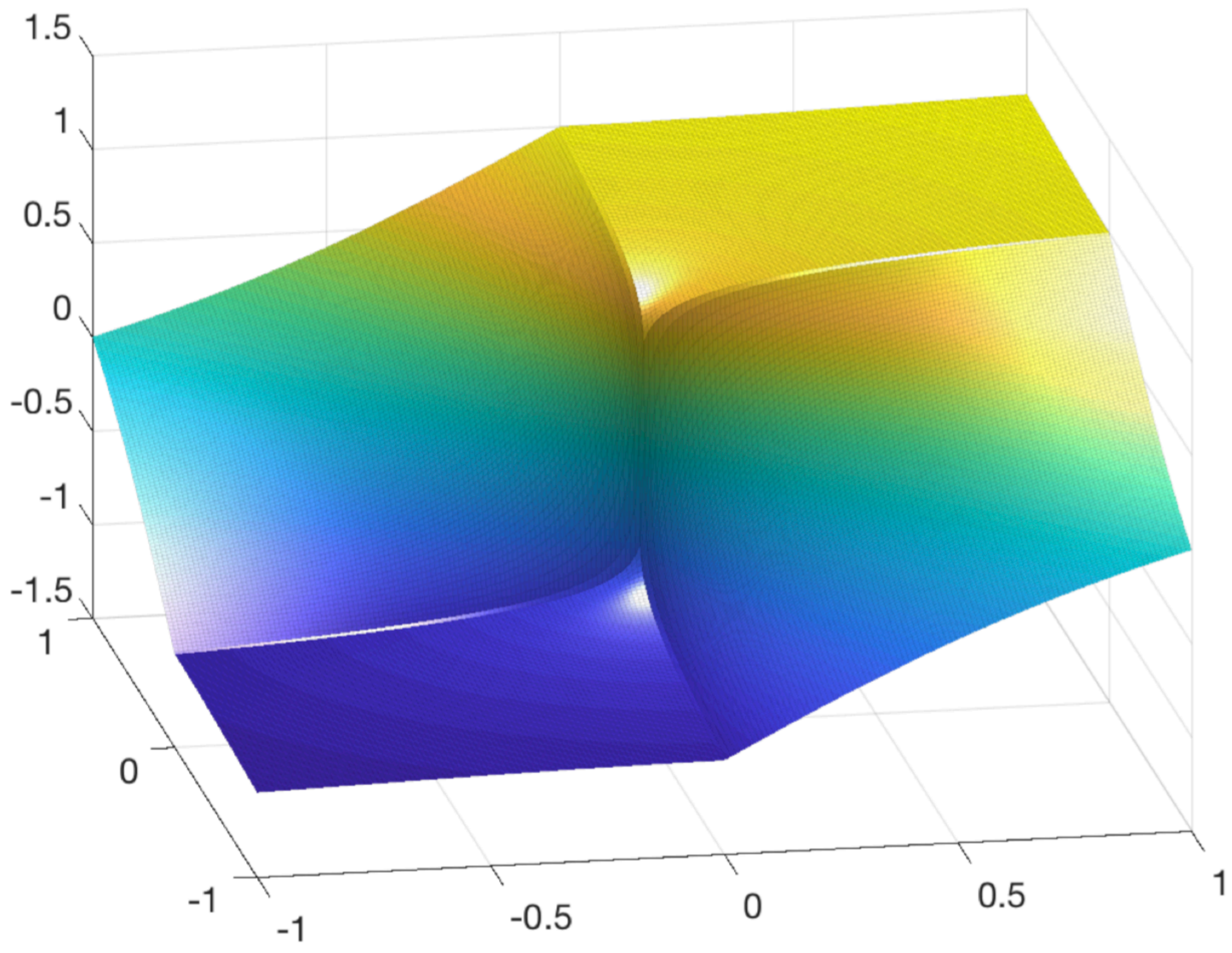
- Consider the domain $\Omega = (-1,1)^2$ divided into 4 subdomains Ω_i, $i = 1,\ldots,4$. The source term $f = 0$ with the exact solution

$$u(r, \theta) = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta)),$$

in each Ω_i. Coefficients a_i, b_i depending on Ω_i and α are given as follows:

a_3

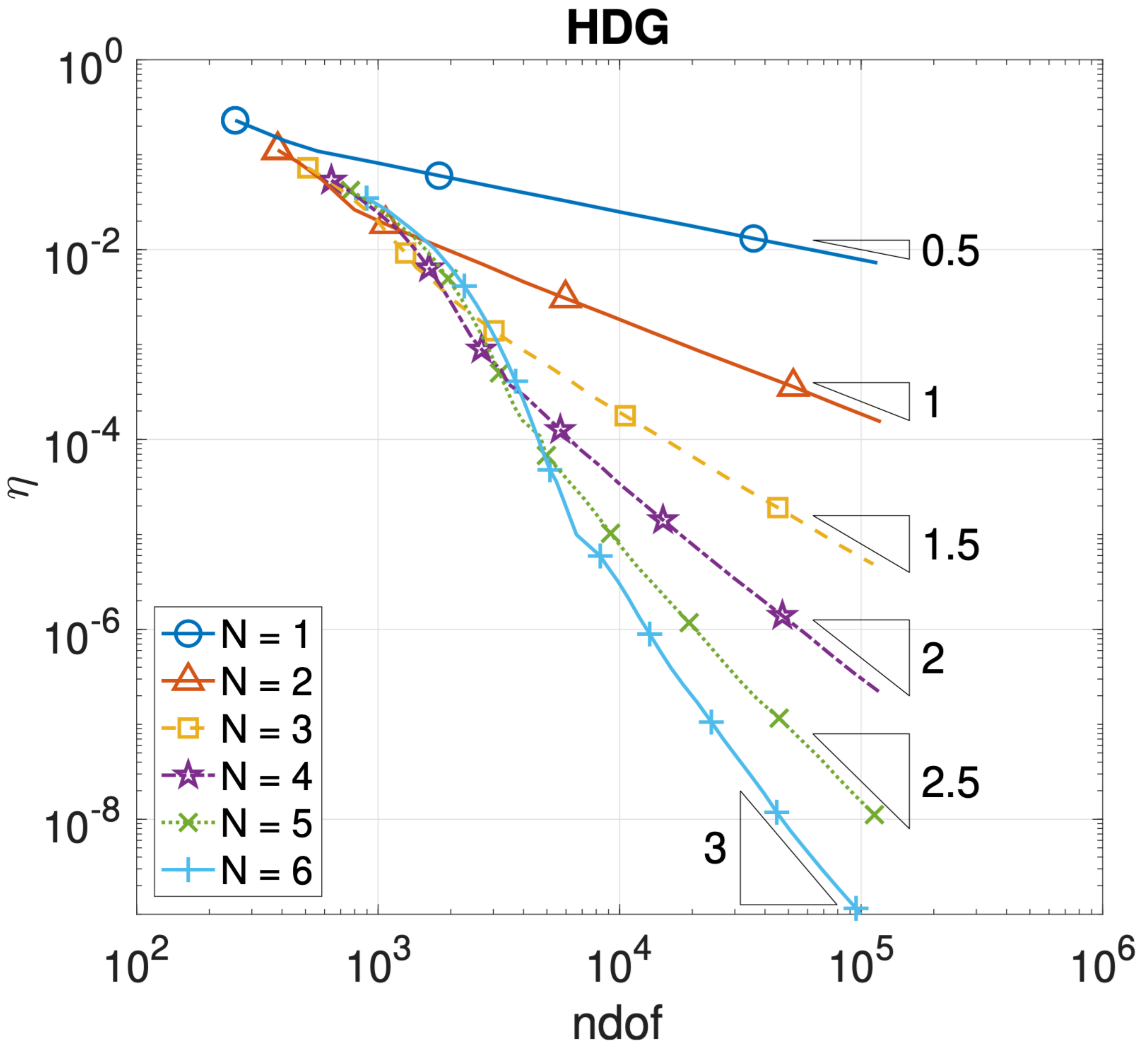




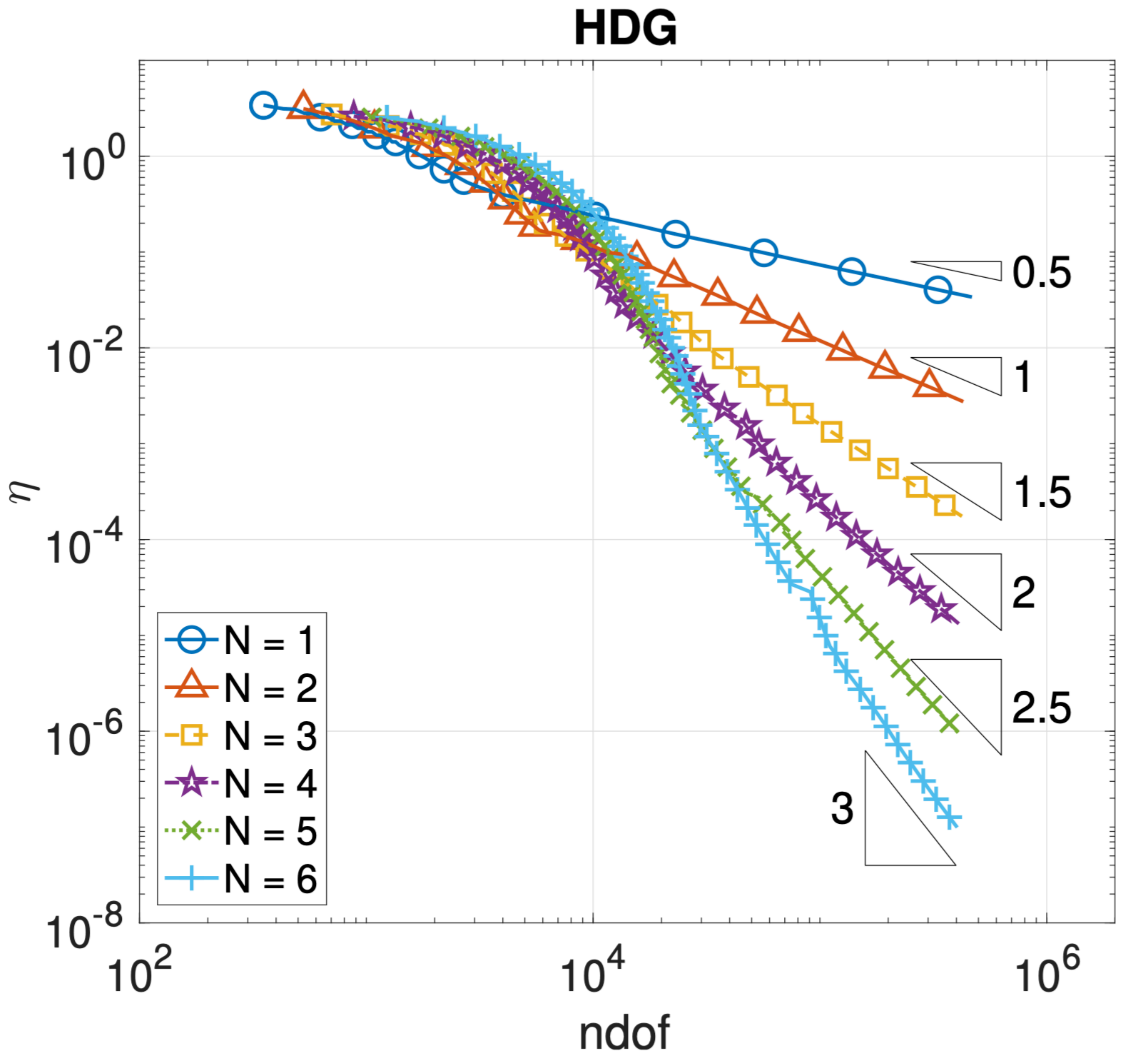
75

Numerical Results

Example 1

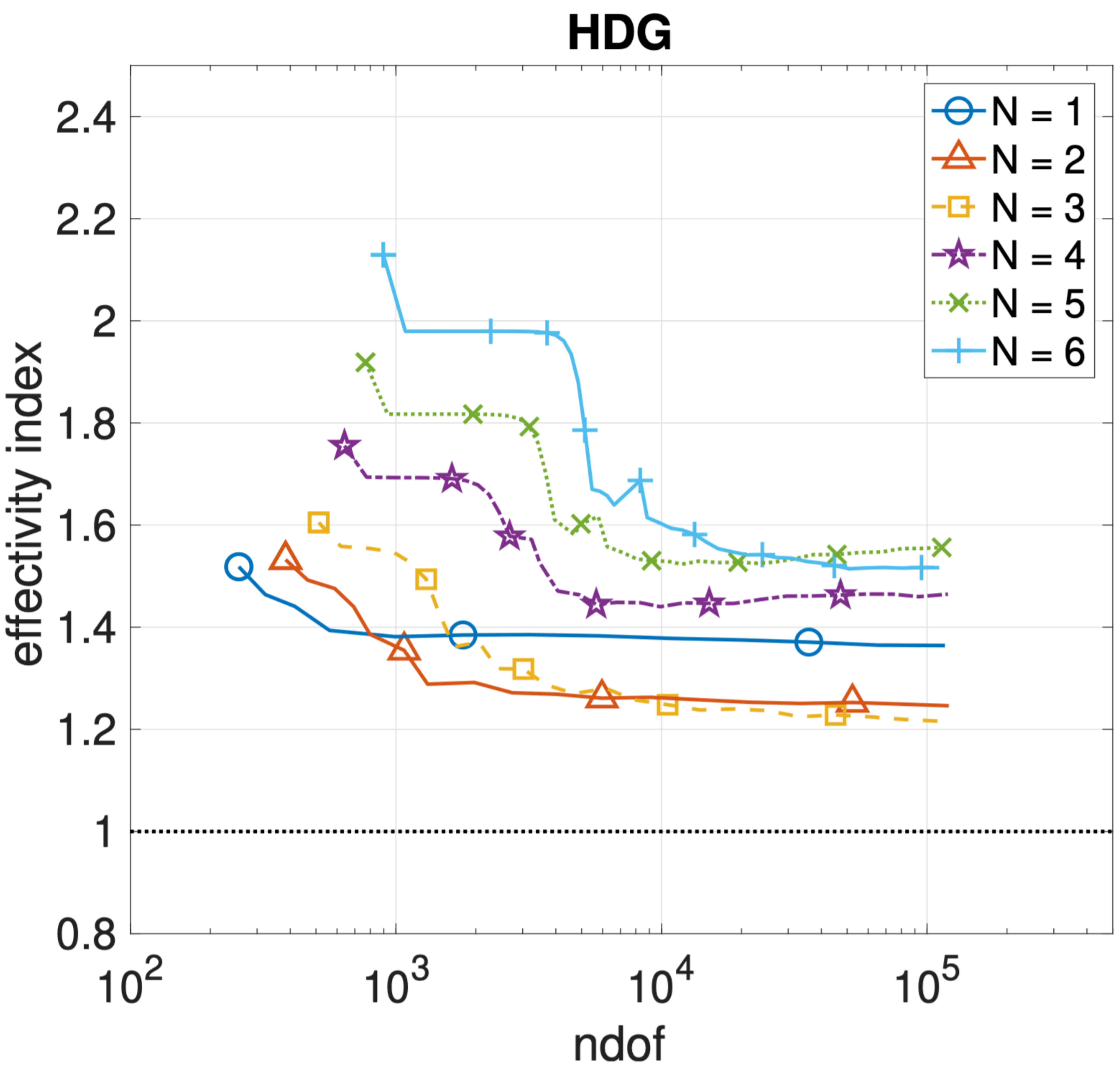


Example 2

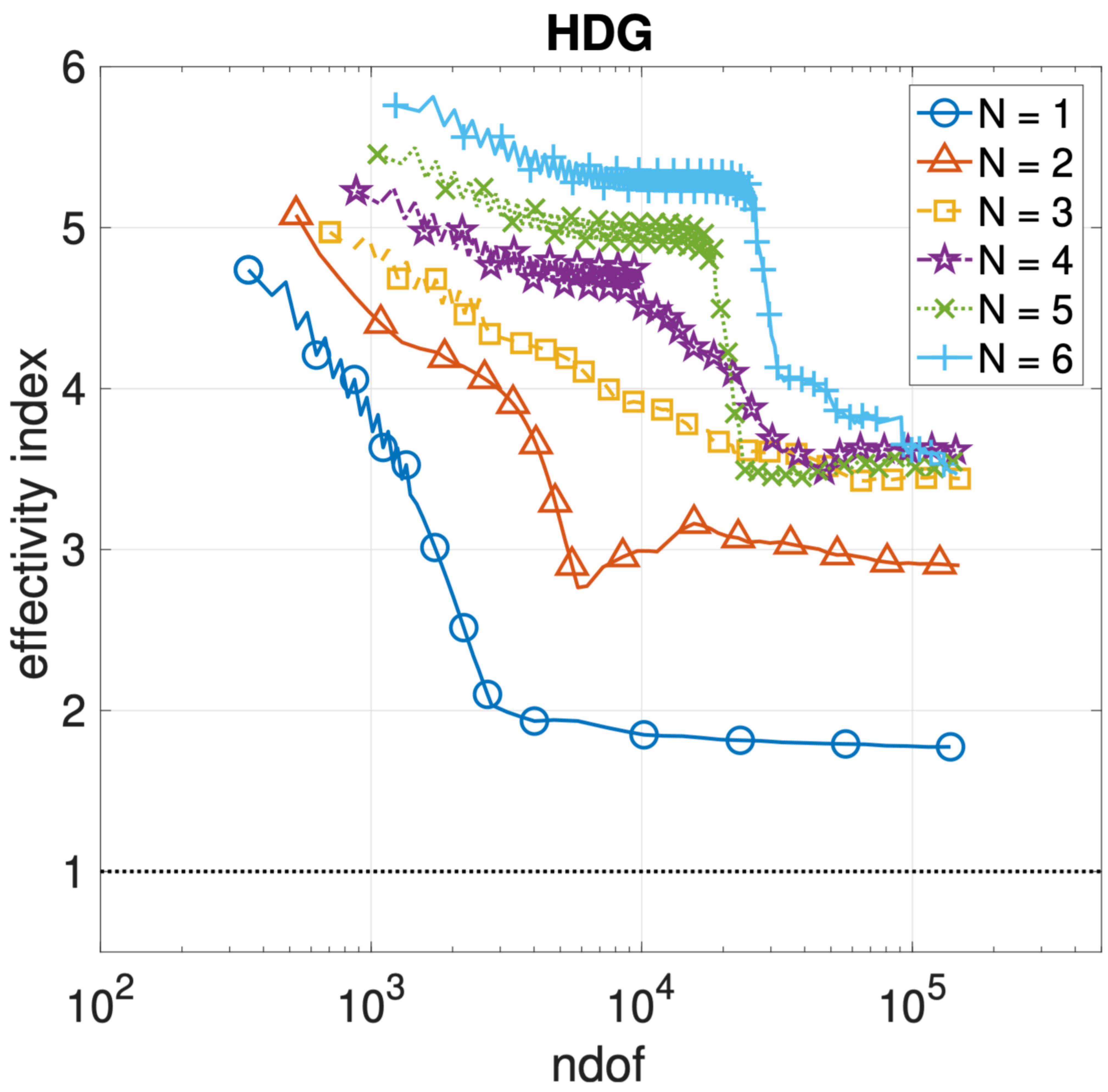


Numerical Results

Example 1



Example 2



To be continued…

- Submit the assignment at the beginning of Lecture 4 (7/8 Mon)

  <HW> Show (V) and (M) are equivalent.

- Supplies
  - Laptop or tablet PC with a keyboard
  - Goggle account
  - Internet connection

Q&A