Finite Element Methods for PDEs

FEM for Poisson's problem in 2D

2024 AJOU-KYUSHU SUMMER SCHOOL ON APPLIED MATHEMATICS

Outline

2D Poisson problem

Consider the two dimensional Poisson problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= u_D && \text{on } \partial\Omega \end{aligned}$$

Weak formulation

Find $u(x, y) \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} = \int_\Omega f v \, d\boldsymbol{x}, \qquad \forall v \in H_0^1(\Omega).$$

Variational formulation

Find $u_h \in V_h^k$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\boldsymbol{x} = \int_\Omega f v_h \, d\boldsymbol{x}, \qquad \forall v_h \in V_h^k$$

where

$$V_h^k = \{v_h \in H_0^1(\Omega) \mid v_h|_T \in P_k(T), \ \forall T \in \mathcal{T}_h\},$$

and $P_k(T)$ is the polynomial function space of degrees $\leq k$.

Affine mapping

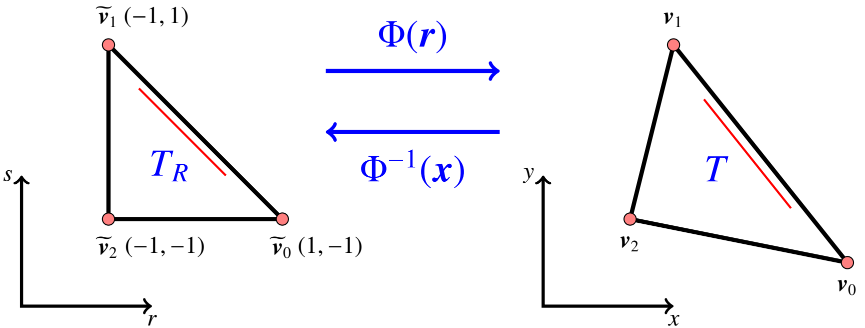


Figure: An affine mapping from the reference triangle T_R to a triangle T.

Barycentric coordinates

Barycentric coordinates $(\widetilde{\lambda}_0,\ \widetilde{\lambda}_1,\ \widetilde{\lambda}_2)$ have the properties

$$\begin{cases} 0 \le \widetilde{\lambda}_i(\boldsymbol{r}) \le 1, & i = 0,\ 1,\ 2 \\ \widetilde{\lambda}_0(\boldsymbol{r}) + \widetilde{\lambda}_1(\boldsymbol{r}) + \widetilde{\lambda}_2(\boldsymbol{r}) = 1. \end{cases}$$

Affine mapping

Let us define a triangle T as

$$T = \mathrm{span}\{\boldsymbol{v}_0,\ \boldsymbol{v}_1,\ \boldsymbol{v}_2\}, \quad \boldsymbol{v}_i = (v_i^{(1)}, v_i^{(2)}),$$

where $\boldsymbol{v}_0,\ \boldsymbol{v}_1,\ \boldsymbol{v}_2$ are vertices of T.

Then, we have an affine mapping Φ such that

$$\Phi(\boldsymbol{r}) = \frac{r+1}{2}\boldsymbol{v}_0 + \frac{s+1}{2}\boldsymbol{v}_1 - \frac{r+s}{2}\boldsymbol{v}_2 = \boldsymbol{x}.$$

Properties

1. $\lambda_0 = \frac{r+1}{2}, \quad \lambda_1 = \frac{s+1}{2}, \quad \lambda_2 = -\frac{r+s}{2}$
2. $x_r = \frac{v_0^{(1)} - v_2^{(1)}}{2}, \quad y_r = \frac{v_0^{(2)} - v_2^{(2)}}{2}, \quad x_s = \frac{v_1^{(1)} - v_2^{(1)}}{2}, \quad y_s = \frac{v_1^{(2)} - v_2^{(2)}}{2}$
3. $r_x = \frac{y_s}{J}, \qquad r_y = -\frac{x_s}{J}, \qquad s_x = -\frac{y_r}{J}, \qquad s_y = \frac{x_r}{J} \qquad (J = x_r y_s - x_s y_r)$
4. $\lambda_i(\boldsymbol{x}) = \widetilde{\lambda}_i(\Phi^{-1}(\boldsymbol{x})), \qquad \widetilde{\lambda}_i(\boldsymbol{r}) = \lambda_i(\Phi(\boldsymbol{r}))$
5. $\frac{d}{dx}\lambda_i(\boldsymbol{x}) = r_x \frac{d}{dr}\widetilde{\lambda}_i(\boldsymbol{r}) + s_x \frac{d}{ds}\widetilde{\lambda}_i(\boldsymbol{r})$
6. $\frac{d}{dy}\lambda_i(\boldsymbol{x}) = r_y \frac{d}{dr}\widetilde{\lambda}_i(\boldsymbol{r}) + s_y \frac{d}{ds}\widetilde{\lambda}_i(\boldsymbol{r})$

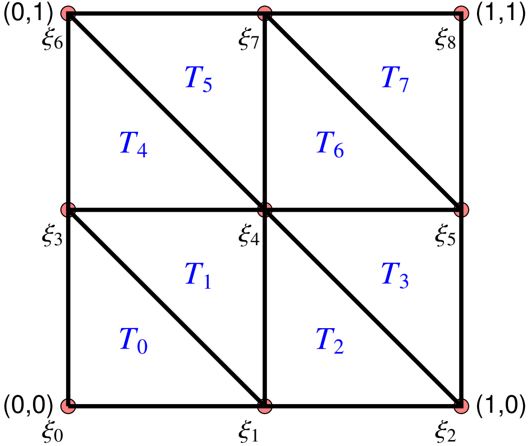
Properties

Using the property $I = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}$, we have

$$I = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} = \begin{bmatrix} x_r & x_s \\ y_r & y_s \end{bmatrix} \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix} = \frac{1}{x_r y_x - s_x y_r} \begin{bmatrix} y_s & -x_s \\ -y_r & x_r \end{bmatrix}.$$

Triangulation data (P_1)

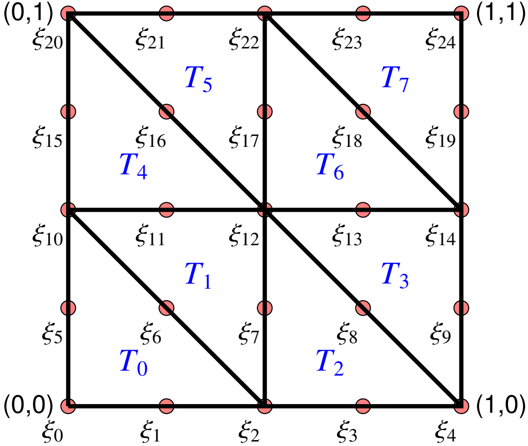
If $\Omega = [0,\ 1]^2$, $M = 2$ and $k = 1$, data are stored as follows.



$$\begin{aligned}
\text{c4n} &= [[0,0],\ [1/2,0],\ [1,0],\\
&\quad [0,1/2],\ [1/2,1/2],\ [1,1/2]\\
&\quad [0,1],\ [1/2,1],\ [1,1]],\\[1em]
\text{n4e} &= [[1,3,0],\ [3,1,4],\ [2,4,1],\ [4,2,5],\\
&\quad [4,6,3],\ [6,4,7],\ [5,7,4],\ [7,5,8]],\\[1em]
\text{ind4e} &= [[0,1,3],\ [4,3,1],\ [1,2,4],\ [5,4,2],\\
&\quad [3,4,6],\ [7,6,4],\ [4,5,7],\ [8,7,5]],\\[1em]
\text{n4db} &= [0,1,2,3,5,6,7,8]
\end{aligned}$$

Triangulation data (P_2)

If $\Omega = [0,\ 1]^2$, $M = 2$ and $k = 2$, data are stored as follows.

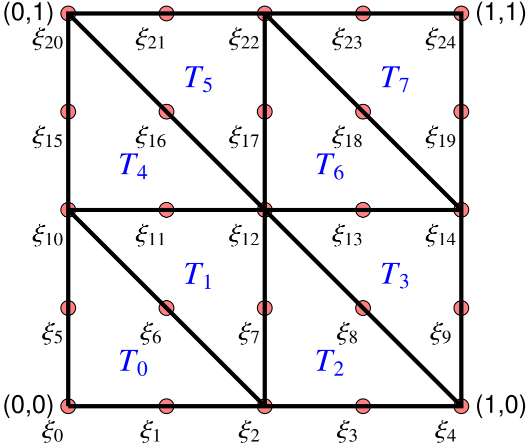


c4n $= [[0,0],\ [1/4,0],\ [1/2,0],\ [3/4,0],\ [1,0],$
$[0,1/4],\ [1/4,1/4],\ [1/2,1/4],\ [3/4,1/4],$
$[1,1/4],\ [0,1/2],\ [1/4,1/2],\ [1/2,1/2],$
$[3/4,1/2],\ [1,1/2],\ [0,3/4],\ [1/4,3/4],$
$[1/2,3/4],\ [3/4,3/4],\ [1,3/4],\ [0,1],$
$[1/4,1],\ [1/2,1],\ [3/4,1],\ [1,1]],$

n4e $= [[2,10,0],\ [10,2,12],\ [4,12,2],\ [12,4,14],$
$[12,20,10],\ [20,12,22],\ [14,22,12],$
$[22,14,24]]$

Triangulation data (P_2)

If $\Omega = [0,\ 1]^2$, $M = 2$ and $k = 2$, data are stored as follows.



$$\begin{aligned}\text{ind4e} \quad &= [[0, 1, 2, 5, 6, 10],\ [12, 11, 10, 7, 6, 2],\\ &\quad [2, 3, 4, 7, 8, 12],\ [14, 13, 12, 9, 8, 4],\\ &\quad [10, 11, 12, 15, 16, 20],\ [22, 21, 20, 17, 16, 12],\\ &\quad [12, 13, 14, 17, 18, 22],\ [24, 23, 22, 19, 18, 14]]\end{aligned}$$

$$\text{n4db} \quad = [0, 1, 2, 3, 5, 9, 10, 14, 15, 19, 20, 21, 22, 23, 24].$$

mesh_FEM2D_Tri

This python code generates an uniform triangular mesh on the domain $[x_\ell, x_r] \times [y_\ell, y_r]$ in 2D with $2M_x$ elements along x-direction and $2M_y$ elements along y-direction. Also this code returns an index matrix for continuous k-th order polynomial approximations.

Basis functions

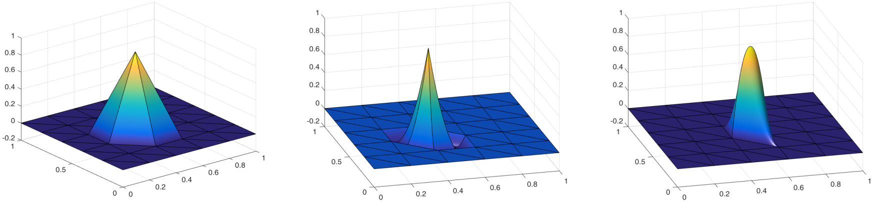
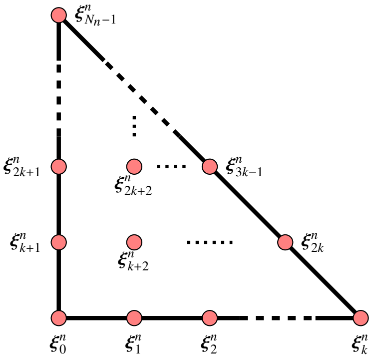


Figure: Global basis functions of V_h^1 (left) and V_h^2 (others) on an interval.

$$\psi_i(\boldsymbol{\xi}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

$$\sum_{i=0}^{N-1} \psi_i(\boldsymbol{x}) = 1 \qquad \forall x \in \Omega$$

Local basis functions



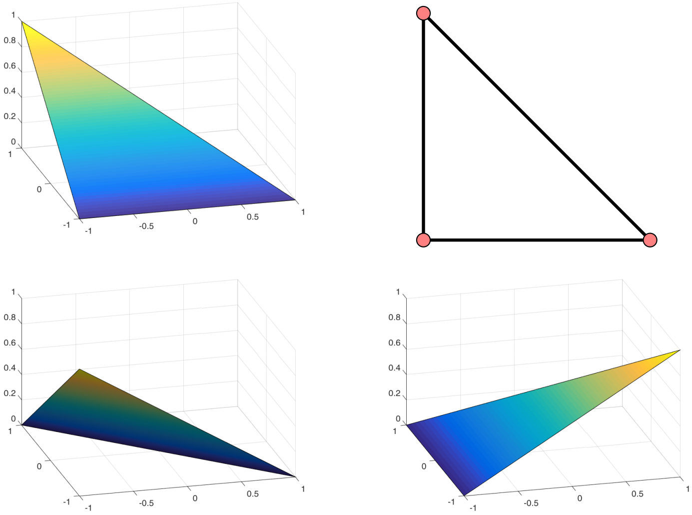
$$N_n = \frac{(k+1)(k+2)}{2}$$

$$\xi^n_j = \xi^n_{(\ell-1)(k+2-\ell/2)+m-1}$$

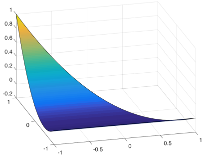
(ℓ : row number, m : column number, $\ell + m \leq k+2$)

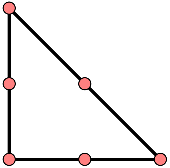
Figure: Node numbering in the n-th triangular element

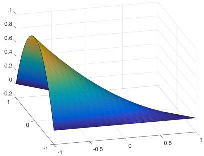
Local basis functions

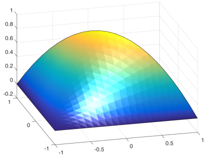


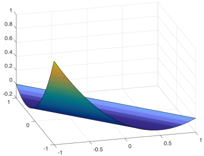
Local basis functions

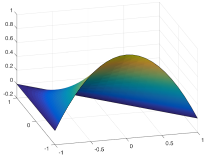


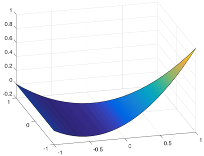












Local basis functions

$$\psi_i^n(\boldsymbol{x}) = 0 \quad \text{if } x \in \Omega \setminus T_n$$

$$\sum_{i=0}^{N_n-1} \psi_i^n(\boldsymbol{x}) = 1 \quad \forall x \in T_n,$$

Interpolation

Using these basis functions, the interpolate function of a function $f(\boldsymbol{x})$ can be written as follows

$$\mathcal{I} f(\boldsymbol{x}) = \sum_{i=0}^{N-1} f_i \psi_i(\boldsymbol{x})$$

where $f_i = f(\boldsymbol{\xi}_i)$.

Numerical solutions

- Numerical solution

$$u_h = \sum_{i=0}^{N-1} u_i \psi_i$$

- Local solution

$$u_h\Big|_{T_n} = \sum_{i=0}^{N_n-1} u_i^n \psi_i^n$$

- Derivatives

$$\nabla u_h = \sum_{i=0}^{N-1} u_i \nabla \psi_i,$$

$$\nabla u_h\Big|_{T_n} = \sum_{i=0}^{N_n-1} u_i^n \nabla \psi_i^n$$

Variational formulation

For a test function $\psi_i(\boldsymbol{x}) \in V_h^k$, the variational formulation can be rewritten as

$$\sum_{j=0}^{N-1} u_j \int_\Omega \nabla\psi_i \cdot \nabla\psi_j \, d\boldsymbol{x} = \int_\Omega f\psi_i \, d\boldsymbol{x}.$$

Finite element system

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$$

where

$$\begin{aligned}
(\boldsymbol{A})_{ij} &= \int_\Omega \nabla\psi_{i-1} \cdot \nabla\psi_{j-1} \, d\boldsymbol{x} \\
(\boldsymbol{b})_i &= \int_\Omega f\psi_{i-1} \, d\boldsymbol{x} \\
(\boldsymbol{u})_j &= u_{j-1}
\end{aligned}$$

Finite element system

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{M}\boldsymbol{f}$$

where

$$(\boldsymbol{A})_{ij} = \int_\Omega \nabla\psi_{i-1} \cdot \nabla\psi_{j-1}\, d\boldsymbol{x}$$

$$(\boldsymbol{u})_j = u_{j-1}$$

$$(\boldsymbol{M})_{ij} = \int_\Omega \psi_{i-1}\psi_{j-1}\, d\boldsymbol{x}$$

$$(\boldsymbol{f})_j = f_{j-1}$$

Matrices

$$\boldsymbol{A} = \sum_{n=1}^{M} \boldsymbol{A}_{T_n},$$

$$\boldsymbol{M} = \sum_{n=1}^{M} \boldsymbol{M}_{T_n}$$

where N_n by N_n matrices $\boldsymbol{A}_{T_n}$ and $\boldsymbol{M}_{T_n}$ are defined as

$$(\boldsymbol{A}_{T_n})_{ij} = \int_{T_n} \nabla\psi^n_{i-1}(x) \cdot \nabla\psi^n_{j-1}\, d\boldsymbol{x}$$

$$(\boldsymbol{M}_{T_n})_{ij} = \int_{T_n} \psi^n_{i-1}\psi^n_{j-1}\, d\boldsymbol{x}$$

where $1 \leq i,\ j \leq N_n$.

Lemma 2.1

For a given interval $T = \mathrm{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2\}$ with barycentric coordinates $\lambda_0,\ \lambda_1$ and λ_2, it holds for $a,\ b,\ c \in \mathbb{N}_0$ that

$$\int_T \lambda_0^a \lambda_1^b \lambda_2^c \, d\boldsymbol{x} = 2|T| \frac{a!\ b!\ c!}{(a+b+c+2)!} \tag{1}$$

where $|T|$ is the area of T.

Differentiation matrix

$$(\boldsymbol{Dx})_{ij} = \frac{\partial \psi_{j-1}}{\partial x}(\boldsymbol{\xi}_{i-1}), \qquad (\boldsymbol{Dy})_{ij} = \frac{\partial \psi_{j-1}}{\partial y}(\boldsymbol{\xi}_{i-1})$$

$\Rightarrow$

$$\frac{\partial}{\partial x}\psi_{i-1}(\boldsymbol{x}) = \sum_{j=0}^{N-1} \frac{\partial \psi_{i-1}}{\partial x}(\boldsymbol{\xi}_j)\psi_j(\boldsymbol{x}) = (\boldsymbol{Dx}^t)_i \boldsymbol{\psi}$$

$$\frac{\partial}{\partial y}\psi_{i-1}(\boldsymbol{x}) = \sum_{j=0}^{N-1} \frac{\partial \psi_{i-1}}{\partial y}(\boldsymbol{\xi}_j)\psi_j(\boldsymbol{x}) = (\boldsymbol{Dy}^t)_i \boldsymbol{\psi}$$

where

$$(\boldsymbol{Dx}^t)_i = [\frac{\partial \psi_{i-1}}{\partial x}(\xi_0) \ \cdots \ \frac{\partial \psi_{i-1}}{\partial x}(\xi_{N-1})],$$

$$(Dy^t)_i = [\frac{\partial \psi_{i-1}}{\partial y}(\xi_0) \ \cdots \ \frac{\partial \psi_{i-1}}{\partial y}(\xi_{N-1})]$$

$$\boldsymbol{\psi} = [\psi_0(x) \ \cdots \ \psi_{N-1}(x)]^t$$

Properties of D

- $$\boldsymbol{Dx} = \sum_{n=0}^{2M^2-1} \boldsymbol{Dx}_{T_n}, \qquad (\boldsymbol{Dx}_{T_n})_{ij} = \frac{\partial \psi^n_{j-1}}{\partial x}(\boldsymbol{\xi}^n_{i-1})$$
- $$\boldsymbol{Dy} = \sum_{n=0}^{2M^2-1} \boldsymbol{Dy}_{T_n}, \qquad (\boldsymbol{Dy}_{T_n})_{ij} = \frac{\partial \psi^n_{j-1}}{\partial y}(\boldsymbol{\xi}^n_{i-1})$$
- $$\nabla \psi^n_i(\boldsymbol{x}) = \left((\boldsymbol{Dx}^t_{T_n})_i \boldsymbol{\psi}^n,\ (\boldsymbol{Dy}^t_{T_n})_i \boldsymbol{\psi}^n\right)$$
- $$\nabla u_h(\boldsymbol{\xi}_m) = \left((\boldsymbol{Dx})_m \boldsymbol{u},\ (\boldsymbol{Dy})_m \boldsymbol{u}\right)$$
- $$\nabla u_h(\boldsymbol{\xi}^n_m) = \left((\boldsymbol{Dx}_{T_n})_m \boldsymbol{u},\ (\boldsymbol{Dy}_{T_n})_m \boldsymbol{u}\right)$$

Mass matrix

$$(\boldsymbol{M})_{ij} = \int_{R_n} \psi_{i-1}(\boldsymbol{x})\psi_{j-1}(\boldsymbol{x})\, d\boldsymbol{x} = J \int_{R_R} \widetilde{\psi}_{i-1}(\boldsymbol{r})\widetilde{\psi}_{j-1}(\boldsymbol{r})\, dr = J(\boldsymbol{M}_R)_{ij}$$

$$\Rightarrow \qquad \boldsymbol{M} = J\boldsymbol{M}_R$$

Stiffness matrix

$$(\boldsymbol{S})_{ij} = J\Big[(r_x^2 + r_y^2)(\boldsymbol{S}_R^{rr})_{ij} + (r_x s_x + r_y s_y)]\Big((\boldsymbol{S}_R^{rs})_{ij} + (\boldsymbol{S}_R^{sr})_{ij}\Big) + (s_x^2 + s_y^2)(\boldsymbol{S}_R^{ss})_{ij}\Big]$$

where

$$(\boldsymbol{S}_R^{rr})_{ij} = (\boldsymbol{Dr}_R^t \boldsymbol{M}_R \boldsymbol{Dr}_R)_{ij}$$

$$(\boldsymbol{S}_R^{rs})_{ij} = (\boldsymbol{Dr}_R^t \boldsymbol{M}_R \boldsymbol{Ds}_R)_{ij}$$

$$(\boldsymbol{S}_R^{sr})_{ij} = (\boldsymbol{Ds}_R^t \boldsymbol{M}_R \boldsymbol{Dr}_R)_{ij}$$

$$(\boldsymbol{S}_R^{ss})_{ij} = (\boldsymbol{Ds}_R^t \boldsymbol{M}_R \boldsymbol{Ds}_R)_{ij}$$

Lemma

For a given interval $T = \mathrm{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2\}$ with barycentric coordinates $\lambda_0,\ \lambda_1$ and λ_2, it holds for $a,\ b,\ c \in \mathbb{N}_0$ that

$$\int_T \lambda_0^a \lambda_1^b \lambda_2^c \, d\boldsymbol{x} = 2|T| \frac{a!\, b!\, c!}{(a+b+c+2)!}$$

where $|T|$ is area of T.

P_1 matrices

$$
\begin{aligned}
\widetilde{\psi}_0(\boldsymbol{r}) &= \widetilde{\lambda}_2(\boldsymbol{r}), & \nabla\widetilde{\psi}_0(\boldsymbol{r}) &= \Big(-\frac{1}{2},\ -\frac{1}{2}\Big), \\
\widetilde{\psi}_1(\boldsymbol{r}) &= \widetilde{\lambda}_0(\boldsymbol{r}), & \nabla\widetilde{\psi}_1(\boldsymbol{r}) &= \Big(\frac{1}{2},\ 0\Big), \\
\widetilde{\psi}_2(\boldsymbol{r}) &= \widetilde{\lambda}_1(\boldsymbol{r}), & \nabla\widetilde{\psi}_2(\boldsymbol{r}) &= \Big(0,\ \frac{1}{2}\Big).
\end{aligned}
$$

P_1 matrices

$$\boldsymbol{M}_R = \frac{1}{6}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

P_1 matrices

$$
\boldsymbol{Dr}_R = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}
$$

$$
\boldsymbol{Ds}_R = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}.
$$

P_1 matrices

$$\boldsymbol{S}_R^{rr} = \boldsymbol{Dr}_R^t \boldsymbol{M}_R \boldsymbol{Dr}_R = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{S}_R^{rs} = \boldsymbol{Dr}_R^t \boldsymbol{M}_R \boldsymbol{Ds}_R = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{S}_R^{sr} = \boldsymbol{Ds}_R^t \boldsymbol{M}_R \boldsymbol{Dr}_R = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{S}_R^{ss} = \boldsymbol{Ds}_R^t \boldsymbol{M}_R \boldsymbol{Ds}_R = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

P_2 matrices

$$
\begin{aligned}
\widetilde{\psi}_0(\boldsymbol{r}) &= \widetilde{\lambda}_2(\boldsymbol{r})(2\widetilde{\lambda}_2(\boldsymbol{r}) - 1), & \nabla\widetilde{\psi}_0(\boldsymbol{r}) &= \left(- 2\widetilde{\lambda}_2(\boldsymbol{r}) + \frac{1}{2},\ -2\widetilde{\lambda}_2(\boldsymbol{r}) + \frac{1}{2}\right), \\
\widetilde{\psi}_1(\boldsymbol{r}) &= 4\widetilde{\lambda}_0(\boldsymbol{r})\widetilde{\lambda}_2(\boldsymbol{r}), & \nabla\widetilde{\psi}_1(\boldsymbol{r}) &= \left(2\widetilde{\lambda}_2(\boldsymbol{r}) - 2\widetilde{\lambda}_0(\boldsymbol{r}),\ -2\widetilde{\lambda}_0(\boldsymbol{r})\right), \\
\widetilde{\psi}_2(\boldsymbol{r}) &= \widetilde{\lambda}_0(\boldsymbol{r})(2\widetilde{\lambda}_0(\boldsymbol{r}) - 1), & \nabla\widetilde{\psi}_2(\boldsymbol{r}) &= \left(2\widetilde{\lambda}_0(\boldsymbol{r}) - \frac{1}{2},\ 0\right), \\
\widetilde{\psi}_3(\boldsymbol{r}) &= 4\widetilde{\lambda}_1(\boldsymbol{r})\widetilde{\lambda}_2(\boldsymbol{r}), & \nabla\widetilde{\psi}_3(\boldsymbol{r}) &= \left(- 2\widetilde{\lambda}_1(\boldsymbol{r}),\ 2\widetilde{\lambda}_1(\boldsymbol{r}) - 2\widetilde{\lambda}_2(\boldsymbol{r})\right), \\
\widetilde{\psi}_4(\boldsymbol{r}) &= 4\widetilde{\lambda}_0(\boldsymbol{r})\widetilde{\lambda}_1(\boldsymbol{r}), & \nabla\widetilde{\psi}_4(\boldsymbol{r}) &= \left(2\widetilde{\lambda}_1(\boldsymbol{r}),\ 2\widetilde{\lambda}_0(\boldsymbol{r})\right), \\
\widetilde{\psi}_5(\boldsymbol{r}) &= \widetilde{\lambda}_1(\boldsymbol{r})(2\widetilde{\lambda}_1(\boldsymbol{r}) - 1), & \nabla\widetilde{\psi}_5(\boldsymbol{r}) &= \left(0,\ 2\widetilde{\lambda}_1(\boldsymbol{r}) - \frac{1}{2}\right),
\end{aligned}
$$

P_2 matrices

$$\boldsymbol{M}_R = \frac{1}{90}\begin{pmatrix} 6 & 0 & -1 & 0 & -4 & -1 \\ 0 & 32 & 0 & 16 & 16 & -4 \\ -1 & 0 & 6 & -4 & 0 & -1 \\ 0 & 16 & -4 & 32 & 16 & 0 \\ -4 & 16 & 0 & 16 & 32 & 0 \\ -1 & -4 & -1 & 0 & 0 & 6 \end{pmatrix},$$

P_2 matrices

$$\boldsymbol{Dr}_R = \frac{1}{2}\begin{pmatrix} -3 & 4 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 & 0 \\ -1 & 2 & -1 & -2 & 2 & 0 \\ 1 & -2 & 1 & -2 & 2 & 0 \\ 1 & 0 & -1 & -4 & 4 & 0 \end{pmatrix}$$

$$\boldsymbol{Ds}_R = \frac{1}{2}\begin{pmatrix} -3 & 0 & 0 & 4 & 0 & -1 \\ -1 & -2 & 0 & 2 & 2 & -1 \\ 1 & -4 & 0 & 0 & 4 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & -2 & 2 & 1 \\ 1 & 0 & 0 & -4 & 0 & 3 \end{pmatrix}$$

P_2 matrices

$$\boldsymbol{S}_R^{rr} = \frac{1}{6}\begin{pmatrix} 3 & -4 & 1 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 & 0 \\ 1 & -4 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & -8 & 0 \\ 0 & 0 & 0 & -8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{S}_R^{rs} = \frac{1}{6}\begin{pmatrix} 3 & 0 & 0 & -4 & 0 & 1 \\ -4 & 4 & 0 & 4 & -4 & 0 \\ 1 & -4 & 0 & 0 & 4 & -1 \\ 0 & 4 & 0 & 4 & -4 & -4 \\ 0 & -4 & 0 & -4 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

P_2 matrices

$$
\boldsymbol{S}_R^{sr} = \frac{1}{6}\begin{pmatrix} 3 & -4 & 1 & 0 & 0 & 0 \\ 0 & 4 & -4 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 4 & -4 & 0 \\ 0 & -4 & 4 & -4 & 4 & 0 \\ 1 & 0 & -1 & -4 & 4 & 0 \end{pmatrix}
$$

$$
\boldsymbol{S}_R^{ss} = \frac{1}{6}\begin{pmatrix} 3 & 0 & 0 & -4 & 0 & 1 \\ 0 & 8 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 8 & 0 & -4 \\ 0 & -8 & 0 & 0 & 8 & 0 \\ 1 & 0 & 0 & -4 & 0 & 3 \end{pmatrix}
$$

MatrixforPoisson_2D_Tri

This python code generates the mass matrix $\boldsymbol{M}_R$, the stiffness matrices $\boldsymbol{S}_R^{rr},\ \boldsymbol{S}_R^{rs},\ \boldsymbol{S}_R^{sr},\ \boldsymbol{S}_R^{ss}$ and the differentiation matrices $\boldsymbol{Dr}_R,\ \boldsymbol{Ds}_R$ for continuous k-th order polynomial approximations on the reference triangle T_R.

Programming

Assignments

1. Add the matrices for the cubic approximations ($k = 3$) in $\boxed{\text{get\_matrices\_2d\_triangle}}$ and check the convergence rate.

2. Modify $\boxed{\text{fem\_for\_poisson\_2d\_triangle\_ex2}}$ to solve the Poisson problem with non-homogeneous Dirichlet boundary condition,

$$\begin{aligned} -\Delta u(\boldsymbol{x}) &= f(\boldsymbol{x}) \qquad \text{in } \Omega \\ u(\boldsymbol{x}) &= u_D(\boldsymbol{x}) \qquad \text{on } \partial\Omega. \end{aligned}$$

3. Modify $\boxed{\text{fem\_for\_poisson\_2d\_ex3}}$ to solve the Poisson problem with mixed boundary condition,

$$\begin{aligned} -\Delta u(\boldsymbol{x}) &= f(\boldsymbol{x}) \qquad \text{in } \Omega \\ u(\boldsymbol{x}) &= u_D(\boldsymbol{x}) \qquad \text{on } \Gamma_D \\ \nabla u(\boldsymbol{x}) \cdot \boldsymbol{n} &= u_N(\boldsymbol{x}) \qquad \text{on } \Gamma_N, \end{aligned}$$

where Γ_D denotes the Dirichlet boundary, Γ_N denotes the Neumann boundary, and $\boldsymbol{n}$ is the outward unit normal vector.

4. Prove Lemma 2.1.