# Four Towns

A newly-settled territory contains four towns that occupy the corners of a square 10 miles on each side. The people of the territory want to build roads that connect all four towns in such a way that they use the least amount of asphalt. What is the minimum total mileage of a road network that connects all four towns?
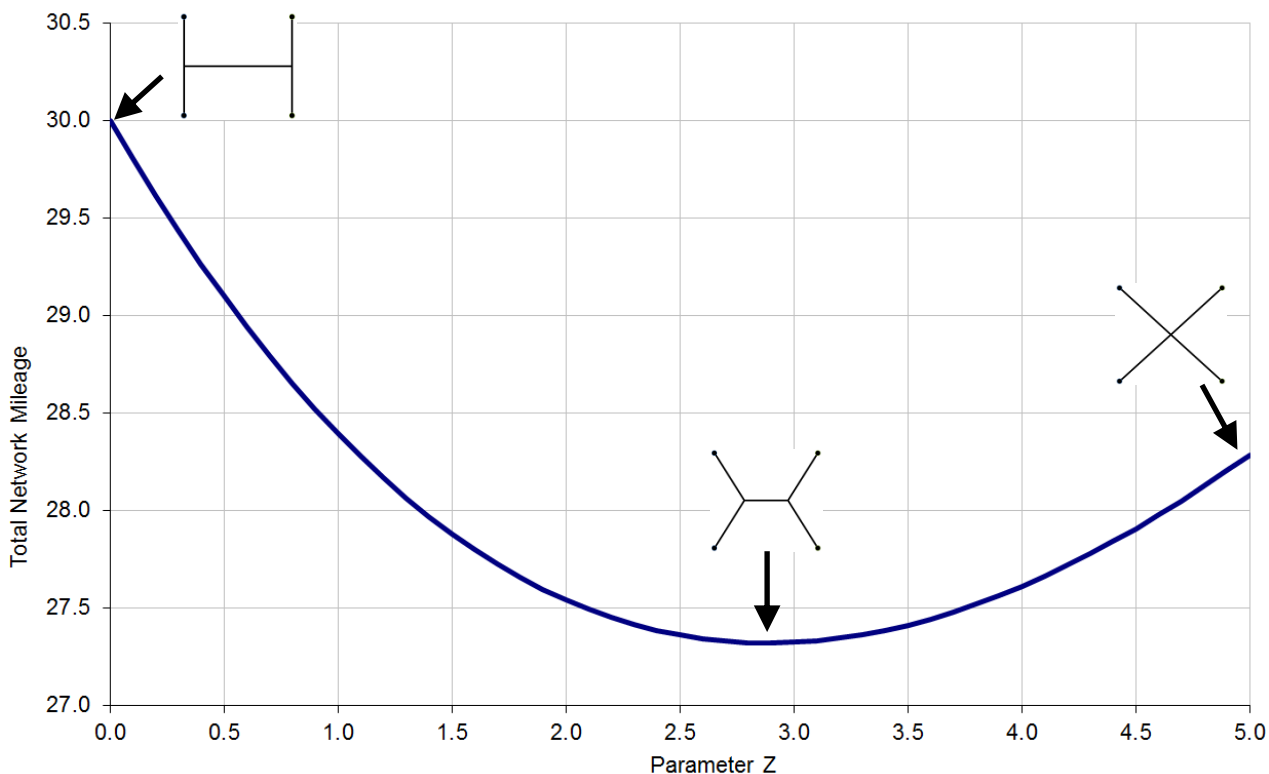
Solution to Four Towns

Consider the road network shown in the diagram to the right. The parameter Z determines the shape of the network. Note that for Z = 0, the network is H-shaped and for Z = 5, the network is X-shaped.

The total network mileage M is given by the formula:

$$M = 4 \times \text{SQRT}(Z^2 + 25) + 10 - 2 \times Z$$
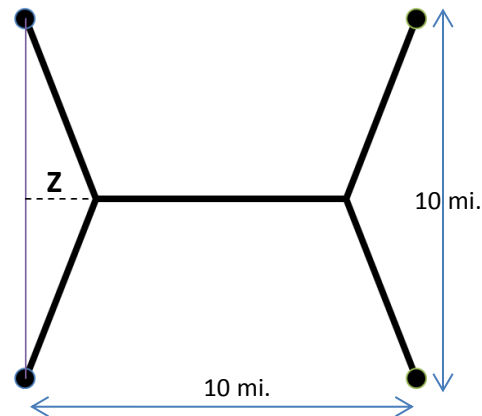
The graph of total network mileage as a function of parameter Z is shown below.

By differentiating the formula for total mileage with respect to parameter Z and setting that equal to zero, we can determine that the minimum total mileage occurs when

$$Z = SQRT(25/3) = 2.89 \text{ mi.}$$

and total network mileage is M = 27.32 mi.

As shown below, this results in a network configuration in which the roads meet at 120° angles. This is called a Steiner Tree and is the configuration that bees use in constructing honeycombs.