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RDT3600 Signal Processing
Monash University
Clayton Campus

Dr. Henry Wu
$\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX by Dougal Scott

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Chapter 1

Outline of Subject

Contact: Two 1 hour lectures and 2 hours practical/tutorial work per week for two semesters.

Syllabus: Sampling of continuous-time signals and sampling rate conversion. Digital signal processign systems. Structures for discrete-time systems. Digital filter design techniques; discrete Fourier transform (DFT) and computation of DFTs. Discrete Hilbert transform and its applications. Quantisation effect in digital signal processing; Fourier analysis of signals using the DFT. Applications of digital signal processing. DSP implementation using, for example, the TMS320C25/C30 digital signal processor(s).

Evaluation:

Practical	40%
Final Exam	60%

All pracs are compulsory unless specified otherwise.

Your performance of the practical work will be evaluated based on

- Attendance
- Demonstration on request
- Quality of the reports

Overdue reports will be penalized. Prac reports must be written on an individual basis.

Excelent prac work will contribute to the overall evaluation of your performance in this subject.

Preliminary Reading:

1. E. W. Kamen “Introduction to Signals and Systems”, MacMillan Publishers, 2nd Edition, 1990
2. R. E. Ziemer, W.H. Tranter and D. R. Fannin “Signals and Systems: Continuous and Discrete”, MacMillan Publishers 1990

Prescribed Texts:

1. A. Oppenheim and R. Schafer “Discrete-Tiem Signal Processing” Prentice-Hall 1989

Recommended Reading:

1. J. G. Proakis and D. G. Manolakis "Introduction to Digital Signal Processing", MacMillan Publishers 1989
2. G. B. Lockharat and B. M Cheetham "BASIC Digital Signal Processing", Butterworths
3. T. W. Parks and C. S. Burrus "Digital Filter Design", John Wiley and Sons 1987

References:

1. L. R. Rabiner and B. Gold "Theory and Application of Digital Signal Processing", Prentice Hall 1975
2. H. Baher "Analog and Digital Signal Processing', John Wiley and Sons 1990
3. M. E. Van Valkenburg, "Analog Filter Design", Holt-Sanders International Editions, CBS College Publishing, 1982
4. A. Bateman and W. Yates, "Digital Processing Design", Pitman Publishing 1988
5. R. W. Hamming "Digital Filters", Prentice Hall International 3rd Edition, 1989
6. Texas Instruments, Linear Circuits Data Book, 1989
7. Signal Processing Associates "SPA-C25-1 Instruction Manual" 1980
8. Texas Instruments "TMS320 Fixed Point DSP Assembly Language Tools User's Guide" 1990

Chapter 2

Introduction

Contents

Definition 2.0-1 *A signal is a physical quantity, or quality which conveys* information*. Signals are used to control and utilize energy and information.*

Definition 2.0-2 *Operations on signals are signal processing. Signal processing is concerned with the* representation, transformation *and* manipulation *of signals and information that they contain.*

2.1 Applications of Signal Processing

- Speech and data communication
- biomedical engineering
- acoustics
- sonar
- radar
- seismology
- oil exploration
- instrumentation
- robotics

- consumer electronics

Further information can be found in

1. IEEE Transactions on Acoustics, Speech and Signal Processing
2. IEEE Transactions on Circuits and Systems
3. IEEE Transactions on Communications
4. IEEE Transactions on Information Theory
5. IEEE Transactions on Automatic Control
6. Automatica

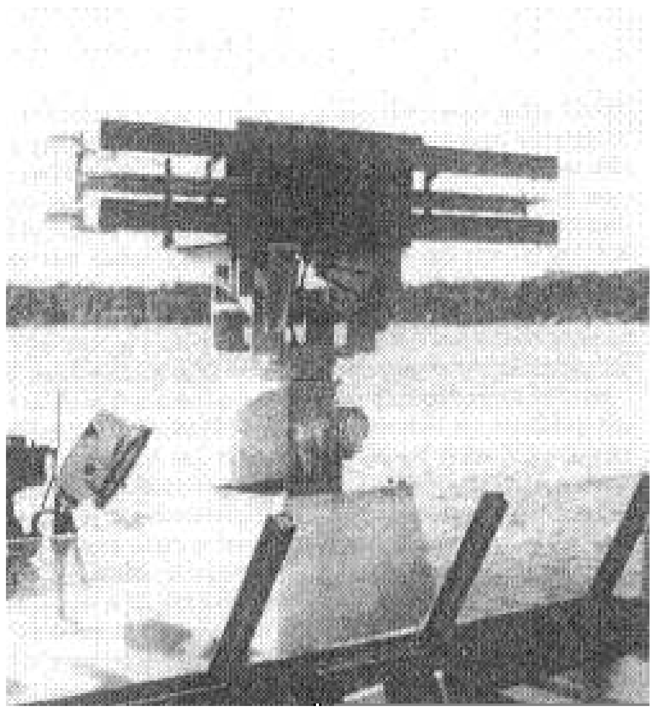


Figure 2.1: PILOT Frequency Modulated Continuous Wave navigation radar — K. Fuller, IEEE Proceedings

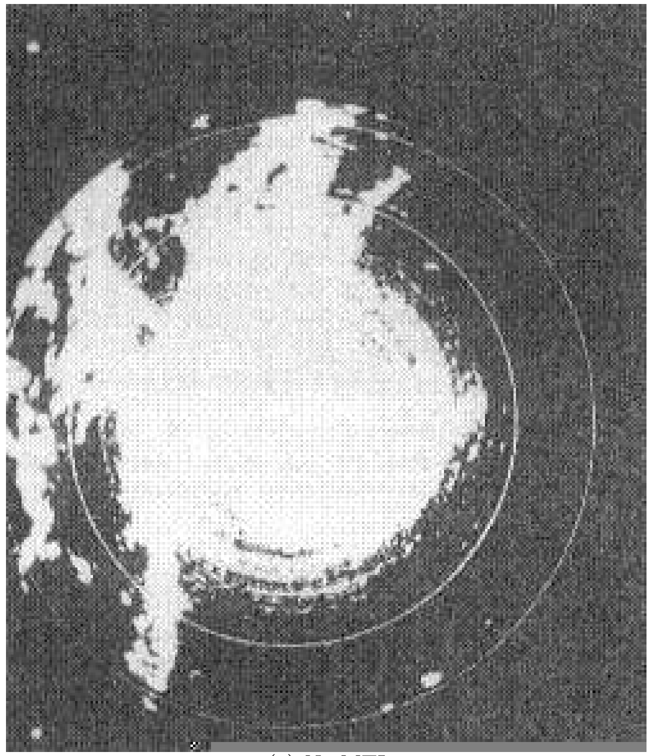
2.2 Classification of Signals

2.2.1 Analog Signals

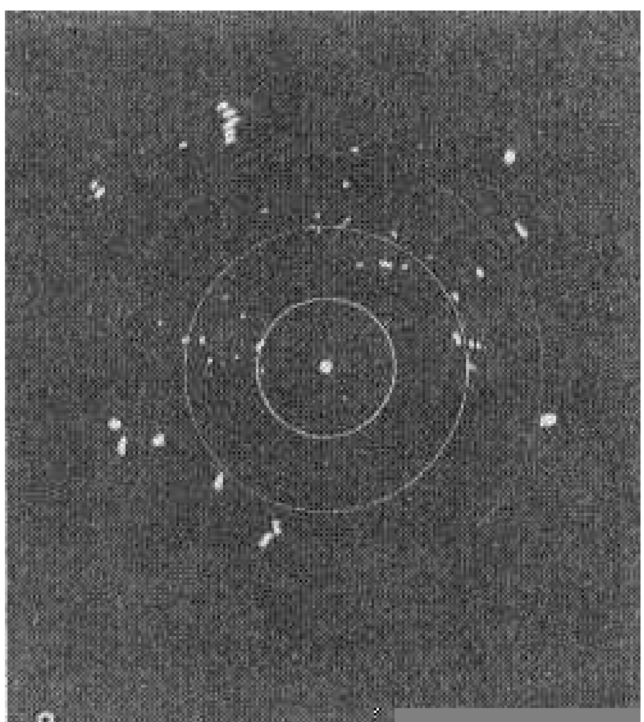
The signal $f(t)$ is defined for all values of the continuous variable t and its amplitude. See Figures 2.7 and 2.8 for examples of analog signals. (continuous-time, continuous)

2.2.2 Discrete-time signals

The signal $f(t)$ is defined for all values of its amplitude but only discrete values of t_i where i is an integer. See Figure 2.9 for an example. Special Case: $t_i = iT$, where T is a constant. (Analog sampled-data)



(a) No MTI



(b) MTI

Figure 2.2: Reducing clutter with Moving Target Indication (MTI)

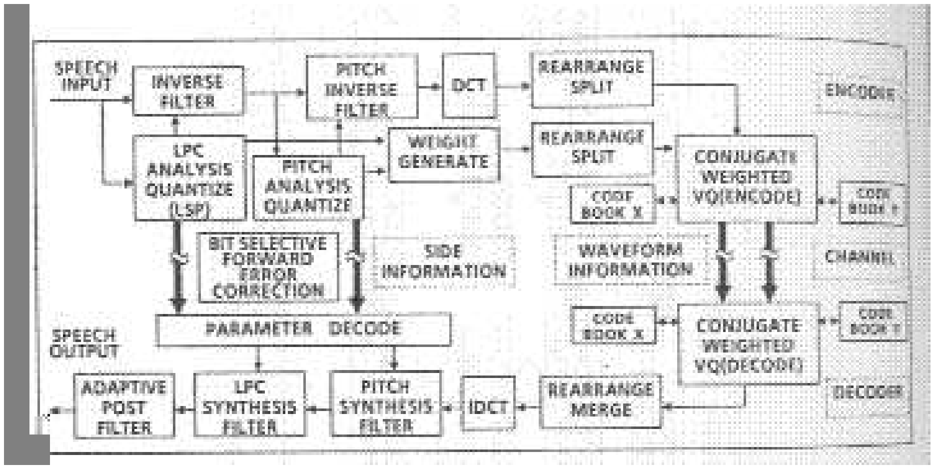
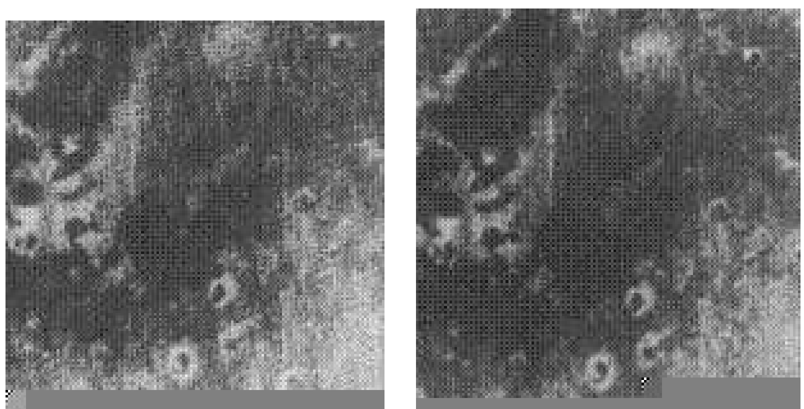
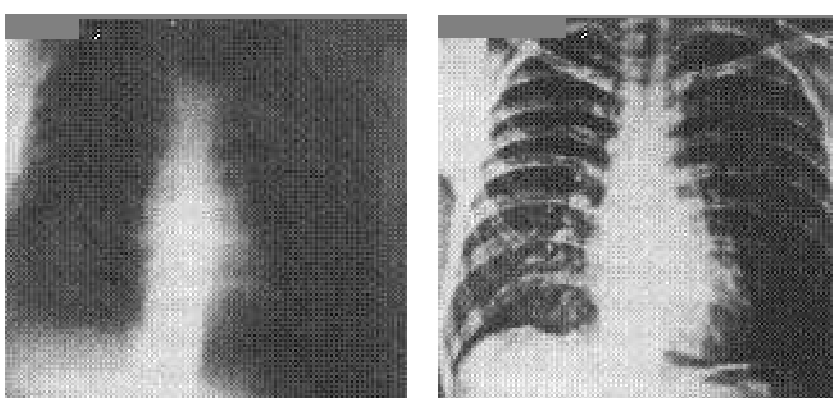


Figure 2.3: TC-WVQ two channel speech coder — Rao & Yip



(a) Original Image (b) Processed Image

Figure 2.4: Examples of image processing — Gonzalez & Wintz



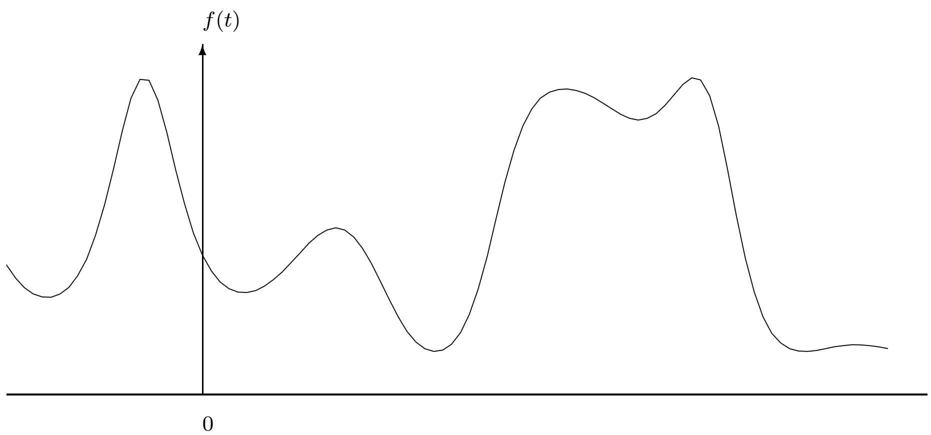
(a) Original Image (b) Processed Image

Figure 2.5: Examples of image processing — Gonzalez & Wintz

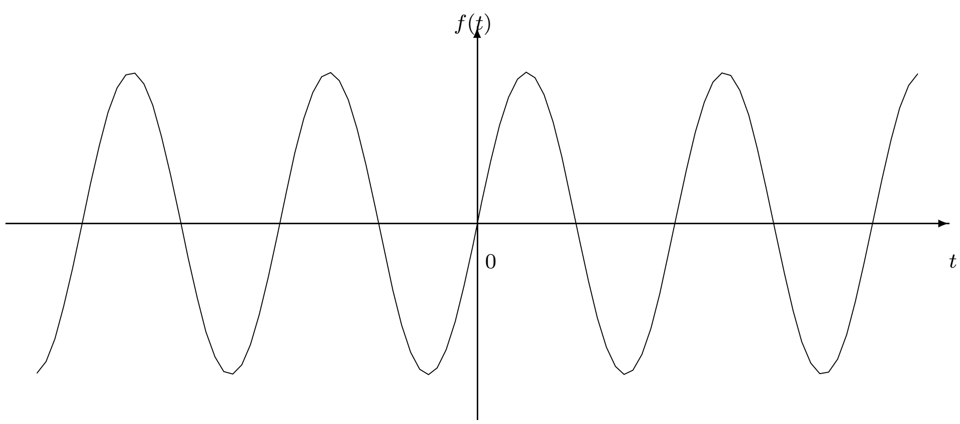
(a) Original Image

(b) Processed Image

Figure 2.6: Examples of image processing — Gonzalez & Wintz



(a) Continuous-Time, Non-periodic



(b) Continuous-Time, Periodic

Figure 2.7: Examples of analog signals

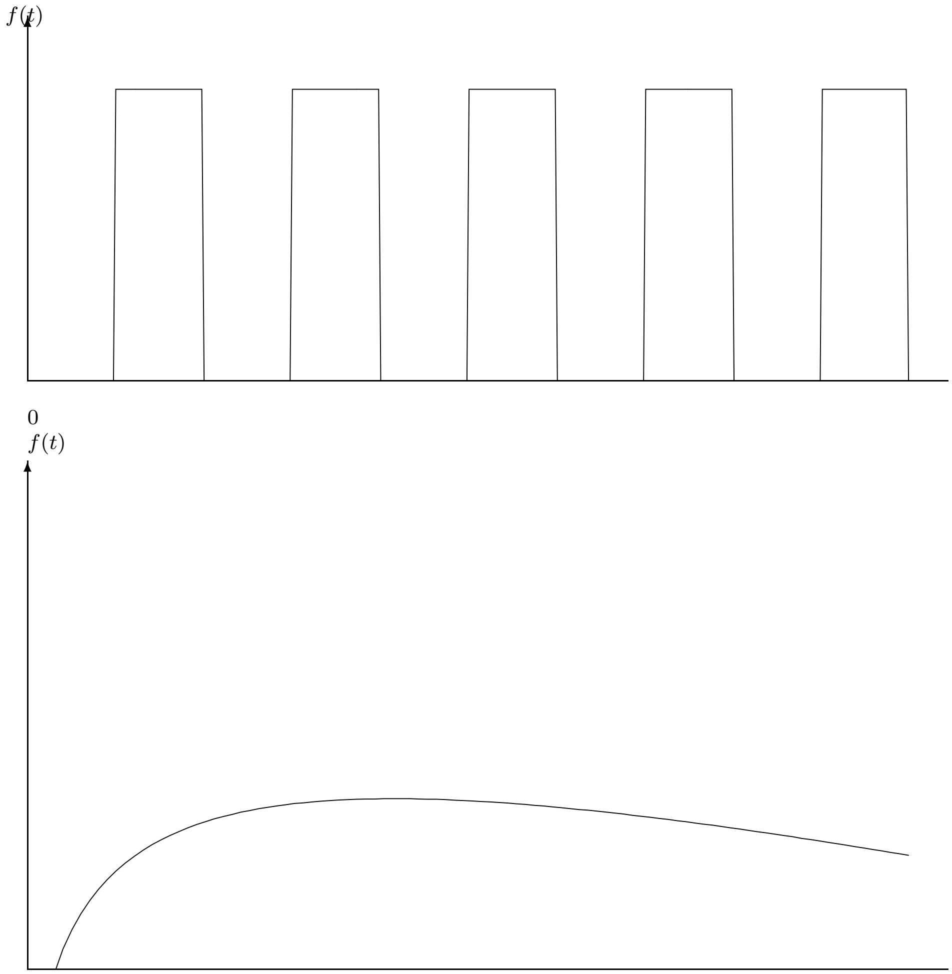


Figure 2.8: Examples of analog signals

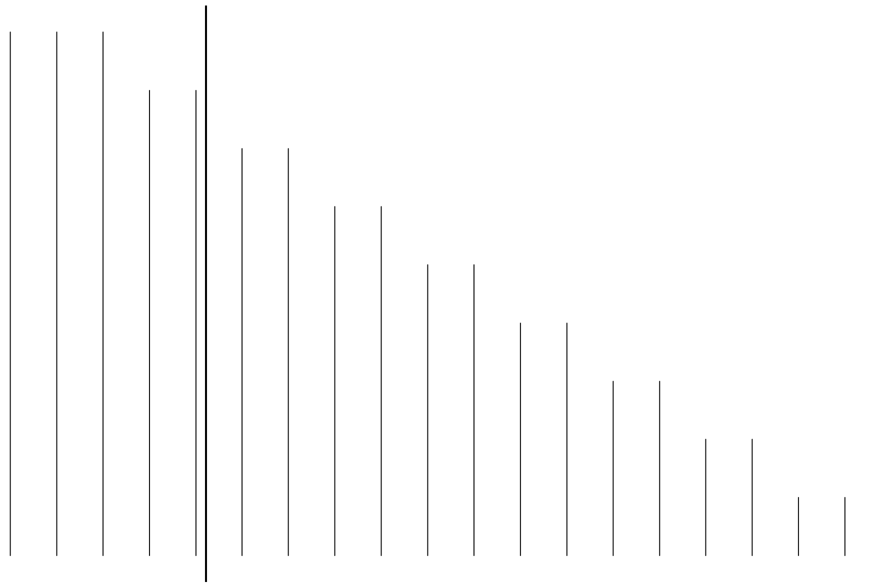


Figure 2.9: Example of a discrete-time signal

2.2.3 Digital Signals

If the discrete-time signal $f(iT)$ amplitudes assume only discrete values which are integral multiples of the same quantity q, the resulting signal is a digital signal. This introduces an error called *quantization error*. See Figure 2.10 for an example of quantization error.

2.2.4 Deterministic and Random Signals

Deterministic signals can be described by a functional relationship or an arbitrary graph or in tabular form.

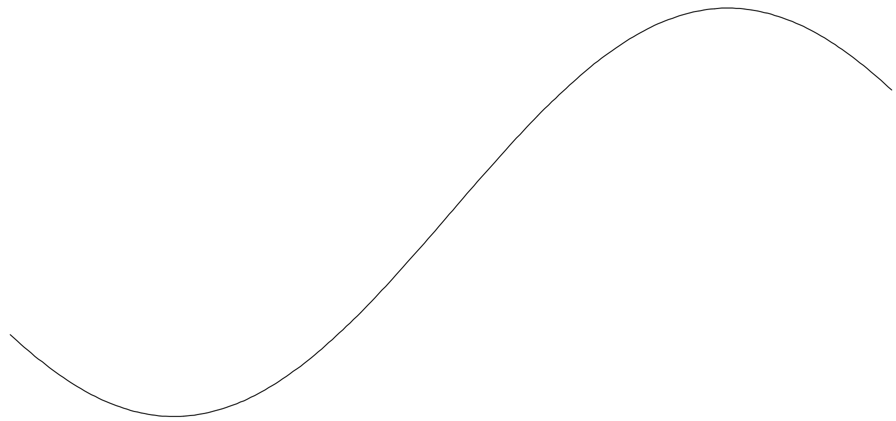
Random signals can only be described by statistical methods.

2.2.5 Electrical vs Non-electrical signals

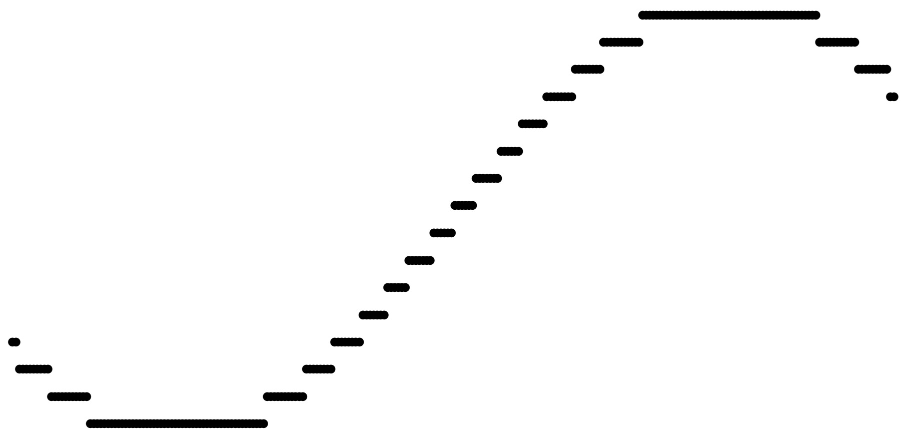
2.3 Signal Processing Systems

Systems which manipulate the required signals for the purpose of

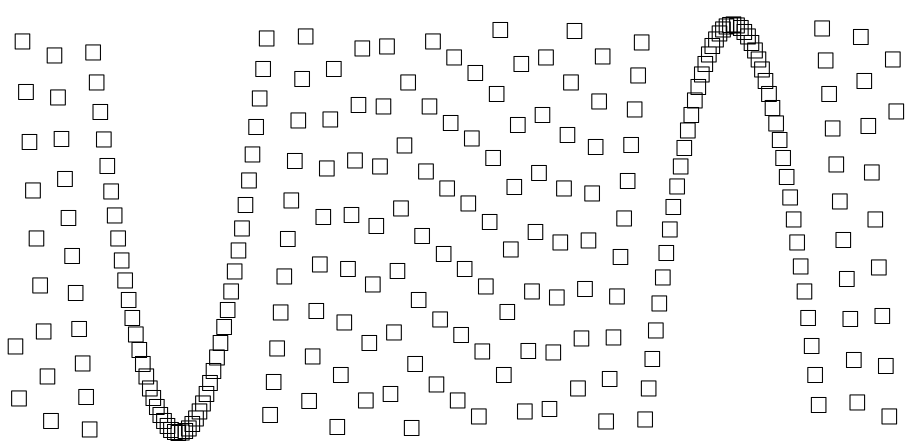
- analysis
- transmission
- detection
- enhancement
- control
- compression
- identification and pattern recognition
- suppresion of noise



(a) Analog input signal

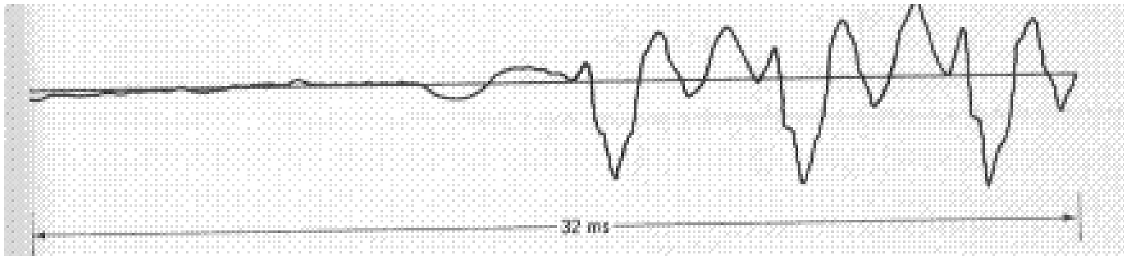


(b) Digitized output signal

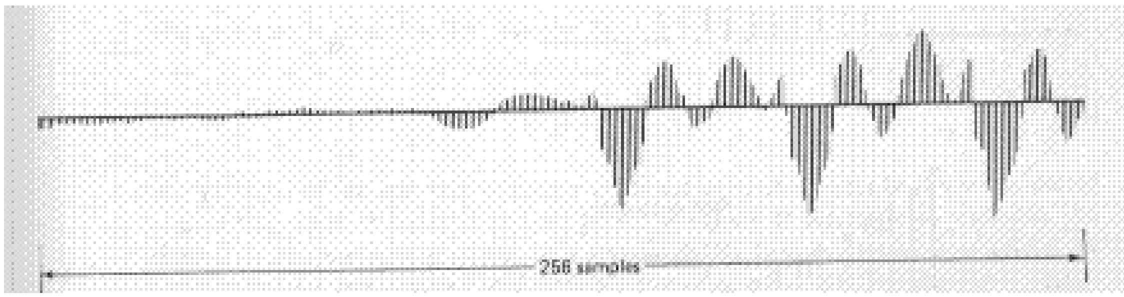


(c) Error introduced by quantization

Figure 2.10: Error introduced by quantization



(a) Continuous time speech signal



(b) Sampled time speech signal

Figure 2.11: Speech digitization — Oppenheim & Schaffer

2.3.1 Analog systems

An analog system is a device, or collection of well defined building blocks. It accepts analog excitation signal $f(t)$ and produces analog response signal $g(t)$, as shown in Figure 2.12.

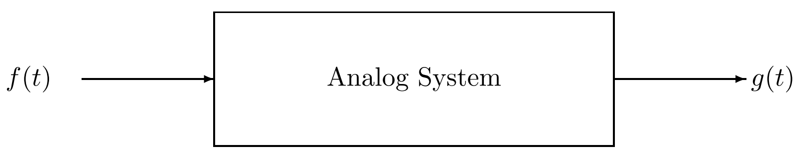
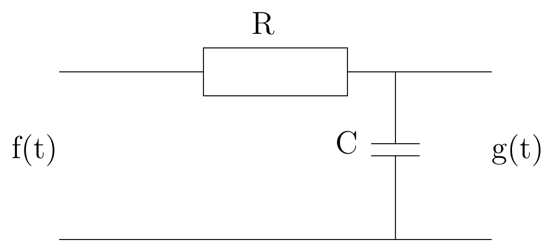


Figure 2.12: Analog System

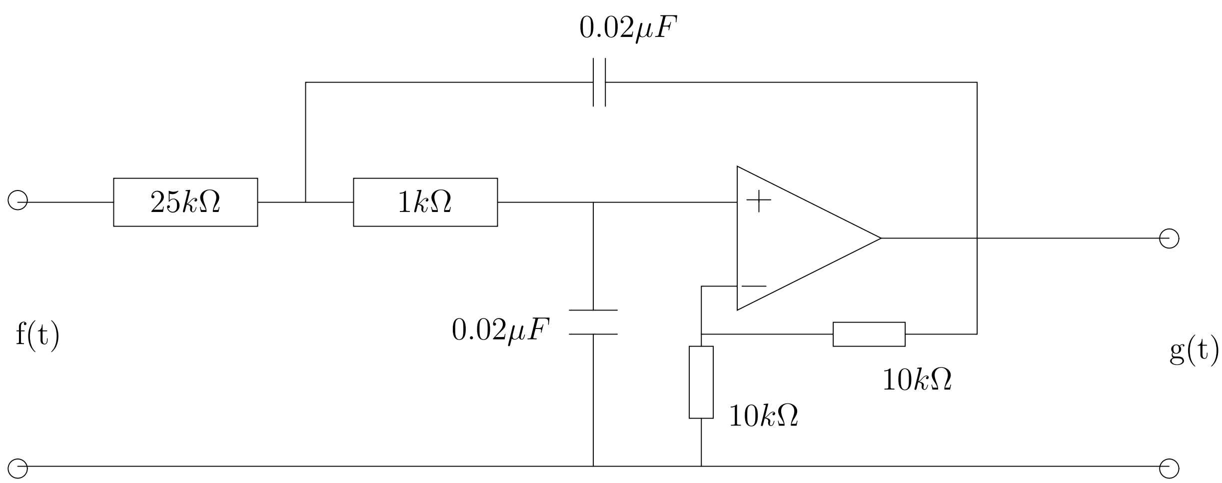
Example 2.3-1 *See Figure 2.13 for examples of analog systems.*

Other analog systems can be simulated by means of an electrical network.

Example 2.3-2 *What can you say about this system*

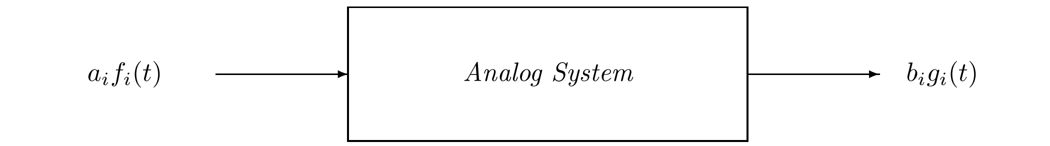


(a) Simple *RC* analog system

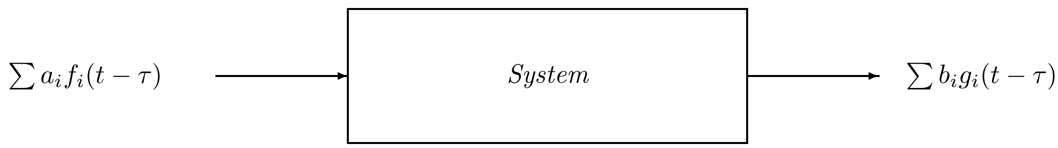


(b) Complex analog system

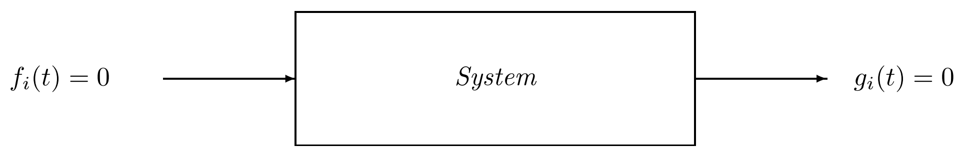
Figure 2.13: Examples of analog systems



if



and



for $t < \tau$ (τ may be zero).

The system is Linear, time-invariant and causal.

The linear systems that are considerd can be modelled by conventional clumped networks and their input and output relations can be represented or described by an ordinary linear differential equation with constant coefficients.

$$\sum_{i=0}^{n} a_i \frac{D^i}{dt} g(t) = \sum_{j=0}^{m} b_j \frac{d^j}{dt} f(t) \tag{2.3-1}$$

where

$$\frac{d^0}{dt} f(t) = f(t) \tag{2.3-2}$$

Basic equaiton and the building blocks for signal processing:

Multiplier:

$$g(t) = K f(t) \tag{2.3-3}$$

See Figure 2.14 for circuit.

Differentiator:

$$g(t) = \alpha \frac{d(f)}{dt} \tag{2.3-4}$$

See Figure 2.15 for circuit.

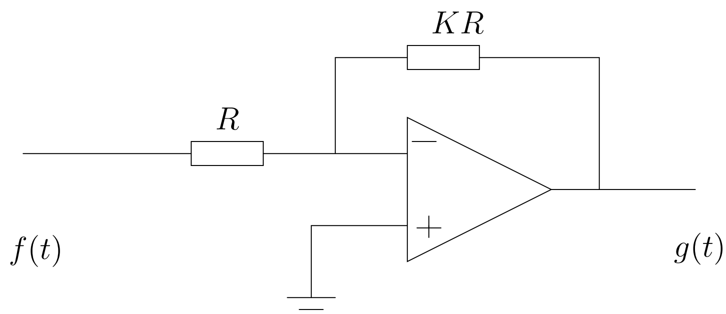


Figure 2.14: Multiplier building block

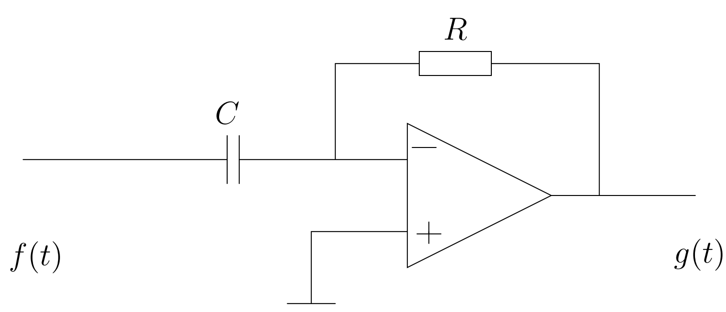


Figure 2.15: Differentiator building block

Integrator:

$$g(t) = \alpha \int_{-\infty}^{t} f(\tau) d\tau \tag{2.3-5}$$

See Figure 2.16 for circuit.

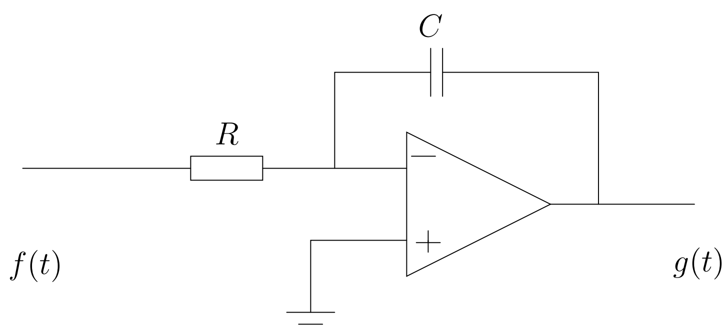


Figure 2.16: Integrator building block

Adder:

$$g(t) = \sum_{i=1}^{n} f_i(t) \tag{2.3-6}$$

See Figure 2.17 for circuit.

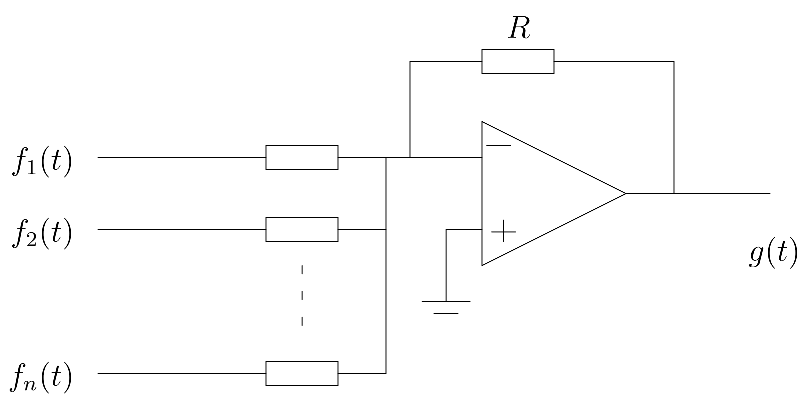


Figure 2.17: Adder building block

2.3.2 Discrete and digital systems

A discrete (or digital) system accepts a discrete (or digital) input $f(nT)$ and produces a discrete (or digital) response $g(nT)$.

$$f(nT) \longrightarrow \boxed{\text{Discrete or Digital System}} \longrightarrow g(nT)$$

EXAMPLE 2.3-3 *1. Discrete-time: switched-capacitor network*

2. DSP based system

3. Given

$$a_i f_i(nT) \longrightarrow \boxed{\textit{Digital System}} \longrightarrow b_i g_i(nT)$$

what can you say about the system if

$$\sum_i a_i f_i(nT - rT) \longrightarrow \boxed{\textit{Digital System}} \longrightarrow \sum_i b_i g_i(nT - rT)$$

2.4 History of Signal Processing

Assignment:

1. Read Chapter 1 of "Discrete-Time Signal Processing" by Oppenheim and Schafer
2. Read "The Acoustics, Speech, and Signal Processing Society — A Historical Perspective" by L. R. Rabiner, in IEEE ASSP Mag, Jan 1984

Analog vs. Discrete (or Digital)
Digital signal processing has many advanteages over analog counterparts in terms of

- reliability
- reproducibility
- high precision
- freedom from ageing and temperature effects

However analog signal processing will accompany digital signal processing for years to come.

1. A typical digital signal processing system

2. Neural network

This subject will reinforce our knowledge of analog and digital signal processing by going through fundamental concepts and theories and basic design methods.

(How and Why)

Chapter 3

The Fourier Series in Spectral Analysis and Function Approximation

Contents

According to Baher, the Fourier integral is at "the heart of the discipline of signal processing."

Large volumes of research publications in the area of signal processing contribute to computation of Fourier co-efficients with *high speed*, *accuracy* and *efficiency*.

3.1 Fundamentals

Definition 3.1-1 *The Fourier series of $f(\theta)$ defined over the interval $[-\pi, \pi]$ are given by the following equations:*

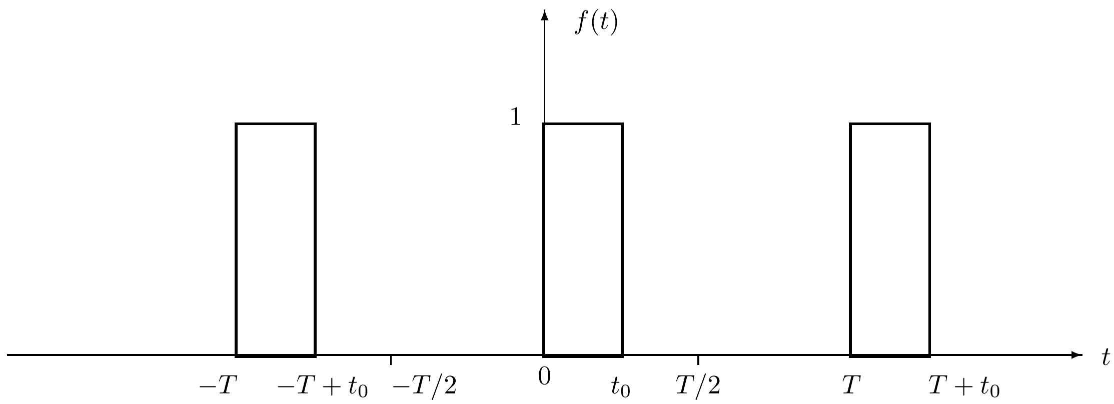
$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin(k\theta)) \tag{3.1-1}$$

with

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(k\theta d\theta) \tag{3.1-2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(k\theta d\theta) \tag{3.1-3}$$

Example 3.1-1 *Find the Fourier Series for the train of rectangular pulses.*



Solution:

1. *Find expression of $f(t)$ using maths equations.*

$$f(t) = \begin{cases} 1 & \textit{for } 0 < t < t_0 \\ 0 & \textit{for } t_0 < t < T \end{cases} \tag{3.1-4}$$

The interval for integration is $[0, T]$.

2. *Use the definition to calculate the coefficients*

$$a_0 = \frac{2}{T}\int_0^{t_0} dt = \frac{2t_0}{T} \tag{3.1-5}$$

$$a_k = \frac{2}{T}\int_0^{t_0} \cos k\omega_0 t dt \tag{3.1-6}$$

$$= \frac{2}{T}\int_0^{t_0} \cos \frac{2\pi k}{T} t dt \tag{3.1-7}$$

$$= \frac{2}{T}\frac{T}{2\pi k}\int_0^{t_0} \cos \frac{2\pi k}{T} t d(\frac{2\pi k}{T} t) \tag{3.1-8}$$

$$= \frac{1}{\pi k}\sin(\frac{2\pi k}{T})\Big|_0^{t_0} \tag{3.1-9}$$

$$= \frac{1}{\pi k}\sin(\frac{2\pi k}{T} t_0) \tag{3.1-10}$$

$$= \frac{1}{\pi k}\sin(k\omega_0 t_0) \tag{3.1-11}$$

$$b_k = \frac{2}{T}\int_0^{t_0} \sin(k\omega_0 t) dt \tag{3.1-12}$$

$$= \frac{2}{T}\int_0^{t_0} \sin(\frac{2\pi k}{T} t) dt \tag{3.1-13}$$

$$= \frac{-1}{\pi k}\left(\cos \frac{2\pi k}{T} t_0 - 1\right) \tag{3.1-14}$$

$$= \frac{1}{\pi k}\left(1 - \cos k\omega_0 t_0\right) \tag{3.1-15}$$

3. Fourier series of $f(t)$ is

$$f(t) = \frac{t_0}{T} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{\sin(k\omega_0 t_0)}{k}\right)\cos k\omega_0 t + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(1-\cos k\omega_0 t_0)}{k}\sin(k\omega_0 t) \tag{3.1-16}$$

Alternatively,

$$f(\theta) = \frac{d_0}{2} + \sum_{k=1}^{\infty} d_k \cos(k\theta + \Phi_k) \tag{3.1-17}$$

or

$$f(\theta) = \frac{d_0}{2} + \sum_{k=1}^{\infty} d_k \sin(k\theta + \psi_k) \tag{3.1-18}$$

where

$$d_k = (a_k^2 + b_k^2)^{1/2} \tag{3.1-19}$$

$$\Phi_k = -\tan^{-1}(b_k/a_k) \tag{3.1-20}$$

$$\psi_k = \Phi_k + \frac{\pi}{2} \tag{3.1-21}$$

If Fourier Series are used to represent functions of time $f(t)$, the following definition applies:

DEFINITION 3.1-2

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \tag{3.1-22}$$

where

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos k\omega_0 t dt \tag{3.1-23}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin k\omega_0 t dt \tag{3.1-24}$$

Note:

$$\theta = \omega_0 t \tag{3.1-25}$$

$$f(t) = f(t \pm rT) \qquad \textit{for } r = 1, 2, \ldots \tag{3.1-26}$$

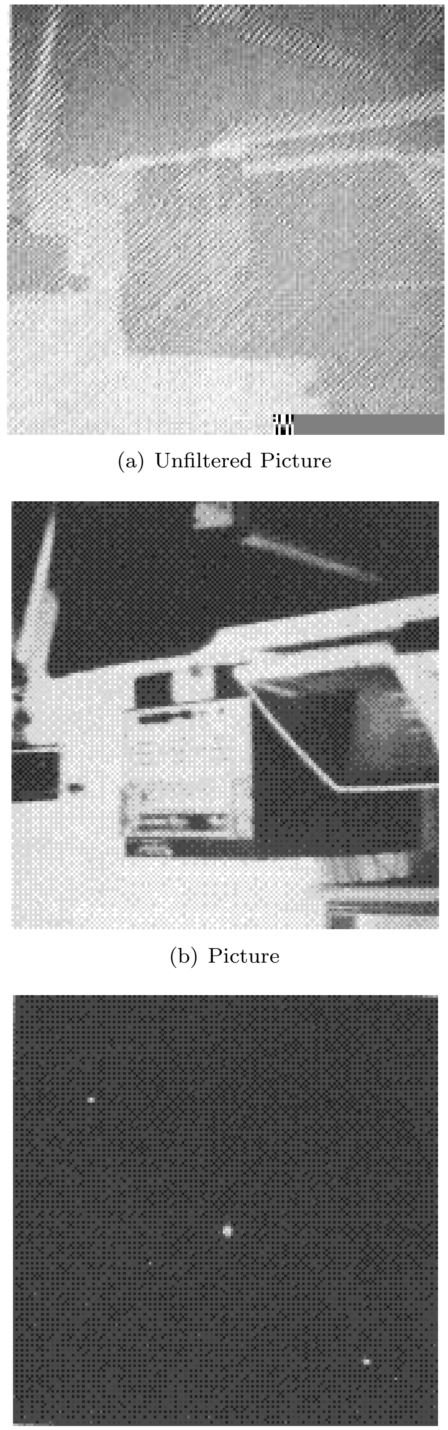
$$\omega_0 = \frac{2\pi}{T} \qquad \textit{Fundamental Radian Frequency} \tag{3.1-27}$$

3.2 Applications of the Fourier series

Two main applications of FS are

- Harmonic analysis and synthesis
- Filtering

They are closely related to each other in real signal processing applications. See Figure 3.1 for an example of Fourier series filtering.



(a) Unfiltered Picture

(b) Picture

(c) Fourier Filter

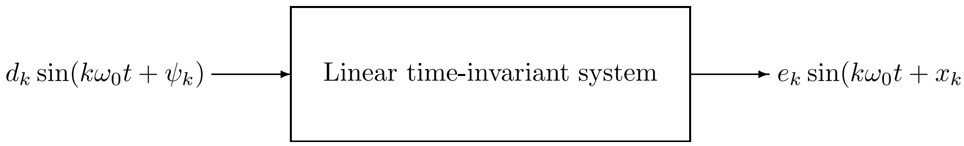
Figure 3.1: Example of filtering images

3.2.1 Harmonic analysis and synthesis

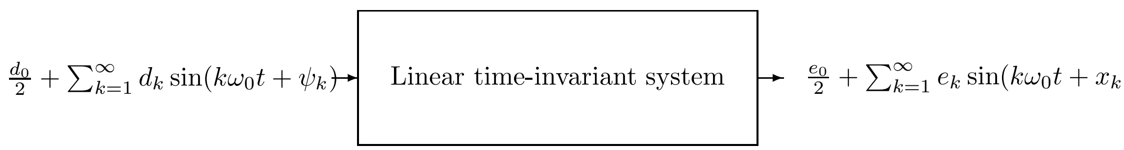
DEFINITION 3.2-1 *The process of finding the FS representation for a signal is called harmonic analysis.*

The harmonic analysis is used for the understanding of signals (what frequency components they have); noise isolation; or preprocessing for filtering of signals.

For a linear time-invariant system, if the input is a sinlge sinusoid $d_k \sin(k\omega_0 t + psi_k)$, the output is also a sinusoid with the same frequency $k\omega_0$ but with difference amplitude and phase, of the form $e_k \sin(k\omega_0 t + x_k)$ as shown in Figure 3.2.



(a) with sinusoidal input



(b) with arbitrary periodic signal input

Figure 3.2: A linear system

If the input is an arbitrary periodic signal $f(t)$ ($\triangleq f(\omega_0 t) \triangleq f(\omega)$), it can be expanded in a FS of the following form:

$$f(t) = \frac{d_0}{2} + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t + \psi_k) \tag{3.2-28}$$

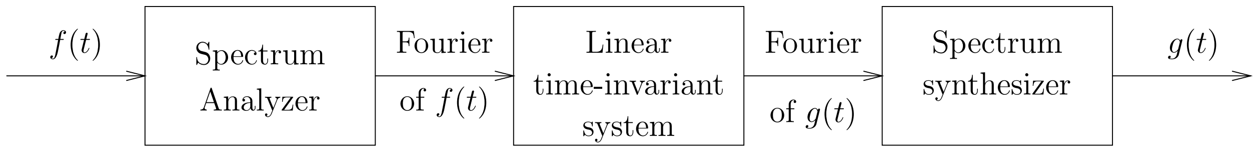
Assume that the system response of the input $\frac{d_0}{2}$ is $\frac{e_0}{2}$ and the output due to each sinusoid $d_k \sin(k\omega_0 t + \psi k)$ is $e_k \sin(k\omega_0 t + x_k)$.

Superposition can be applied because the system in linear to obtain the system output $g(t)$ due to the input signal $f(t)$.

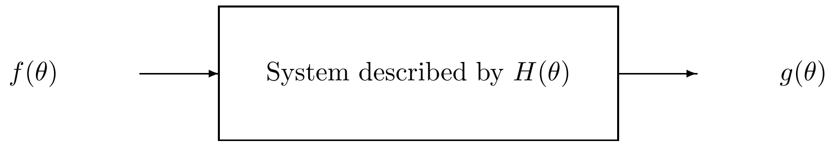
$$g(t) = \frac{e_0}{2} + \sum_{k=1}^{\infty} e_k \sin(k\omega_0 t + x_k) \tag{3.2-29}$$

DEFINITION 3.2-2 *The process of constructing a signal from its Fourier series is called harmonic (or spectral) synthesis.*

Harmonic synthesis is used to construct signlas with desired frequency components for signal generation, identification, noise cancellation; for post filtering (or signal processing).



(a) Illustrating harmonic (spectral) analysis and synthesis



(b) General system

Figure 3.3: Harmonic Synthesis

An example is shown in Figure 3.3.

In the practical situation, only finite terms of the FS. are used to represent signals. ie.

$$ {}_n(t) = \frac{d_0}{2} + \sum_{k=1}^{n} d_k \sin(k\omega_0 t + \psi_k) \tag{3.2-30} $$

$$ f(t) = f_n(t) + \Delta_n \tag{3.2-31} $$

where

$$ \Delta_n = \sum_{k=n+1}^{\infty} d_k \sin(k\omega_0 t + \psi_k) \tag{3.2-32} $$

being the error. $f_n(t)$ is used to approximate $f(t)$. i.e.

$$ f_n(t) \sim f(t) \tag{3.2-33} $$

Ideally we should like to approximate $f(t)$ by $f_n(t)$ using as small an n as possible to give a small Δ_n (error). Hence, the convergence considerations. (Convergent? Convergence rate or fast?)

3.2.2 Filtering of signals

In general, a system input-output relation can be represented by

$$ \hat{g}(\theta) = \hat{H}(\theta) f(\theta) \tag{3.2-34} $$

where $f(\theta)$ and $\hat{g}(\theta)$ are the input and output of the system represented by $\hat{H}(\theta)$.

$$ \hat{H}(\theta) = \frac{a_0}{w} + \sum_{k=1}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin k\theta \tag{3.2-35} $$

In practice, we would like to approximate $\hat{H}(\theta)$ by $H(\theta)$ which only has a finite number of terms of $\hat{H}(\theta)$.

$$H(\theta) = \frac{a_0}{2} + \sum_{k=1}^{n} a_n \cos k\theta + \sum_{k=1}^{n} b_k \sin k\theta \tag{3.2-36}$$

so that

$$H(\theta) \sim \hat{H}(\theta) \tag{3.2-37}$$

and

$$g(\theta) \sim \hat{g}(\theta) \tag{3.2-38}$$

where

$$g(\theta) = H(\theta) f(\theta) \tag{3.2-39}$$

Yet again, the convergence is a very important issue.

EXAMPLE 3.2-1 *Approximate the function $\hat{H}(\theta)$ as shown in Figure 3.4 by a truncated Fourier Series.*

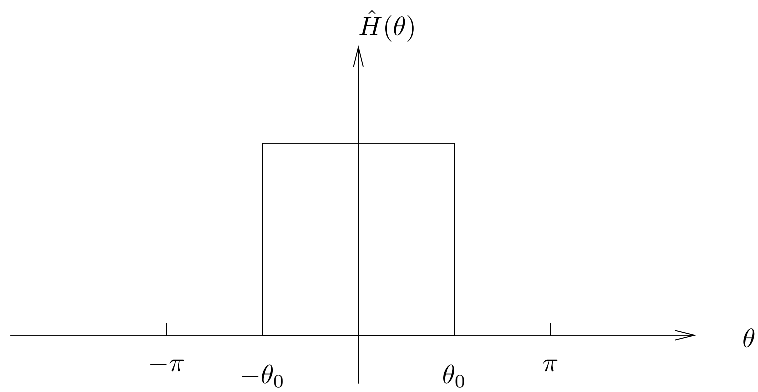


Figure 3.4: Function $\hat{H}(\theta)$

Solution:

1.

$$\hat{H}(\theta) = \begin{cases} 1 & \text{for } 0 \leqslant |\theta| \leqslant \theta_0 \\ 0 & \text{for } \theta_0 < |\theta| < \pi \end{cases} \tag{3.2-40}$$

Even function.

2.

$$b_k = 0 \qquad (\text{as } \hat{H}(\theta) \text{ even}) \tag{3.2-41}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \hat{H}(\theta) d\theta \tag{3.2-42}$$

$$= \frac{2}{\pi} \int_0^{\theta_0} d\theta = \frac{2}{\pi}\theta_0 \tag{3.2-43}$$

and

$$a_k = \frac{2}{\pi}\int_0^{\pi} \hat{H}(\theta)\cos k\theta d\theta \tag{3.2-44}$$
$$= \frac{2}{\pi}\int_0^{\theta_0} \cos k\theta d\theta \tag{3.2-45}$$
$$= \frac{2}{\pi k}\int_0^{\theta_0} \cos k\theta d(k\theta) \tag{3.2-46}$$
$$= \frac{2}{\pi k}\sin\theta_0 \tag{3.2-47}$$

where $k = 0, 1, 2, \ldots$

3.
$$\hat{H}(\theta) = \frac{\theta_0}{\pi} + \frac{2\theta_0}{\pi}\sum_{k=1}^{\infty}\frac{\sin(k\theta_0)}{(k\theta_0)}\cos(k\theta) \tag{3.2-48}$$

4. *Approximate* $\hat{H}(\theta)$ *by* $H_n(\theta)$.

$$H_n(\theta) = \frac{\theta_0}{\pi} + \frac{2\theta_0}{\pi}\sum_{k=1} n\frac{\sin(k\theta_0)}{(k\theta_0)}\cos(k\theta) \tag{3.2-49}$$

Assume $\theta_0 = \frac{\pi}{2}$

$$H_0(\theta) = \frac{1}{2} + \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}}\cos(k\theta) \tag{3.2-50}$$
$$= \frac{1}{2} + \frac{2}{\pi}\cos(\theta) \tag{3.2-51}$$
$$H_3(\theta) = \frac{1}{2} + \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}}\cos(\theta) + \frac{\sin(\pi)}{\pi}\cos(2\theta)$$
$$+ \frac{\sin(\frac{3\pi}{2})}{3\frac{\pi}{2}}\cos(3\theta) \tag{3.2-52}$$
$$= \frac{1}{2} + \frac{2}{\pi}\cos(\theta) + 0 + \left(-\frac{2}{3\pi}\right)\cos(3\theta) \tag{3.2-53}$$
$$= H_1(\theta) - \frac{2}{3\pi}\cos(3\theta) \tag{3.2-54}$$
$$H_5(\theta) = H_3(\theta) + \frac{2}{5\pi}\cos(5\theta) \tag{3.2-55}$$
$$\vdots$$

They are shown in Figure 3.5 in graphic forms. Each harmonic is shown individually in Figure 3.6.

Note:

1. Approximate fairly well using only a few terms (or harmonics)
2. The basic magnitude and shape can be formed by using low frequency harmonics whilst the sharp changes like the edge of the functions waveform is represented by high frequency harmonics.

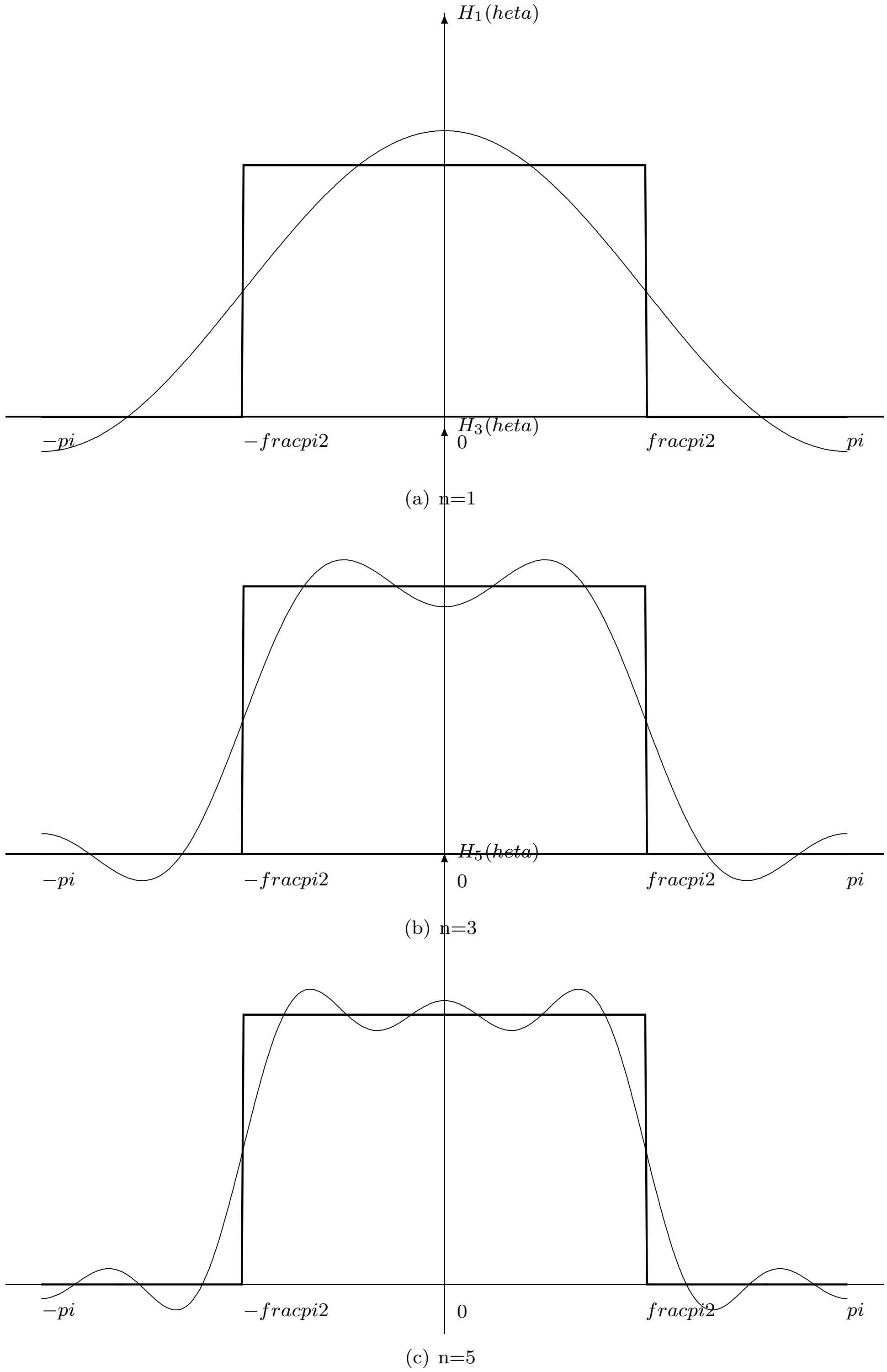


Figure 3.5: Examples of function approximation

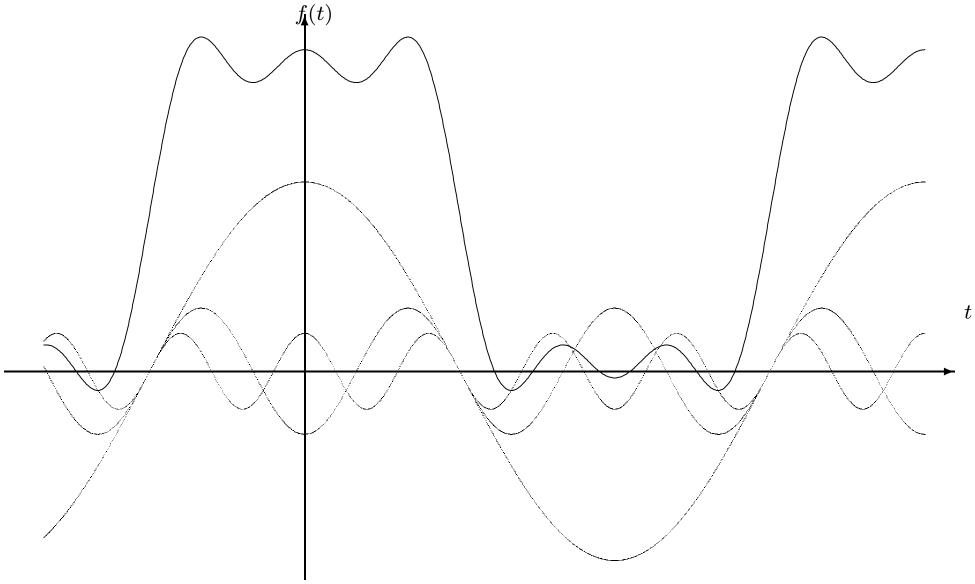


Figure 3.6: Harmonic content of a square wave

3.3 Dirichlet's Conditions

For the Fourier Series representation of a given function $f(\theta)$ to be valid, $f(\theta)$ must satisfy certain conditions.

Sufficient conditions for the convergence of the F.S. to the function $f(\theta)$ given by Dirichlet (1837) accommondate all the functions generated or encountered in engineering and applied physics.

For a real periodic function $f(\theta)$ of the real variable θ with period 2π to satisfy Dirichlet's conditions, it either:

A. is bounded in the period $[-\pi, \pi]$ and has at most a finite number of maxima, and a finite number of minima, and a finite number of discontinuities.

B. has a finite number of isolated points in $[-\pi, \pi]$ at which the function becomes infinite, but when the arbitrarily small neighbourhoods of these points are excluded $f(\theta)$ satisfies condition A in the remainder of the interval. Furthermore, the integral of $f(\theta)$ over a period must be absolutely convergent, namely

$$\int_{-\pi}^{\pi} |f(\theta)| d\theta \qquad \text{is finite} \tag{3.3-56}$$

3.4 The complex fourier series

A more compact form of F.S. of a given function $f(\theta)$ is defined by

$$f(\theta) = \sum_{k=-\infty}^{\infty} C_k e^{jk\theta} \tag{3.4-57}$$

where

$$e^{jk\theta} = \cos k\theta + j \sin k\theta \tag{3.4-58}$$

$$C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-jk\theta} d\theta \tag{3.4-59}$$

EXAMPLE 3.4-1 *Given a periodic function $f(\theta)$ with 2π period and*

$$f(\theta) = \begin{cases} A & 0 < \theta < \pi \\ -A & -\pi < \theta < 0 \end{cases} \tag{3.4-60}$$

Find the complex Fourier coefficients.
Solution:

$$C_0 = \frac{1}{2\pi}\left\{\int_{-\pi}^{0}(-A)d\theta + \int_0^{\pi} Ad\theta\right\} \tag{3.4-61}$$

$$= 0 \tag{3.4-62}$$

$$\int_{-\pi}^{0}(-A)d\theta = \int_0^{\pi}(-A)d\theta' \tag{3.4-63}$$

$$\theta' = \theta + \pi \tag{3.4-64}$$

$$C_k = \frac{1}{2\pi}\left\{\int_{-\pi}^{0}(-A)e^{-jk\theta}d\theta + \int_0^{\pi} Ae^{-jk\theta}d\theta\right\} \tag{3.4-65}$$

$$= \frac{1}{2\pi}\left\{\frac{A}{jk}\int_{-\pi}^{0} e^{-jk\theta}d(-jk\theta) + \frac{-A}{jk}\int_0^{\pi} e^{-jk\theta}d(-jk\theta)\right\} \tag{3.4-66}$$

$$= \frac{A}{2jk\pi}\left\{e^{-jk\theta}\big|_{-\pi}^{0} - e^{-jk\theta}\big|_{0}^{\pi}\right\} \tag{3.4-67}$$

$$= \frac{A}{2jk\pi}\left\{1 - e^{jk\pi} + 1 - e^{-jk\pi}\right\} \tag{3.4-68}$$

If k is even, $e^{\pm jk\pi} = 1$ then $C_k = 0$; if k is odd, $e^{\pm jk\pi} = -1$ then $C_k = +\frac{2A}{jk\pi} = -\frac{2jA}{k\pi}$

Whe the function is given as a function of time t, it is assumed that

$$\theta = \omega_0 t \tag{3.4-69}$$

where ω_0 is the fundamental radian frequency.

Periodic function is defined as

$$f(t) = f(t \pm rT)\text{for } r = 0, 1, \ldots \tag{3.4-70}$$

and

$$w_0 = \frac{2\pi}{T} \tag{3.4-71}$$

The complex Fourier Series takes the form

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \tag{3.4-72}$$

with

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi k}{T}t}dt \tag{3.4-73}$$

or

$$= \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)e^{-jk\omega_0 t}dt \tag{3.4-74}$$

Define

$$\omega_k = k\omega_0 \qquad k^{\text{th}} \text{ harmonic} \tag{3.4-75}$$

$$C_k = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)e^{-j\omega_k t}dt \tag{3.4-76}$$

Notice that

time domain	$f(t)$	continuous time function
discrete frequency domain	C_k	complex function of discrete frequency variable

As C_k is complex,

$$C_k = |C_k|e^{j\phi_k} \tag{3.4-77}$$

where

$$\phi_k = \tan^{-1}\frac{\Im(C_k)}{\Re(C_k)} \tag{3.4-78}$$

The plots of $|C_k|$ against $\omega_k = k\omega_0$ and ϕ_k against ω_k are called the amplitude spectrum and the phase spectrum respectively. Often, $|C_k|^2$ is used in the place of $|C_k|$ and the plot is called the *power spectrum*.

3.4.1 Bessel's inequality and its meaning (or implementation)

Bessel's inequality states that

$$\sum_{k=-n}^{n}|C_k|^2 < \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2 d\theta \tag{3.4-79}$$

It means that if $f(\theta)$ is bounded (or at least its square-integral is finite), the sum of the magnitudes of the Fourier coefficients is bounded.

Since the right-hand side of inequality is independant of n, each coefficient magnitude must have a limit as $n \to \infty$. Thus

$$\lim_{k\to\infty}|C_k| = 0 \tag{3.4-80}$$

3.5 The Gibb's Phenomenon and Windowing Functions

The Gibb's phenomenon occurs whenever Fourier series is used to approoximate a discontinuous function. (As shown in the cases of a square wave function and a saw-tooth function in Figure 3.7.)

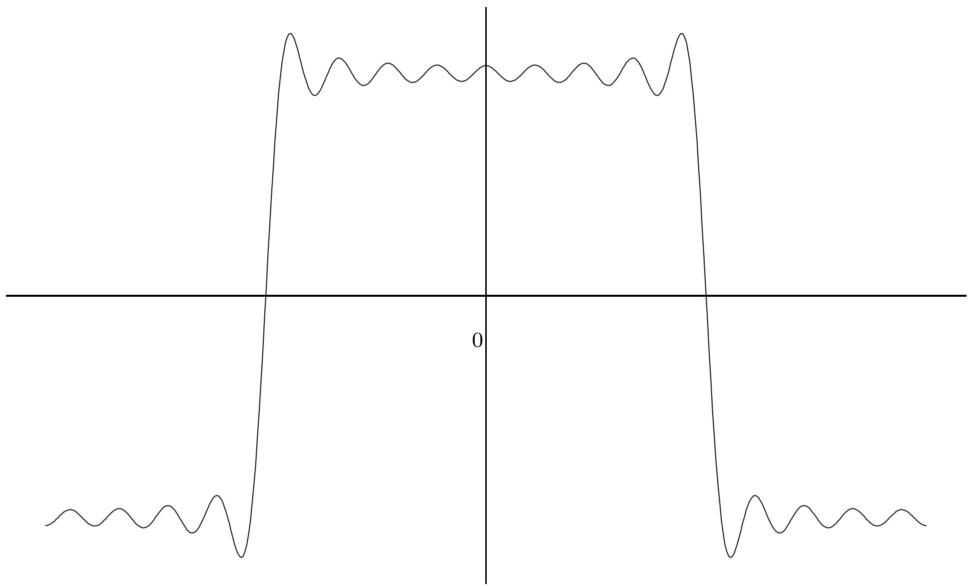
Since the ideal filter characteristics always contain discontinuities and the approximating series must be finite in order to be practical, methods must be found to suppress the Gibb's phenomenon when the truncated Fourier series is used.

Using a truncated Fourier Series $f_n(\theta)$ to approximate the original function $f(\theta)$ represented by Fourier Series is equivalent to applying *a rectangular window* to the Fourier coefficients.

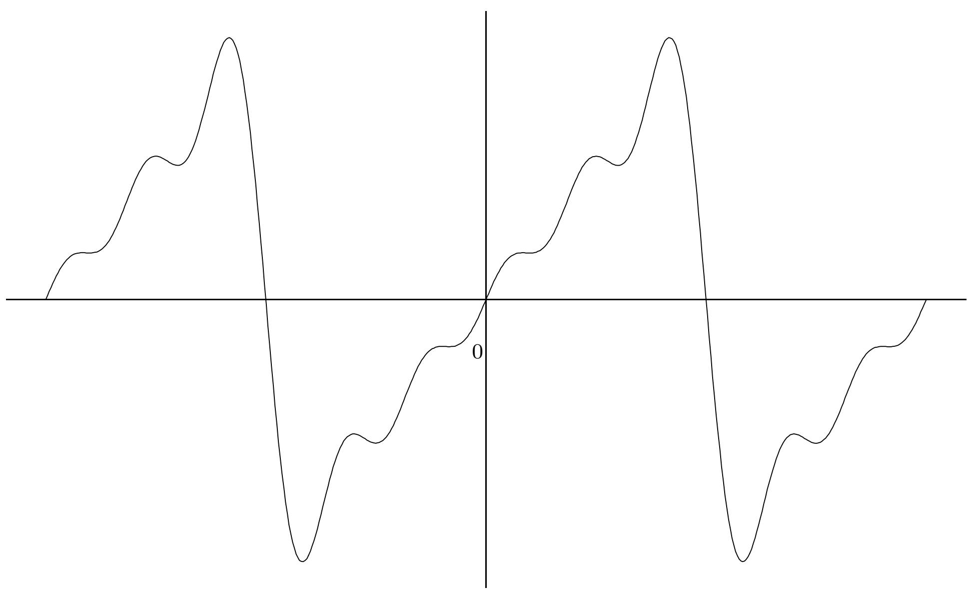
$$f(\theta) = \sum_{k=-\infty}^{\infty} C_k e^{jk\theta} \tag{3.5-81}$$

$$f_n(\theta) = \sum_{k=-n}^{n} C_k e^{jk\theta} \tag{3.5-82}$$

$$= \sum_{k=-\infty}^{\infty} W_k C_k e^{jk\theta} \tag{3.5-83}$$



(a) Square Wave



(b) Saw-Tooth Wave

Figure 3.7: Illustration of Gibb's phenomena

where

$$W_k = \begin{cases} 1 & |k| \leqslant n \\ 0 & |k| > n \end{cases} \tag{3.5-84}$$

$$\{C_k\} = \{\dots, C_{-n-1}, C_{-n}, C_{-n+1}, \dots, C_0, \dots, C_n, C_{n+1}, \dots\} \tag{3.5-85}$$

$$\{W_k\} = \{\underline{0, \dots, 0}|\overline{1, 1, \dots, 1, 1}|\underline{0, \dots}\} \tag{3.5-86}$$

$$\{W_k C_k\} = \{0, \dots, 0, C_{-n}, C_{-n+1}, \dots, C_0, \dots, C_n, 0, \dots, 0\} \tag{3.5-87}$$

The function w_k defined by Equation 3.5-84 is called a rectangular window. (A window on the fourier coefficients)

To understand the effects caused by W_k in the original domain, *the fourier sereies of the convolution of the two functions* has to be discussed.

If

$$f_1(\theta) = \sum_{k=-\infty}^{\infty} W_k e^{jk\theta} \tag{3.5-88}$$

$$f_2(\theta) = \sum_{k=-\infty}^{\infty} C_k e^{jk\theta} \tag{3.5-89}$$

and the convolution of $f_1(\theta)$ with $f_2(\theta)$ is defined as

$$g(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\theta - \psi) f_2(\psi) d\psi \tag{3.5-90}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} W_k e^{jk(\theta-\psi)} f_2(\psi) d\psi \tag{3.5-91}$$

$$= \sum_{k=-\infty}^{\infty} W_k e^{jk\theta} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} f_2(\psi) e^{-jk\psi} d\psi}_{C_k} \tag{3.5-92}$$

$$= \sum_{k=-\infty}^{\infty} W_k C_k e^{jk\theta} \tag{3.5-93}$$

In other words, applying a rectangular window on FS. is equivalent to convolution of the original function $f(\theta)$ with a window function.

The rectangular window W_k (in FS domain) can be represented in the original domain as Dirichlet's kernel.

$$C_n(\theta) = \sum_{k=-n} n e^{jk\theta} \tag{3.5-94}$$

$$= \frac{\sin(n + \frac{1}{2})\theta}{\sin \frac{\theta}{2}} \tag{3.5-95}$$

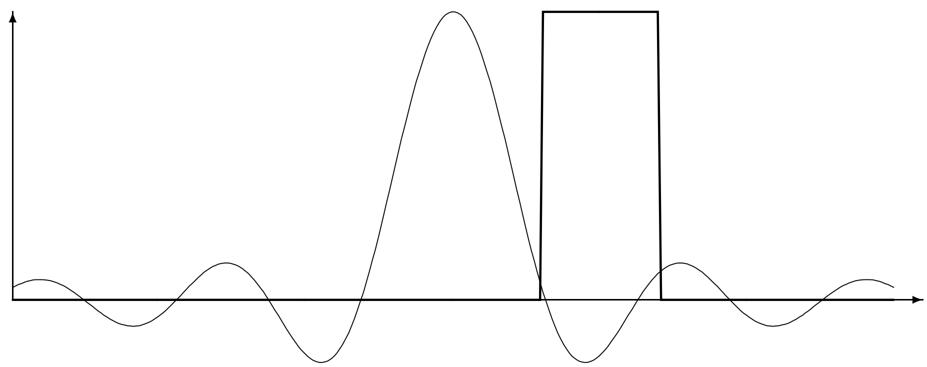
The convolution of $C_n(\theta)$ with the square wave within one period is shown in Figure 3.8.

A lot of effort has been made to design different windows from the rectangular window to suppress the Gibb's phenomenon. The improvement is shown in Figure 3.9.

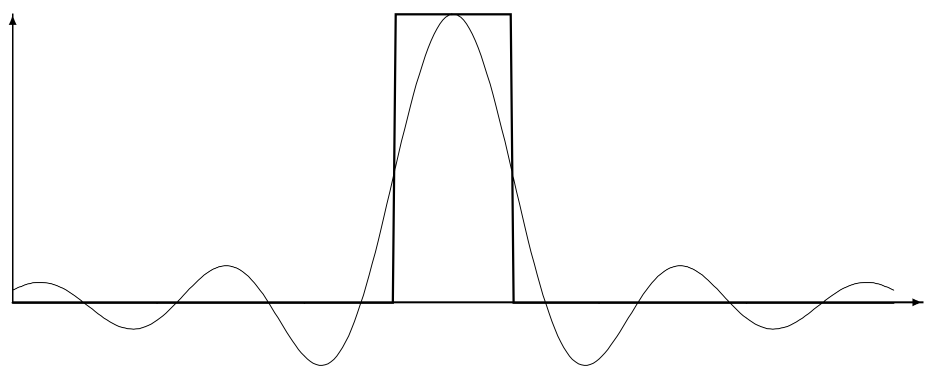
Some of the commonly used windows are listed as follows. ($W_k = 0$ for $|k| > n$)

i. The rectangular window

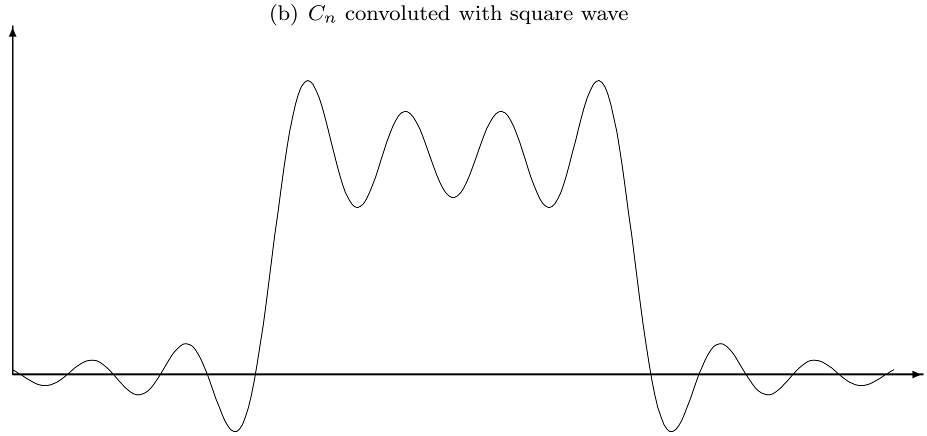
$$W_k = 1 \tag{3.5-96}$$



(a) C_n convoluted with square wave



(b) C_n convoluted with square wave



(c) Partial sum showing Gibb's phenomenon

Figure 3.8: Interpretation of the partial sum as the convolution of $C_n(\theta)$ with the function

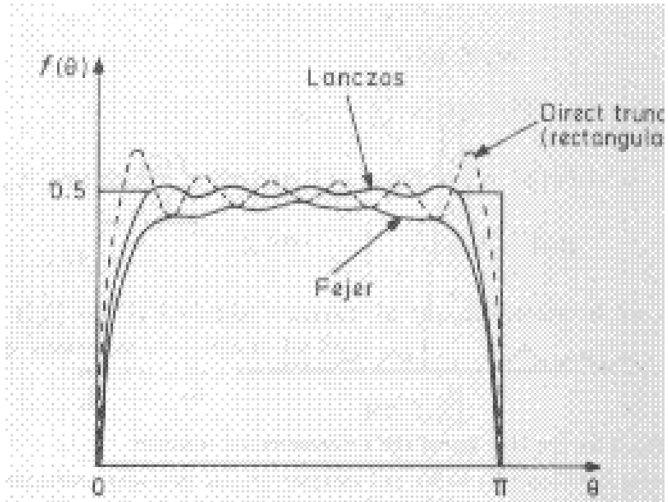


Figure 3.9: Illustration of the various types of windows — Someone

ii. the Fejer window

$$W_k = 1 - k/n \tag{3.5-97}$$

iii. The Lanczos

$$W_k = \frac{\sin \frac{k\pi}{n}}{k\frac{\pi}{n}} \tag{3.5-98}$$

iv. The von Hann window

$$W_k = 0.5[1 + \cos \frac{k\pi}{n} \tag{3.5-99}$$

v. The Hamming window

$$W_k = 0.54 + 0.46 \cos \frac{k\pi}{n} \tag{3.5-100}$$

vi. The Kaiser window

$$W_k = \frac{I_0\{\beta[1 - (\frac{k}{n})^2]^{\frac{1}{2}}\}}{I_0\beta} \tag{3.5-101}$$

where β is a parameter, I_0 is the zero-order modified Bessel function of the first kind, which is given by

$$I_0(x) = 1 + \sum_{r=1}^{\infty} \left[\frac{1}{r!} \left(\frac{x}{2} \right)^r \right]^2 \tag{3.5-102}$$

$r = 15 \sim 20$ are sufficient.

Chapter 4

The Fourier Transformation and Generalized Signals

Contents

In this chapter, the Fourier transforms will be discussed with respect to the Fourier series coefficients for the periodic functions. Basic properties of the Fourier transform wil be presented. The Unit Impulse function and its properties will also be examined.

4.1 Fourier Transform and its Properties

So far, we have only dealt with periodic functions of time, eg. $f(t)$, and their frequency domain representation using the Fourier series. It is important to notice that the frequency spectrum of a periodic function is discrete (C_k).

However, many functions that exit in the real applications are non-periodic. The Fourier integral is introduced to find the frequency domain representation for non-periodic functions. The evolution from the Fourier series to the Fourier integral can be found in Chapter 3 of Baher's book, or other references.

DEFINITION 4.1-1 *Given a function $f(t)$, provided that $f(t)$ has only a finite number of maxima and minima as well as a finite number of discontinuities over $(-\infty, \infty)$, in addition $f(t)$ is bounded, or*

$$\int_{-\infty}^{\infty} |f(t)| dt \qquad \textit{is finite} \tag{4.1-1}$$

the fourier transform (FT) of $f(t)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \tag{4.1-2}$$

and the inverse Fourier transform of $F(\omega)$ is defined by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \tag{4.1-3}$$

$f(t)$ and $F(\omega)$ constitute a Fourier transform pain and may be expressed by

$$f(t) \leftrightarrow F(\omega) \tag{4.1-4}$$

By convention, the following notations are used:

$$F(\omega) = \mathcal{F}[\{(\sqcup)] \tag{4.1-5}$$

$$f(t) = \mathcal{F}^{-\infty}[\mathcal{F}(\omega)] \tag{4.1-6}$$

In general, the Fourier Transform $F(\omega)$ of a function $f(t)$ is a complex function of ω.

$$F(\omega) = |F(\omega)| e^{j\phi(\omega)} \tag{4.1-7}$$

$$= R(\omega) + jx(\omega) \tag{4.1-8}$$

where ω is a continuous variable and $|F(\omega)|$ plotted against ω is the continuous amplitude spectrum of $f(t)$ and $\phi(\omega)$ plotted against ω presents the continuous phase spectrum.

Note:

function:periodic→nonperiodic
spectrum:discrete→continuous
Fourier: series →integral (transform)

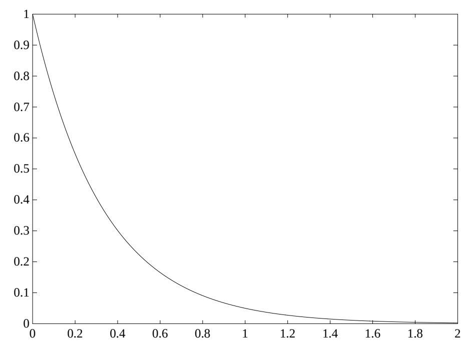
EXAMPLE 4.1-1 *Given a function*

$$f(t) = \begin{cases} e^{-bt} & t \geqslant 0 \\ 0 & t < 0 \end{cases} \tag{4.1-9}$$

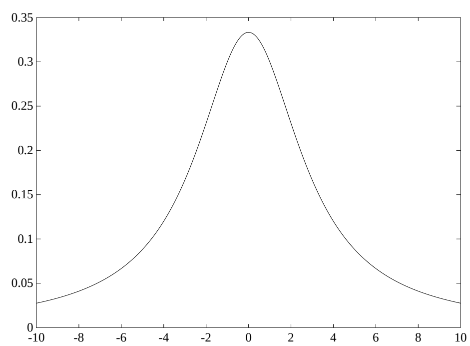
where $b > 0$, find its Fourier Transform.

Solution:

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt && (4.1\text{-}10)\\
&= \int_{-\infty}^{0} 0e^{-j\omega t}dt + \int_{0}^{\infty} e^{-bt}e^{-j\omega t}dt && (4.1\text{-}11)\\
&= \int_{0}^{\infty} e^{-(b_j\omega)t}dt && (4.1\text{-}12)\\
&= -\frac{1}{b+j\omega}e^{-(b+j\omega)t}\Big|_{0}^{\infty} && (4.1\text{-}13)\\
&= \frac{1}{b+j\omega} && (4.1\text{-}14)
\end{aligned}
$$



$\leftrightarrow$



The F.T. possesses linearity: if

$$f_1(t) \leftrightarrow F_1(\omega) \tag{4.1-15}$$

and

$$f_2(t) \leftrightarrow F_2(\omega) \tag{4.1-16}$$

then

$$a_1f_1(t) + a_2f_2(t) \leftrightarrow a_1F_1(\omega) + a_2F_2(\omega) \tag{4.1-17}$$

where $a_i (i = 1, 2)$ are arbitrary constants.

4.1.1 Fourier Transform of Real Functions

The definition of the Fourier Transform can be further simplified if $f(t)$ is a real function or real even or real odd function.

As mentioned previously that the Fourier Transform of a function is generally complex, it can be expressed as

$$F(\omega) = |F(\omega)|e^{j\phi(\omega)} \tag{4.1-18}$$

or

$$F(\omega) = R(\omega) + jx(\omega) \tag{4.1-19}$$

where $R(\omega)$ and $x(\omega)$ are the real part and imaginary part of the $F(\omega)$

$$|F(\omega)| = \sqrt{R^2(\omega) + x^2(\omega)} \tag{4.1-20}$$

$$\phi(\omega) = \tan^{-1}\frac{x(\omega)}{R(\omega)} \tag{4.1-21}$$

If $f(t)$ is real,

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)\,(\cos\omega t - j\sin\omega t)\,dt && (4.1\text{-}22)\\ &= \int_{-\infty}^{\infty} f(t)\cos\omega t dt - j\int_{-\infty}^{\infty} f(t)\sin\omega t dt && (4.1\text{-}23)\\ &= R(\omega) + jx(\omega) && (4.1\text{-}24) \end{aligned}$$

where

$$R(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t dt \tag{4.1-25}$$

$$x(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t dt \tag{4.1-26}$$

Furthermore, $R(\omega)$ is even and $x(\omega)$ is odd. Therefore

$$F^* = F(-\omega) \tag{4.1-27}$$

where $*$ (the asterisk) denotes the complex conjugate.

And $|F(\omega)|$ is an even function of ω and $\phi(\omega)$ is an odd function of ω.

If the function $f(t)$ is real and even, i.e.

$$f(-t) = f(t) \tag{4.1-28}$$

then

$$\begin{aligned} F(\omega) = R(\omega) &= \int_{-\infty}^{\infty} f(t)\cos\omega t dt && (4.1\text{-}29)\\ &= 2\int_{0}^{\infty} f(t)\cos\omega t dt && (4.1\text{-}30) \end{aligned}$$

(Because $x(\omega) = -\int_{-\infty}^{\infty} \underbrace{f(t)\sin\omega t}_{\text{odd}}\, dt = 0.$)

If the function $f(t)$ is odd, ie.

$$f(-t) = -f(t) \tag{4.1-31}$$

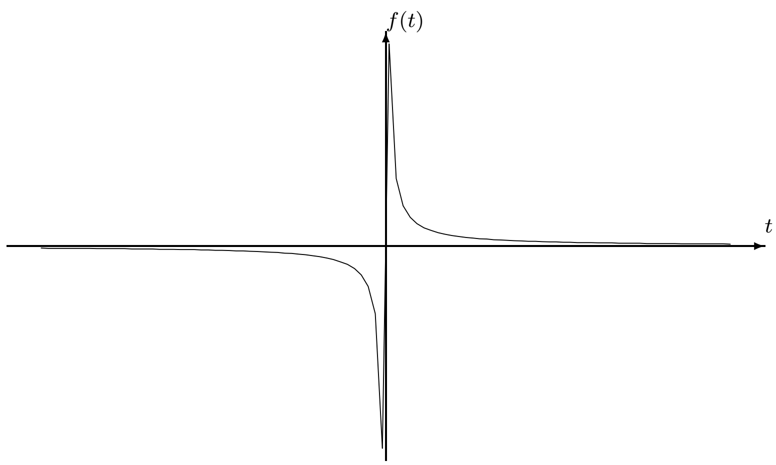
then

$$F(\omega) = jx(\omega) = -j\int_{-\infty}^{\infty} \sin\omega t dt \tag{4.1-32}$$

$$= -2j\int_{0}^{\infty} \sin\omega t dt \tag{4.1-33}$$

(Because $R(\omega) = \int_{-\infty}^{\infty} \underbrace{f(t)\cos\omega t}_{\text{odd}}\, dt = 0.$)

EXAMPLE 4.1-2 *Find the fourier transform of the function $f = \frac{1}{t}$.*



Solution:

1.

$$f(-t) = -\frac{1}{t} = -f(t) \tag{4.1-34}$$

The function is odd.

2.

$$F(\omega) = -2j\int_{0}^{\infty} \frac{\sin\omega t}{t} dt \tag{4.1-35}$$

$$= -2j\omega\int_{0}^{\infty} \frac{\sin\omega t}{\omega t} dt \tag{4.1-36}$$

$$= -2j\int_{0}^{\infty} \frac{\sin\omega t}{\omega t} d(\omega t) \tag{4.1-37}$$

Now $\psi \triangleq \omega t$

$$\int_0^\infty \frac{\sin \omega t}{\omega t} d\omega t = \int_0^\infty \tag{4.1-38}$$

$$= \frac{\sin \psi}{\psi} d\psi \tag{4.1-39}$$

is called the sine integral and

$$\int_0^\infty \frac{\sin \psi}{\psi} d\psi = \begin{cases} \frac{\pi}{2} & \omega > 0 \\ -\frac{\pi}{2} & \omega < 0 \\ 0 & \omega = 0 \end{cases} \tag{4.1-40}$$

$$F(\omega) = -2j\frac{\pi}{2}\,\mathrm{sgn}(\omega) \tag{4.1-41}$$

$$= -j\pi\,\mathrm{sgn}(\omega) \tag{4.1-42}$$

where

$$\mathrm{sgn}(\omega) = \begin{cases} 1 & \omega > 0 \\ -1 & \omega < 0 \end{cases} \tag{4.1-43}$$

3. *Thus*

$$\frac{1}{t} \leftrightarrow -j\pi\,\mathrm{sgn}(\omega) \tag{4.1-44}$$

Generally, for a real function $f(t)$

$$F(\omega) = \mathcal{F}[\{(\sqcup)] \tag{4.1-45}$$

$$\mathcal{F}[\{(-\sqcup)] = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt \tag{4.1-46}$$

$$t' \triangleq -t \tag{4.1-47}$$

$$\left(\begin{array}{ll} t{\to}-\infty & t'{\to}+\infty \\ t{\to}+\infty & t'{\to}-\infty \end{array} \right. \tag{4.1-48}$$

$$\mathcal{F}[\{(-\sqcup)] = F[f(t')] \tag{4.1-49}$$

$$= \int_{\infty}^{\infty} f(t')e^{j\omega t'}d(-t') \tag{4.1-50}$$

$$= -\int_{-\infty}^{\infty} f(t')e^{j\omega t'}dt' \tag{4.1-51}$$

$$= \int_{-\infty}^{\infty} f(t')e^{-j\omega t'}dt' \tag{4.1-52}$$

$$= F(\omega') \tag{4.1-53}$$

$$= F(-\omega) \tag{4.1-54}$$

As

$$F(-\omega) = R(-\omega) + jx(-\omega) \tag{4.1-55}$$

$$= R(\omega) - jx(\omega) \tag{4.1-56}$$

(If $f(t)$ is real, $R(\omega)$ even, $x(\omega)$ odd.) it follows that

$$f(t) \leftrightarrow R(\omega) + jx(\omega) \tag{4.1-57}$$

$$f(-t) \leftrightarrow R(\omega) - jx(\omega) \tag{4.1-58}$$

For a given real function $f(t)$

$$f(t) = \underbrace{\text{even part}}_{\frac{1}{2}[f(t)+f(-t)]} + \underbrace{\text{odd part}}_{\frac{1}{2}[f(t)-f(t)]} \tag{4.1-59}$$

Using linearity of F.T.

$$\frac{1}{2}[f(t) + f(-t)] \leftrightarrow \frac{1}{2}\left\{[R(\omega) + jx(\omega)] + [R(\omega) - jx(\omega)]\right\} \tag{4.1-60}$$
$$= \frac{1}{2}\{2R(\omega)\} \tag{4.1-61}$$
$$= R(\omega) \tag{4.1-62}$$
$$\frac{1}{2}[f(t) - f(-t)] \leftrightarrow \frac{1}{2}\left\{[R(\omega) + jx(\omega)] - [R(\omega) - jx(\omega)]\right\} \tag{4.1-63}$$
$$= j\frac{1}{2}\{2x(\omega)\} \tag{4.1-64}$$
$$= jx(\omega) \tag{4.1-65}$$

It is concluded that:

The even part of a real function is transformed into the real part of the F.T. and the odd part of the function in transformed into the imaginary part of the F.T. See Figure 4.1 for an example.

4.1.2 Fourier Transform of Causal Signals

For a real causal function

$$f(-t) = 0 \qquad t > 0 \tag{4.1-66}$$

it can be expressed as

$$\begin{array}{rccc} f(t) = & \frac{1}{2}[f(t) + f(-t)] & + & \frac{1}{2}[f(t) - f(-t)] \\ & \Updownarrow & & \Updownarrow \\ & \frac{1}{2}f(t) & + & \frac{1}{2}f(t) \end{array} \tag{4.1-67}$$

or

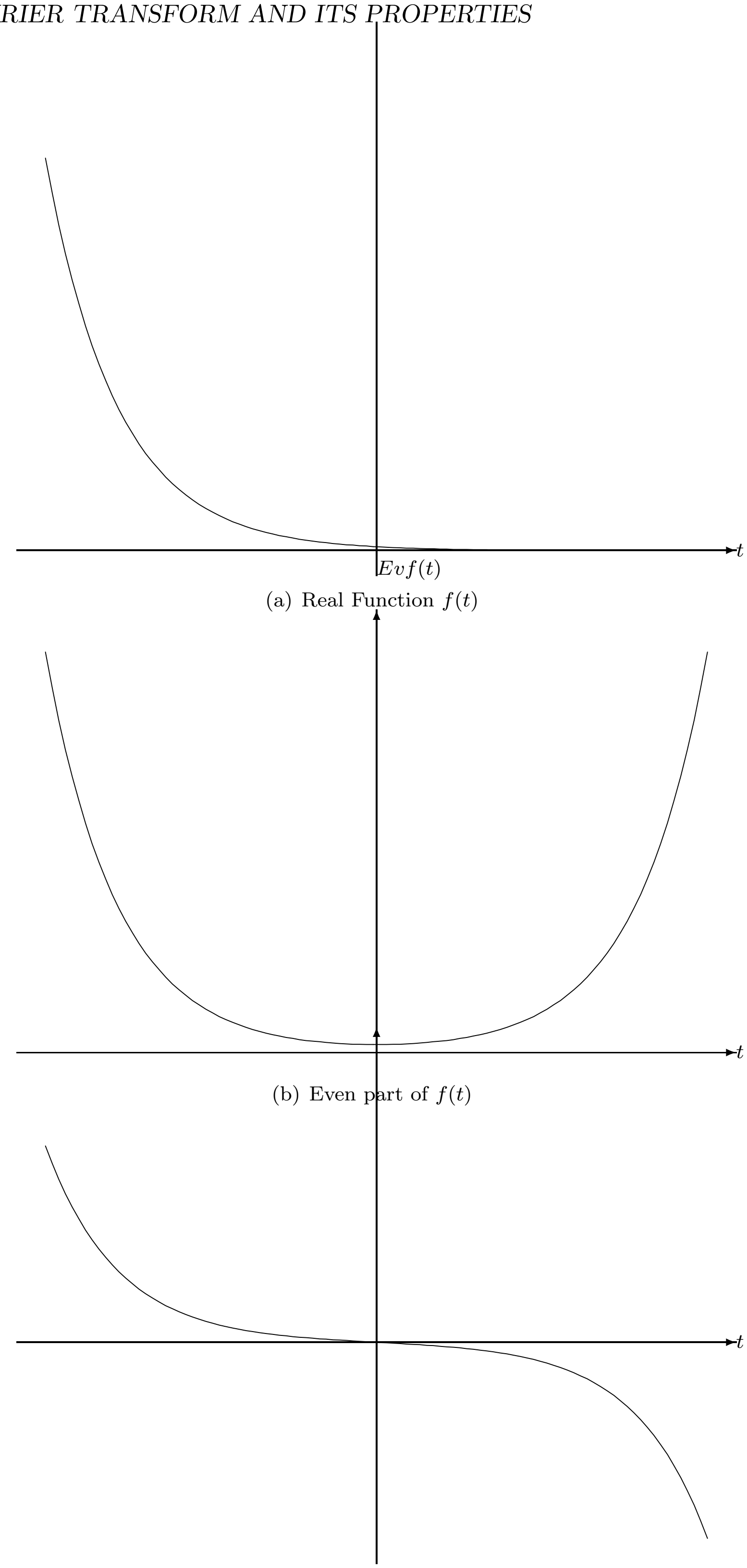
$$f(t) = 2 \times \text{even part of } f(t) \tag{4.1-68}$$
$$= 2 \times \text{odd part of } f(t) \tag{4.1-69}$$

Using linearity of the F.T.

$$f(t) \leftrightarrow 2R(\omega) \tag{4.1-70}$$
$$f(t) \leftrightarrow 2jx(\omega) \tag{4.1-71}$$

provided that $f(t)$ is casual and real. Equation 4.1-71 states that a real causal function is uniquely determined by either the real or imaginary part of the Fourier Transform.

The relation between $R(\omega)$ and $x(\omega)$ of the $F(\omega)$ of a real causal function $f(t)$ is stated by the Hilbert Transform.



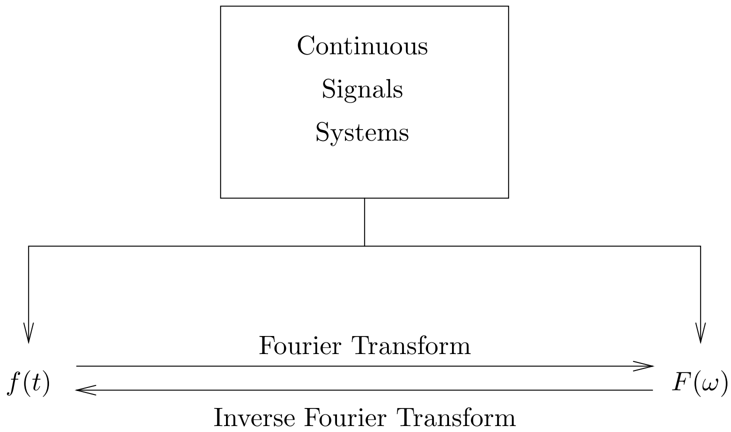
(a) Real Function $f(t)$

(b) Even part of $f(t)$

(c) Odd part of $f(t)$

Figure 4.1: Even and Odd parts of a real function

4.2 Basic Properties of the Fourier Transform



- Representation
- Understanding
- Operation and Manipulation

4.2.1 Symmetry

if

$$f(t) \leftrightarrow F(\omega) \tag{4.2-72}$$

$$F(\pm t) \leftrightarrow f(\mp\omega) \tag{4.2-73}$$

Proof:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{4.2-74}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \tag{4.2-75}$$

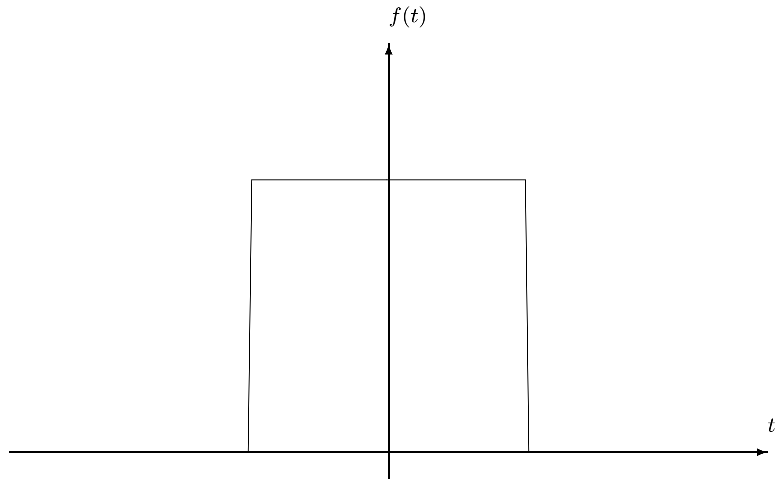
In Equation 4.2-75 $t \to -\omega$, $\omega \to t$ and multiply by 2π

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t)e^{-j\omega t}dt \tag{4.2-76}$$

$$= \mathcal{F}[\mathcal{F}(\sqcup)] \tag{4.2-77}$$

Example 4.2-1 *Given*

$$f(t) = \begin{cases} 1 & |t| < \frac{a}{2} \\ 0 & \textit{otherwise} \end{cases}$$



The Fourier Transform of $f(t)$ is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{4.2-78}$$

$$= \int_{-a/2}^{a/2} 1e^{-j\omega t}dt \tag{4.2-79}$$

$$= -\frac{1}{j\omega}\ e^{-j\omega t}\Big|_{-a/2}^{a/2} \tag{4.2-80}$$

$$= -\frac{1}{j\omega}\left\{e^{-j\omega\frac{a}{2}} - e^{j\omega\frac{a}{2}}\right\} \tag{4.2-81}$$

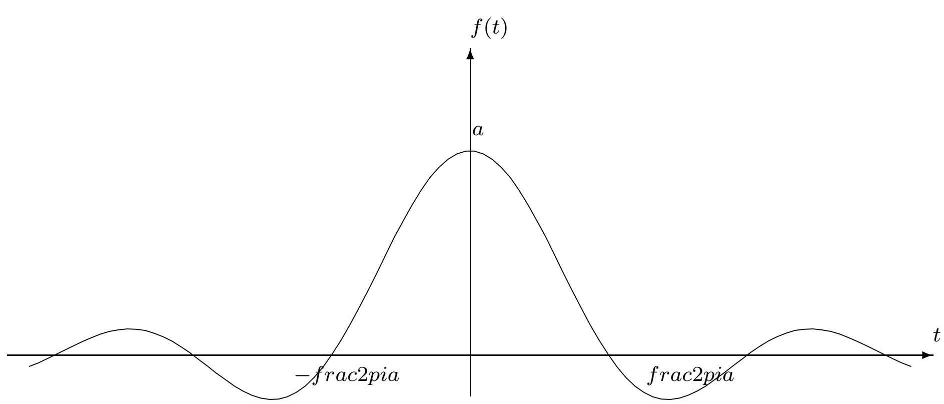
$$= -\frac{1}{j\omega}\left\{\cos\frac{\omega a}{2} - j\sin\frac{\omega a}{2} - \cos\frac{\omega a}{2} - j\sin\frac{\omega a}{2}\right\} \tag{4.2-82}$$

$$= \frac{2}{\omega}\sin\frac{\omega a}{2} \tag{4.2-83}$$

$$= a\frac{\sin\frac{\omega a}{2}}{\frac{\omega a}{2}} \tag{4.2-84}$$

That is

$$f(t) \leftrightarrow a\frac{\sin\frac{\omega a}{2}}{\frac{\omega a}{2}} \tag{4.2-85}$$



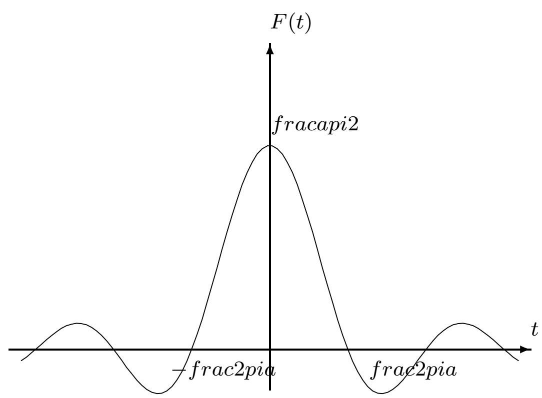
Using the symmetry property

$$a\frac{\sin\frac{ta}{2}}{\frac{ta}{2}} \leftrightarrow 2\pi f(-\omega) \tag{4.2-86}$$

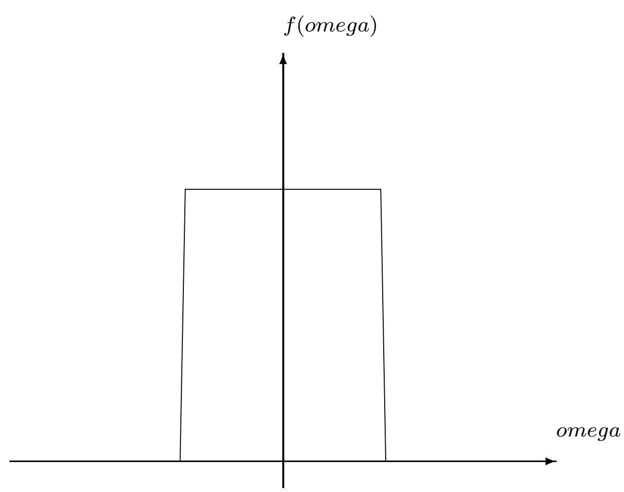
or

$$\frac{\sin\frac{at}{2}}{\pi t} \leftrightarrow f(-\omega) \tag{4.2-87}$$

$$f(-\omega) = \begin{cases} 1 & |\omega| < \frac{a}{2} \\ 0 & otherwise \end{cases} \tag{4.2-88}$$



$\leftrightarrow$



4.2.2 Conjugate

If $f(t)$ is a complex function and $\mathcal{F}[\{(\sqcup)] = \mathcal{F}(\omega)$,

$$\mathcal{F}[\{^*(\{)] = \int_{-\infty}^{\infty} f^*(t)e^{-j\omega t}dt \tag{4.2-89}$$

$$= \left\{\int_{-\infty}^{\infty} f(t)[e^{-j\omega t}]^* dt\right\}^* \tag{4.2-90}$$

$$= \left\{\int_{-\infty}^{\infty} f(t)e^{j\omega t}]dt\right\}^* \tag{4.2-91}$$

$$= \left\{\int_{-\infty}^{\infty} f(t)e^{-j\omega t'}]dt\right\}^* \tag{4.2-92}$$

$$= F^*(\omega') \tag{4.2-93}$$

$$= F^*(-\omega) \tag{4.2-94}$$

That is $f^* \leftrightarrow F^*(\omega)$

However, if $f(t)$ is real

$$f^*(t) = f(t) \tag{4.2-95}$$

it follows that

$$f(t) \leftrightarrow F^*(-\omega) \tag{4.2-96}$$

and

$$F(\omega) = F^*(-\omega) \tag{4.2-97}$$

4.2.3 Linearity

If

$$a_i F_i(t) \leftrightarrow a_i F_i(\omega) \tag{4.2-98}$$

$$\sum_i a_i f_i(t) \leftrightarrow \sum_i a_i F_i(\omega) \tag{4.2-99}$$

where a_i are arbitrary constants.

4.2.4 Scaling

If α is a real constant,

$$f(\alpha t) \leftrightarrow \frac{1}{|\alpha|} F \frac{\omega}{\alpha} \tag{4.2-100}$$

The scaling property states that the compression of the time scale corresponds to the expansion of te frequency scale. The magnitude is contracted while the horizontal expands. As shown in Figure 4.2.

4.2.5 Shifting in Time

If $f(t) \leftrightarrow F(\omega)$

$$f(t-\alpha) \leftrightarrow e^{-j\alpha\omega)} F(\omega) \tag{4.2-101}$$

It states that $f(t-\alpha)$ has the same amplitude as $f(t)$ and only differs in the phase spectrum by $(-\alpha\omega)$. See Figure 4.3 for an example.

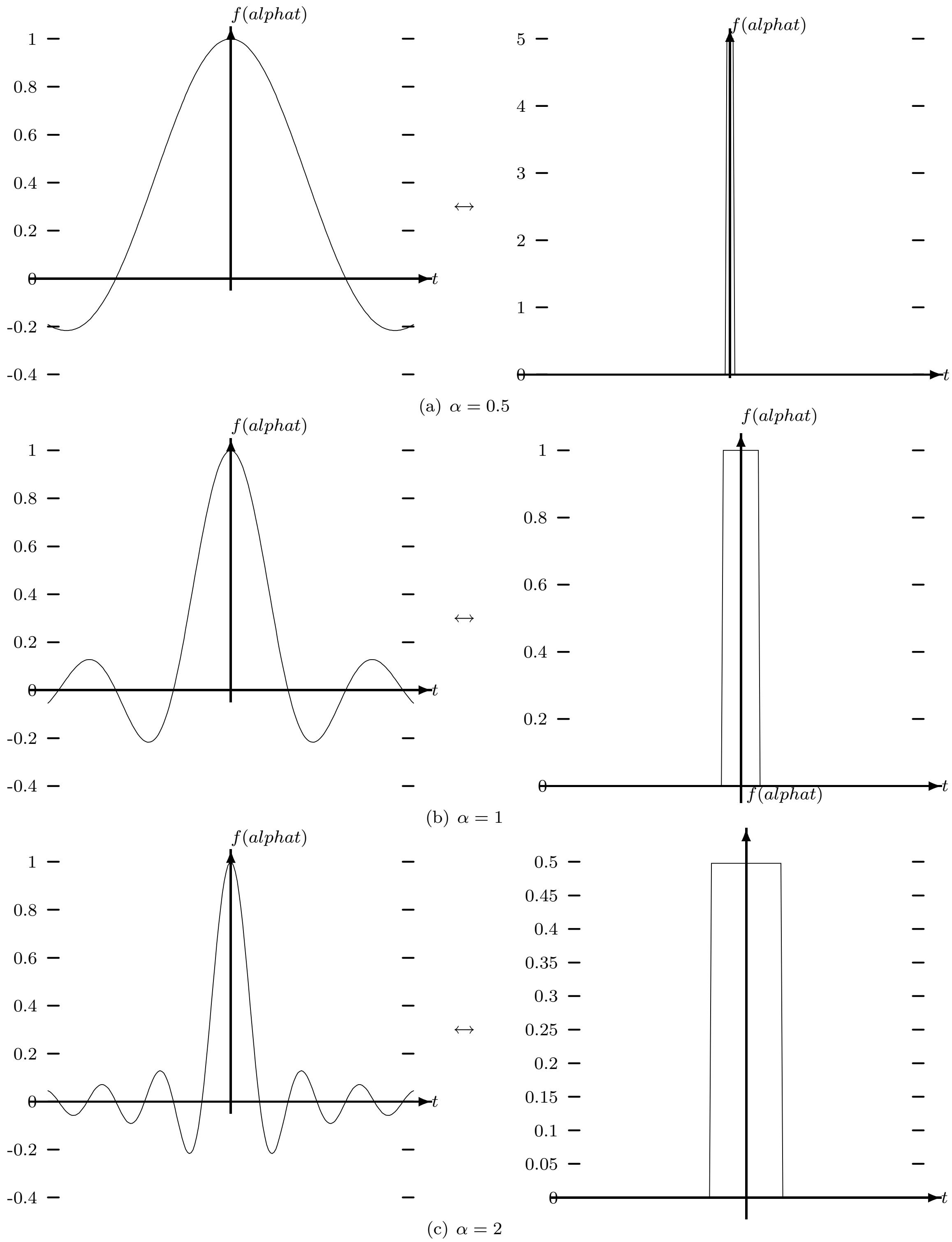
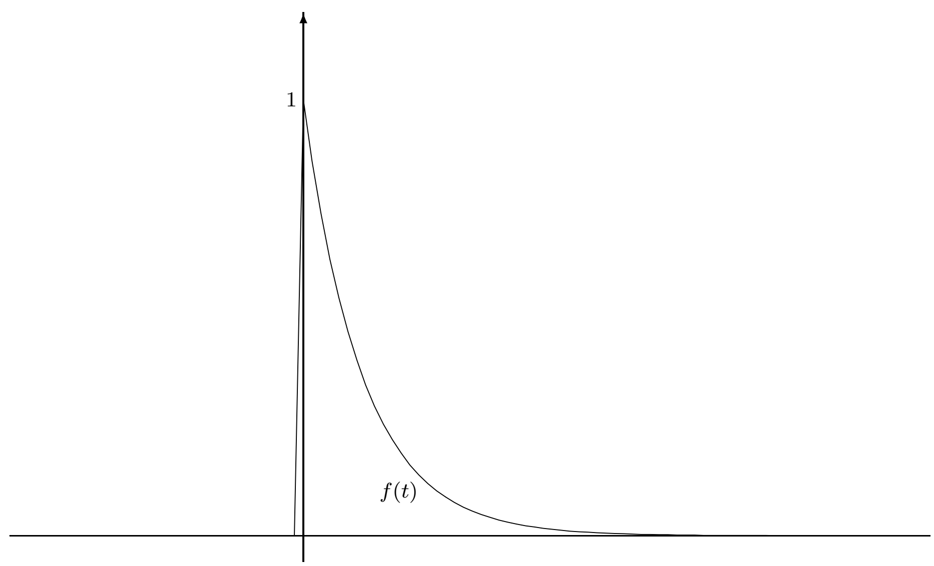
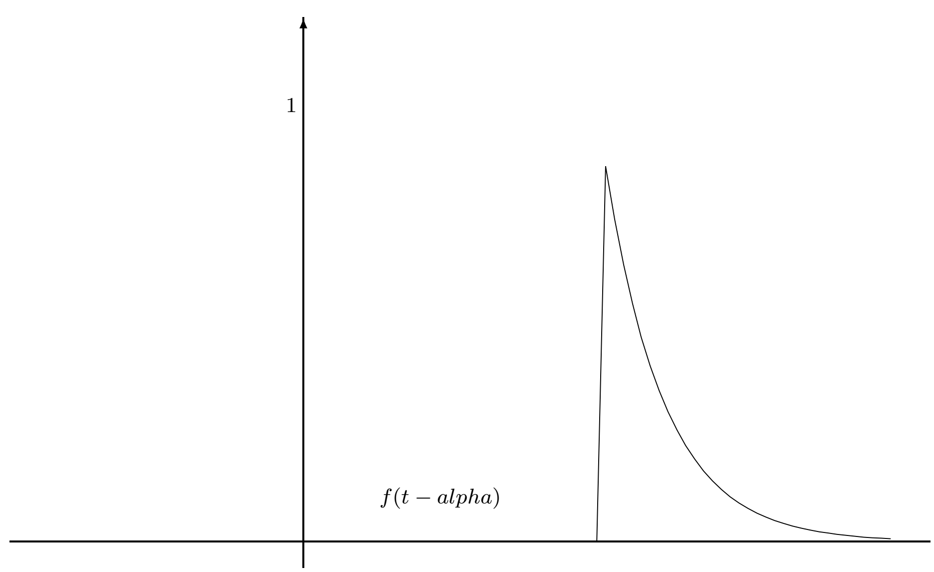


Figure 4.2: Time-scaling property of the Fourier transform



(a) e^{-bt}



(b) e^{-bt} shifted by α

Figure 4.3: Example of Shifting in Time

4.2.6 Shifting in Frequency

If

$$f(t) \leftrightarrow F(\omega) \tag{4.2-102}$$
$$f(t)e^{\pm j\omega_0 t} \leftrightarrow F(\omega \mp \omega_0) \tag{4.2-103}$$
$$\mathcal{F}\left[\{(\sqcup)\rceil^{\pm|\omega,\sqcup}\right] \tag{4.2-104}$$
$$= \int_{-\infty}^{\infty} f(t)e^{\pm j\omega_0 t}e^{-j\omega t}dt \tag{4.2-105}$$
$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega \mp \omega_0)t}dt \tag{4.2-106}$$
$$= \int_{-\infty}^{\infty} f(t)e^{(-j\omega')t}dt \tag{4.2-107}$$
$$= F(\omega') \tag{4.2-108}$$
$$= F(\omega \mp \omega_0) \tag{4.2-109}$$

4.2.7 Modulation

Using the linearity and the shifting in frequency properties of the Fourier Transform.

$$\frac{1}{2}f(t)\left[e^{j\omega_0 t)} + e^{-j\omega_0 t}\right] \leftrightarrow \frac{1}{2}\left[F(\omega-\omega_0) + F(\omega+\omega_0)\right] \tag{4.2-110}$$
$$\Updownarrow \tag{4.2-111}$$
$$f(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}\left[F(\omega-\omega_0) + F(\omega+\omega_0)\right] \tag{4.2-112}$$

In this case $\cos\omega_0 t$ is called the carrier and $f(t)$ is called modularity or baseband signal. The function $\cos\omega_0 t$ is said to be modulated in amplitude by the signal $f(t)$,

The spectrum of the modulated signal is obtained from that of $f(t)$ by shifting the baseband spectrum by $-\omega_0$ and ω_0 producing two side-bands as shown in Figure 4.4.

EXAMPLE 4.2-2 *Examine the amplitude spectrum of a $\cos\omega_0 t$ function modulated by a pulse signal with a width of $2d$.*

The baseband spectrum of the pulse function is shown in (b).

$$P_{2d} \leftrightarrow \frac{2\sin d\omega}{\omega} \tag{4.2-113}$$

Using the modualtion property of the F.T.

$$P_{2d}\cos\omega_0 t \leftrightarrow \frac{\sin d(\omega-\omega_0)}{\omega-\omega_0} + \frac{\sin d(\omega+\omega_0)}{\omega+\omega_0} \tag{4.2-114}$$

as shown in (c) and (d).

4.2.8 Differentiation with respect to time

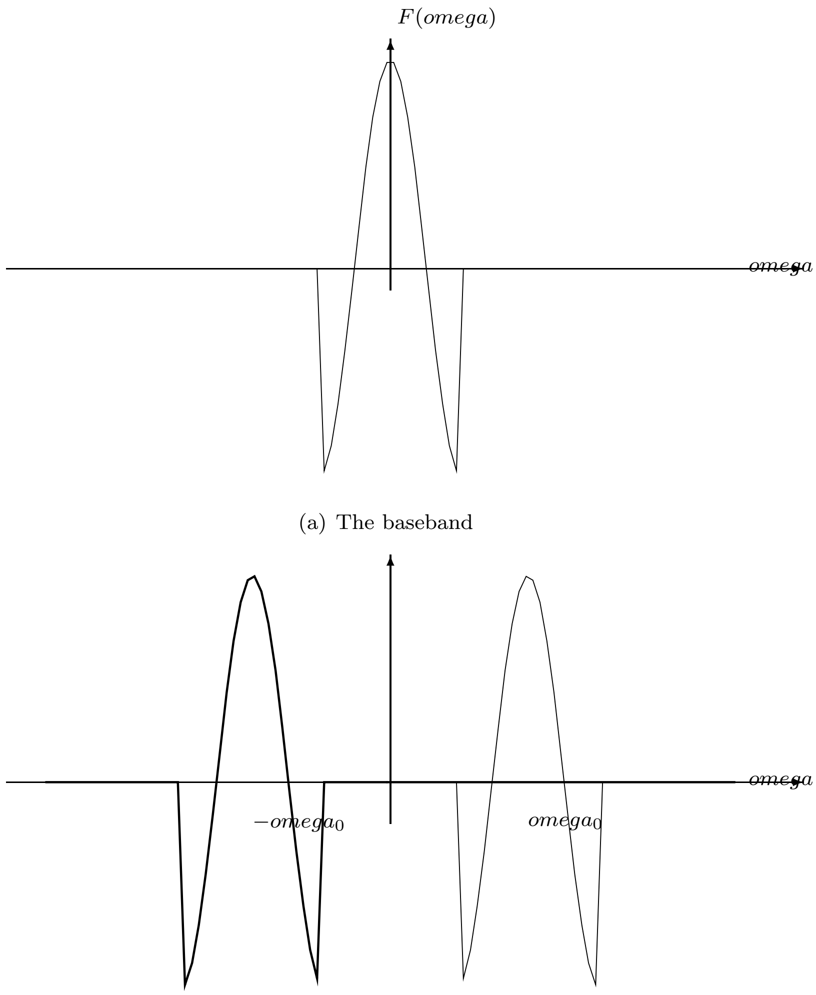
If

$$f(t) \leftrightarrow F(\omega) \tag{4.2-115}$$
$$\frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(\omega) \tag{4.2-116}$$

provided that

$$\frac{d^n f(t)}{dt^n} \tag{4.2-117}$$

still satisfies the Fourier Transform conditions.



(a) The baseband

(b) The two side bands

Figure 4.4: Illustration of the amplitude modulation property of the Fourier transform

4.2.9 Differentiation with respect to frequency

If

$$f(t) \leftrightarrow F(\omega) \tag{4.2-118}$$

$$(-jt)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n} \tag{4.2-119}$$

4.2.10 Time convolution

If

$$f_1(t) \leftrightarrow F_1(\omega) \tag{4.2-120}$$

$$f_2(t) \leftrightarrow F_2(\omega) \tag{4.2-121}$$

and define the convolution of $f_1(t)$ and $f_2(t)$ as

$$f_1(t) * f_2(t) \triangleq \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau \tag{4.2-122}$$

$$\triangleq \int_{-\infty}^{\infty} f_1(\tau) f(t-\tau) d\tau \tag{4.2-123}$$

$$f_1(t) * f_2(t) \leftrightarrow F_1(\omega) F_2(\omega) \tag{4.2-124}$$

4.2.11 Frequency convolution

If

$$f_1(t) \leftrightarrow F_1(\omega) \tag{4.2-125}$$

and

$$f_2(t) \leftrightarrow F_2(\omega) \tag{4.2-126}$$

$$f_1(t) f_2(t) \leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega) \tag{4.2-127}$$

where

$$F_1(\omega) * F_2(\omega) \triangleq \int_{-\infty}^{\infty} F_1(\mu) F_2(\omega-\mu) d\mu \tag{4.2-128}$$

$$\triangleq \int_{-\infty}^{\infty} F_1(\omega-\mu) F_2(\mu) d\mu \tag{4.2-129}$$

Proof:

$$\mathcal{F}^{-\infty}[\mathcal{F}_{\infty}(\omega) * \mathcal{F}_{\in}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega) * F_2(\omega) e^{j\omega t} d\omega \tag{4.2-130}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F_1(\mu) F_2(\omega-\mu) d\mu \right\} e^{j\omega t} d\omega \tag{4.2-131}$$

As $\omega = \mu + x$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F_1(\mu) F_2(x) d\mu \right\} e^{j(\mu+x)t} dx \tag{4.2-132}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\mu) e^{j\mu t} d\mu \int_{-\infty}^{\infty} F_2(x) e^{jxt} dx \tag{4.2-133}$$

$$= f_1(t) 2\pi f_2(t) \tag{4.2-134}$$

Thus

$$\frac{1}{2\pi}\mathcal{F}^{-\infty}[\mathcal{F}_{\infty}(\omega) * \mathcal{F}_{\in}(\omega)] = f_1(t)f_2(t) \tag{4.2-135}$$

4.3 Parseval's Theorem and Energy Spectrum

The frequecny-convolution property of the Fourier Transform can be used to derive Parseval's theorem presented as follows:

$$\int_{-\infty}^{\infty} f(t)f^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F^*(\omega)d\omega \tag{4.3-136}$$

or

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \tag{4.3-137}$$

For a real function $f(t)$, $|f(t)|$ can be replaced by $f(t)$ in the above equation.

The theorem states that the energy contents of $f(t)$ is given by $\frac{1}{2\pi}$ times the area under the *energy spectrum* which is the plot of $|F(\omega)|^2$ against ω.

Energy spectrum (or energy spectral density of the signal $f(t)$) can bne defined as

$$\mathcal{E}(\omega) \triangleq |\mathcal{F}(\omega)|^{\in} \tag{4.3-138}$$

Equation 4.3-137 states the face that the total energy in the spectrum is equal to the total energy in the $f(t)$, and $f(t)$ is a *finite energy signal* as well.

4.4 Correlations Functions and the Wiener-Kintchine Theorem

In some applications we would like to know the resemblance of signals $f(t)$ and $g(t)$ or the resemblance of a section (segment) of a signal with its neighbourhood.

DEFINITION 4.4-1 *The autocorrelation function of a real finite energy signal $f(t)$ is defined as*

$$\rho_{ff}(\tau) = \int_{-\infty}^{\infty} f(t)f(t+\tau)dt \tag{4.4-139}$$

THEOREM 4.4-1 (WIENER-KINTCHINE THEOREM 1) *If*

$$f(t) \leftrightarrow F(\omega) \tag{4.4-140}$$

$$\rho_{ff}(\tau) \leftrightarrow F^*(\omega)F(\omega) \tag{4.4-141}$$

or

$$\rho_{ff}(\tau) \leftrightarrow |F(\omega)|^2 \tag{4.4-142}$$

This means that the autocorrelation function and the energy spectrum constitute a Fourier transform pair.

Proof:

$$\mathcal{F}[\rho\{\{(\tau)] = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)f(t+\tau)dt\right\} e^{-j\omega\tau}d\tau \tag{4.4-143}$$

as $t + \tau = x$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t) f(x) dx \right\} e^{-j\omega(x-t)} d\tau \tag{4.4-144}$$

$$= \underbrace{\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}_{F^*(\omega)} \underbrace{\int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx}_{F(\omega)} \tag{4.4-145}$$

DEFINITION 4.4-2 *The cross-correlation between two finite energy signals $f(t)$ and $g(t)$ is defined as*

$$\rho_{fg}(\tau) = \int_{-\infty}^{\infty} f(t) g(t + \tau) dt \tag{4.4-146}$$

THEOREM 4.4-2 (WIENER-KINTCHINE THEOREM 2) *If*

$$f(t) \leftrightarrow F(\omega) \tag{4.4-147}$$

and

$$g(t) \leftrightarrow G(\omega) \tag{4.4-148}$$

$$\rho_{fg}(\tau) \leftrightarrow F^*(\omega) G(\omega) \tag{4.4-149}$$

or

$$\rho_{fg}(\tau) \leftrightarrow \mathcal{E}_{\{\}}(\omega) \tag{4.4-150}$$

where $\mathcal{E}_{\{\}}(\omega) \triangleq \mathcal{F}^*(\omega)\mathcal{G}(\omega)$ is caled the Cross-energy spectrum of the two given signals. That is the cross-correlation and the cross-energy spectrum constitute a Fourier transform pair.

If $\rho_{fg}(\tau)$ is zero for all τ, the two signals are said to be uncorrelated.

4.5 The Unit impulse and Generalized Functions

What is the problem?

Given a unit step function

$$U(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \tag{4.5-151}$$

try to calculate its Fourier transform

$$\mathcal{F}[\mathcal{U}(\sqcup)] = \int_{-\infty}^{\infty} U(t) e^{-j\omega t} dt \tag{4.5-152}$$

$$= \int_{0}^{\infty} 1 e^{-j\omega t} dt \tag{4.5-153}$$

$$= \frac{1}{-j\omega} \int_{0}^{\infty} e^{-j\omega t} d(-j\omega t) \tag{4.5-154}$$

$$= \frac{1}{-j\omega} e^{-j\omega t} \Big|_{0}^{\infty} \tag{4.5-155}$$

$$= \frac{1}{-j\omega} e^{-j\omega \infty} + \frac{1}{j\omega} \tag{4.5-156}$$

It does not converge!

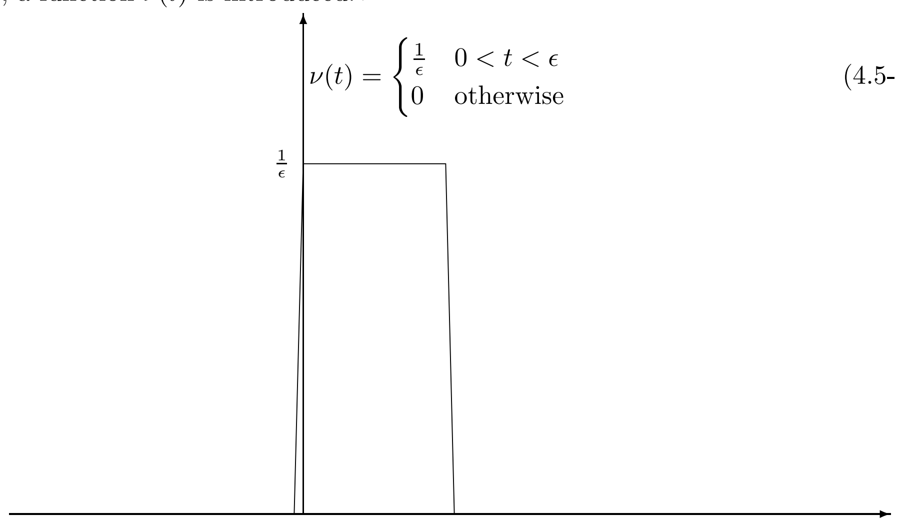
In otherwords, a simple and very useful function like $U(t)$ does not have a Fourier transform.

The need arises to introduce the unit impulse function (or Dirac delta function) and definition of generalized functions.

4.5.1 A Heuristic introduction of the Unit Impulse functions

Since the unit step function $U(t)$ cannot be differentiated at $t = 0$ (its discontinuity point), a function $\nu(t)$ is introduced

$$\nu(t) = \begin{cases} \frac{1}{\epsilon} & 0 < t < \epsilon \\ 0 & \text{otherwise} \end{cases} \tag{4.5-157}$$



The integral of $\nu(t)$ is almost equal to $U(t)$ except for the $[0, \epsilon]$ as

$$\int_{-\infty}^{t} \nu(t)dt = \begin{cases} 0 & t \leqslant 0 \\ \frac{t}{\epsilon} & 0 < t < \epsilon \\ 1 & \epsilon \leqslant t < \infty \end{cases} \tag{4.5-158}$$

$$\int_{-\infty}^{\infty} \nu(t)dt = \int_{0^-}^{\epsilon} \nu(t)dt = 1 \tag{4.5-159}$$

when $\epsilon \to 0$, $\nu(t)$ approaches the Dirac delta function (Unit Impulse) $\delta(t)$ defined as

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1 \tag{4.5-160}$$

A more rigorous derivation of the impulse $\delta(t)$ may be found in Baher's book (p.p. 88–93) using the concept of distribution or the limit of a sequence.

4.5.2 Properties of the Unit Impulse

Property 1

$$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t) \tag{4.5-161}$$

where α is a real constant

Property 2

$$\delta(t) = \delta(-t) \tag{4.5-162}$$

Property 3

$$\delta(t) \leftrightarrow 1 \tag{4.5-163}$$
$$1 \leftrightarrow 2\pi\delta(\omega) \tag{4.5-164}$$

Property 4

$$\int_{-\infty}^{\infty} \phi(\tau)\left(\frac{d^n\delta(\tau)}{d\tau^n}\right)d\tau = (-1)^n \left.\frac{d^n\phi(t)}{dt^n}\right|_{t=0} \tag{4.5-165}$$

Proof:

$$\int_{-\infty}^{\infty} \phi(\tau)\delta(\tau - t)d\tau = \phi(t) \tag{4.5-166}$$

Differentiate two sides of the equation n times with respect to t

$n = 1$:

$$\int_{-\infty}^{\infty} \phi(\tau)\frac{d\delta(\tau - t)}{d(\tau - t)}.\frac{d(\tau - t)}{dt}d\tau = \frac{d\phi(t)}{dt} \tag{4.5-167}$$
$$\int_{-\infty}^{\infty} \phi(\tau)\frac{d\delta(\tau - t)}{d(\tau - t)}.(-1)d\tau = \frac{d\phi(t)}{dt} \tag{4.5-168}$$

As $t \rightarrow 0$

$$\int_{-\infty}^{\infty} \phi(\tau)\frac{d\delta(\tau)}{d\tau}d\tau = (-1) \left.\frac{d\phi(t)}{dt}\right|_{t=0} \tag{4.5-169}$$

n:

$$\int_{-\infty}^{\infty} \phi(\tau)\frac{d^n\delta(\tau - t)}{d(\tau - t)^n}\left(\frac{d(\tau - t)}{dt}^n\right)d\tau = \frac{d^n\phi(t)}{dt^n} \tag{4.5-170}$$

As $t \rightarrow 0$

$$\int_{-\infty}^{\infty} \phi(\tau)\frac{d^n\delta(\tau - t)}{d\tau^n}d\tau = (-1)^n \left.\frac{d^n\phi(t)}{dt}\right|_{t=0} \tag{4.5-171}$$

Property 5

$$g(t)\delta(t - \alpha) = g(\alpha)\delta(t - \alpha) \tag{4.5-172}$$

Property 6

$$\delta(t) = \frac{dU(t)}{dt} \tag{4.5-173}$$

With the introduction of the $\delta(t)$ function, Fourier transforms of many useul functions become possible.

1.

$$K\delta(t) \leftrightarrow K \tag{4.5-174}$$
$$K \leftrightarrow 2\pi K\delta(\omega) \tag{4.5-175}$$

2.

If

$$U(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t) \tag{4.5-176}$$
$$U(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega} \tag{4.5-177}$$

Proof:

Recalling

$$\frac{1}{t} \leftrightarrow \frac{1}{-j\pi}2\pi\left(\frac{1}{-\omega}\right) = \frac{2}{j\omega} \tag{4.5-178}$$

use symmetry property

$$\mathcal{F}[\mathcal{U}(\sqcup)] = \mathcal{F}[\frac{\infty}{\in} + \mathcal{F}[\frac{\infty}{\in}\,\mathrm{sgn}(\sqcup)] \tag{4.5-179}$$
$$= 2\pi\frac{1}{2}\delta(\omega) + \frac{1}{2}\frac{2}{j\omega} \tag{4.5-180}$$
$$= \pi\delta(\omega) + \frac{1}{j\omega} \tag{4.5-181}$$

3. The relation between the Fourier Transform of $f(t)$ and that of its integral is

$$\int_{-\infty}^{\infty} f(\tau)d\tau \leftrightarrow \pi F(0)\delta(\omega) + \frac{F(\omega)}{j\omega} \tag{4.5-182}$$

4. Use time shifting property

$$\delta(t-T) \leftrightarrow e^{-j\omega T} \tag{4.5-183}$$
$$\delta(t+T) \leftrightarrow e^{j\omega T} \tag{4.5-184}$$

Therefore

$$\frac{1}{2}[\delta(t-T) + \delta(t+T)] \leftrightarrow \cos\omega T \tag{4.5-185}$$

and

$$\frac{1}{2j}[\delta(t+T) - \delta(t-T)] \leftrightarrow \sin\omega T \tag{4.5-186}$$

5. Using the symmetry property

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0) \tag{4.5-187}$$
$$e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega+\omega_0) \tag{4.5-188}$$

therefore

$$\cos\omega_0 t \leftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \tag{4.5-189}$$
$$\sin\omega_0 t \leftrightarrow j\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \tag{4.5-190}$$

4.6 The Impulse Response and System Function

A linear time-invariant system may be represented by

$$f(t) \longrightarrow \boxed{\quad h(\cdot) \quad} \longrightarrow g(t)$$

where

$$g(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau) d\tau \tag{4.6-191}$$

or

$$F(\omega) \longrightarrow \boxed{\quad H(\omega) \quad} \longrightarrow G(\omega)$$

where

$$F(\omega) = \mathcal{F}[\{(\sqcup)] \tag{4.6-192}$$
$$G(\omega) = \mathcal{F}[\}(\sqcup)] \tag{4.6-193}$$

and

$$H(\omega) = \mathcal{F}[\langle(\sqcup)] \tag{4.6-194}$$

If $f(t) = \delta(t)$

$$g(t) = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau = h(t) \tag{4.6-195}$$

Therefore $h(t)$ is called the impulse response of the system.

In this case:

$$\delta(t) \longrightarrow \boxed{\quad h(\cdot) \quad} \longrightarrow h(t)$$

$$1 \longrightarrow \boxed{\quad H(\omega) \quad} \longrightarrow H(\omega)$$

If $f(t) = e^{j\omega_0 t}$,

$$e^{j\omega_0 t} \longrightarrow \boxed{\quad \text{linear, time-invariant} \quad} \longrightarrow g(t)$$

$$g(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega_0 (t-\tau)} d\tau \tag{4.6-196}$$

$$= e^{j\omega_0 t} \int_{-\infty}^{\infty} \underbrace{h(\tau) e^{-j\omega_0 \tau} d\tau}_{H(\omega_0)} \tag{4.6-197}$$

or

$$g(t) = H(\omega)e^{j\omega t} \tag{4.6-198}$$

This means that if $f(t)$ is an exponential function, the response $g(t)$ is also one with a difference in the amplified $|H(\omega)|$ and the phase $\phi(\omega)$ where $H(\omega) = |H(\omega)|e^{j\phi(\omega)}$.

If

$$f(t) = \cos(\omega t) = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t} \tag{4.6-199}$$

$$= \Re(e^{j\omega t}) \tag{4.6-200}$$

$$g(t) = \frac{1}{2}H(\omega)e^{j\omega t} + \frac{1}{2}H(-\omega)e^{-j\omega t} \tag{4.6-201}$$

$$= \frac{1}{2}A(\omega)e^{j\phi(\omega)}e^{j\omega t} + \frac{1}{2}A(-\omega)e^{j\phi(-\omega)}e^{-j\omega t} \tag{4.6-202}$$

$$= \frac{1}{2}A(\omega)e^{j\omega t + j\phi(\omega)} + \frac{1}{2}A(\omega)e^{-j\omega t - j\phi(\omega)} \tag{4.6-203}$$

$$= \frac{1}{2}A(\omega)\left\{\cos(\omega t + \phi(\omega)) + j\sin(\omega t + \phi(\omega)) + \frac{1}{2}\cos(\omega t + \phi(\omega)) - j\sin(\omega t + \phi(\omega))\right\} \tag{4.6-204}$$

$$= A(\omega)\cos(\omega t + \phi(\omega)) \tag{4.6-205}$$

4.7 Causal Functions and the Hilbert Transform

Definition 4.7-1 *A system is causal if its impulse response $h(t)$ is a causal function.*

$$h(t) = 0 \qquad \textit{for } t < 0 \tag{4.7-206}$$

As a result, if the input $f(t)$ is causal,

$$g(t) = \int_{-\infty}^{\infty} f(t)h(t-\tau)d\tau \tag{4.7-207}$$

$$= \int_{-\infty}^{t} h(t-\tau)d\tau + \underbrace{\int_{t}^{\infty} f(t)h(t-\tau)d\tau}_{=0} \tag{4.7-208}$$

for $t - \tau < 0$ or $\tau > t$, $h(t-\tau) = 0$

$$= \int_{-\infty}^{t} f(t)h(t-\tau)d\tau \tag{4.7-209}$$

$g(t)$ is a ‘product’ of the past $f(t)$ contribution, therefore causal.

and if

$$f(t) = 0 \qquad \text{for } t < 0 \tag{4.7-210}$$

$$g(t) = 0 \qquad \text{for } t < 0 \tag{4.7-211}$$

For a causal function $f(t)$, we showed previously that $f(t)$ is uniquely determined by $R(\omega)$ or $x(\omega)$, which is the real or imaginary part of its Fourier transform

$$f(t) \leftrightarrow 2R(\omega) \tag{4.7-212}$$
$$f(t) \leftrightarrow 2jx(\omega) \tag{4.7-213}$$

If

$$f(t) = \sin\omega t \tag{4.7-214}$$

it can be proven that

$$g(t) = A(\omega)\sin(\omega t + \phi(\omega)) \tag{4.7-215}$$

Note:

$$f(t) = \sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \tag{4.7-216}$$

Now it has been proven that the repsonse of a linear time-invariant system to a sinusoid is another sinusoid with the same frequency but with a different amplitude and phase.

It is concluded that the system frequency response function $H(\omega)$ can be defined (or obtained) as one of the following:

1. $H(\omega)$ is the Fourier Transform of the $h(t)$;
2. $g(t) = H(\omega)e^{j\omega t}$;
3. $H(\omega) = \frac{G(\omega)}{F(\omega)} = \frac{\text{FT of the output}}{\text{FT of the input}}$

Furthermore $f(t)$ satisfies

$$\text{even part of } f(t) = \text{odd part of } f(t) \qquad \text{for } t > 0 \tag{4.7-217}$$

and

$$f(t) = 0 \qquad t < 0 \tag{4.7-218}$$

If we adopt Baher's symbol

$$\text{Ev}\, f(t) = \text{Od}\, f(t)\, \text{sgn}(t) \tag{4.7-219}$$

or

$$\text{Od}\, f(t) = \text{Ev}\, f(t)\, \text{sgn}(t) \tag{4.7-220}$$

Using the previous results

$$\text{Ev}\, f(t) \leftrightarrow R(\omega) \tag{4.7-221}$$
$$\text{Od}\, f(t) \leftrightarrow jx(\omega) \tag{4.7-222}$$

and

$$\text{sgn}\, t \leftrightarrow \frac{2}{j\omega} \tag{4.7-223}$$
$$\mathcal{F}(\text{Ev}\,\{(\sqcup)) = \mathcal{F}(\text{Od}\,\{(\sqcup)\,\text{sgn}\,\sqcup) \tag{4.7-224}$$
$$R(\omega) = \frac{1}{2\pi} jx(\omega) * \frac{2}{j\omega} \tag{4.7-225}$$

That is

$$R(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\mu)}{\omega - \mu} d\mu \tag{4.7-226}$$

Equation 4.7-226 expresses the relation between $R(\omega)$ and $x(\omega)$ of a causal (real) function $f(t)$.

Taking Fourier Transform of both side Equation 4.7-220 gives

$$x(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(\mu)}{\omega - \mu} d\mu \tag{4.7-227}$$

Equations 4.7-226 and 4.7-227 constitute a Hilbert trasnform pair.

If the real part of the Fourier Transform of a real (causal) function is given, the Hilbert transform can be used to generate the imaginary part to complete Fourier Transform if the given function and vice versa.

4.8 The Impulse Train and its applications

The impulse train is defined as

$$\delta_n(t) = \sum_{k=-n} n\delta(t + kT) \tag{4.8-228}$$

using the time-shift property of the Fourier Transform, the Fourier transform $\delta_n(t)$ is given by

$$\hat{C}_n(\omega) = \sum_{k=-n}^{n} e^{jkT\omega} \tag{4.8-229}$$

ie

$$\delta_n(t) \leftrightarrow \hat{C}_n(\omega) \tag{4.8-230}$$

A diagram of an impulse train and its transform can be seen in Figure 4.5.

if $n \to \infty$

$$\delta_\infty = \lim_{n\to\infty} \delta_n(t) \tag{4.8-231}$$

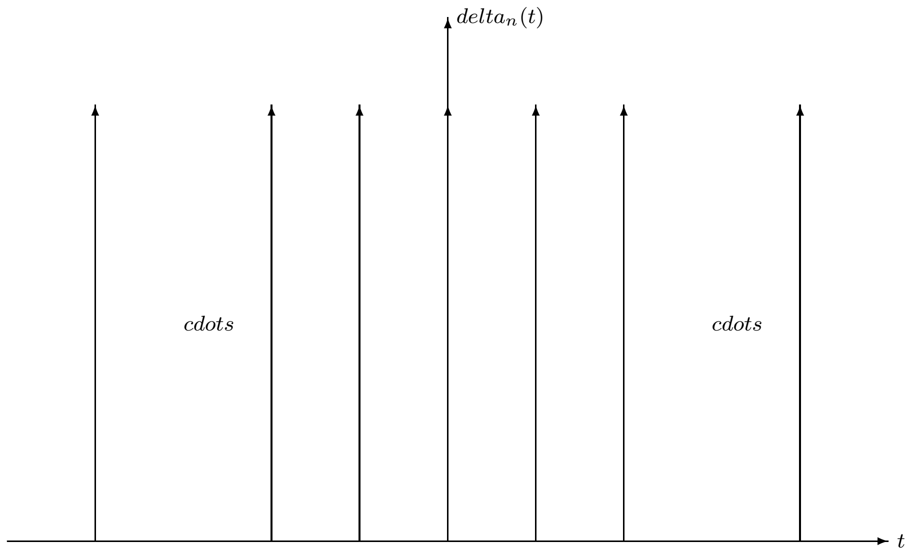
and it can be shown that

$$\delta_\infty(t) \leftrightarrow \omega_0 \delta_\infty(\omega) \tag{4.8-232}$$

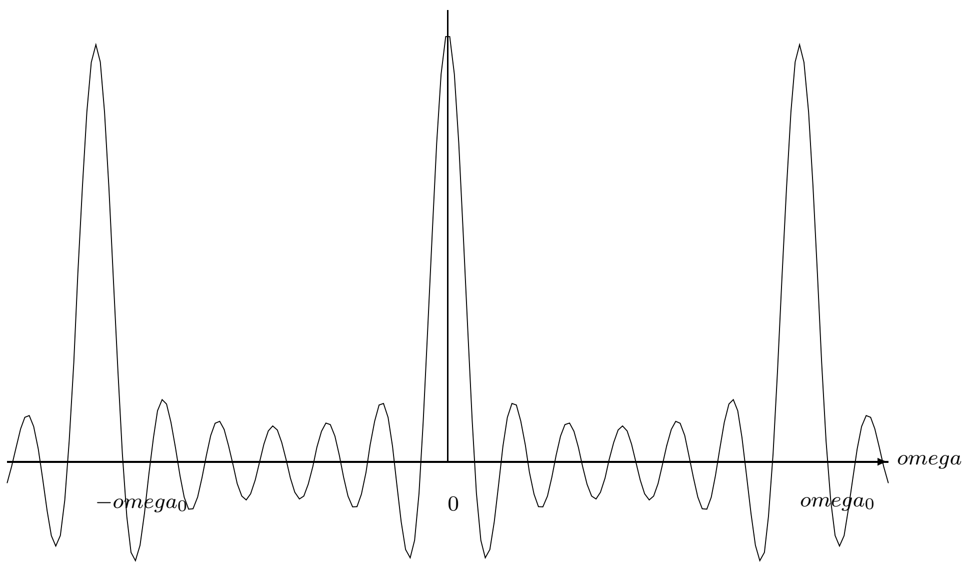
where

$$\delta_\infty(\omega) = \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \tag{4.8-233}$$

A diagram of an infinite impulse train and its transform can be seen in Figure 4.6.



(a) Train of $2n + 1$ impulses



(b) and its spectrum

Figure 4.5: Finite impulse train and spectrum

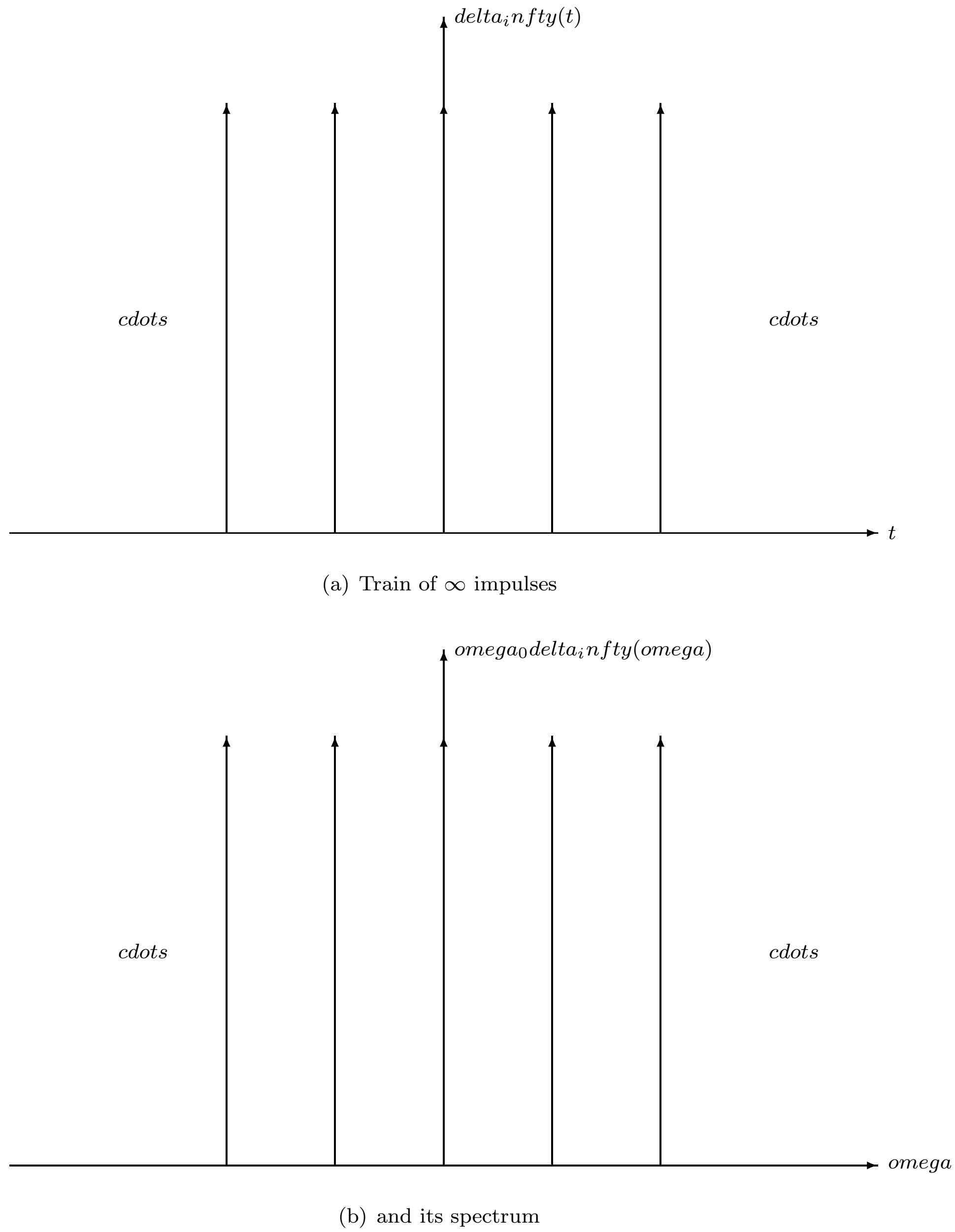


Figure 4.6: Infinite impulse train and spectrum

4.8.1 Extending a function $F(t)$ using $\delta_n(t)$

Given a function $f(t)$, another function $f_n(t)$ can be defined as

$$f_n(t) = \sum_{k=-n}^{n} f(t - kT) \tag{4.8-234}$$

or

$$f_n(t) = f(t) * \delta_n(t) \tag{4.8-235}$$

noticing

$$f(t - kT) = f(t) * \delta(t - kT) \tag{4.8-236}$$

and

$$\delta_n(t) = \sum_{k=-n}^{n} \delta(t - kT) \tag{4.8-237}$$

If

$$f(t) \leftrightarrow F(\omega) \tag{4.8-238}$$
$$f_n(t) \leftrightarrow F(\omega)\hat{C}_n(\omega) \tag{4.8-239}$$

where

$$\hat{C}_n(\omega) = \sum_{k=-n}^{n} e^{jkT\omega} = \frac{\sin(n + \frac{1}{2})T\omega}{\sin\frac{T\omega}{2}} \tag{4.8-240}$$

An example of signal extension can be seen in Figure 4.7.

4.8.2 The Fourier Transform of a Periodic Function

Given a periodic function $f_p(t)$, form a non-periodic function $f(t)$, as shown in Figure 4.8 such that

$$f(t) = \begin{cases} f_p(t) & \text{for } |t| < \frac{T}{2} \\ 0 & \text{for } |t| > \frac{T}{2} \end{cases} \tag{4.8-241}$$

Assume $f(t) \leftrightarrow F(\omega)$ so that

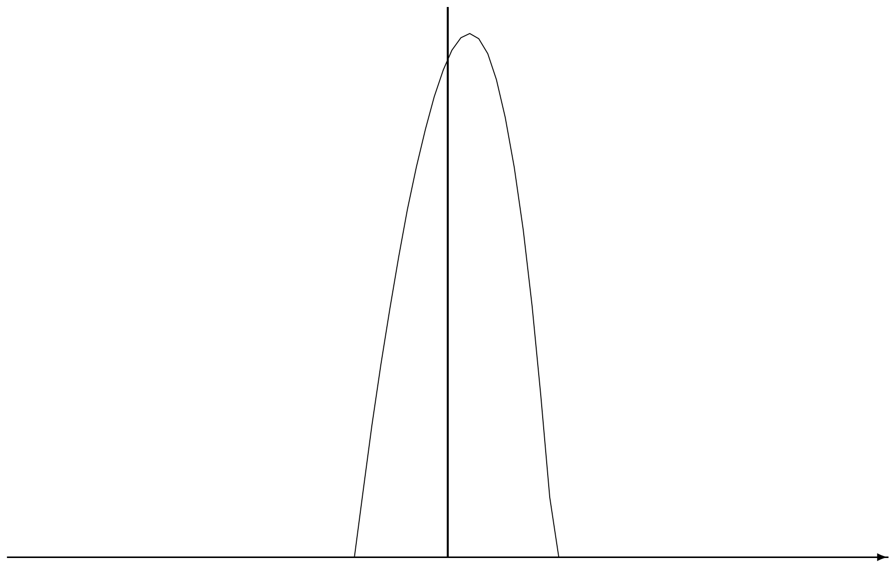
$$F(\omega) = \int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt \tag{4.8-242}$$

Because elsewhere $f(t) = 0$. It is obvious that

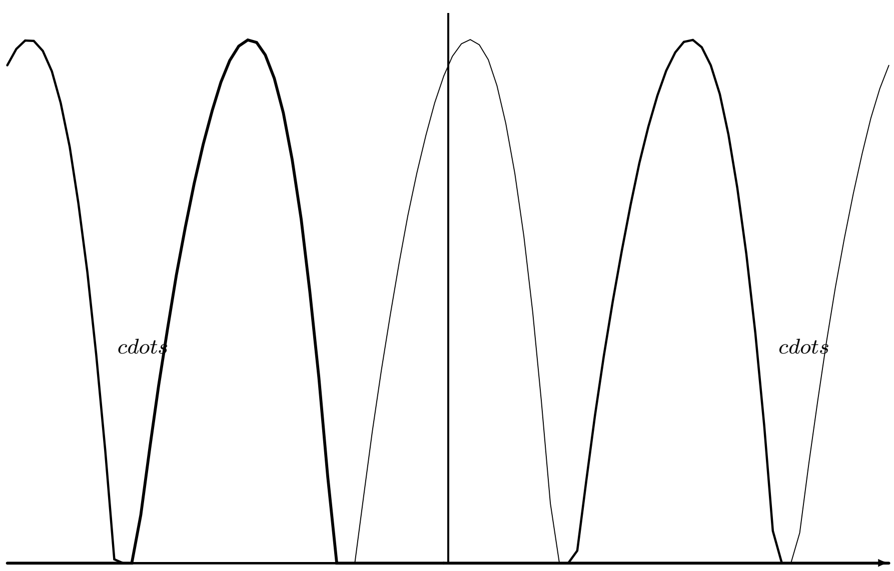
$$f_p(t) = f(t) * \delta_\infty(t) \tag{4.8-243}$$

To find the Fourier Transform $F_p(\omega)$ of $f_p(t)$ we have

$$F_p(\omega) = \mathcal{F}[\{_{\surd}(\sqcup)] = \mathcal{F}(\{(\sqcup) * \delta_\infty(\sqcup)) \tag{4.8-244}$$
$$= F(\omega)[\omega_0\delta_\infty(t)] \tag{4.8-245}$$
$$= \omega_0 F(\omega) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \tag{4.8-246}$$



(a) A signal $f(t)$



(b) the signal obtained by extension $f(t)$ every T seconds

Figure 4.7: Example of signal extension using δ_n

For $\omega \neq k\omega_0$

$$\delta(\omega - k\omega_0) = 0 \tag{4.8-247}$$

Thus

$$F_p(\omega) = \omega_0 \sum_{k=-\infty}^{\infty} F(k\omega_0)\delta(\omega_k \omega_0) \tag{4.8-248}$$

where

$$F(k\omega_0) = \int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t} dt \tag{4.8-249}$$

and

$$\omega_0 = \frac{2\pi}{T} \tag{4.8-250}$$

Therefore, the Fourier transform spectrum $F_p(\omega)$ of a periodic function $f_p(t)$ with period T, is the infinite train of equidistant impulses, (ω_0 apart). Each impulse is of strength $F(k\omega_0)$ which is the value at $k\omega_0$ of the Fourier Transform of the non-periodic function $f(t)$ defining $f_p(t)$ over one period.

If we define

$$C_k(k\omega_0) = \frac{1}{T} F(k\omega_0) \tag{4.8-251}$$

$$= \frac{1}{T} \int_{T/2}^{T/2} f(t)e^{-jk\omega_0 t} dt \tag{4.8-252}$$

the Fourier Transform of the periodic function $f_p(t)$ is:

$$F_p(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0) \tag{4.8-253}$$

The inverse Fourier Transform of the $F_p(\omega)$ is given by

$$f_p(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \tag{4.8-254}$$

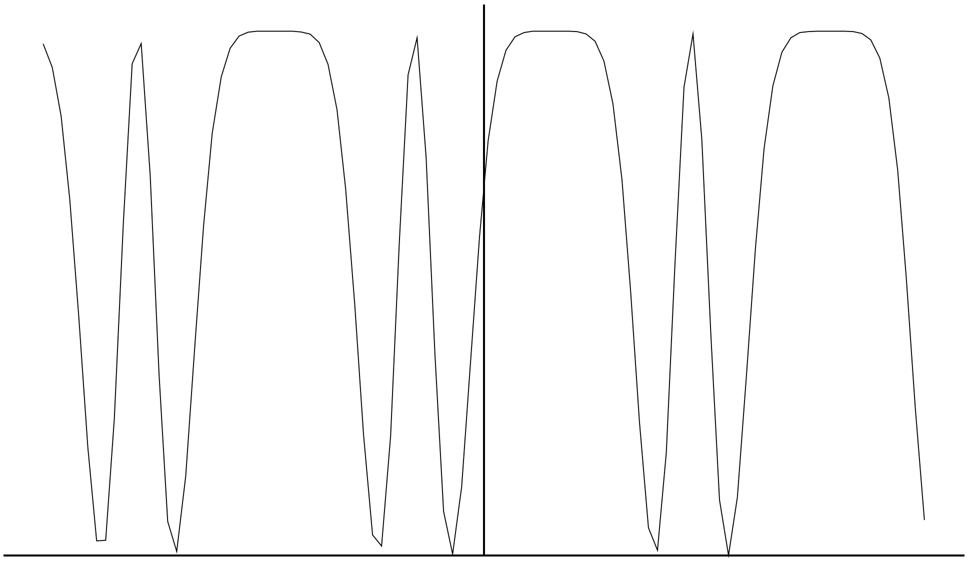
(Comparing Equation 4.8-253 and 4.8-254 with the Fourier series coefficients

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \tag{4.8-255}$$

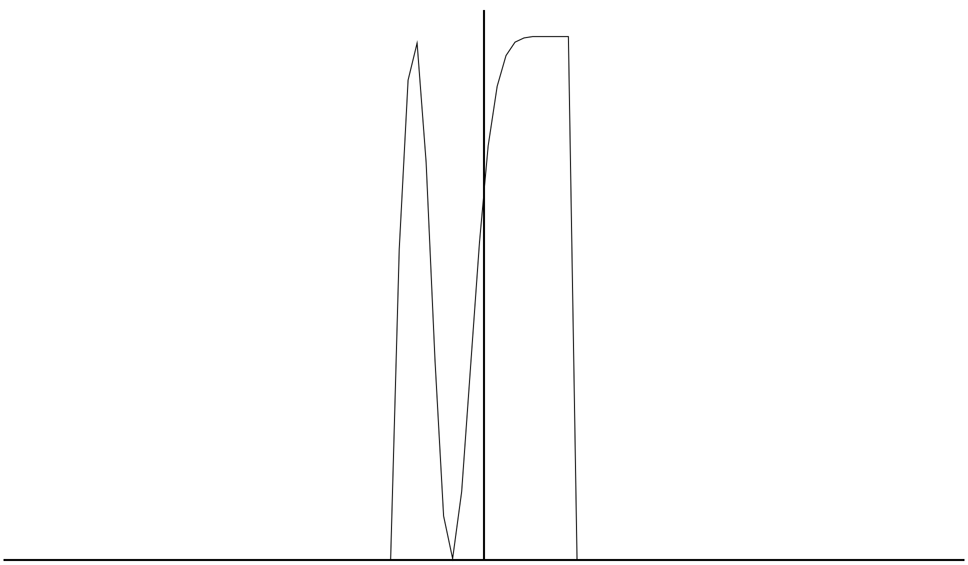
$$C_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t)E^{-jk\omega_0 t} dt) \tag{4.8-256}$$

Fourier series is a special case of the Fourier transform.

Remark 4.8-1 *Any periodicity in the time domain results in impulses in the Fourier Spectrum*

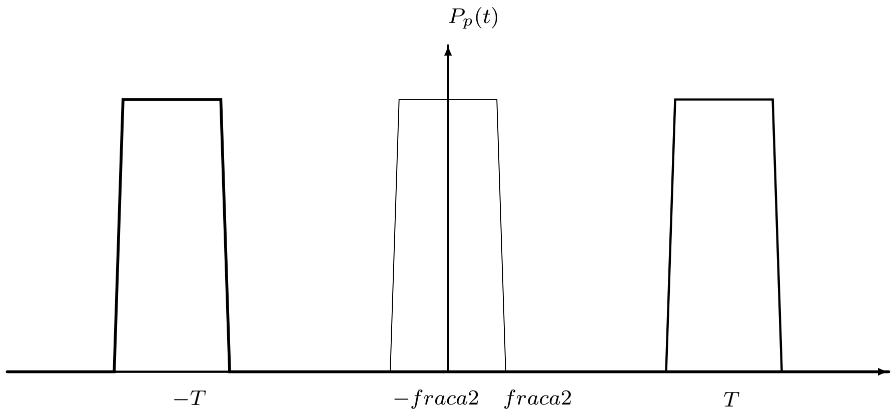


(a) A periodic signal $f_p(t)$

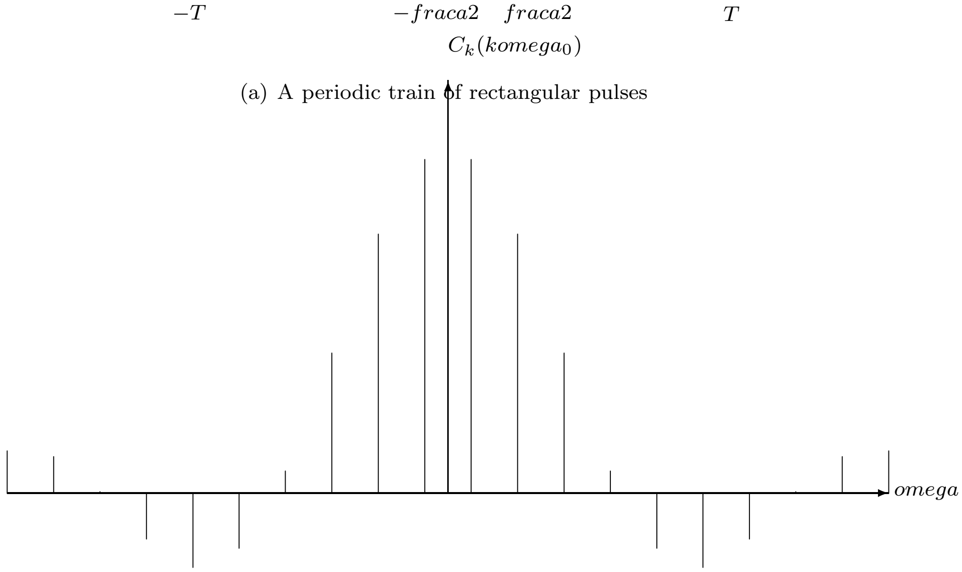


(b) A non-periodic signal obtained $f_p(t)$

Figure 4.8: Generating a non-periodic signal from a periodic signal



(a) A periodic train of rectangular pulses



(b) Its spectrum as a train of impulses

Figure 4.9: A periodic train of rectangular pulses and its fourier series coefficients

EXAMPLE 4.8-1 *Find the Fourier spectrum of the periodic train of rectangular pulses show in Figure 4.9.*

Solution:

Method 1: The Fourier coefficients

$$
\begin{aligned}
C_k &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t}dt && (4.8\text{-}257)\\
&= \frac{1}{T}\int_{-T/2}^{T/2} P_p e^{-jk\omega_0 t}dt && (4.8\text{-}258)\\
&= \frac{1}{T}\int_{-a/2}^{a/2} e^{-jk\omega_0 t}dt && (4.8\text{-}259)\\
&= \frac{1}{T}\frac{1}{-jk\omega_0}e^{-jk\omega_0 t}dt\Big|_{-a/2}^{a/2} && (4.8\text{-}260)\\
&= \frac{1}{-jk\omega_0}\left\{e^{-jk\omega_0\frac{a}{2}} - e^{jk\omega_0\frac{a}{2}}\right\} && (4.8\text{-}261)\\
&= \frac{1}{T}\frac{1}{-jk\omega_0}\left\{\cos\left(-k\omega_0\frac{a}{2}\right) - \sin\left(k\omega_0\frac{a}{2}\right) - \cos\left(k\omega_0\frac{a}{2}\right) - j\sin\left(k\omega_0\frac{a}{2}\right)\right\} && (4.8\text{-}262)\\
&= \frac{1}{T}\frac{1}{-jk\omega_0}\left\{-2j\sin\left(k\omega_0\frac{a}{2}\right)\right\} && (4.8\text{-}263)\\
&= \frac{1}{T}\frac{2\sin\left(k\omega_0\frac{a}{2}\right)}{k\omega_0} && (4.8\text{-}264)\\
&= \frac{a}{T}\frac{\sin\left(k\omega_0\frac{a}{2}\right)}{k\omega_0\frac{a}{2}} && (4.8\text{-}265)
\end{aligned}
$$

Method 2:

1) Find the fourier transform for the non-periodic pulse function $P_a(t)$

$$
\begin{aligned}
\mathcal{F}[\mathcal{P}_\dashv(\sqcup)] &= F(\omega) && (4.8\text{-}266)\\
&= \int_{-\infty}^{\infty} P_a(t)e^{-j\omega t}dt && (4.8\text{-}267)\\
&= \int_{-a/2}^{a/2} e^{-j\omega t}dt && (4.8\text{-}268)\\
&= \frac{1}{-j\omega}e^{-j\omega t}\Big|_{-a/2}^{a/2} && (4.8\text{-}269)\\
&= \frac{1}{-j\omega}\left\{e^{-j\omega\frac{a}{2}} - e^{j\omega\frac{a}{2}}\right\} && (4.8\text{-}270)\\
&= \frac{1}{-j\omega}\left\{-2j\sin\left(\omega\frac{a}{2}\right)\right\} && (4.8\text{-}271)
\end{aligned}
$$

2)

$$C_k(k\omega_0) = \frac{1}{T}F(k\omega_0) \tag{4.8-272}$$
$$= \frac{a}{T}\frac{\sin\left(k\omega_0\frac{a}{2}\right)}{k\omega_0\frac{a}{2}} \tag{4.8-273}$$

3)

$$F_p(\omega) = 2\pi\sum_{k=-\infty}^{\infty} C_k\delta\left(\delta\omega - k\omega_0\right) \tag{4.8-274}$$
$$= \frac{2\pi a}{T}\sum_{k=-\infty}^{\infty}\frac{\sin\left(k\omega_0\frac{a}{2}\right)}{k\omega_0\frac{a}{2}}\delta\left(\omega - k\omega_0\right) \tag{4.8-275}$$

REMARK 4.8-2 *Fourier coefficients of the Fourier series and Fourier transform $F_p(\omega)$ of the periodic function $P_p(t)$ are two equivalent representations in the Fourier domain.*

4.8.3 Power Spectra and Correlation Functions

A periodic signal is a finite power signal rather that a finite energy one becuase for a signal to be of finite energy, it has to be square integratable over $[-\infty, \infty]$ (recalling

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega \tag{4.8-276}$$

to be finite.) for a signal to be of finite power, it has to satsify

$$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2 dt = \sum_{k=-\infty}^{\infty}|C_k|^2 \tag{4.8-277}$$

being finite.

Define the autocorrelation function $\rho_{ff}(\tau)$ of the periodic signal $f_\rho(t)$ as

$$\rho_{ff}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f_r ho(t) f_r ho(t+\tau)dt \tag{4.8-278}$$

Assume

$$f_\rho(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \tag{4.8-279}$$

then

$$f_\rho(t+\tau) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0(t+\tau)} \tag{4.8-280}$$
$$\rho_{ff}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0(t+\tau)} f_\rho(t)\right)dt \tag{4.8-281}$$
$$= \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0\tau}\frac{1}{T}\int_{-T/2}^{T/2} f_\rho(t)e^{jk\omega_0 t}dt \tag{4.8-282}$$
$$= \sum_{k=-\infty}^{\infty} C_k C_k^* e^{jk\omega_0\tau} \tag{4.8-283}$$

(Notice that

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t}dt \tag{4.8-284}$$

and

$$C_k^* = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jk\omega_0 t}dt \tag{4.8-285}$$

)

$$\rho_{ff} = \sum_{k=-\infty}^{\infty} |C_k|^2 e^{jk\omega_0\tau} \tag{4.8-286}$$

By the definition, the power spectral amplitudes $|C_k|^2$ are the coefficients of the Fourier series of the autocorrelation function $\rho_{ff}(\tau)$. (Notice the definition of the inverse Fourier Transform

$$f_\rho(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}) \tag{4.8-287}$$

If $\tau = 0$ in Equation 4.8-286 then

$$\rho_{ff} = \frac{1}{T}\int_{-T/2}^{T/2} f_\rho(t)f_\rho(t)dt \tag{4.8-288}$$

$$= \sum_{k=-\infty}^{\infty} |C_k|^2 \tag{4.8-289}$$

It is obvious that $\rho_{ff}(0)$ is equal to the average power of the signal. Equation 4.8-289 is Parsevals theorem for periodic functions.

If we define the power spectrum $P(\omega)$ of a periodic signal $F_\rho(t)$ as the Fourier Transform of its autocorrelation $\rho_{ff}(t)$,

$$P(\omega) = \mathcal{F}\left[\sum_{\|=-\infty}^{\infty} |\mathcal{C}_{\|}|^{\in}\rceil^{\||\omega\prime\sqcup}\right] \tag{4.8-290}$$

$$= 2\pi \sum_{k=-\infty}^{\infty} |C_k|^2\delta(\omega - k\omega_0) \tag{4.8-291}$$

Noticing that:

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0) \tag{4.8-292}$$

or

$$e^{jk\omega_0 t} \leftrightarrow 2\pi\delta(\omega - k\omega_0) \tag{4.8-293}$$

Therefore, the power spectrum of a periodic function consists of impulses at ω_0 and its harmonics. The strength of each impulse is $2\pi|C_k|^2$.

The cross-correlation $\rho_{fg}(\tau)$ of two periodic signals $f(t)$ and $g(t)$ is defined as

$$\rho_{fg}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)g(t+\tau)dt \tag{4.8-294}$$

Assume

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \tag{4.8-295}$$

and

$$g(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t} \tag{4.8-296}$$

$$\rho_{fg}(\tau) = \sum_{k=-\infty}^{\infty} C_k d_k^* e^{jk\omega_0 \tau} \tag{4.8-297}$$

The cross-power spectrum of the two signals is defined as

$$\rho_{fg}(\omega) = \mathcal{F}[\rho_{\{\}}(\tau)] \tag{4.8-298}$$

$$= 2\pi \sum_{k=-\infty}^{\infty} C_k d_k^* \delta(\omega - k\omega_0) \tag{4.8-299}$$

Equation 4.8-297 and 4.8-299 constitute the Wiener-Kintchine theorem for finite power signals.

4.8.4 Poisson's summation formula

Given a non-periodic function $f(t)$, a periodic function $f_\rho(t)$ may be defined as

$$f_\rho(t) = f(t) * \delta_\infty(t) \tag{4.8-300}$$

where

$$\delta_\infty(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{4.8-301}$$

Since

$$f_\rho(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \tag{4.8-302}$$

or

$$f_\rho(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(k\omega_0) e^{jk\omega_0 t} \tag{4.8-303}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} F(k\omega_0) e^{jk\omega_0 t} \tag{4.8-304}$$

If $t = 0$, we have

$$\sum_{k=-\infty}^{\infty} f(kT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(k\omega_0) \tag{4.8-305}$$

$$f_\rho(t) = f(t) * \delta_\infty = \sum_{k=-\infty}^{\infty} f(t + kT) \tag{4.8-306}$$

$$\tag{4.8-307}$$

Equation 4.8-305 is called *Poisson's summation formula* which states that if $f(t)$ is an arbitrary function with Fourier transform $F(\omega)$, then the sum of the values of the function $f(t)$ at discrete values of $t = kT(k = 0, 1, \ldots)$ is related ot the sum of the values of the $F(\omega)$ evaluated at discrete values of $\omega = k\omega_0(\omega_0 = w\pi/T)$.

4.8.5 Schwartz's inequality

Schwartz's inequality states that

$$\left(\int_{-\infty}^{\infty} f^2(t)dt\right)\left(\int_{-\infty}^{\infty} g^2(t)dt\right) \geqslant \left(\int_{-\infty}^{\infty} f(t)g(t)dt\right)^2 \tag{4.8-308}$$

4.8.6 The Uncertainty Principle

Assume the Fourier transform pair

$$f(t) \leftrightarrow F(\omega) \tag{4.8-309}$$

and $f(t)$ is of duration Δt and $F(\omega)$ has a band-width of $\Delta\omega$.

$$(\Delta t)^2 = \frac{1}{\epsilon} \int_{-\infty}^{\infty} (t - \hat{(t)})^2 f^2(t)dt \tag{4.8-310}$$

where

$$\epsilon = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \tag{4.8-311}$$

$$\hat{t} = \frac{1}{\epsilon} \int_{-\infty}^{\infty} tf^2(t)dt \tag{4.8-312}$$

And,

$$(\Delta\omega)^2 = \frac{1}{2\pi\epsilon} \int_{-\infty}^{\infty} (\omega - \hat{\omega})^2 |F(\omega)|^2 d\omega \tag{4.8-313}$$

where

$$\hat{\omega} = \frac{1}{2\pi\epsilon} \int_{-\infty}^{\infty} \omega |F(\omega)|^2 d\omega \tag{4.8-314}$$

The *Uncertainty Principle* states that:

If

$$\sqrt{t} f(t) \to 0 \qquad \text{as } |t| \to \infty \tag{4.8-315}$$

$$\Delta t \Delta\omega \geqslant \frac{1}{2} \tag{4.8-316}$$

It means that the duration Δt of $f(t)$ and its spectral band-width $\Delta\omega$ cannot be simultaneously small.

4.8.7 Time-limited and Band-limited signals

DEFINITION 4.8-1 *A function is time-limited if*

$$f(t) = 0 \qquad \textit{for } |t| > \tau \tag{4.8-317}$$

$f(t)$ is called τ-limited.

DEFINITION 4.8-2 *A function $f(t)$ is band-limited if the spectrum $F(\omega)$ of the $f(t)$ satisfies:*

$$F(\omega) = 0 \qquad \textit{for } |\omega| > \omega_0 \tag{4.8-318}$$

It is called ω_0-limited.

THEOREM 4.8-1 *A fucntion $f(t)$ cannot be both band-limited and time-limited.*

Chapter 5

The Laplace Transformation and its applications

Contents

It is a revision chapter of its definition, properties and its applications to analysis of electrical networks.

5.1 Definition and Properties of the Laplace Transform

Definition 5.1-1 *The Laplace transform $F(s)$ of a function $f(t)$ is defined by*

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \tag{5.1-1}$$

where s is the complex frequency variable taking the place of $j\omega$ in the Fourier transform, $s = \sigma + j\omega$. It is also denoted by

$$F(s) = \mathcal{L}[\{(\sqcup)] \tag{5.1-2}$$

The inverse transform in defined by

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \tag{5.1-3}$$

It is also denoted by

$$f(t) = \mathcal{L}^{-\infty}[\mathcal{F}(\int)] \tag{5.1-4}$$

REMARK 5.1-1 *The Laplace transform has a wider validity than the Fourier transform.*

EXAMPLE 5.1-1 *Find the Laplace transform for $e^{\alpha t}$ where α is a positive real constant.*

Solution:

$$\begin{aligned}
F(s) &= \mathcal{L}[\rceil^{\alpha\sqcup}] && (5.1\text{-}5)\\
&= \int_{0^-}^{\infty} e^{\alpha t}e^{-st}dt && (5.1\text{-}6)\\
&= \int_{0^-}^{\infty} e^{(\alpha-s)t}dt && (5.1\text{-}7)\\
&= \frac{1}{\alpha-s}e^{(\alpha-s)t}\Big|_0^{\infty} && (5.1\text{-}8)\\
&= \frac{1}{s-\alpha} && (5.1\text{-}9)\\
F(s) &= \frac{1}{s-\alpha} \qquad \text{if } \Re(s) = \sigma > \alpha && (5.1\text{-}10)
\end{aligned}$$

EXAMPLE 5.1-2 *Find the Laplace transform of the unit step function $U(t)$*

Solution

$$\begin{aligned}
F(s) &= \mathcal{L}[\mathcal{U}(\sqcup)] && (5.1\text{-}11)\\
&= \int_{0^-}^{\infty} U(t)e^{-st}dt && (5.1\text{-}12)\\
&= -\frac{1}{s}e^{-st}\Big|_0^{\infty} = \frac{1}{s} && (5.1\text{-}13)\\
F(s) &= \frac{1}{s} \qquad \text{if } \Re(s) > 0 && (5.1\text{-}14)
\end{aligned}$$

EXAMPLE 5.1-3

$$\begin{aligned}
\mathcal{L}[\delta(\sqcup)] &= \int_{0^-}^{\infty} \delta(t)e^{-st}dt && (5.1\text{-}15)\\
&= \int_{0^-}^{0^+} \delta(t)e^{-st}dt && (5.1\text{-}16)\\
&= e^{-s0} && (5.1\text{-}17)\\
&= 1 && (5.1\text{-}18)
\end{aligned}$$

EXAMPLE 5.1-4

$$\mathcal{L}[\rceil^{\pm|\omega_\prime\sqcup}] = \int_{0^-}^{\infty} e^{\pm j\omega_0 t}e^{-st}dt \tag{5.1-19}$$

$$= \int_{0^-}^{\infty} e^{-(s\mp j\omega_0)t}dt \tag{5.1-20}$$

$$= \frac{-1}{s\mp j\omega_0}e^{-(s\mp j\omega_0)t}\Big|_0^\infty \tag{5.1-21}$$

$$= \frac{1}{s\mp j\omega_0} \tag{5.1-22}$$

5.2 Properties of the Laplace Transform

5.2.1 Linearity

If $\mathcal{L}[\{_\rangle(\sqcup)] = \mathcal{F}_\rangle(\int)$ for $i = 1, 2, \ldots, n$,

$$\mathcal{L}\left[\sum_{\rangle=\infty}^{\backslash} \dashv_\rangle \{_\rangle(\sqcup)\right] = \sum_{\rangle=\infty}^{\backslash} \dashv_\rangle \mathcal{F}_\rangle(\int) \tag{5.2-23}$$

EXAMPLE 5.2-1 *Find the Laplace transform of* $\cos(\omega_0 t)$ *and* $\sin(\omega_0 t)$.

Solution:

Since

$$\mathcal{L}\left[\rceil^{\pm|\omega_\prime\sqcup}\right] = \frac{1}{s\mp j\omega_0} \tag{5.2-24}$$

and

$$\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}] \tag{5.2-25}$$

$$\sin(\omega_0 t) = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] \tag{5.2-26}$$

$$\mathcal{L}[\cos\omega_\prime\sqcup] = \frac{1}{2}\left\{\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right\} \tag{5.2-27}$$

$$= \frac{s}{s^2+\omega_0^2} \tag{5.2-28}$$

$$\mathcal{L}[\sin(\omega_\prime\sqcup)] = \frac{1}{2j}\left\{\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right\} \tag{5.2-29}$$

$$= \frac{\omega_0}{s^2+\omega_0^2} \tag{5.2-30}$$

5.2.2 Scaling

$$\mathcal{L}[\{(\alpha\sqcup)] = \frac{\infty}{\alpha}\mathcal{F}\left(\frac{\int}{\alpha}\right) \tag{5.2-31}$$

5.2.3 Differentiation in the Time Domain

$$\mathcal{L}\left[\frac{\lceil\{(\sqcup)}{\lceil\sqcup}\right] = \int\mathcal{F}(\int) - \{(\prime^-) \tag{5.2-32}$$

where $\mathcal{L}[\{(\sqcup)] = \mathcal{F}(\int)$ and $f(0^-)$ is also called the initial value of $f(t)$. And

$$\mathcal{L}\left[\{^{(\backslash)}(\sqcup)\right] = \int^{\backslash}\mathcal{F}(\int) - \int^{\backslash-\infty}\{(\prime^-) - \int^{\backslash-\in}\{'(\prime^-) - \cdots \{^{(\backslash-\infty)}(\prime^-) \tag{5.2-33}$$

where $f^{(n-1)}(0^-)$ is the $(n-1)^{\text{th}}$ derivative of $f(t)$ evaluated at $t = 0^-$.

5.2.4 Integration in the Time Domain

$$\mathcal{L}\left[\int_{\prime^-}^{\prime^+} \{(\tau)\lceil\tau\right] = \frac{\mathcal{F}(\int)}{\int} \tag{5.2-34}$$

where $\mathcal{L}[\{(\sqcup)] = \mathcal{F}(\int)$.

Remark 5.2-1 *The above two properties can be used to convert an integro-differential equation into an algebraic equation.*

5.2.5 Differentiation in the Frequency Domain

If $\mathcal{L}[\{(\sqcup)] = \{(\int)$, then

$$l^{-1}\left[\frac{dF(s)}{ds}\right] = -tf(t) \tag{5.2-35}$$

Example 5.2-2

Find $\mathcal{L}^{-\infty}\left[\frac{\int}{(\int^{\in}+\omega_\prime^{\in})^{\in}}\right]$

Solution:

1: Since

$$-\frac{1}{2\omega_0}\frac{d}{ds}\left(\frac{\omega_0}{s^2+\omega_0^2}\right) = \frac{s}{(s^2+\omega_0^2)^2} \tag{5.2-36}$$

and

$$\mathcal{L}[\sin(\omega_\prime\sqcup)] = \frac{\omega_0}{s^2+\omega_0^2} \tag{5.2-37}$$

2:

$$\mathcal{L}^{-\infty}\left[\frac{\int}{(\int^{\in}+\omega_\prime^{\in})^{\in}}\right] = -\frac{1}{2\omega_0} - t\sin(\omega_0 t) \tag{5.2-38}$$

$$= \frac{1}{2\omega_0}t\sin(\omega_0 t) \tag{5.2-39}$$

5.2.6 Integration in the Frequency Domain

If $\mathcal{L}[\{(\sqcup)] = \mathcal{F}(\int)$ then

$$\mathcal{L}^{-\infty}\left[\int_{\int}^{\infty}\mathcal{F}(\int)\lceil\int\right] = \frac{\{(\sqcup)}{\sqcup} \tag{5.2-40}$$

5.2.7 Translation in the Time Domain

If $\mathcal{L}[\{(\sqcup)] = \mathcal{F}(\int)$ then

$$\mathcal{L}[\{(\sqcup - \alpha)\mu(\sqcup - \alpha)] = \rceil^{-\alpha\int}\mathcal{F}(\int) \tag{5.2-41}$$

EXAMPLE 5.2-3 *Find the* $\mathcal{L}[_{\surd}(\sqcup)]$ *where* $p(t)$ *is shown in Figure 5.1.*

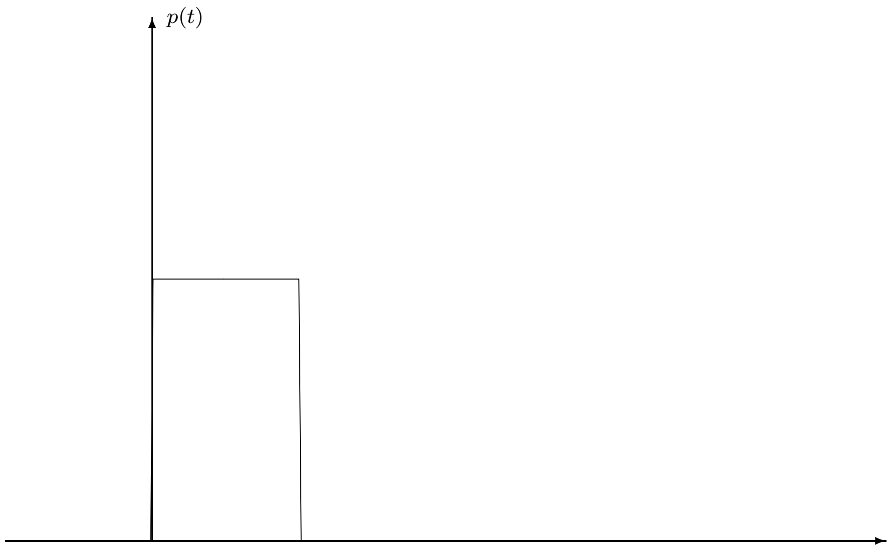


Figure 5.1: Function $p(t)$

Solution:

$$p(t) = U(t) - U(t - T) \tag{5.2-42}$$

See Figure 5.2

$$\mathcal{L}[\mathcal{U}(\sqcup)] = \frac{1}{s} \tag{5.2-43}$$

and

$$\mathcal{L}[\mathcal{U}(\sqcup - \mathcal{T})] = e^{-Ts}\frac{1}{s} \tag{5.2-44}$$

$$\mathcal{L}[_{\surd}(\sqcup)] = \mathcal{L}[\mathcal{U}(\sqcup) - \mathcal{U}(\sqcup - \mathcal{T})] \tag{5.2-45}$$

$$= \mathcal{L}[\mathcal{U}(\sqcup)] - \mathcal{L}[\mathcal{U}(\sqcup - \mathcal{T})] \tag{5.2-46}$$

$$= \frac{1}{s} - \frac{1}{s}e^{-Ts} \tag{5.2-47}$$

5.2.8 Translation in the Frequency Domain

If $\mathcal{L}[\{(\sqcup)] = \mathcal{F}(\int)$

$$\mathcal{L}^{-\infty}[\mathcal{F}(\int - \alpha)] = \rceil^{\alpha\sqcup}\{(\sqcup) \tag{5.2-48}$$

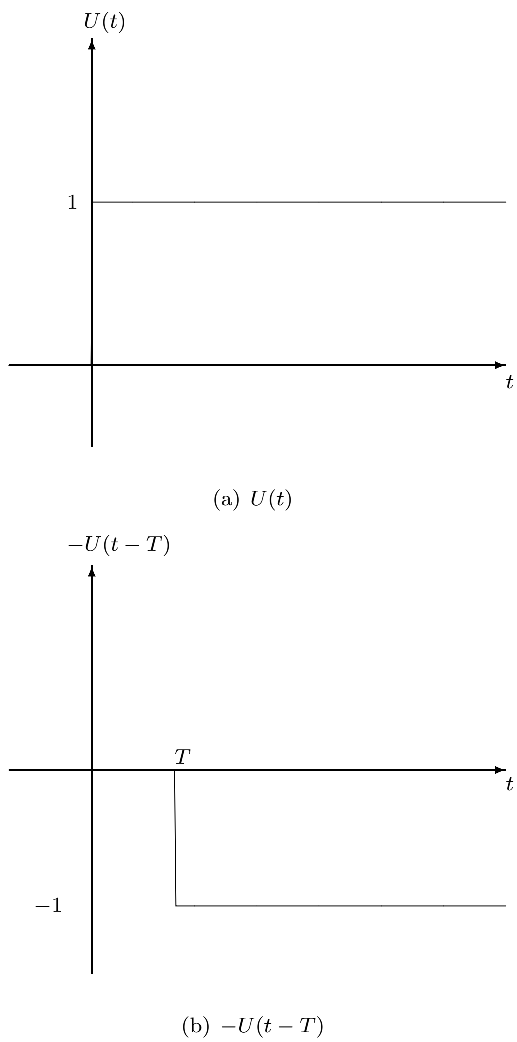


Figure 5.2: Frequency domain integration example

5.2.9 The Laplace Transform of Periodic Signals

If $f(t)$ is a periodic function with period T

$$f(t-rT) = f(t) \qquad \text{for } r = 1, 2, \ldots \tag{5.2-49}$$

$$\mathcal{L}\left[\{(\sqcup)\right] = \int_{0^-}^{\infty} f(t)e^{-st}dt \tag{5.2-50}$$

$$= \int_{0^-}^{T} f(t)e^{-st}dt + \int_{T}^{2T} e^{-st}dt + \cdots \tag{5.2-51}$$

$$+ \int_{rT}^{(r+1)T} f(t)e^{-st}dt + \cdots \tag{5.2-52}$$

Define

$$\hat{f}(t) = \begin{cases} f(t) & 0 \leqslant t \leqslant T \\ 0 & \text{otherwise} \end{cases} \tag{5.2-53}$$

$$\mathcal{L}\left[\hat{\{}(\sqcup)\right] = \hat{F}(s) = \int_{0^-}^{T} f(t)e^{-st}dt \tag{5.2-54}$$

and

$$f(t) = \sum_{r=0}^{\infty} \hat{f}(t-rT)U(t-rT) \tag{5.2-55}$$

Then

$$\mathcal{L}\left[\{(\sqcup)\right] = \mathcal{L}\left[\sum_{\nabla=\prime}^{\infty} \hat{\{}(\sqcup-\nabla\mathcal{T})\mathcal{U}(\sqcup-\nabla\mathcal{T})\right] \tag{5.2-56}$$

$$= \hat{F}(s) + e^{-Ts}\hat{F}(s) + e^{-2Ts}\hat{F}(s) + \cdots \tag{5.2-57}$$

$$= \hat{F}(s)\left(\sum_{r=0}^{\infty} e^{-rTs}\right) \tag{5.2-58}$$

$$= \frac{\hat{F}(s)}{1-e^{-Ts}} \qquad \text{for } |e^{-Ts}| < 1 \tag{5.2-59}$$

5.2.10 Convolution

If $f_1(t)$ and $f_2(t)$ are casual, the convolution of $f_1(t)$ and $f_2(t)$ is defined as:

$$f_1(t) * f_2(t) \triangleq \int_{0^-}^{\infty} f_1(t-\tau)f_2(\tau)d\tau \tag{5.2-60}$$

$$\triangleq \int_{0^-}^{t} f_1(\overbrace{t-\tau}^{t'})f_2(\tau)d\tau \tag{5.2-61}$$

$(f_1(t-\tau) = 0$ when $\tau > t)$

$$f_1(t) * f_2(t) = f_2(t) * f_1(t) \tag{5.2-62}$$

Assume that

$$\mathcal{L}[\{_\infty(\sqcup)] = F_1(s) \tag{5.2-63}$$

and

$$\mathcal{L}[\{_{\in}(\sqcup)] = F_2(s) \tag{5.2-64}$$

$$\mathcal{L}[\{_{\infty}(\sqcup) * \{_{\in}(\sqcup)] = F_1(s)F_2(s) \tag{5.2-65}$$

Proof:

$$\mathcal{L}\left[\int_{\prime -}^{\sqcup} \{_{\infty}(\sqcup - \tau)\{_{\in}(\tau)\lceil\tau\right] \tag{5.2-66}$$

$$= \int_{0^-}^{\infty}\left(\int_{0^-}^{t} f_1(t-\tau)f_2(\tau)d\tau\right)e^{-st}dt \tag{5.2-67}$$

$$= \int_{0^-}^{\infty}\left(\int_{0^-}^{\infty} f_1(t-\tau)f_2(\tau)d\tau\right)e^{-st}dt \tag{5.2-68}$$

$$= \int_{0^-}^{\infty}\left(\int_{0^-}^{\infty} f_1(x)f_2(\tau)d\tau\right)e^{-s(x+\tau)}dx \qquad x = t-\tau \tag{5.2-69}$$

$$= \int_{0^-}^{\infty}\int_{0^-}^{\infty} f_1(x)e^{-sx}dx \int_{0^-}^{\infty} f_2(\tau)e^{-s\tau}d\tau \tag{5.2-70}$$

$$= F_1(s)F_s(2) \tag{5.2-71}$$

5.2.11 The Initial and Final Value Theorems

THEOREM 5.2-1 (INITIAL VALUE THEOREM) *Assume that $f(t)$ is continuous at $t = 0$ or has a finite discontinuity at this point, its Laplace transform $F(s)$ is a proper function. Then*

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) \tag{5.2-72}$$

EXAMPLE 5.2-4

Given

$$F(s) = \frac{2(s+1)}{s^2+2s+5} \tag{5.2-73}$$

$$f(0^+) = \lim_{s\to\infty} sF(s) \tag{5.2-74}$$

$$= \lim_{s\to\infty} \frac{2s(s+1)}{s^2+2s+5} \tag{5.2-75}$$

$$= \lim_{s\to\infty}\left(2 - \frac{2s+10}{s^2+2s+5}\right) \tag{5.2-76}$$

$$= 2 \tag{5.2-77}$$

THEOREM 5.2-2 (FINAL VALUE THEOREM) *Assume that a function $f(t)$ whose transform $F(s)$ has poles only in the left half-plane or on the $j\omega$-axis. Then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \tag{5.2-78}$$

EXAMPLE 5.2-5

Given

$$F(s) = \frac{(5s+3)}{s(s+1)} \tag{5.2-79}$$

which has poles at $s = 0$ and $s = -1$

$$f(\infty) = \lim_{s \to 0} sF(s) \tag{5.2-80}$$
$$= \lim_{s \to 0} s\frac{5s+3}{s(s+1)} \tag{5.2-81}$$
$$= \lim_{s \to 0} \frac{5s+3}{s+1} \tag{5.2-82}$$
$$= 3 \tag{5.2-83}$$

5.3 The Inverse Laplace Transform of a Rational Function

In this section, methods will be examined which find a function $f(t)$ from its Laplace transform $F(s)$.

DEFINITION 5.3-1 *$F(s)$ is a real rational function if*

$$F(s) = \frac{N(s)}{D(s)} \tag{5.3-84}$$

where $N(s)$ and $D(s)$ are polynomials with real coefficients.

DEFINITION 5.3-2 *$F(s)$ is a proper function if $F(s)$ is a real rational function and*

$$\deg N(s) < \deg D(s) \tag{5.3-85}$$

EXAMPLE 5.3-1

$$\deg(s^1 0 + s^5 - 3s^2 + 1) = 10 \tag{5.3-86}$$

REMARK 5.3-1 *A real rational function $F(s)$ can be represented by a polynomial of s and a proper function $F'(s)$. ie.*

$$F(s) = P(s) + F'(s) \tag{5.3-87}$$

where $F(s) = \frac{N(s)}{D(s)}$ and $F'(s) = \frac{N_1(s)}{D(s)}$.

So to find the inverse Laplace transform of a real rational function is to find the inverse Laplace transform of a proper function (where $\deg N(s) < \deg D(s)$) and that of a polynomial of s (when $\deg N(s) \geqslant \deg D(s)$).

Since

$$P(s) = \sum_{i=0}^{n} p_i s^i \tag{5.3-88}$$

where each p_i is a real constant, and

$$\mathcal{L}^{-\infty}[\infty] = \delta(t) \tag{5.3-89}$$
$$\mathcal{L}^{-\infty}[\mathcal{S}^\backslash] = \frac{d^n \delta(t)}{dt^n} \tag{5.3-90}$$

to find the inverse Laplace transform for $P(s)$ is trivial.

Now the problem of finding the inverse Laplace transform of a real rational function becomes that of finding the inverse Laplace transform of a proper function.

In general, finding inverse Laplace transforms of $F(s)$ by definition is not a trivial task. However, a "Look Up Table" approach can be used to facilitate this operation. For instance we know that

$$\mathcal{L}^{-\infty}\left[\frac{\infty}{\int}\right] = U(t) \tag{5.3-91}$$

$$\mathcal{L}^{-\infty}\left[\frac{\infty}{\int+\alpha}\right] = e^{-\alpha t} \tag{5.3-92}$$

$$\mathcal{L}^{-\infty}\left[\frac{\omega_\prime}{\int^\in+\omega_\prime^\in}\right] = \sin(\omega_0 t) \tag{5.3-93}$$

$$\mathcal{L}^{-\infty}\left[\frac{\int}{\int^\in+\omega_\prime^\in}\right] = \cos(\omega_0 t) \tag{5.3-94}$$

$$\mathcal{L}^{-\infty}\left[\frac{\backslash !}{\int^{\backslash+\infty}}\right] = t^n \qquad n \text{ is an integer} \tag{5.3-95}$$

$$\mathcal{L}^{-\infty}\left[\mathcal{F}(\int-\alpha)\right] = e^{\alpha t} f(t) \qquad \mathcal{L}[\{(\sqcup)] = \mathcal{F}(\int) \tag{5.3-96}$$

$$\mathcal{L}^{-\infty}\left[\frac{\beta}{(\int-\alpha)^\in+\beta^\in}\right] = e^{\alpha t}\sin(\beta t) \tag{5.3-97}$$

$$\mathcal{L}^{-\infty}\left[\frac{(\int-\alpha)}{(\int-\alpha)^\in+\beta^\in}\right] = e^{\alpha t}\cos(\beta t) \tag{5.3-98}$$

If a given $F(s)$ can be expressed by a sum of the above trivial terms for which the inverse Laplace transform is trivial, the inverse of $F(s)$ will also be trivial.

The method used here is the partial fraction expansion.

It is known that the frequencies s_k that makes $F(s) = \infty$ (or $D(s) = 0$) are the poles of $F(s)$. Then

$$F(s) = \frac{N(s)}{\prod_{k=1}^{n}(\delta - s_k)^{m_k}} \tag{5.3-99}$$

where $S_k(k = 1, 2, \ldots, n)$ are poles of $F(s)$ and m_k may be any positive integer value.

EXAMPLE 5.3-2

$$F(s) = \frac{2s^3+1}{s^4+5s^3+8s^2-4s} \tag{5.3-100}$$

$F(s)$ is a proper function of s and

1.

$$D(s) = s^4 - 5s^3 + 8s^2 - 4s \tag{5.3-101}$$

$$D(0) = 0 \tag{5.3-102}$$

$$D(1) = 1 - 5 + 8 - 4 = 0 \tag{5.3-103}$$

$$D(2) = 16 - 40 + 32 - 8 = 0 \tag{5.3-104}$$

2.

$$D(s) = s(s^3 - 5s^2 + 8s - 4) \tag{5.3-105}$$

3.

$$\begin{array}{r|l} & s^2 \;-4s \;+4 \\ \hline s-1 & s^3 - 5s^2 + 8s \quad -4 \\ & s^3 \;-s^2 \\ & \quad -4s^2 + 8s \\ & \quad \underline{-4s^2 + 4s} \\ & \qquad 4s - 4 \\ & \qquad \underline{4s - 4} \\ & \qquad\quad 0 \end{array} \tag{5.3-106}$$

$$D(s) = s(s-1)(s^2 - 4s + 4) \tag{5.3-107}$$

4.

$$\begin{array}{r|l} & \quad s \; -2 \\ \cline{2-2} s-2 & \; s^2-4s+4 \\ & \; \underline{s^2-2s} \\ & \quad\; -2s+4 \\ & \quad\; \underline{-2s+4} \\ & \qquad\quad 0 \end{array} \tag{5.3-108}$$

So,

$$D(s) = s(s-1)(s-2)^2 \tag{5.3-109}$$

or

$$F(s) = \frac{2s^3+1}{s(s-1)(s-2)^2} \tag{5.3-110}$$

$$= \frac{?}{s} + \frac{?}{s-1} + \frac{?}{(s-2)^2} + \frac{?}{s-2} \tag{5.3-111}$$

In Equation 5.3-99, poles of $F(s)$ may take different forms.

i. A simple pole

$$(s - s_k) \tag{5.3-112}$$

where s_k is either real of complex. If s_k is complex,

$$s_k = \alpha + j\beta \tag{5.3-113}$$

there must be another pole of $F(s)$ which is the conjugate of s_k or

$$s_k^* = \alpha - j\beta \tag{5.3-114}$$

to make $D(s)$ a polynomial with real coefficients. Therefore

$$(s-s_k)(s-s_k^*) = [s-(\alpha+j\beta)][s-(\alpha-j\beta)] \tag{5.3-115}$$

$$= s^2 - 2\alpha s + (\alpha^2 + \beta^2) \tag{5.3-116}$$

$$= s^2 - cs + d \tag{5.3-117}$$

$$= (s-\alpha)^2 + \beta^2 \tag{5.3-118}$$

where $c = -2\alpha$ and $d = \alpha^2 + \beta^2$. If $\alpha = c = 0$ then $(s-s_k)(s-s_k^*) = s^2 + \beta^2$ which corresponds to a pair of pure imaginary poles.

ii. A multiple pole

$$(s - s_k)^r \tag{5.3-119}$$

where $r > 1$.

If s_k and s_k^* form two complex poles of $F(s)$

$$(s-s_k)^r(s-s_k^*)^r = (s^2 + cs + d)^r \tag{5.3-120}$$

So, in general, a proper function of s

$$F(s) = \frac{N(s)}{(s-a)(s-b)^r(s^2+cs+d)(s^2+es+f)^m} \tag{5.3-121}$$

The partial fraction expansion of $F(s)$ is given as

$$F(s) = \frac{A}{s-a} + \left(\frac{B_1}{(s-b)^r} + \frac{B_2}{(s-b)^{r-1}} + \cdots + \frac{B_r}{s-b}\right) + \frac{Cs+D}{s^2+cs+d} + \left(\frac{E_1 s + F_1}{(s^2+es+f)^m} + \frac{E_2 s + F_2}{(s^2+es+f)^{m+1}} + \cdots + \frac{E_m s + F_m}{s^2+es+f}\right) \tag{5.3-122}$$

Now the problem is to determine the constants in the numerator of each term.

5.3.1 Algebraic Method

EXAMPLE 5.3-3 *Find the inverse Laplace transform of the function*

$$F(s) = \frac{s^2-6}{s^7+4s^3+3s} \tag{5.3-123}$$

1.

$$D(s) = S^3 + 4s^2 + 3s = s(s^2+4s+3) \tag{5.3-124}$$

$$D(0) = 0 \tag{5.3-125}$$

$$D(-1) = 0 \tag{5.3-126}$$

$$\begin{array}{r|l} & \underline{\quad s \ +3} \\ s+1 & \ s^2+4s+3 \\ & \underline{\ s^2 \ +s \qquad} \\ & \qquad 3s \ +3 \\ & \qquad \underline{3s \ +3} \\ & \qquad\qquad 0 \end{array} \tag{5.3-127}$$

$$D(s) = s(s+1)(s+3) \qquad i.e.\ D(-3) = 0 \tag{5.3-128}$$

2.

$$F(s) = \frac{s^2-6}{s(s+1)(s+3)} \tag{5.3-129}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} \tag{5.3-130}$$

Multiply both sides of the Equation 5.3-130 by $s(s+1)(s+3)$

$$s^2 - 6 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1) \tag{5.3-131}$$

$$= (A+B+C)s^2 + (4A+3B+C)s + 3A \tag{5.3-132}$$

To make the two polynomials of Equation 5.3-132 equal we have

$$s^0: \qquad -6 = 3A \tag{5.3-133}$$

$$A = -\frac{6}{3} = -2 \tag{5.3-134}$$

$$s^1: \qquad 4A + 3B + C = 0 \tag{5.3-135}$$

Substitute $A = -2$ into the above equation

$$-8 + 3B + C = 0 \tag{5.3-136}$$

$$3B + C = 8 \tag{5.3-137}$$

$$s^2: \qquad A + B + C = 1 \tag{5.3-138}$$

$$B + C = 3 \tag{5.3-139}$$

$$C = 3 - B \tag{5.3-140}$$

Substitute Equation 5.3-140 into Equation 5.3-137.

$$3B + 3 - B = 8 \tag{5.3-141}$$

$$B = \frac{5}{2} \tag{5.3-142}$$

Substitute $B = \frac{5}{2}$ into Equation 5.3-140.

$$C = \frac{1}{2} \tag{5.3-143}$$

$$F(s) = -\frac{2}{5} + \frac{5}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s+3} \tag{5.3-144}$$

3.

$$\mathcal{L}^{-\infty}[\mathcal{F}(\int)] = \mathcal{L}^{-\infty}\left[-\frac{\in}{\int}\right] + \mathcal{L}^{-\infty}\left[\frac{\nabla}{\in}\frac{\infty}{\int+\infty}\right] + \mathcal{L}^{-\infty}\left[\frac{\infty}{\in}\frac{\infty}{\int+\ni}\right] \tag{5.3-145}$$

$$= -2U(t) + \frac{5}{2}e^{-t}U(t) + \frac{1}{2}e^{-3t}U(t) \tag{5.3-146}$$

Note: $\mathcal{L}[\mathcal{U}(\sqcup)] = \frac{\infty}{\int}$, and the property of translation in the frequency domain.

EXAMPLE 5.3-4 *Find the inverse Laplace transform of the function*

$$F(s) = \frac{16}{s(s^2+4)^2} \tag{5.3-147}$$

Solution:

$$F(s) = \frac{16}{s(s^2+4)^2} = \frac{A}{s} + \frac{B_1 s + C_1}{s^2+4} + \frac{B_2 s + C_2}{(s^2+4)^2} \tag{5.3-148}$$

$$16 = A(s^2+4)^2 + (B_1 s + C_1)s(s^2+4) + s(B_2 s + C_2) \tag{5.3-149}$$

$$= (A + B_1)s^4 + C_1 s^3 + (8A + 4B_1 + B_2)s^2 + (4C_1 + C_2)s + 16A \tag{5.3-150}$$

$$s^0: \quad 16A = 16 \tag{5.3-151}$$

$$\therefore A = 1 \tag{5.3-152}$$

$$s^1: \quad 4C_1 + C_2 = 0 \tag{5.3-153}$$

$$s^2: \quad 8A + 4B_1 + B_2 = 0 \tag{5.3-154}$$

$$s^3: \quad C_1 = 0 \tag{5.3-155}$$

$$\therefore C_2 = 0 \tag{5.3-156}$$

$$s^4: \quad A + B_1 = 0 \tag{5.3-157}$$

$$B_1 = -1 \tag{5.3-158}$$

Therefore

$$8A + 4B_1 + B_2 = 0 \tag{5.3-159}$$

$$B_2 = -(4B_1 + 8A) \tag{5.3-160}$$

$$= -4 \tag{5.3-161}$$

$$F(s) = \frac{1}{s} + \frac{-s}{s^2+4} + \frac{-4s}{(s^2+4)^2} \tag{5.3-162}$$

$$\mathcal{L}^{-\infty} = \mathcal{L}^{-\infty}\left[\frac{\infty}{\int}\right] - \mathcal{L}^{-\infty}\left[\frac{\int}{\int^{\in}+\triangle}\right] - \triangle\mathcal{L}\left[\frac{\int}{(\int^{\in}+\triangle)^{\in}}\right] \tag{5.3-163}$$

$$= U(t) - \cos(2t)U(t) - t\sin(2t)U(t) \tag{5.3-164}$$

(Note:

$$-\frac{1}{2\omega_0}\frac{d}{ds}\left(\frac{\omega_0}{s^2+\omega_0^2}\right) = \frac{s}{(s^2+\omega_0^2)^2} \tag{5.3-165}$$

and

$$\mathcal{L}^{-\infty}\left[\frac{\int}{(\int^{\in}+\omega_{,}^{\in})^{\in}}\right] = -\frac{1}{2\omega_0}(-tf(t)) \tag{5.3-166}$$

$$= -\frac{1}{2\omega_0}\left(-t\mathcal{L}^{-\infty}\left[\frac{\omega_{,}}{\int^{\in}+\omega_{,}^{\in}}\right]\right) \tag{5.3-167}$$

$$= -\frac{1}{2\omega_0}\left(-t\cos(\omega_0 t)\right) \tag{5.3-168}$$

$$= \frac{t}{2\omega_0}\cos(\omega_0 t) \tag{5.3-169}$$

where $\omega_0 = 2$.)

5.3.2 Formulae for the residues

5.3.2.1 Simple Real Poles

If

$$F(s) = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \frac{A_3}{s-a_3} + \cdots \tag{5.3-170}$$

$$A_i = \lim_{s\to a_i}(s-a_i)F(s) \qquad \text{where } i = 1, 2, 3, \ldots \tag{5.3-171}$$

(Note:

if $i = 1$

$$A_1 = \lim_{s\to a_1}\left\{(s-a_1)\frac{A_1}{s-a_1} + \underbrace{\frac{(s-a_1)A_2}{s-a_2}}_{=0} + \underbrace{\frac{(s-a_1)A_3}{s-a_3}}_{=0}\right\} \tag{5.3-172}$$

$$= \lim_{s\to a_1}(s-a_1)F(s) \tag{5.3-173}$$

)

EXAMPLE 5.3-5 *Find the inverse Laplace transform of*

$$F(s) = \frac{s^2-6}{s(s+1)(s+3)} \tag{5.3-174}$$

Solution:

$$F(s) = \frac{s^2-6}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} \tag{5.3-175}$$

$$A = \lim_{s\to 0} sF(s) = \frac{-6}{1\cdot 3} = -2 \tag{5.3-176}$$

$$B = \lim_{s\to -1}(s+1)F(s) = \frac{1-6}{(-1)(-1+3)} = \frac{-5}{-2} = \frac{5}{2} \tag{5.3-177}$$

$$C = \lim_{s\to -3}(s+3)F(s) = \frac{9-6}{(-3)(-3+1)} = \frac{3}{6} = \frac{1}{2} \tag{5.3-178}$$

The rest of the procedure is the same as that in Example 5.3-3.

5.3.2.2 Multiple Real Poles

Assume $F(s)$ has an r^{th} order pole at $s = \alpha$ and

$$F(s) = \frac{N(s)}{D(s)} = \frac{\phi(s)}{(s-\alpha)^r} \tag{5.3-179}$$

where

$$\phi(s) = (s-\alpha)^r \frac{N(s)}{D(s)} \tag{5.3-180}$$

Expand $\phi(s)$ in the series at $s = \alpha$

$$\begin{aligned}\phi(s) &= \Phi(\alpha) + (s-\alpha)\Phi'(\alpha) + \frac{(s-\alpha)^2}{2!}\Phi''(\alpha) + \cdots \\ &+ \frac{(s-\alpha)^{r-1}}{(r-1)!}\Phi^{(r-1)}(\alpha) + (s-\alpha)^r G(s)\end{aligned} \tag{5.3-181}$$

then

$$\begin{aligned}F(s) = \frac{\phi(s)}{(s-\alpha)^r} &= \frac{\Phi(\alpha)}{(s-\alpha)^r} + \frac{\Phi'(\alpha)}{(s-\alpha)^{r-1}} + \frac{\Phi''(\alpha)}{2!(s-\alpha)^{r-2}} + \cdots \\ &+ \frac{\Phi^{r-1}(\alpha)}{(r-1)!(s-\alpha)} + G(s)\end{aligned} \tag{5.3-182}$$

Assume

$$F(s) = \frac{k_0}{(s-\alpha)^r} + \frac{k_1}{(s-\alpha)^{r-1}} + \cdots + \frac{k_{r-1}}{(s-\alpha)} + G(s) \tag{5.3-183}$$

where

$$k_i = \frac{1}{i!}\frac{d^i}{ds^i}\Phi(s)\bigg|_{s=\alpha} \qquad \text{for } i = 0, 1, \ldots, (r-1) \tag{5.3-184}$$

EXAMPLE 5.3-6 *Find the inverse Laplace transform of the function*

$$F(s) = \frac{s-2}{s(s+1)^3} \tag{5.3-185}$$

Solution:

1.

$$F(s) = \frac{A}{s} + \frac{k_0}{(s+1)^3} + \frac{k_1}{(s+1)^2} + \frac{k_2}{(s+1)} \tag{5.3-186}$$

2.

$$A = \lim_{s \to 0} sF(s) = \left. \frac{s-2}{(s+1)^3} \right|_{s=0} = -2 \tag{5.3-187}$$

3.

$$\Phi(s) = (s+1)^3 F(s) = \frac{s-2}{s} = 1 - \frac{2}{s} \tag{5.3-188}$$

$$\Phi'(s) = \frac{2}{s^2} \tag{5.3-189}$$

$$\Phi''(s) = -\frac{4}{s^3} \tag{5.3-190}$$

$$k_0 = \Phi(-1) = 1 - \frac{2}{-1} = 3 \tag{5.3-191}$$

$$k_1 = \Phi'(-1) = 2 \tag{5.3-192}$$

$$k_2 = \frac{\Phi''(-1)}{2!} = \frac{-4}{2(-1)^3} = 2 \tag{5.3-193}$$

4.

$$F(s) = -\frac{2}{s} + \frac{3}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)} \tag{5.3-194}$$

5.

$$f(t) = \mathcal{L}^{-\infty}[\mathcal{F}(\int)] = -\in\mathcal{U}(\sqcup) + \left\{\rceil^{-\sqcup}\left(\frac{\ni}{\in}\sqcup^{\in}\right) + \rceil^{-\sqcup}(\in\sqcup) + \in\right\}\mathcal{U}(\sqcup) \tag{5.3-195}$$

Note:

$$\mathcal{L}[\sqcup^{\backslash}] = \frac{\backslash !}{\int^{\backslash+\infty}} \tag{5.3-196}$$

5.3.2.3 A pair of simple complex conjugate poles

Assume that

$$F(s) = \frac{N(s)}{D(s)} \tag{5.3-197}$$

$$= \frac{\phi(s)}{(s-a)^2+b^2} = \frac{As+B}{(s-a)^2+b^2} + G(s) \tag{5.3-198}$$

$$\lim_{s \to a+jb}((s-a)^2+b^2)F(s) = \lim_{s \to a+jb} \Phi(s) \tag{5.3-199}$$

$$= A(a+jb) + B \tag{5.3-200}$$

$$= (Aa+B) + jAB \tag{5.3-201}$$

$$= \Phi_1 + j\Phi_2 \tag{5.3-202}$$

where $\Phi_1 = Aa + B$ and $\Phi_2 = Ab$

$$A = \frac{\Phi_2}{b} = \frac{\Im\left[\lim_{s\to a+jb}((s-a)^2+b^2)F(s)\right]}{b} \tag{5.3-203}$$

$$B = \Phi_1 - \Phi_2 \frac{a}{b} = \frac{b\Phi_1 - a\Phi_2}{b} \tag{5.3-204}$$

$$= \frac{b\Re\left[\lim_{s\to a+jb}((s-a)^2+b^2)F(s)\right] - a\Im\left[\lim_{s\to a+jb}((s-a)^2+b^2)F(s)\right]}{b} \tag{5.3-205}$$

$$\frac{As+b}{(s-a)^2+b^2} = \frac{(\Phi_2/b)s + (b\Phi_1 - a\Phi_2)/b}{(s-a)^2+b^2} \tag{5.3-206}$$

$$= \frac{1}{b}\frac{(s-a)\Phi_2 + b\Phi_1}{(s-a)^2+b^2} \tag{5.3-207}$$

$$\mathcal{L}^{-\infty}\left[\frac{\infty}{\lfloor}\frac{(\int-\dashv)\oplus_\in + \lfloor\oplus_\infty}{(\int-\dashv)^\in + \lfloor^\in}\right] = \frac{1}{b}e^{at}\left\{\Phi_2\cos(bt) + \Phi_1\sin(bt)\right\} \tag{5.3-208}$$

$$= \frac{1}{b}e^{at}(\Phi_1^2+\Phi_2^2)^{\frac{1}{2}}\left\{\frac{\Phi_2}{(\Phi_1^2+\Phi_2^2)}\cos(bt) + \frac{\Phi_1}{(\Phi_1^2+\Phi_2^2)^{\frac{1}{2}}}\sin(bt)\right\} \tag{5.3-209}$$

$$= \frac{1}{b}e^{at}(\Phi_1^2+\Phi_2^2)^{\frac{1}{2}}\left\{\sin(\theta)\cos(bt) + \cos(\theta)\sin(bt)\right\} \tag{5.3-210}$$

$$= \frac{1}{b}e^{at}(\Phi_1^2+\Phi_2^2)^{\frac{1}{2}}\sin(bt+\theta) \tag{5.3-211}$$

where

$$\theta = \tan^{-1}\left(\frac{\Phi_2}{\Phi_1}\right) \tag{5.3-212}$$

It follows that

$$\mathcal{L}^{-\infty}[\mathcal{F}(\int)] = \frac{\infty}{\lfloor}\rceil^{\dashv\sqcup}(\oplus_\infty^\in + \oplus_\in^\in)^{\frac{\infty}{\in}}\sin(\lfloor\sqcup+\theta) + \mathcal{L}^{-\infty}[\mathcal{G}(\int)] \tag{5.3-213}$$

It is obvious that a pair of poles of $F(s)$ at $s = a \pm jb$ gives rise in the time domain to

1. Steady sinusoidal oscillation if $a = 0$
2. damped oscillation with the damping factor $e^{-|a|t}$ if $a < 0$
3. oscillation with indefinately growing amplitudes if $a > 0$.

5.3.2.4 Multiple Complex Conjugate Pole pairs

$$F(s) = \frac{N(s)}{[(s-a)^2+b^2]^r\sum_{k=0}^{n-2r}(\delta - s_k)} \tag{5.3-214}$$

Either algebraic methods or that used in Section 5.3.2.2 for multiple real poles may be used in this case.

5.4 The Significance of Poles and Zeros

The locations of poles of $F(s)$ determine the time behaviour of $f(t)$ and those of zeros determine the magnitude of each term of $f(t)$.

5.5 Solution of Linear Differential Equations using the Laplace Transform

EXAMPLE 5.5-1 *Solve the following differential equation.*

$$f'''(t) + 6f''(t) + 11f'(t) + 6f(t) = 1 \tag{5.5-215}$$

with the zero inital conditions. i.e.

$$f''(0^-) = f'(0^-) = f(0^-) = 0 \tag{5.5-216}$$

1. *Take the Laplace tranform of Equation 5.5-215.*

$$\begin{aligned} s^3F(s) - s^2f(0) - sf'(0) - f''(0) + 6[s^2F(s) - sf(0) \\ - f'(0)] + 11[sF(s) - f(0)] + 6F(s) = \frac{1}{s} \end{aligned}$$

$$s^3F(s) + 6s^2F(s) + 11sF(s) + 6F(s) = \frac{1}{s} \tag{5.5-217}$$

$$F(s) = \frac{1}{s(s^3 + 6s^2 + 11s + 6)} \tag{5.5-218}$$

2.

$$F(0) = F(-1) = F(-2) = F(-3) = 0 \tag{5.5-219}$$

$$F(s) = \frac{1}{s(s+1)(s+2)(s+3)} \tag{5.5-220}$$

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3} \tag{5.5-221}$$

$$A = \lim_{s\to 0} sF(s) \qquad = \lim_{s\to 0} \frac{a}{s^3 + 6s^2 + 11s + 6} \tag{5.5-222}$$

$$= \frac{1}{6} \tag{5.5-223}$$

$$B = \lim_{s\to -1} (s+1)F(s) \qquad = \lim_{s\to -1} \frac{1}{s(s+2)(s+3)} \tag{5.5-224}$$

$$= -\frac{1}{2} \tag{5.5-225}$$

$$C = \lim_{s\to -2} (s+2)F(s) \qquad = \lim_{s\to -2} \frac{1}{s(s+1)(s+3)} \tag{5.5-226}$$

$$= \frac{1}{2} \tag{5.5-227}$$

$$D = lim_{s\to -3}(s+3)F(s) \qquad = \lim_{s\to -3} \frac{1}{s(s+1)(s+2)} \tag{5.5-228}$$

$$= -\frac{1}{6} \tag{5.5-229}$$

Therefore

$$F(s) = \frac{1}{6s} + \frac{-1}{2(s+1)} + \frac{1}{2(s+2)} + \frac{-1}{6(s+3)} \tag{5.5-230}$$

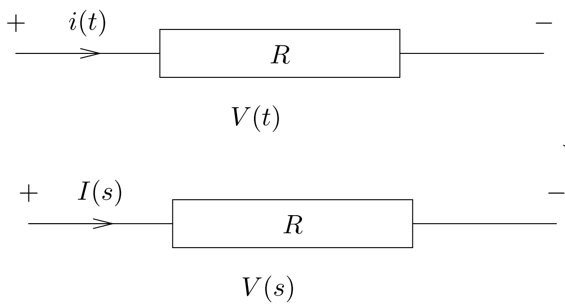
3.

$$f(t) = \mathcal{L}^{-\infty}[\mathcal{F}(\int)] = \frac{\infty}{/}\mathcal{U}(\sqcup) + \left(-\frac{\infty}{\in}\right)\rceil^{-\sqcup}\mathcal{U}(\sqcup) + \frac{\infty}{\in}\rceil^{-\in\sqcup}\mathcal{U}(\sqcup) - \frac{\infty}{/}\rceil^{-\ni\sqcup}\mathcal{U}(\sqcup) \tag{5.5-231}$$

5.6 Network Analysis Using the Laplace Transform

5.6.1 The Basic Building Blocks

5.6.1.1 The Resistor

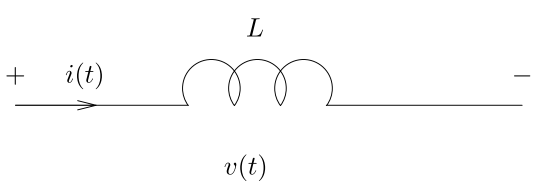


$\Downarrow \quad V(t) = Ri(t) \quad \Rightarrow \quad V(s) = RI(s)$

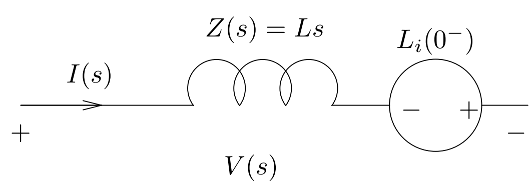
$$z(s) = \frac{V(s)}{I(s)} = R \tag{5.6-232}$$

$$y(s) = \frac{I(s)}{V(s)} = G = \frac{1}{R} \tag{5.6-233}$$

5.6.1.2 The Inductor



$\Rightarrow$



$v(t) = L\frac{di(t)}{dt} \quad \Rightarrow \quad V(s) = L(sI(s) - i(0^-))$

The impendance is defined as the ratio of $V(s)$ to $I(s)$ under zero initial conditions.

$$Z(s) = \frac{V(s)}{I(s)} = Ls \tag{5.6-234}$$

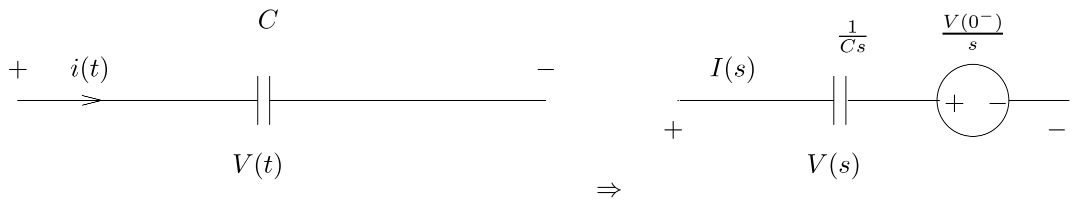
$$Y(s) = \frac{1}{Z(s)} = \frac{1}{Ls} \tag{5.6-235}$$

5.6.1.3 The Capacitor

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \tag{5.6-236}$$

$$= \frac{1}{C}\int_{-\infty}^{0^-} i(\tau)d\tau + \frac{1}{C}\int_{0^-}^{t} i(\tau)d\tau \tag{5.6-237}$$

$$= v(0^-) + \frac{1}{C}\int_{0^-}^{t} i(\tau)d\tau \tag{5.6-238}$$



$$V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s} \tag{5.6-239}$$

5.6.2 The Transformed Network

To find the impedance of the RLC series circuit shown in Figure 5.3 we start with:

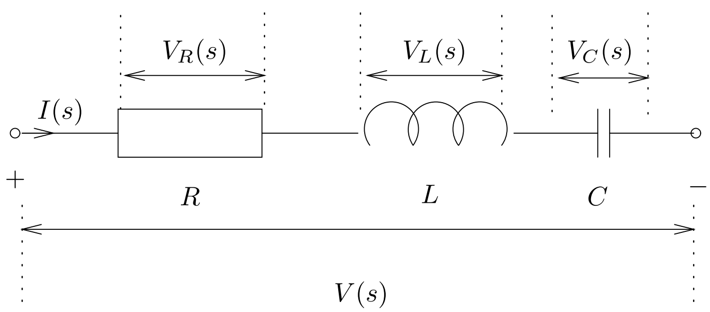


Figure 5.3: Series RLC circuit

$$V(s) = V_R(s) + V_L(s) + V_C(s) \tag{5.6-240}$$

$$= RI(s) + L_s I(s) + \frac{1}{C_s} I(s) \tag{5.6-241}$$

$$Z(s) = \frac{V(s)}{I(s)} = R + Ls + \frac{1}{Cs} \tag{5.6-242}$$

For a series connection of n impedances as shown in Figure 5.4,

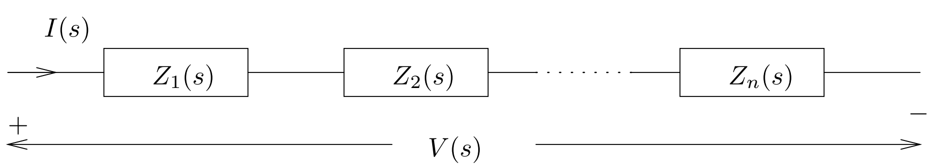


Figure 5.4: Series connection of n impedences

$$V(s) = Z_1(s)I(s) + Z_2(s)I(s) + \cdots + Z_n(s)I(s) \tag{5.6-243}$$

$$= \sum_{i=1}^{n} Z_i(s)I(s) \tag{5.6-244}$$

So

$$Z(s) = \frac{V(s)}{I(s)} = \sum_{i=1}^{n} Z_i(s) \tag{5.6-245}$$

5.6.3 Single Loop Network

Solve $i(t)$ with the excitation $v(t)$ as shown in Figure 5.5 using the Laplace transform.

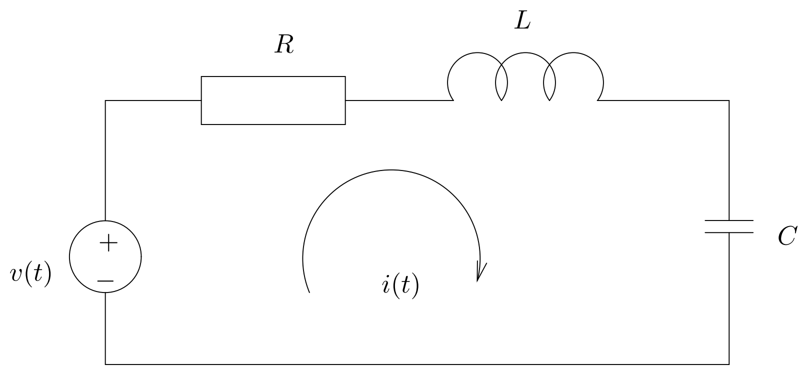


Figure 5.5: Single Loop Network

1. Convert the given network into a transformed network using the basic building blocks giving Figure 5.6.

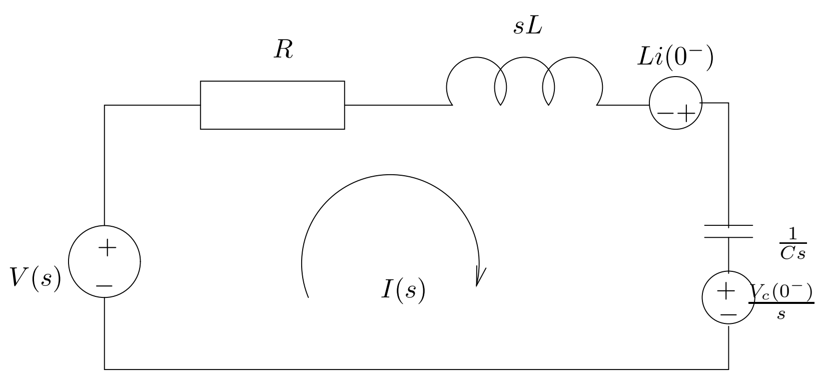


Figure 5.6: Transformed Single Loop Network

2. Solve the transformed network using the standard methods.

$$V(s) = RI(s) + LsI(s) - Li(0^-) + \frac{1}{Cs}I(s) + \frac{v_c(0^-)}{s} \tag{5.6-246}$$

$$I(s) = \frac{a}{R + Ls + \frac{1}{Cs}}\left(V(s) + Li(0^-) - \frac{v_c(0^-)}{s}\right) \tag{5.6-247}$$

3.

$$i(t) = \mathcal{L}^{-\infty}[\mathcal{I}(\int)] \tag{5.6-248}$$

EXAMPLE 5.6-1 *Given $R = 1\Omega$, $L = 2H$, $C = 1F$ and $i(0^-) = 1A$, $V_c(0^-) = 5V$ in the RLC single loop network shown in Figure 5.5. Assume $v(t) = U(t)$, a unit step function.*

Find the current $i(t)$.

Solution:

1. *In the transformed network*

$$V(s) = \mathcal{L}[\mathcal{U}(\sqcup)] = \frac{\infty}{\int} \tag{5.6-249}$$

2.

$$\frac{1}{s} = I(s) + 2sI(s) - 2 + \frac{1}{s}I(s) + \frac{5}{s} \tag{5.6-250}$$

$$\begin{aligned}
I(s) &= \frac{1}{1 + 2s + \frac{1}{5}}\left(\frac{1}{s} + 2 - \frac{5}{s}\right) && (5.6\text{-}251)\\
&= \frac{2s-4}{2s^2+s+1} && (5.6\text{-}252)\\
&= \frac{s-2}{s^2+0.5s+0.5} && (5.6\text{-}253)\\
&= \frac{s-2}{(s+0.25)^2 + \underbrace{0.5 - 0.0625}_{0.4375}} && (5.6\text{-}254)
\end{aligned}$$

3.

$$\begin{aligned}
I(s) &= \frac{s-2}{(s+0.25)^2+0.4375} && (5.6\text{-}255)\\
&= \frac{s+0.25-0.25-2}{(s+0.25)^2+0.4375} && (5.6\text{-}256)\\
&= \frac{(s+0.25)}{(s+0.25)^2+(\sqrt{0.4735})^2} - \frac{2.25}{(s+0.25)^2+(\sqrt{0.4375})^2} && (5.6\text{-}257)\\
i(t) &= \mathcal{L}^{-\infty}[\mathcal{I}(\int)] && (5.6\text{-}258)\\
&= \mathcal{L}^{-\infty}\left[\frac{\int + \prime.\in\bigtriangledown}{(\int + \prime.\in\bigtriangledown)^{\in} + \prime.\notin} - \frac{\in.\in\bigtriangledown}{\prime./}\,\frac{\prime./}{(\int + \prime.\in\bigtriangledown)^{\in} + \prime.\notin}\right] && (5.6\text{-}259)\\
&= e^{-0.25}\left[\cos(0.66t) - 3.41\sin(0.66t)\right] && (5.6\text{-}260)
\end{aligned}$$

EXAMPLE 5.6-2 *Find $V_0(t)$ in the network shown in Figure 5.7 with zero initial conditions and the input conditions shown in Figure 5.8.*

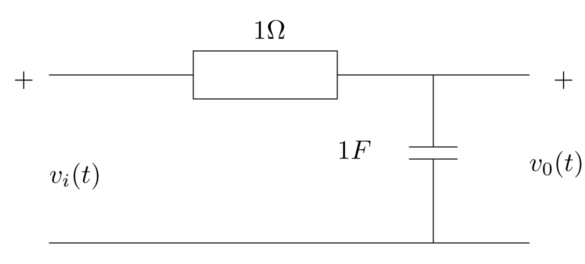


Figure 5.7: Network for Example 5.6-2

Solution:

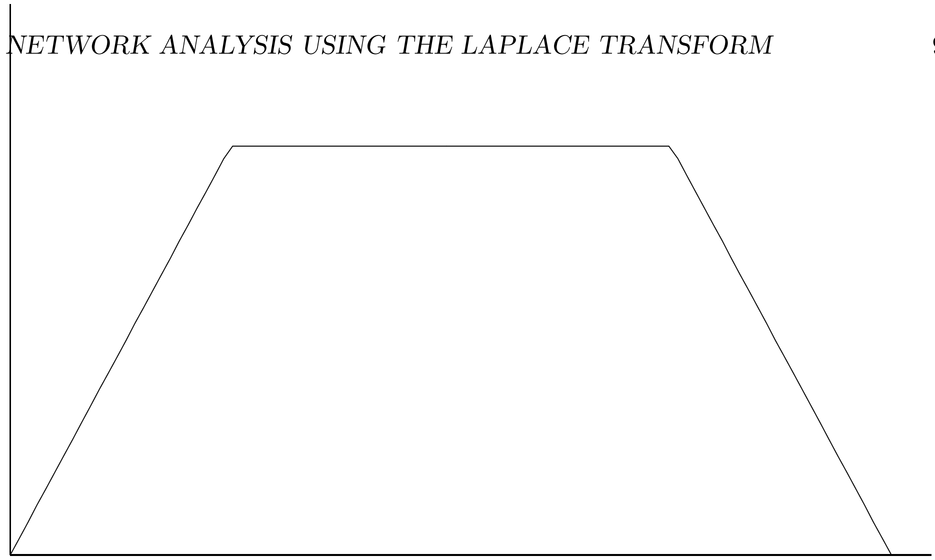


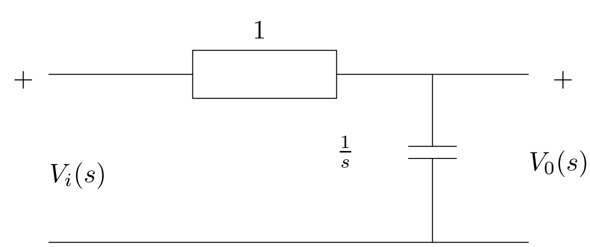
Figure 5.8: Excitation for Example 5.6-2

1.

$$v_i(t) = tU(t) - (t-1)U(t-1) - (t-3)U(t-3) + (t-4)U(t-4) \tag{5.6-261}$$

$$V_i(s) = \frac{1}{s^2}\left(1 - e^{-s} - e^{-3s} + e^{-4s}\right) \tag{5.6-262}$$

2. Transformed network



$$V_O(s) = \frac{\frac{1}{s}}{1+\frac{1}{s}} V_i(s) \tag{5.6-263}$$

$$= \frac{1}{s+1}\frac{1}{s^2}\left(1 - e^{-s} - e^{-3s} + e^{-4s}\right) \tag{5.6-264}$$

$$\frac{1}{s^2(s+1)} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{B}{s+1} \tag{5.6-265}$$

Since

$$\begin{aligned}
B &= \lim_{s\to -1}(s+1)\frac{1}{s^2(s+1)} &&= \frac{1}{s^2} = 1\\
A_1 &= \lim_{s\to 0} s^2\frac{1}{s^2(s+1)} &&= \lim_{s\to 0}\frac{1}{s+1} = 1\\
A_2 &= \lim_{s\to 0}\left(s^2\frac{1}{s^2(s+1)}\right)' &&= \lim_{s\to 0}\left(\frac{1}{s+1}\right)'\\
&= \lim_{s\to 0}\frac{-1}{(s+1)^2} &&= -1
\end{aligned}$$

So that

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \tag{5.6-266}$$

$$\mathcal{L}^{-\infty}\left[\frac{\infty}{\int^{\in}(\int+\infty)}\right] = \mathcal{L}^{-\infty}\left[\frac{\infty}{\int^{\in}} - \frac{\infty}{\int} + \frac{\infty}{\int+\infty}\right] \tag{5.6-267}$$

$$= (t-1+e^{-t})U(t) \tag{5.6-268}$$

$$\triangleq v_o'(t) \tag{5.6-269}$$

$$v_O(t) = \mathcal{L}^{-\infty}\left\{\frac{\infty}{\int^{\in}(\int+\infty)}[\infty - \rceil^{-\int} - \rceil^{-\ni\int} + \rceil^{-\triangle\int}]\right\} \tag{5.6-270}$$

$$= v_O'(t) - v_O(t-4)U(t-4) \tag{5.6-271}$$

$$= (t-1-e^{-t})U(t) - (t-2-e^{-(t-1)})U(t-1) \tag{5.6-272}$$

$$-(t-4-e^{-(t-3)})U(t-3) + (t-5-e^{-(t-4)})U(t-4) \tag{5.6-273}$$

Note:

$$\mathcal{L}[\{(\sqcup-\alpha)\mathcal{U}(\sqcup-\alpha)] = \rceil^{-\alpha\sqcup}\mathcal{F}(\int) \tag{5.6-274}$$

EXAMPLE 5.6-3 *To analyze a given network, we may use nodal or loop methods. From Figure 5.9 use the nodal analysis method.*

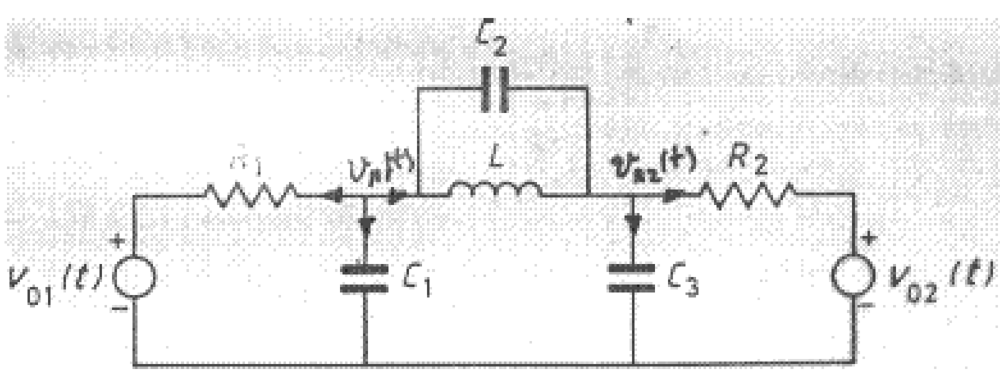
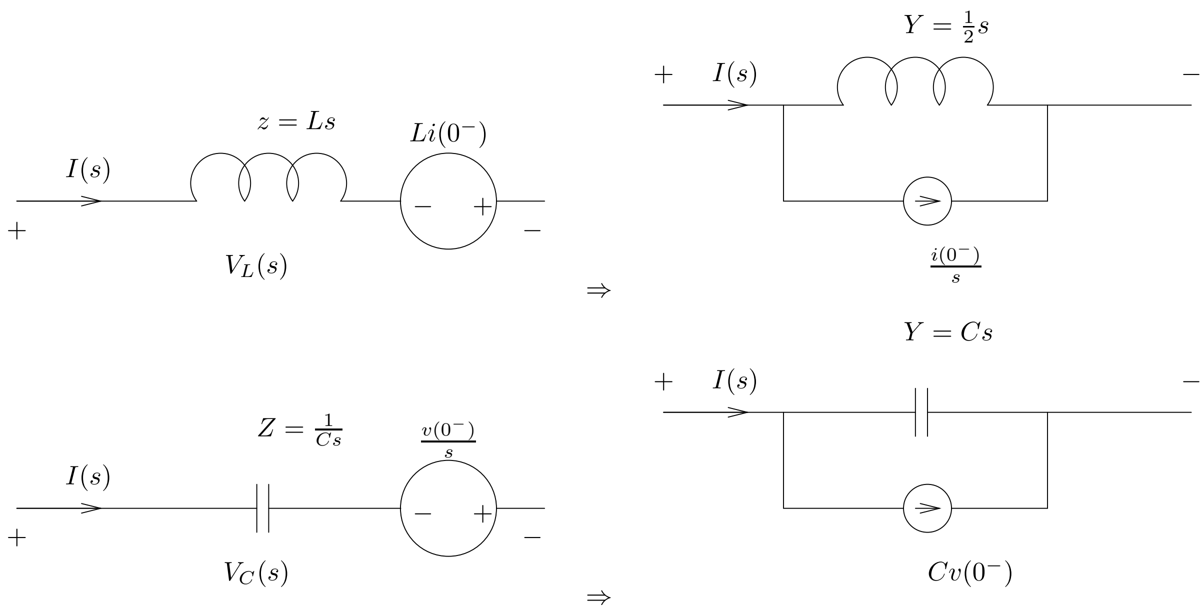


Figure 5.9: Multi-node multi-source network — Baher

Basic building blocks:



$$\left(\frac{1}{R_1} + C_1 s + C_2 s + \frac{1}{Ls}\right) V_{n_1}(s) - \left(C_2 s + \frac{1}{Ls}\right) V_{n_2}(s) = \frac{V_{O_1}(s)}{R_1} + C_1 v_{C_2}(0^-) + C_2 v_{C_2}(0^-) - \frac{i_L(0^-)}{s} \tag{5.6-275}$$

$$-\left(C_2 s + \frac{1}{Ls}\right) V_{n_1}(s) - \left(\frac{1}{R_2} + C_2 s + C_3 s + \frac{1}{Ls}\right) V_{n_2}(s) = \frac{V_{O_2}(s)}{R_2} - C_2 v_{C_2}(0^-) + C_3 v_{C_3}(0^-) - \frac{i_L(0^-)}{s} \tag{5.6-276}$$

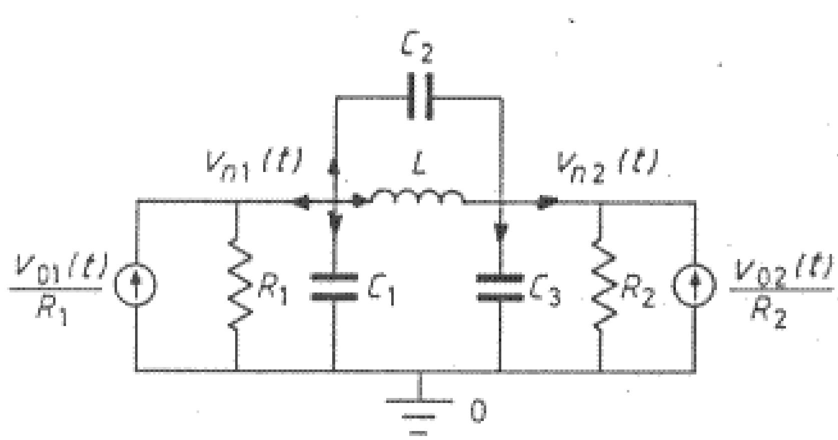


Figure 5.10: Network in Figure 5.9 with voltage sources converted into current sources

When all the parameters and initial conditions and excitations of the network are given the above is nothing but two linear algebraic equations with two unknowns $V_{n_1}(s)$ and $V_{n_2}(s)$. When $V_{n_1}(s)$ and $V_{n_2}(s)$ are solved, $v_{n_1}(t) = \mathcal{L}^{-\infty}[\mathcal{V}_{\backslash\infty}(\int)]$ and $v_{n_2}(t) = \mathcal{L}^{-\infty}[\mathcal{V}_{\backslash\in}(\int)]$.

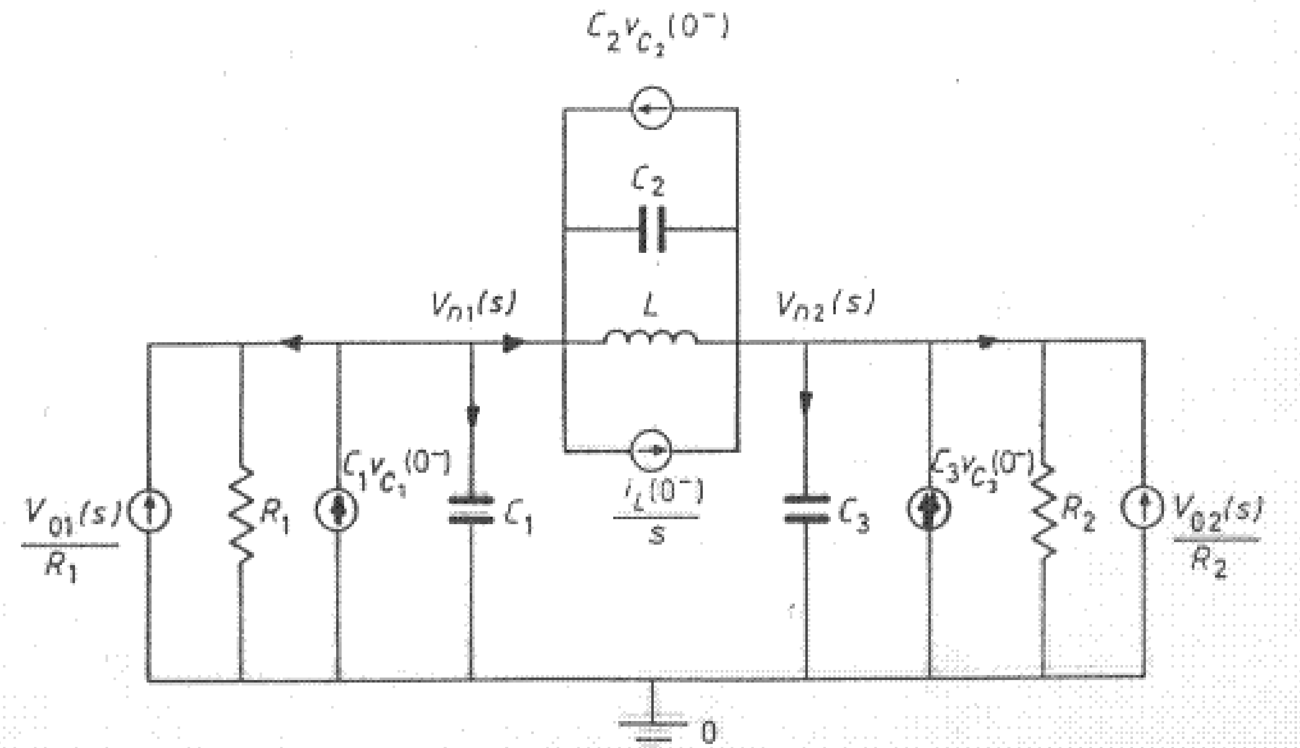


Figure 5.11: Transformed network with initial condition generators — Baher

Chapter 6

Analog Signal Processing Systems

Contents

- General system function of linear time-invariant analog systems;
- System stability
- Realization of system functions (analog);
- Analog filter design

6.1 The System Function and its Stability

A linear time-invariant system with excitatio $f(t)$ and response $g(t)$ can be represented by an ordinary linear differential equation with constant coefficients.

$$\sum_{i=0}^{m} a_i \frac{d^i}{dt^i}[f(t)] = \sum_{j=0}^{n} b_j \frac{d^j}{dt^j}[g(t)] \tag{6.1-1}$$

where a_i and b_j are constants and $\frac{d^0}{dt^0}[f(t)] = f(t)$ and $\frac{d^0}{dt^0}[g(t)] = g(t)$.

Assume the system is initially relaxed, i.e. all initial conditions are zero, and

$$\mathcal{L}[\{(\sqcup)] = F(s) \tag{6.1-2}$$

$$\mathcal{L}[\}(\sqcup)] = G(s) \tag{6.1-3}$$

apply the Laplace transform on both sides of Equation 6.1-1 to obtain

$$\sum_{i=0}^{m} a_i s^i F(s) = \sum_{j=0}^{n} b_j s^j G(s) \tag{6.1-4}$$

which is an algebraic equation.

It follows that

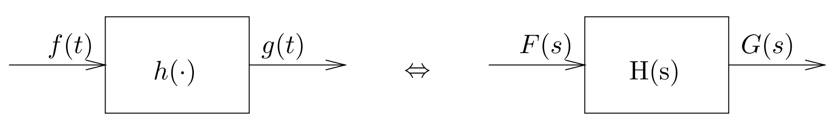
$$F(s) \sum_{i=0}^{m} a_i s^i = G(s) \sum_{j=0}^{n} b_j s^j \tag{6.1-5}$$

And the Laplace transform of the system response due to the excitation is

$$G(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{j=0}^{n} b_j s^j} F(s) \tag{6.1-6}$$

The system function $H(s)$ is

$$H(s) = \frac{G(s)}{F(s)} = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{j=0}^{n} b_j s^j} \tag{6.1-7}$$



It is known that the impulse response of the system

$$h(t) = \mathcal{L}^{-\infty}[\mathcal{H}(\int)] \tag{6.1-8}$$

is

$$h(t) = \mathcal{L}^{-\infty}[\mathcal{H}(\int) \cdot \infty] \tag{6.1-9}$$

noticing $\mathcal{L}[\delta(\sqcup)] = \infty$.

If $H(s)$ or $h(t)$ is known, the system response to any excitation can be determined. (The significance of the study of $H(s)$ or $h(t)$ therefore, becomes obvious.)

Example 6.1-1 *Given the system function*

$$H(s) = \frac{2}{(s+2)^3} \tag{6.1-10}$$

find the system response $g(t)$ to a unit step excitation $u(t)$.

Solution:

1.

$$f(t) = U(t) \tag{6.1-11}$$

$$F(s) = \mathcal{L}[\{(\sqcup)] = \mathcal{L}[\mathcal{U}(\sqcup)] = \frac{\infty}{\int} \tag{6.1-12}$$

2.

$$G(s) = H(s) \cdot F(s) \tag{6.1-13}$$

$$= \frac{2}{(s+2)^3} \cdot \frac{1}{s} \tag{6.1-14}$$

3. Partial fraction expansion

$$G(s) = \frac{A}{s} + \frac{B_0}{(s+2)^3} + \frac{B_1}{(s+2)^2} + \frac{B_2}{s+2} \tag{6.1-15}$$

$$A = \lim_{s\to 0} s \frac{2}{(s+2)^3} \cdot \frac{1}{s} = \lim_{s\to 0} \frac{2}{(s+2)^3} = \frac{1}{4} = 0.25 \tag{6.1-16}$$

$$\Phi(s) = (s+2)^3 G(s) = \frac{2}{s} \Rightarrow B_0 = \Phi(-2) = -1 \tag{6.1-17}$$

$$\Phi'(s) = -\frac{2}{s^2} \Rightarrow B_1 = \frac{1}{1!}\Phi'(-2) = -\frac{1}{2} = -0.5 \tag{6.1-18}$$

$$\Phi''(s) = \frac{4}{s^3} \Rightarrow B_2 = \frac{1}{2!}\Phi''(-2) = -\frac{1}{4} = -0/25 \tag{6.1-19}$$

Thus

$$g(t) = \mathcal{L}^{-\infty}[\mathcal{G}(\int)] \tag{6.1-20}$$

$$= 0.25U(t) - 0.25e^{-2t}U(t) - 0.5te^{-2t}U(t) - 0.5t^2e^{-2t}U(t) \tag{6.1-21}$$

$$= 0.25\left(1 - e^{-2t} - 2te^{-2t} - 2t^2e^{-2t}\right)U(t) \tag{6.1-22}$$

(Note:

$$\mathcal{L}[\sqcup^\backslash] = \frac{n!}{s^{n+1}} \tag{6.1-23}$$

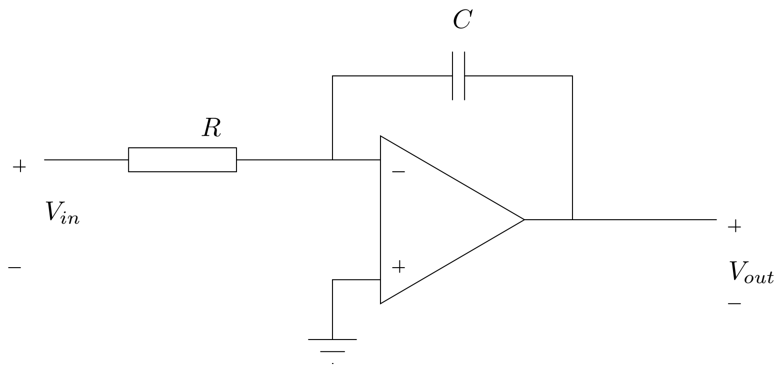
$$\mathcal{L}\left[\rceil^{\alpha\sqcup}\{(\sqcup)\right] = F(s-\alpha) \tag{6.1-24}$$

)

6.1.1 System Stability

Intuitively speaking, it is not desirable to have a system output growing unbounded or uncontrollable.

For example an integrator



$$H(s) = -\frac{K}{s} \qquad \text{where } K = \tfrac{1}{RC} \tag{6.1-25}$$

Input and output waveforms of an integrator are shown in Figure 6.1.

The pole of the integrator is at the origin of the s-plane. (We are on the edge of a stable/unstable system).

Notice that in the partial fraction expansion, all the poles of the system function have corresponding terms and will contribute to the system response (or output of the system) after the inverse Laplace transform. This is why the system stability can be defined or determined in the s-plane.

Definition 6.1-1 (Wide-Sense Stable and Strictly Stable) *A system is said to be wide-sense stable if all lthe poles of the system function $H(s)$ lie in the closed left-half plane ($\Re[s] \leqslant 0$) with those on the $j\omega$ axis being simple.*

The system is said to be strictly stable if the poles on the $j\omega$ axis are excluded, i.e. all the poles of $H(s)$ occur only in the open left-half plane ($\Re[s] < 0$).

Assume the system is causal and

$$h(t) = 0 \qquad \text{for } t < 0 \tag{6.1-26}$$

and

$$H(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{j=0}^{n} a_j s^j} = \frac{N_m(s)}{D_n(s)} \tag{6.1-27}$$

Strict stability requires that $H(s)$ has only poles with negative real parts excluding poles on the $j\omega$ axis and at infinity, i.e.

$$m \leqslant n \tag{6.1-28}$$

$$D(s) \neq 0 \qquad \text{for } \Re[s] \geqslant 0 \tag{6.1-29}$$

The impulse response is

$$h(t) = \mathcal{L}^{-\infty}[\mathcal{H}(\int)] = \mathcal{L}^{-\infty}\left[\underbrace{k_0}_{\Updownarrow=\backslash} + \underbrace{\sum_{r=1}^{n_1} \frac{k_r}{s - s_r}}_{\text{real poles}} + \underbrace{\sum_{r=1}^{n_2} \frac{A_r s + B_r}{(s - a_r)^2 + b_r^2}}_{\text{complex pole pairs}}\right] \tag{6.1-30}$$

where $n = n_1 + 2n_2$, $s_r < 0$, $a_r < 0$, $b_r > 0$. So that

$$h(t) = k_0 \delta(t) + \sum_{r=1}^{n_1} E_r e^{s_r t} + \sum_{r=1}^{n_2} C_r e^{a_r t} \sin(b_r t + \theta_r) \tag{6.1-31}$$

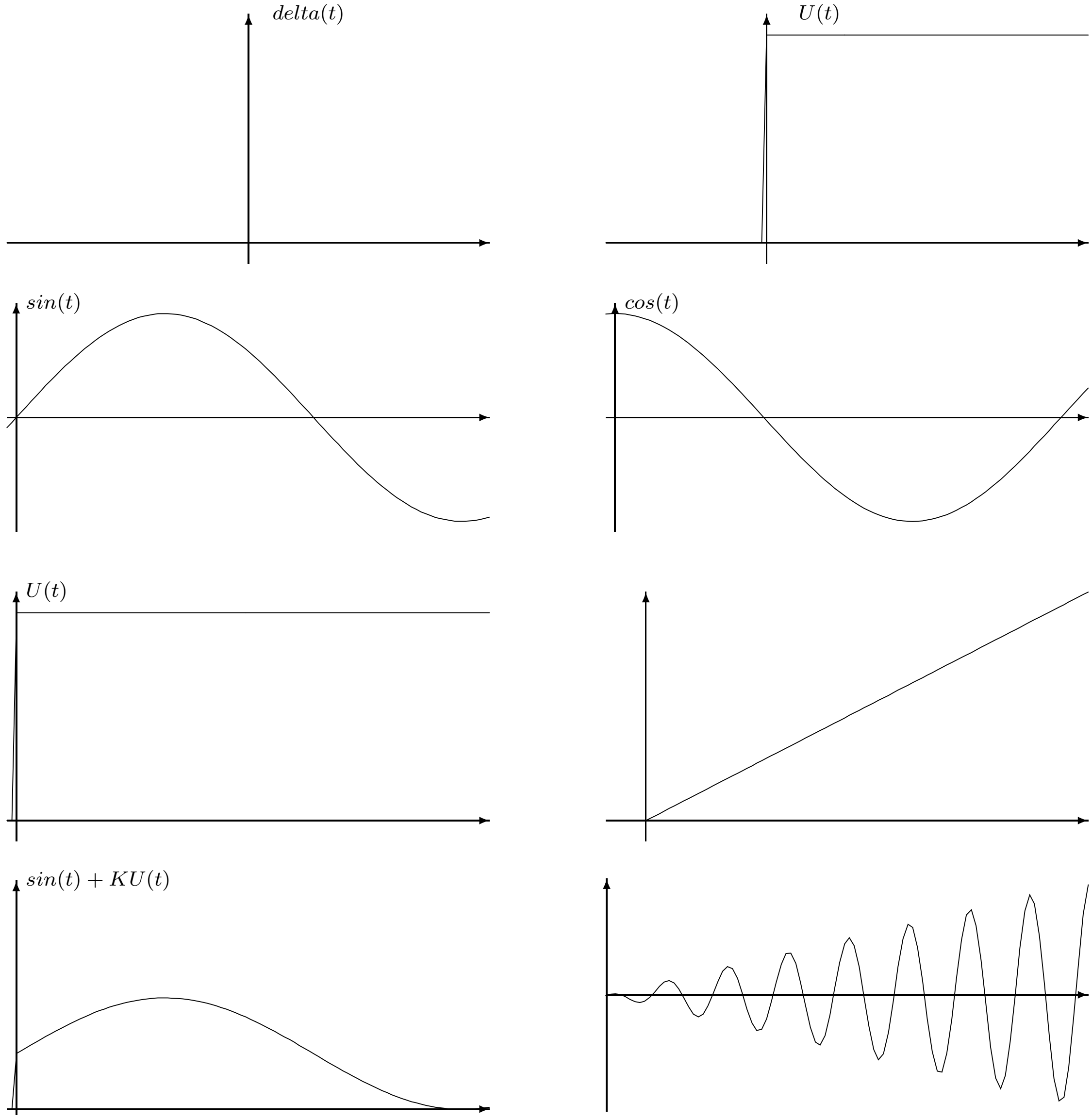


Figure 6.1: Stability of an integrator

From Equation 6.1-31, strict stability in the time domain means that apart from a possible impulse $k_0\delta(t)$, the impulse response of the system is a bounded function, i.e.

$$\int_0^\infty |h(t)|dt < \infty \tag{6.1-32}$$

The strict stability is also referred to as Bounded Input-Bouded Output (or BIBO). It can be proven that

$$\begin{aligned}
|g(t)| &= \left|\int_0^t h(\tau)f(t-\tau)d\tau\right| && (6.1\text{-}33)\\
&= \left|\int_0^\infty h(\tau)f(t-\tau)d\tau\right| && (6.1\text{-}34)\\
&\leqslant \int_0^\infty |h(\tau)||f(t-\tau)|d\tau && (6.1\text{-}35)\\
&\leqslant K\int_0^\infty |h(\tau)|d\tau && (6.1\text{-}36)
\end{aligned}$$

If the input is bounded

$$|f(t)| < K < \infty \qquad \text{for all } t \tag{6.1-37}$$

and the system is strictly stable, i.e.

$$\int_0^\infty |h(t)|dt < M < \infty \qquad \text{for all } t \tag{6.1-38}$$

$|g(t)| \leqslant KM$ and the output is bounded.

Wide-sense stability:

1.
$$\frac{1}{s} \leftrightarrow \text{unit step} \tag{6.1-39}$$

2.
$$\frac{1}{s^2+\omega_0^2} \leftrightarrow \sin(\omega_0 t) or \cos(\omega_0 t) \tag{6.1-40}$$

3.
$$s \leftrightarrow \delta(t) \tag{6.1-41}$$

Remark 6.1-1 *A system function $H(s)$ which has multiple poles on $j\omega$-axis is not stable. For example*

$$\mathcal{L}^{-\infty}\left[\frac{\infty}{(\int^{\in}+\omega_{/}^{\in})^{\in}}\right] = \frac{t}{\omega_0}\sin(\omega_0 t) \tag{6.1-42}$$

$$\mathcal{L}^{-\infty}\left[\frac{\infty}{\int^{\in}}\right] = t \tag{6.1-43}$$

If the system response possesses these terms in its partial fraction expansion, the output signal will not be a bounded one.

6.2 A Stability Test — Hurwitz Polynomials

A simple method using Hurwitz Polynomials can be used to determine whether a system is stable or not.

Definition 6.2-1 *A real polynomial $Q(s)$ is termed a Hurwitz polynomial if all its zeros lie in the closed left half-plane ($\Re[s] \leqslant 0$) with those on the imaginary axis being simple. $Q(s)$ is said to be strictly Hurwitz if $j\omega$-axis zeros are also excluded, i.e. all the zeros occur only in the left-half plane ($\Re[s] < 0$).*

Remark 6.2-1

1. *If a system is wide-sense stable, the denominator of its system function $H(s)$ must be a Hurwitz polynomial.*
2. *If a system is to be strictly stable (or BIBO stable), the denominator of $H(s)$ must be a strictly Hurwitz polynomial.*

Testing procedure for checking the Hurwitz character of a given polynomial:

1. Given

$$Q(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots a_1 s + a_0 \tag{6.2-44}$$

$$= \sum_{i=0}^{n} a_i s^i \tag{6.2-45}$$

2. Seperate the even and odd parts of $Q(s)$.

$$Q(s) = M(s) + N(s) \tag{6.2-46}$$

$$M(s) = a_n s^n + a_{n-2}s^{n-2} + \cdots \tag{6.2-47}$$

$$N(s) = a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + \cdots \tag{6.2-48}$$

where

- for n even, $M(s)$ is all even and $N(s)$ is all odd
- for n odd, $M(s)$ is odd, and $N(s)$ is even

3. Use successive division and inversion method to represent $\frac{M(s)}{N(s)}$ in the following form

$$\frac{M(s)}{N(s)} = \alpha_1 s + \cfrac{1}{\alpha_2 s + \cfrac{1}{\alpha_3 s + \cfrac{1}{\alpha_4 s + \cfrac{1}{\ddots \alpha_n s}}}} \tag{6.2-49}$$

4. The necessary and sufficient conditions for $Q(s) = M(s) + N(s)$ to be strictly Hurwitz is that all the n quotients in the continued fraction expansion (Equation 6.2-49) be strictly positive, i.e.

$$\alpha_i > 0 \qquad \text{for } i = 1, 2, \ldots, n \tag{6.2-50}$$

If all the quotients α_i after α_k ($k < n$) are zero, $M(s)$ and $N(s)$ have an even polynomial $E(s)$ which cannot be strictly Hurwitz.

To test whether or not $E(s)$ possesses only simple pure imaginary zeros, in which case $Q(s)$ is Hurwitz, we form a new polynomial

$$E(s) + E'(s) \tag{6.2-51}$$

where

$$E'(s) = \frac{d}{ds}E(s) \tag{6.2-52}$$

which is an odd polynomial.

If $E(s)+E'(s)$ is strictly Hurwitz, $E(s)$ possesses only simple pure imaginary zeros and $Q(s)$ is Hurwitz.

EXAMPLE 6.2-1 *Test the system stability of a system represented by the system function*

$$H(s) = \frac{s^3+5}{s^5+s^4+6s^3+4s^2+8s+3} \tag{6.2-53}$$

Solution:

1. *Use the Hurwitz polynomial method, the denominator of $H(s)$ is subject ot test.*

$$\begin{aligned} D(s) &= s^5+s^4+6s^3+4s^2+8s+3 && (6.2\text{-}54)\\ &= M(s)+N(s) && (6.2\text{-}55)\end{aligned}$$

where

$$M(s) = s^5+6s^3+8s \tag{6.2-56}$$

and

$$N(s) = s^4+4s^2+3 \tag{6.2-57}$$

2. *Use successive division and inversion*

$$\begin{array}{r|l} & s \\ \hline s^4+4s^2+3 & s^5+6s^3+8s \\ & s^5+4s^3+3s \\ \hline & 2s^3+5s \end{array}$$

$$\begin{array}{r|l} & \frac{1}{2}s \\ \hline 2s^3+5s & s^4+4s^2+3 \\ & s^4+\frac{5}{2}s^2 \\ \hline & \frac{3}{2}s^2+3 \end{array}$$

$$\begin{array}{r|l} & \frac{4}{3}s \\ \hline \frac{3}{2}s^2+3 & 2s^3+5s \\ & 2s^3+4s \\ \hline & s \end{array}$$

$$\begin{array}{r|l} & \frac{3}{2}s \\ \hline s & \frac{3}{2}s^2+3 \\ & \frac{3}{2}s^2 \\ \hline & 3 \end{array}$$

$$\begin{array}{r|l} & \frac{1}{3}s \\ \hline 3 & s \\ & s \\ \hline & 0 \end{array}$$

i.e.

$$\frac{M(s)}{N(s)} = s + \cfrac{1}{\frac{1}{2}s + \cfrac{1}{\frac{4}{3}s + \cfrac{1}{\frac{3}{2}s + \cfrac{1}{\frac{1}{3}s}}}} \tag{6.2-58}$$

3. *All five quotients are positive, therefore $D(s)$ is strictly Hurwitz, the system is BIBO stable.*

EXAMPLE 6.2-2 *Test the stability of a system that has a system function $H(s)$.*

$$H(s) = \frac{s^5 + 6s^2 + s + 1}{s^8 + 3s^7 + 10s^6 + 24s^5 + 35s^4 + 57s^3 + 50s^2 + 36s + 24} \tag{6.2-59}$$

Solution:

1.

$$D(s) = s^8 + 3s^7 + 10s^6 + 24s^5 + 35s^4 + 57s^3 + 50s^2 + 36s + 24 \tag{6.2-60}$$
$$M(s) = s^8 + 10s^6 + 35s^4 + 50s^2 + 24 \tag{6.2-61}$$
$$N(s) = 3s^7 + 24s^5 + 57s^3 + 36s \tag{6.2-62}$$

2.

$$\begin{array}{r|l} & \frac{1}{3}s \\ \hline 3s^7 + 24s^5 + 57s^3 + 36s & s^8 + 10s^6 + 35s^4 + 50s^2 + 24 \\ & s^8 + 8s^6 + 19s^4 + 12s^2 \\ \hline & 2s^6 + 16s^4 + 38s^2 + 24 \end{array}$$

$$\begin{array}{r|l} & \frac{3}{2}s \\ \hline 2s^6 + 16s^4 + 38s^2 + 24 & 3s^7 + 24s^5 + 57s^3 + 36s \\ & 3s^7 + 24s^5 + 57s^3 + 36s \\ \hline & 0 \end{array}$$

$D(s)$ is no strict Hurwitz, and $M(s)$ and $N(s)$ have an even polynomial $2s^6 + 16s^4 + 38s^2 + 24$ and $D(s)$ has an even factor

$$E(s) = s^6 + 8s^4 + 19s^2 + 12 \tag{6.2-63}$$

3.

$$E'(s) = 6s^5 + 32s^3 + 38s \tag{6.2-64}$$

4. *$\frac{E(s)}{E'(s)}$ has positive partial quotients $\frac{1}{6}s$, $\frac{9}{4}s$, $\frac{16}{21}s$, $\frac{49}{60}s$, $\frac{15}{7}s$ and $\frac{1}{10}s$. Therefore $E(s)+E'(s)$ is strict Hurwitz, $D(s)$ is Hurwitz and the system is wide-sense stable.*

6.3 Realization of System Functions

The process of finding the fundamental block-diagram representation for a given system function $H(s)$ is called the simulation of the system or the realization of the system function.

Find a realization for the second order transfer function:

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + 1} \tag{6.3-65}$$

$$H(s) = \underset{\displaystyle H_1(s)}{\underset{\Updownarrow}{(b_2 s^2 + b_1 s + b_0)\cdot}} \quad \underset{\displaystyle H_2(s)}{\underset{\Updownarrow}{\tfrac{a}{a_2 s^2 + a_1 s + 1}}} \tag{6.3-66}$$

Assume

$$H_1(s) = \frac{G_1(s)}{F_1(s)} \tag{6.3-67}$$

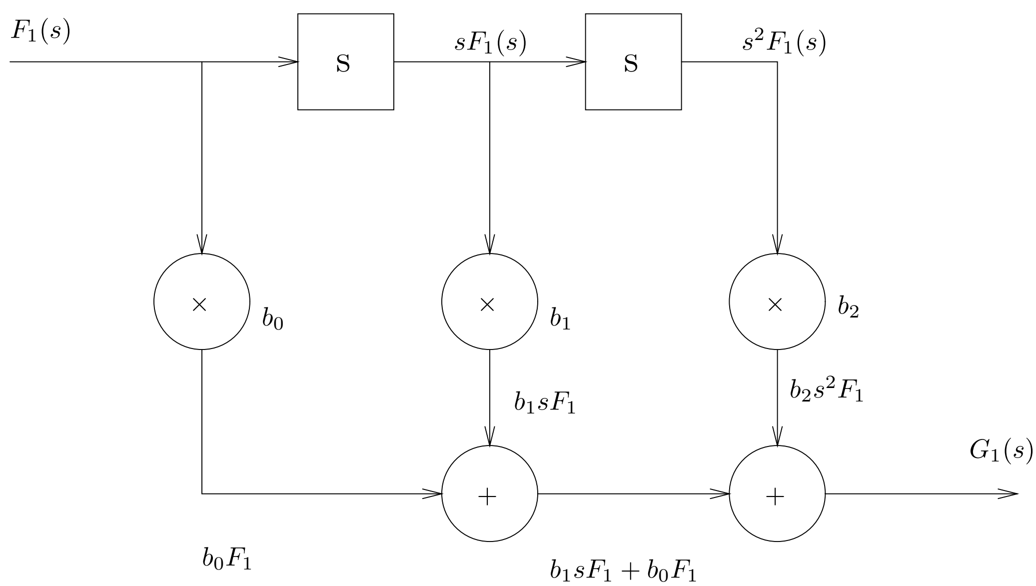
and

$$H_2(s) = \frac{G_2(s)}{F_2(s)} \tag{6.3-68}$$

$$G_1(s) = H_1(s)F_1(s) = (b_2 s^2 + b_1 s + b_0)F_1(s) \tag{6.3-69}$$

$$= b_2 s^2 F_1(s) + b_1 s F_1(s) + b_0 F_1(s) \tag{6.3-70}$$

A realization of $H_1(s)$ is

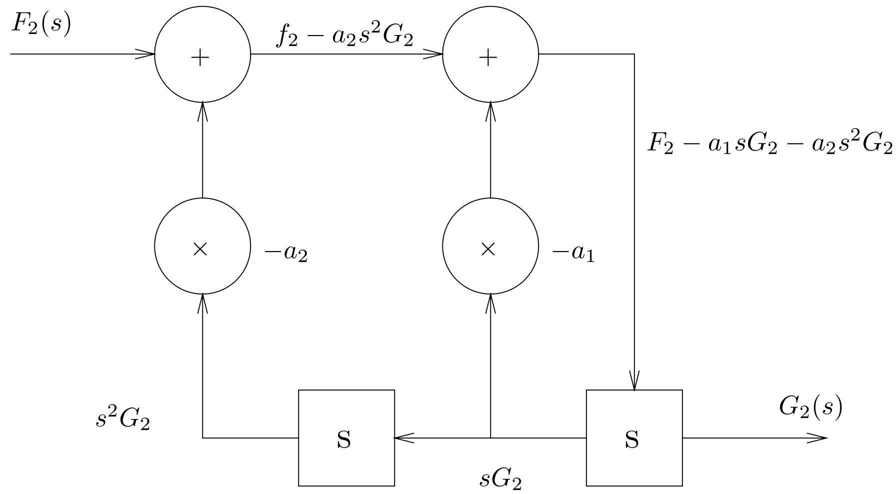


$$G_2(s) = H_2(s)F_2(s) = \frac{F_2(s)}{a_2 s^2 + a_1 s + 1} \tag{6.3-71}$$

$$G_2(s) + a_1 s G_2(s) + a_2 s^2 G_2(s) = F_2(s) \tag{6.3-72}$$

$$G_2(s) = F_2(s) - a_1 s G_2(s) - a_2 s^2 G_2(s) \tag{6.3-73}$$

A realization of $H_2(s)$ is



If

$$H_1(s)H_2(s) = \frac{G_1(s)}{F_1(s)} \cdot \frac{G_2(s)}{F_2(s)} \tag{6.3-74}$$

make

$$G_2(s) = F_1(s) \qquad F_2(s) = F(s) \qquad G_1(s) = G(s) \tag{6.3-75}$$

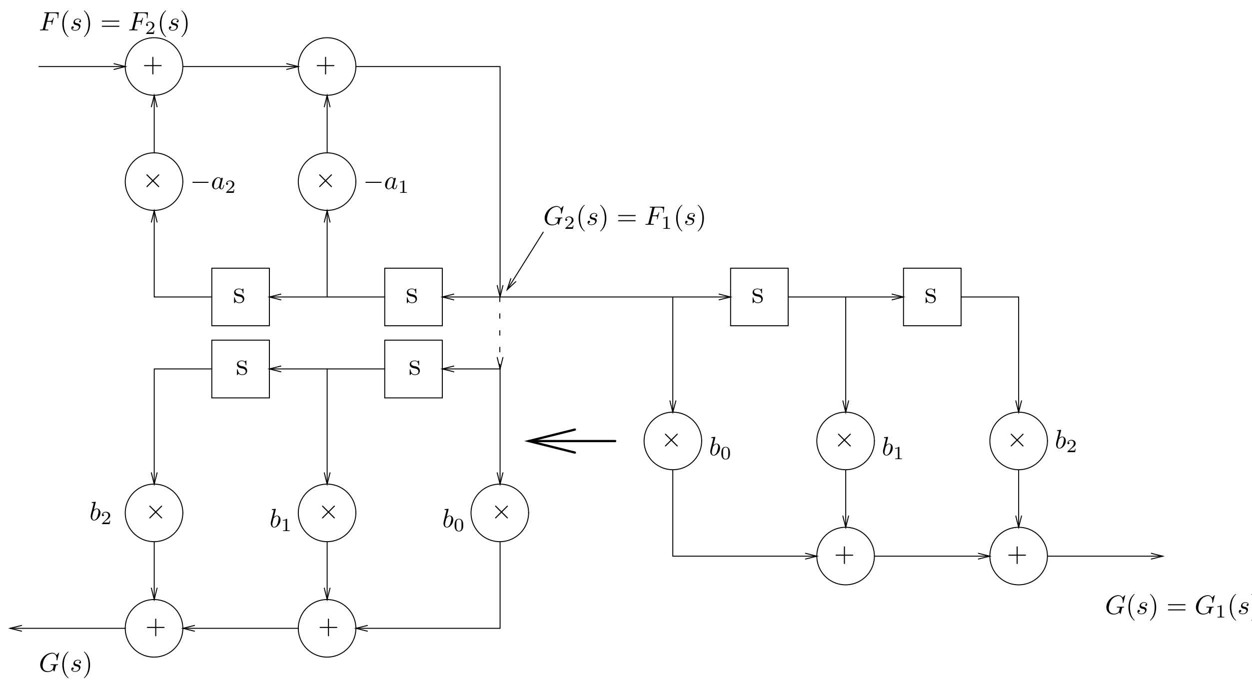
$$H(s) = H_1(s) \cdot H_2(s) = \frac{G(s)}{F(s)} \tag{6.3-76}$$

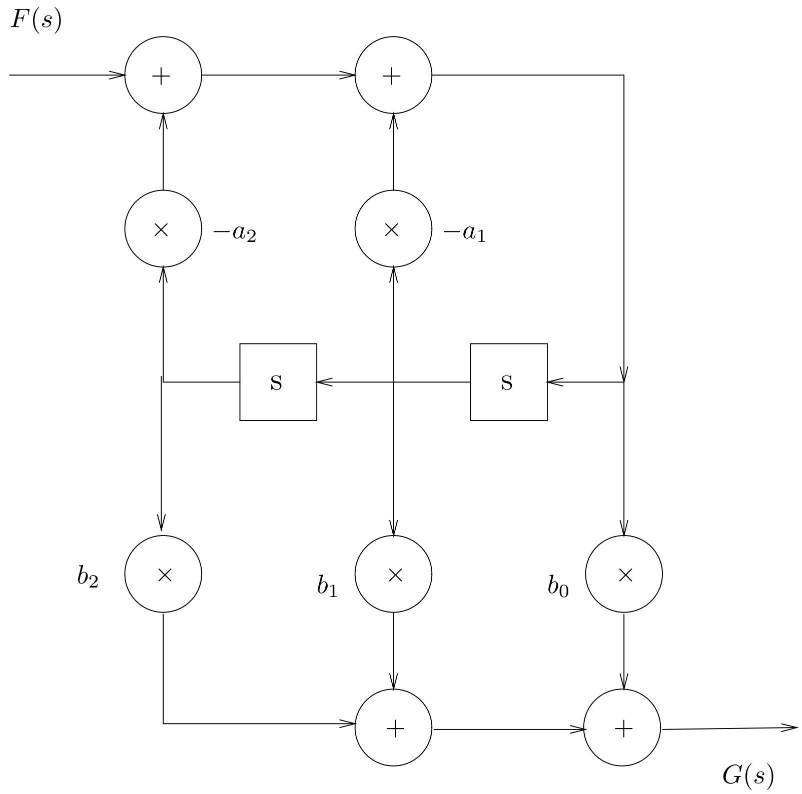
and

$$G(s) = (b_2 s^2 + b_1 s + b_0) G_2(s) \tag{6.3-77}$$

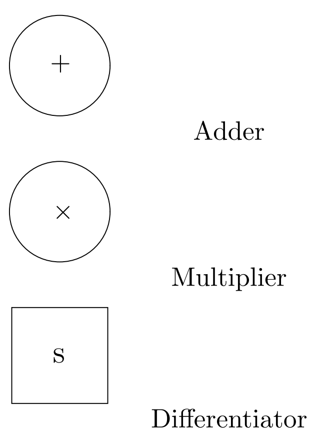
$$G_2(s) = F(s) - a_1 s G(s) - a_2 s^2 G(s) \tag{6.3-78}$$

A realization of $H(s)$ is

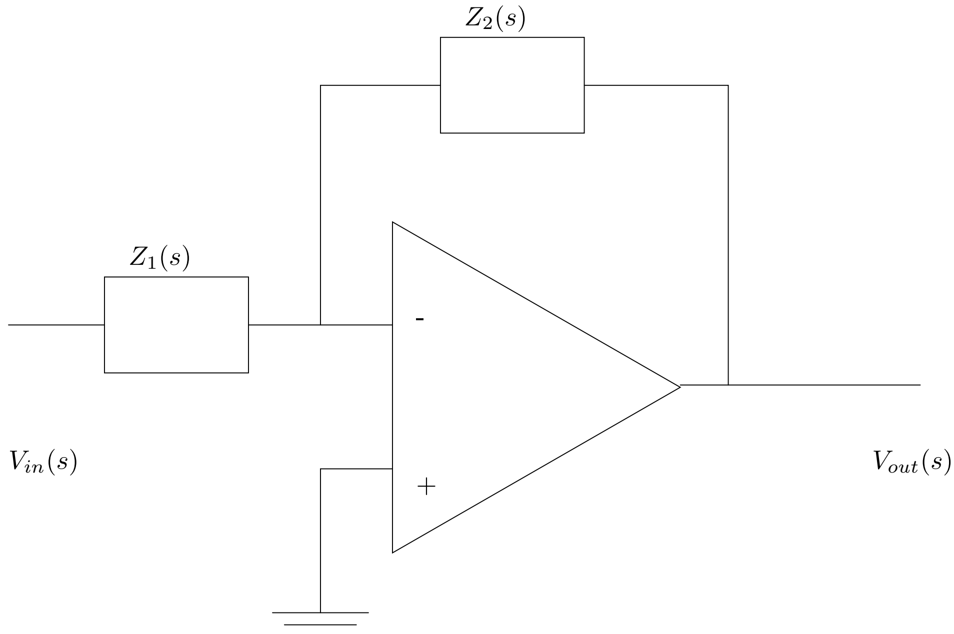




In the realization diagram for $H(s)$, all the elementary operations:



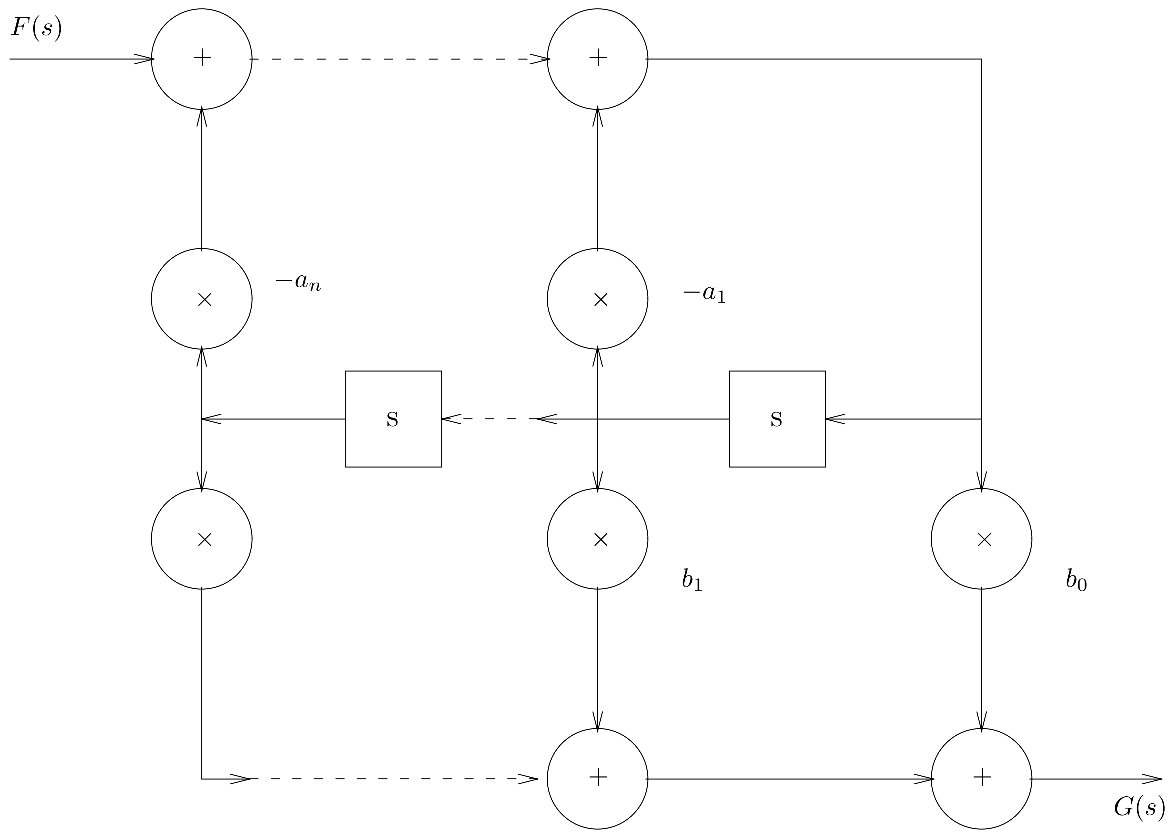
can be implemented using active circuits.



In general, a system transform function

$$H(s) = \frac{b_n s^n + b_{n-1} s^{n-q} + \cdots + b_2 s^2 + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + 1} \tag{6.3-79}$$

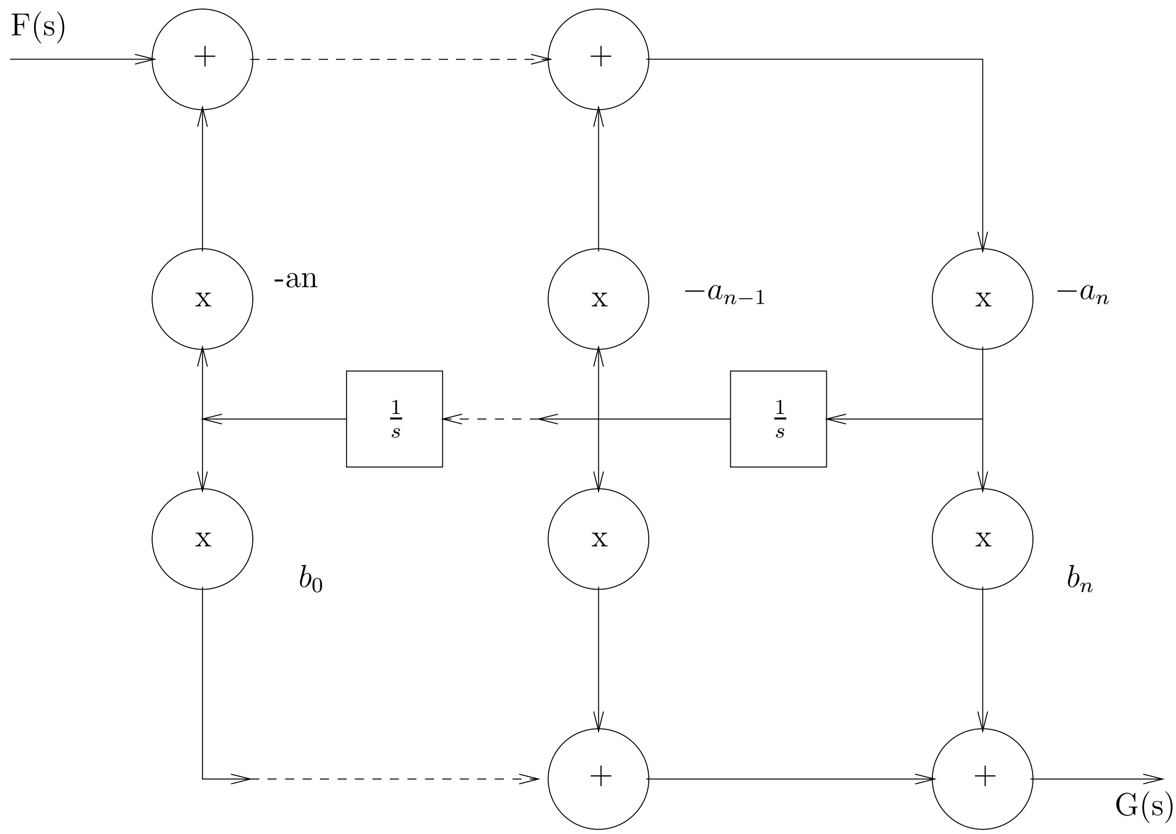
can be simulated or realized by the following diagram:



Since the system transfer function $H(s)$ may be also represented in the following form,

$$H(s) = \frac{b_n + b_{n-1}s^{n-1} + \cdots + b_0 s^{-n}}{a_n + a_{n-1}s^{-1} \cdots + s^{-n}} \tag{6.3-80}$$

another system realization is shown as follows.

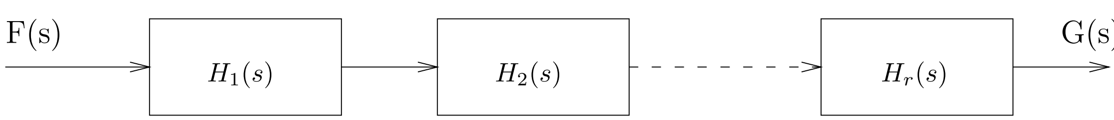


Each $\frac{1}{s}$ represents an integrator.

Since the realization and implementation of first and second order transfer functions are comparatively easy, the system transfer function $H(s)$ can be factorized as the product of second order factos with a possible first order factor for an odd degree function i.e.

$$H(s) = \prod_{k=1}^{r} H_k(s) \tag{6.3-81}$$

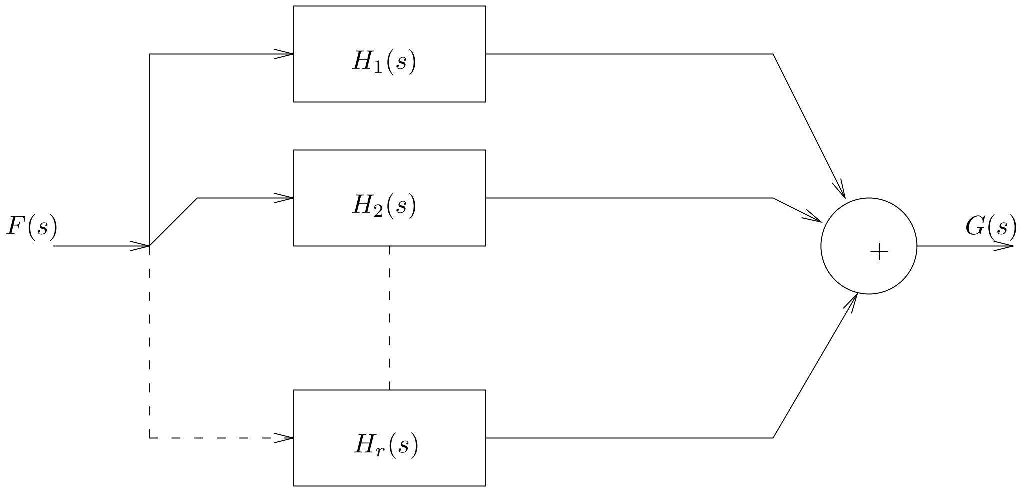
Therefore a realization of $H(s)$ is



If the partial fraction expansion is used for $H(s)$,

$$H(s) = \sum_{k=1}^{r} H_k(s) \tag{6.3-82}$$

Therefore, a parallel realization of $H(s)$ is



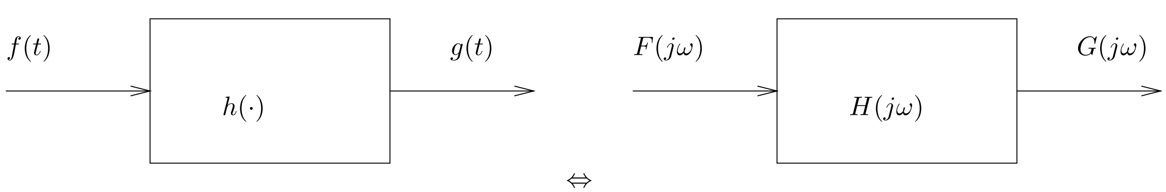
From the above realizations of system functions it can be seen that the first order and second order transfer functions or filters are the basic building blocks or sub-systems. This is probably the reason why the majority of filter design and implementation has been directed to first and second order filters.

6.4 Introduction to analog filter design

DEFINITION 6.4-1 (FILTER) *A filter is a signal processing system whose response differs from the excitation in a prescribed manner. It is a frequency selective system or network which accepts an input singal with a certain spectrum, and attenuates certain frequenct components while passing the rest of the spectrum without change.*

6.4.1 Ideal Filters and Types of Filters

Consider a system



The frequency response transfer function is given by

$$H(j\omega) = \frac{G(j\omega)}{F(j\omega)} = |H(j\omega)|e^{j\Psi(\omega)} \tag{6.4-83}$$

where $|H(j\omega)|$ is the amplitude response and $\Psi(\omega)$ is the phase response.

For example, consider a low-pass filter the specifications of which may be given as shown in Figure 6.2.

The tolerance scheme shown in Figure 6.2 is such that in the passband and stopband the repsonse lies within the shaded area.

Pass-band $0 < \omega < \omega_0$

Transition-band $\omega_0 < \omega < \omega_s$

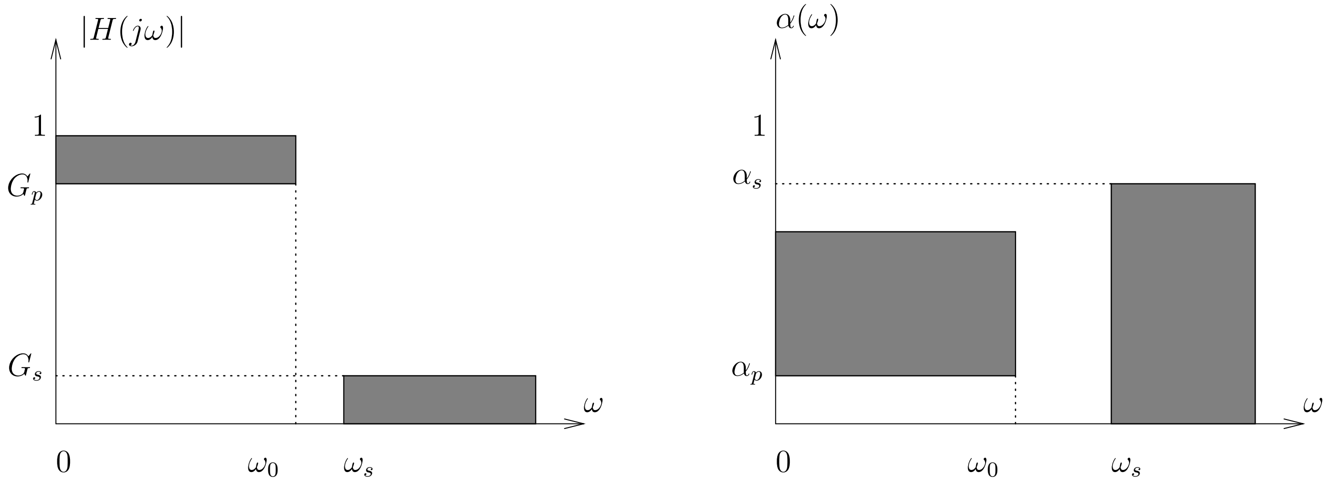


Figure 6.2: Specification for a low-pass filter

Stop-band $\omega > \omega_s$

It is desirable to make the transition band $\Delta\omega = \omega_s - \omega_0$ as small as possible in order to approximate the ideal low-pass filter.

As we discussed previously, if $H(s) = H_1(s)H_2(s)H_3(s)$ the frequency response $H(j\omega)$ of the filter may be obtained by superpositioning the frequency repsonses $H_1(j\omega)$, $H_2(j\omega)$ and $H_3(j\omega)$ as shown in Figure 6.3.

If the design specifications are given by the magnitude frequency response (as shown in Figure 6.4), to find the system transfer function becomes a cut and try excersise.

$$H(j\omega) = 100 \frac{\left(\frac{j\omega}{\omega_2} + 1\right)}{\left(\frac{j\omega}{\omega_1} + 1\right)\left(\frac{j\omega}{\omega_3} + 1\right)^2} \tag{6.4-84}$$

A similar design approach may also be used for filter design based on the phase response.

6.4.2 Amplitude-Oriented Design

Specifications of a filter required may be given by a magnitude function $|H(j\omega)|$ or a magnitude squared function $|H(j\omega)|^2$, or by a boss function

$$\alpha(\omega) = 20 \log \frac{1}{|H(j\omega)|} dB \tag{6.4-85}$$

as mentioned previously.

Definition 6.4-2 *The gain of the system defined by $H(j\omega)$ is given as*

$$A = 20 \log |H(j\omega)| dB \tag{6.4-86}$$

Definition 6.4-3 *The loss of a system described by $H(j\omega)$ is defined as*

$$\alpha = 20 \log \frac{1}{|H(j\omega)|} \tag{6.4-87}$$

$$= -20 \log |H(j\omega)| dB \tag{6.4-88}$$

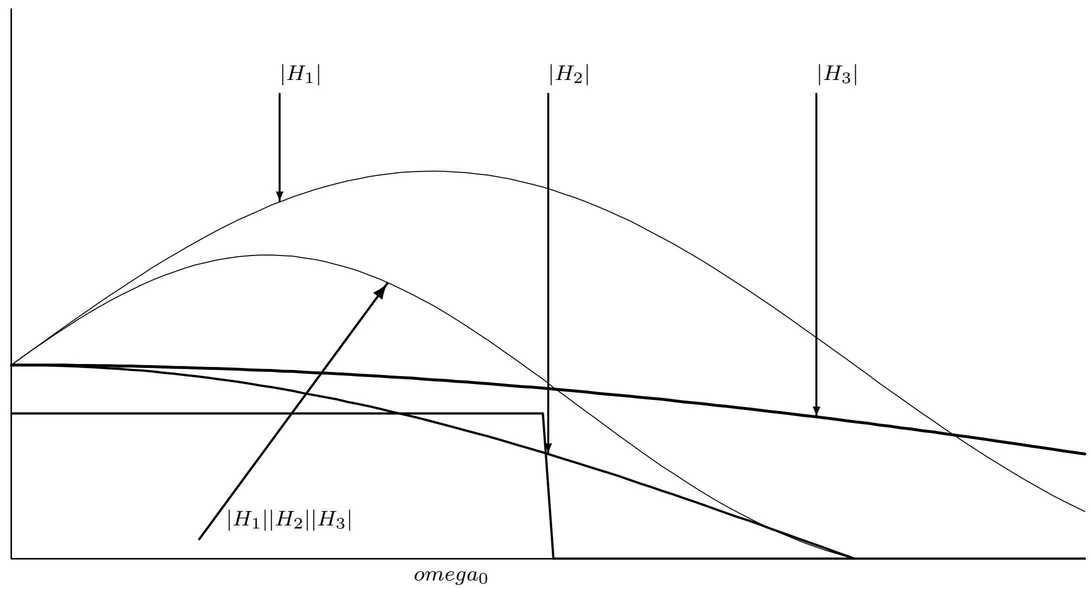


Figure 6.3: Superposition of frequency response

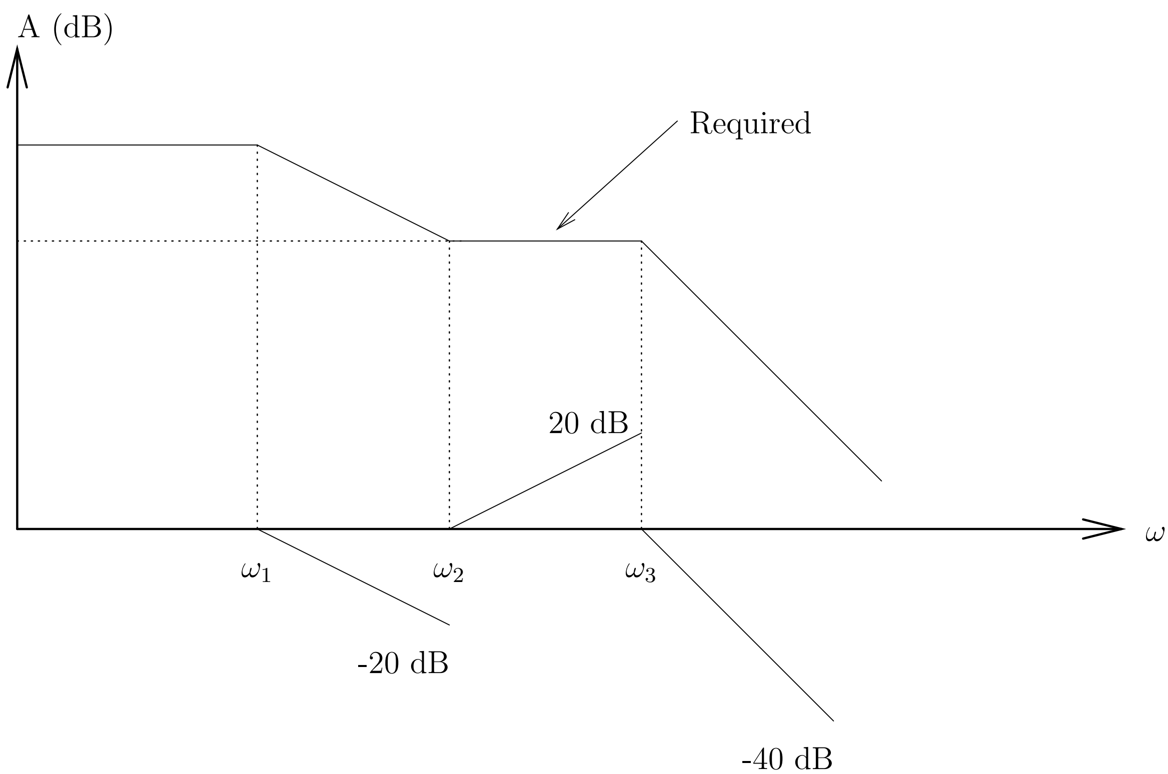


Figure 6.4: Filter Design

α is also called the attenuation function of the filter.

Filters are classified according to their amplitude response as low-pass, high-pass, band-pass and band-stop (or notch) filters.

Amplitude characterstics for the ideal filters are shown in Figure 6.5.

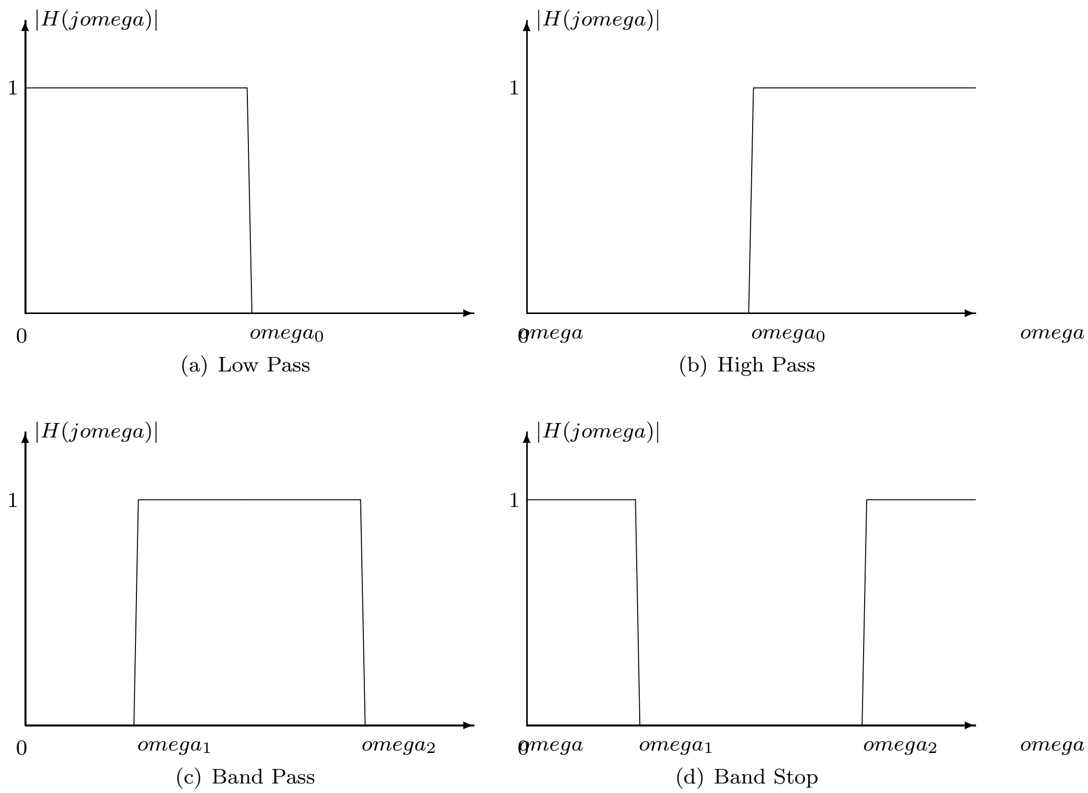


Figure 6.5: Ideal filter amplitude characteristics

Filters can also be characterized according to their phase response. For instance an all-pass filter is a phase-shifting filter, since its amplitude is unchanged while its phase may be varied or shifted by changing the frequency.

An ideal linear phase-shifting filter is given by

$$\Psi(\omega) = -\tau\omega \tag{6.4-89}$$

where τ is a constant.

The ideal low-pass filter's phase response is given in Figure 6.6. It is a linear phase-shifting filter.

DEFINITION 6.4-4 *The time delay $T(\omega)$ of a filter is defined to be the negative of the slope of the phase response, i.e.*

$$T(\omega) = \frac{d}{d\omega}\Psi(\omega) \tag{6.4-90}$$

REMARK 6.4-1 *The more non-linear the phase response, the more distorted will be the output signal.*

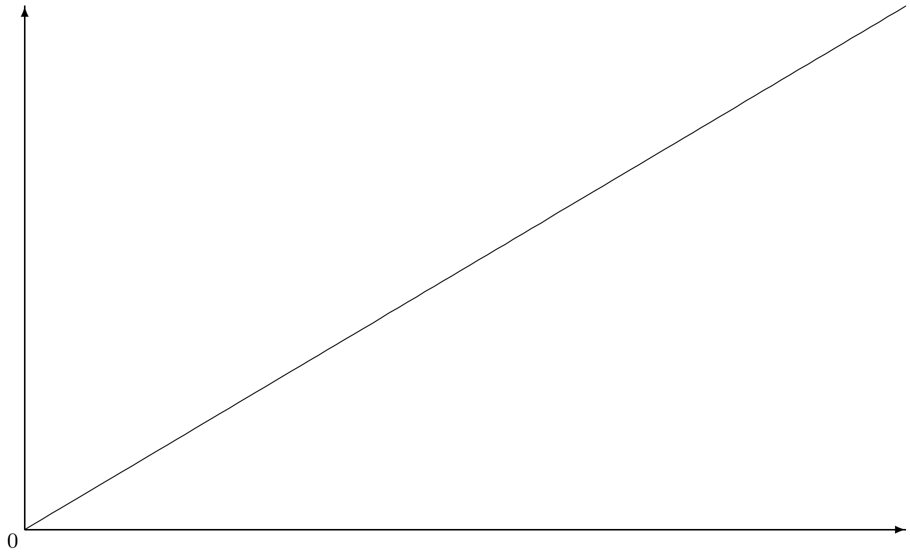


Figure 6.6: Ideal low-pass filter phase characteristic

Unfortunately, as the amplitude response improves (approaches the ideal case), the phase response deteriorates, and vice versa. Thus filter design involves a compromise between good amplitude response and good phase response. (Prac # 3).

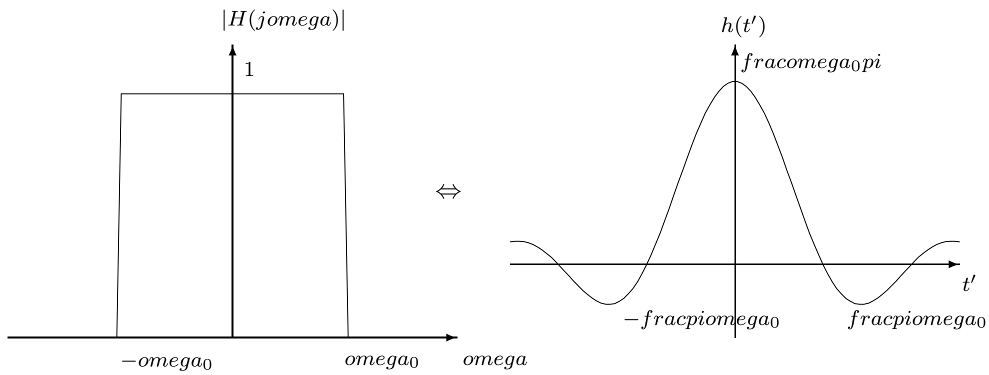
The ideal filter characteristics cannot be obtained (or implemeted) using realizable (causal) transfer functions and must, therefore, be approximated.

EXAMPLE 6.4-1 *Given an ideal low-pass filter defined as*

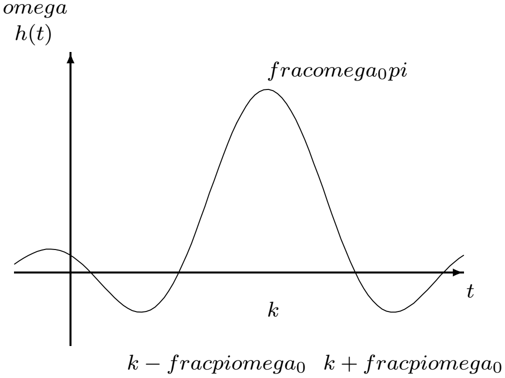
$$H(j\omega) = \begin{cases} e^{-jk\omega} & 0 \leqslant |\omega| \leqslant \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases} \tag{6.4-91}$$

the impulse response of the filter can be obtained by taking the inverse Fourier transform of $H(j\omega)$.

$$
\begin{aligned}
h(t) = \mathcal{F}^{-\infty}[\mathcal{H}(|\omega)] &= \frac{\infty}{\in\pi}\int_{-\infty}^{\infty}\mathcal{H}(|\omega)\rceil^{|\omega\sqcup}\lceil\omega && (6.4\text{-}92)\\
&= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{-jk\omega}e^{j\omega t}d\omega && (6.4\text{-}93)\\
&= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j(t-k)\omega}d\omega && (6.4\text{-}94)\\
&= \frac{1}{2\pi}\cdot\frac{1}{j(t-k)}e^{j(t-k)\omega}\Big|_{-\omega_0}^{\omega_0} && (6.4\text{-}95)\\
&= \frac{1}{2\pi j(t-k)}\left[e^{j(t-k)\omega_0} - e^{-j(t-k)\omega_0}\right] && (6.4\text{-}96)\\
&= \frac{1}{2\pi j(t-k)}\left\{2j\sin((t-k)\omega_0)\right\} && (6.4\text{-}97)\\
&= \frac{\sin((t-k)\omega_0)}{\pi(t-k)} \qquad \textit{for all } t && (6.4\text{-}98)\\
&= \frac{\omega_0}{\pi}\cdot\frac{\sin(\omega_0(t-k))}{\omega_0(t-k)} = \frac{\omega_0}{\pi}\frac{\sin\left(2\omega_0\frac{t'}{2}\right)}{2\omega_0\frac{t'}{2}} \qquad \textit{where } t' = t-k && (6.4\text{-}99)
\end{aligned}
$$



Because $h(t)$ is not causal, therefore, cannot be implemented using analog devices in practice.



6.5 Design Method using the Bode Diagrams

The asymptotic Bode diagrams:

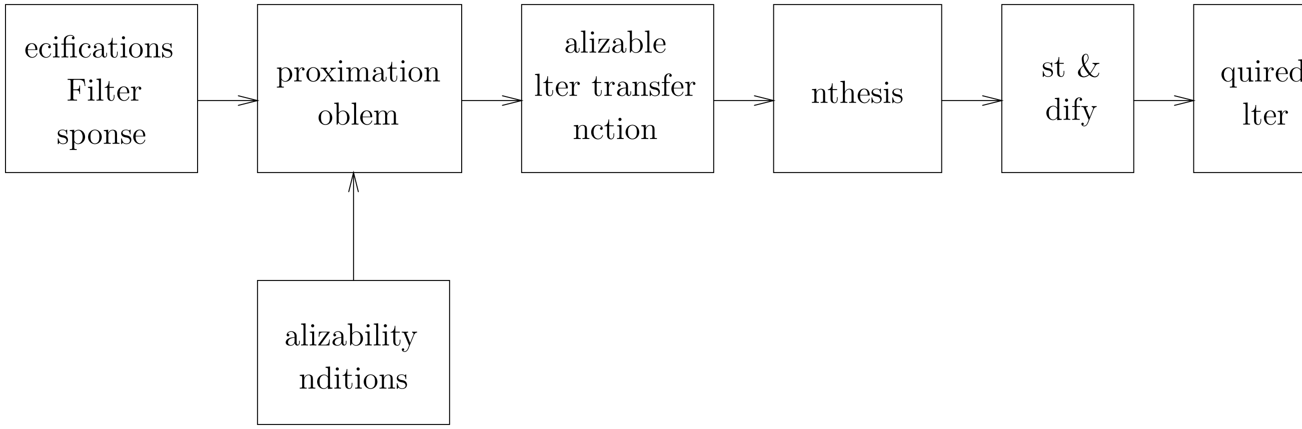


Figure 6.7: Basic design procedure of a filter

6.5.0.1 Simple first order (real) zero

See Figure 6.8.

$$H_1(j\omega) = \frac{j\omega}{\omega_0} + 1 = |H_1(j\omega)|e^{j\Psi_1(\omega)} \tag{6.5-100}$$

$$A_1 = 20\log|H_1(j\omega)| = 20\log\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \tag{6.5-101}$$

To draw the asymptotic Bode diagram for A_1:

(a).

$$\frac{\omega}{\omega_0} \ll 1, \qquad A_1 = 20\log\sqrt{1} = 0dB \tag{6.5-102}$$

(b).

$$\frac{\omega}{\omega_0} \gg 1, \qquad A_1 = 20\log\sqrt{\left(\frac{\omega}{\omega_0}\right)^2} = 20\log\frac{\omega}{\omega_0}dB \tag{6.5-103}$$

(c). Noticing ω_0 is the break frequency.

To draw the asymptotic Bode phase plot $\Psi_1(\omega) = \tan^{-1}\frac{\omega}{\omega_0}$

(a').

$$\omega = 0 \qquad \Psi_1(\omega) = 0 \tag{6.5-104}$$

$$\omega = \frac{\omega_0}{10} \qquad \Psi_1(\omega) = \tan^{-1} 0.1 = 5.7^\circ \approx 0^\circ \tag{6.5-105}$$

(b').

$$\omega = \omega_0 \qquad \Psi_1(\omega) = \tan^{-1} 1 = 45^\circ \tag{6.5-106}$$

(c').

$$\omega = \infty \qquad \Psi_1(\omega) = \tan^{-1}\infty = 90^\circ \tag{6.5-107}$$

$$\omega = 10\omega_0 \qquad \Psi_1(\omega) = \tan^{-1} 10 = 84.3^\circ \approx 90^\circ \tag{6.5-108}$$

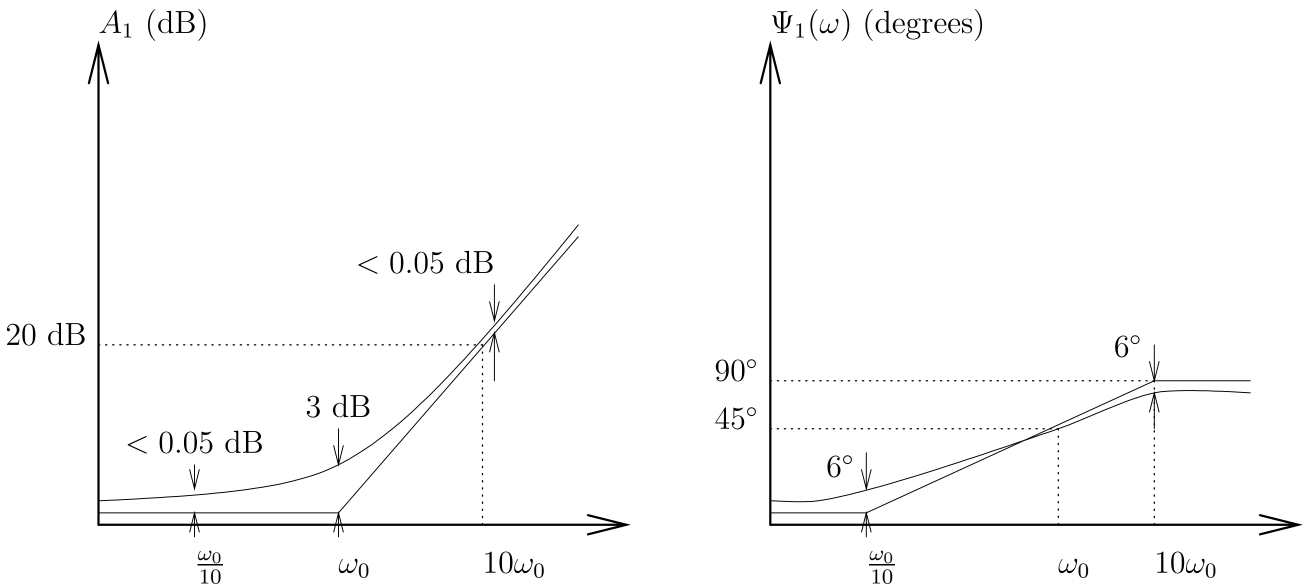


Figure 6.8: Simple first order zero

6.5.0.2 Simple first order (real) poles

See Figure 6.9.

$$H_2(j\omega) = \frac{1}{j\frac{\omega}{\omega_0} + 1} = \frac{1}{|H_2(j\omega)|e^{j\Psi_2(\omega)}} = |H_2(j\omega)|e^{j\Psi_2(\omega)} \tag{6.5-109}$$

$$A_2 = 20\log|H_2(j\omega)| = 20\log\left|\frac{1}{j\frac{\omega}{\omega_0} + 1}\right| = -20\log\sqrt{\left(\frac{\omega}{\omega_0}\right)^2 + 1} \tag{6.5-110}$$

To draw the asymptotic Bode diagram for A_2:

(a).

$$\frac{\omega}{\omega_0} \ll 1, \qquad A_2 = -20\log\sqrt{1} = 0dB \tag{6.5-111}$$

$$\frac{\omega}{\omega_0} = 0.1, \qquad A_2 = -20\log\sqrt{1.1} = 0.8 \approx 0dB \tag{6.5-112}$$

(b).

$$\frac{\omega}{\omega_0} \gg 1, \qquad A_2 \approx -20\log\sqrt{\left(\frac{\omega}{\omega_0}\right)^2} = 20\log\frac{\omega}{\omega_0} \tag{6.5-113}$$

$$\frac{\omega}{\omega_0} = 10, \qquad A_2 \approx -20\log 10 = -20dB \tag{6.5-114}$$

(c). Noticing ω_0 is the corner frequency.

To draw the asymptotic Bode phase plot Ψ_2

$$\Psi_2(\omega) = -\Psi_2(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right) \tag{6.5-115}$$

(a').

$$\omega = 0 \qquad \Psi_2(\omega) = -\tan^{-1} 0 = 0^\circ \tag{6.5-116}$$

$$\omega = \frac{\omega_0}{10} \qquad \Psi_2(\omega) = -\tan^{-1} 0.1 = -5.7^\circ \approx 0^\circ \tag{6.5-117}$$

(b').

$$\omega = \omega_0 \qquad \Psi_2(\omega) = -\tan^{-1} 1 = -45^\circ \tag{6.5-118}$$

(c').

$$\omega = \infty \qquad \Psi_2(\omega) = -\tan^{-1} \infty = -90^\circ \tag{6.5-119}$$

$$\omega = 10\omega_0 \qquad \Psi_2(\omega) = -\tan^{-1} 10 = -84.3^\circ \approx -90^\circ \tag{6.5-120}$$

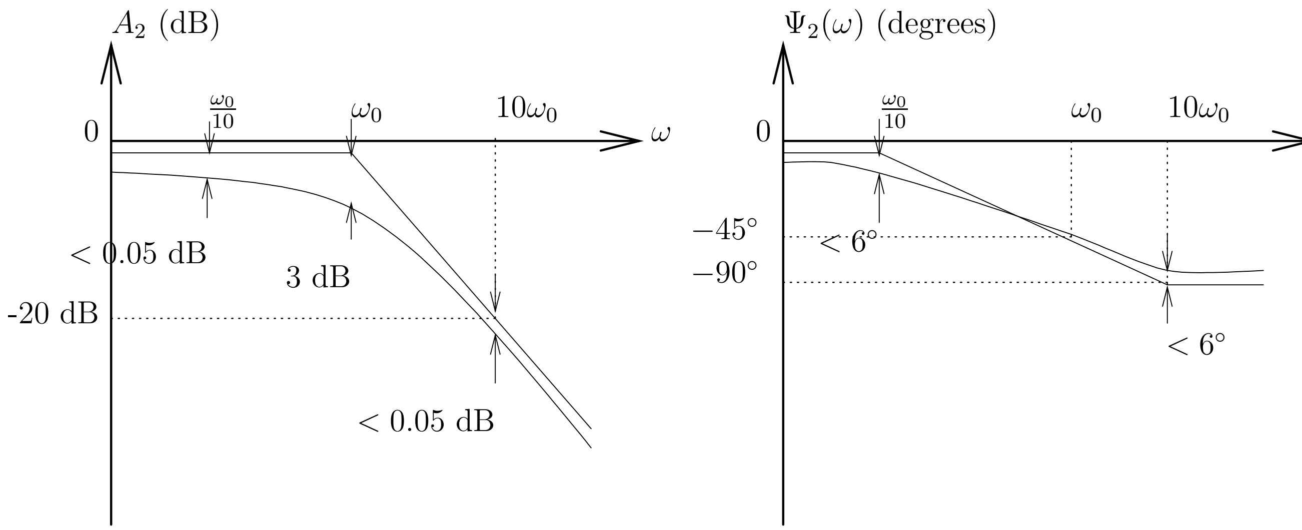


Figure 6.9: Simple first order pole

6.5.0.3 Simple second order zero

See Figure 6.10.

$$H_3(s) = H_3(0)\left[\left(\frac{s}{\omega_0}\right)^2 + \frac{\alpha}{\omega_0}s + 1\right] \tag{6.5-121}$$

$$H_3(j\omega) = H_3(0)\left[\left(\frac{j\omega}{\omega_0}\right)^2 + \frac{\alpha}{\omega_0}(j\omega) + 1\right] = H_3(0)\left[\alpha j\frac{\omega}{\omega_0} + 1 - \left(\frac{\omega}{\omega_0}\right)^2\right] \tag{6.5-122}$$

Assume $H_3(0) = 1$

$$A_3 = 20\log|H_3(j\omega)| = 20\log\sqrt{\left\{1 - \left(\frac{\omega}{\omega_0}\right)^2\right\}^2 + \left(\frac{\alpha}{\omega_0}\omega\right)^2} \tag{6.5-123}$$

To draw the asymptotic Bode diagram for A_3:

(a).

$$\frac{\omega}{\omega_0} \ll 1, \qquad A_3 \approx 20\log\sqrt{1} = 0dB \tag{6.5-124}$$

(b).

$$\frac{\omega}{\omega_0} \gg 1, \qquad A_3 \approx 20\log\sqrt{\left(\frac{\omega}{\omega_0}\right)^4} = 20\log\left(\frac{\omega}{\omega_0}\right)^2 \tag{6.5-125}$$

$$= 40\log\frac{\omega}{\omega_0}dB \tag{6.5-126}$$

(c). Noticing ω_0 is the corner frequency.

To draw the asymptotic Bode phase diagram for Ψ_3

$$\Psi_3(\omega) = \tan^{-1}\frac{\alpha\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2} \tag{6.5-127}$$

(a').

$$\omega = 0, \qquad \Psi_3 = \tan^{-1}0 = 0^\circ$$

(b').

$$\omega = \omega_0, \qquad \Psi_3 = \tan^{-1}\frac{\alpha}{0^+} = 90^\circ \tag{6.5-128}$$

(c').

$$\omega \to \infty, \qquad \Psi_3 = \tan^{-1}\frac{\alpha\frac{\infty}{\omega_0}}{1-\left(\frac{\infty}{\omega_0}\right)^2} = 180^\circ \tag{6.5-129}$$

6.5.0.4 Second Order Pole

$$H_4(s) = \frac{H(0)}{\left(\frac{s}{\omega_0}\right)^2 + \frac{\alpha}{\omega_0}j\omega + 1} \tag{6.5-130}$$

$$H_4(j\omega) = \frac{H(0)}{\left(\frac{j\omega}{\omega_0}\right)^2 + \frac{\alpha}{\omega_0}j\omega + 1} \tag{6.5-131}$$

$$= |H_4(j\omega)|e^{j\Psi_4(\omega)} \tag{6.5-132}$$

Use $A_4 = -A_3$ and $\Psi_4(\omega) = -\Psi_3(\omega)$. The asymptotic Bode diagrams are shown in Figure 6.11.

6.5.0.5 General Transfer Function $H(s)$

$$H(s) = \frac{\left(\frac{s}{\omega_1}+1\right)\left(\frac{s}{\omega_3}+1\right)\left(\left(\frac{s}{\omega_5}\right)^2 + \frac{\alpha_5}{\omega_5}s + 1\right)}{\left(\frac{s}{\omega_2}+1\right)\left(\frac{s}{\omega_4}+1\right)\left(\left(\frac{s}{\omega_6}\right)^2 + \frac{\alpha_6}{\omega_6}s + 1\right)} \tag{6.5-133}$$

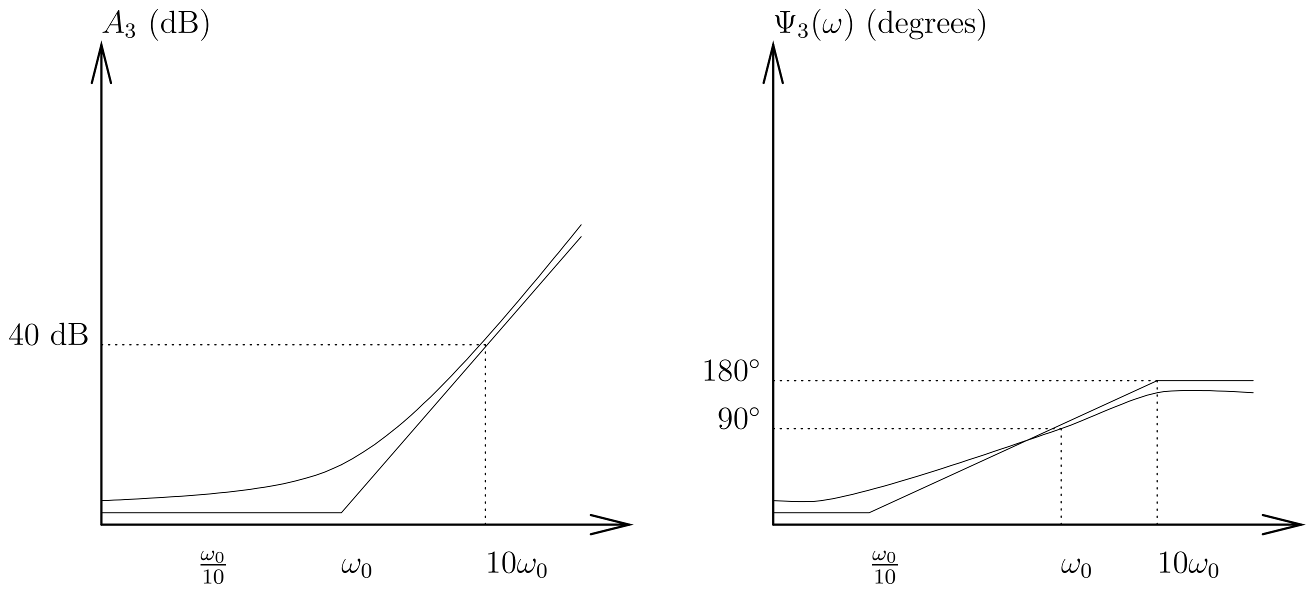


Figure 6.10: Simple second order zero

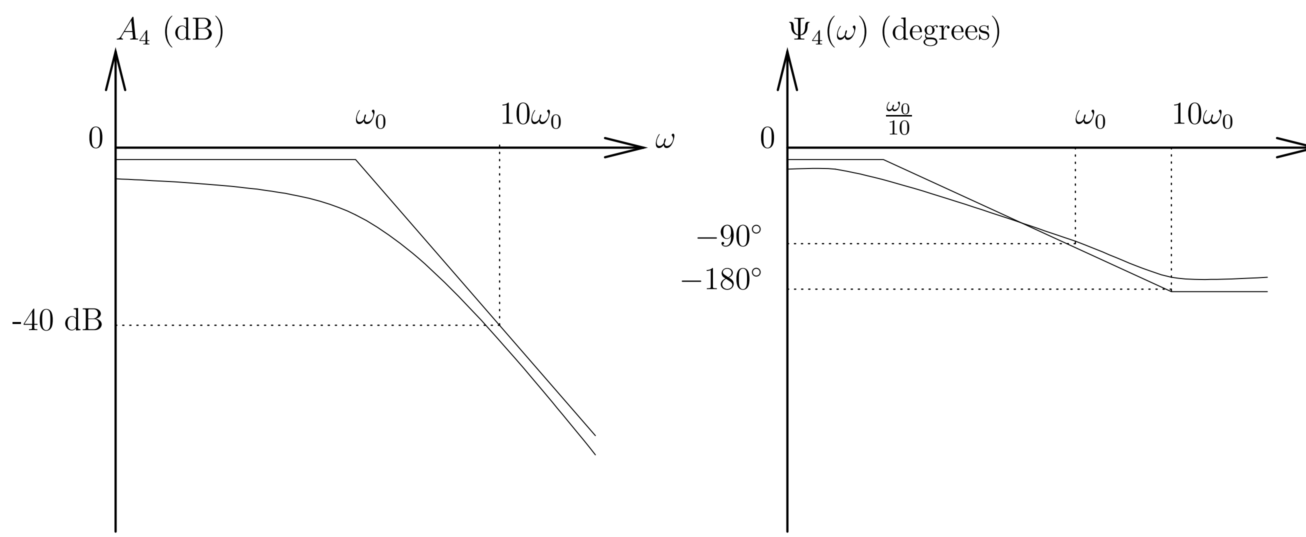


Figure 6.11: Second order pole

Define

$$H_1(s) = \frac{s}{\omega_1} + 1 \tag{6.5-134}$$
$$H_2(s) = \frac{s}{\omega_2} + 1 \tag{6.5-135}$$
$$H_3(s) = \frac{s}{\omega_3} + 1 \tag{6.5-136}$$
$$H_4(s) = \frac{s}{\omega_4} + 1 \tag{6.5-137}$$
$$H_5(s) = \left[\left(\frac{s}{\omega_5}\right)^2 + \frac{\alpha_5}{\omega_5}s + 1\right] \tag{6.5-138}$$
$$H_6(s) = \left[\left(\frac{s}{\omega_6}\right)^2 + \frac{\alpha_6}{\omega_6}s + 1\right] \tag{6.5-139}$$

Thus

$$H(s) = \frac{H_1(s)H_3(s)H_5(s)}{H_2(s)H_4(s)H_6(s)} \tag{6.5-140}$$
$$H(j\omega) = |H(j\omega)|e^{j\Psi(\omega)} \tag{6.5-141}$$

and

$$H_i(j\omega) = |H_i(j\omega)|e^{j\Psi_i(\omega)} \tag{6.5-142}$$

Therefore

$$H(j\omega) = |H(j\omega)|e^{j\Psi(\omega)} \tag{6.5-143}$$
$$= \frac{|H_1(j\omega)|e^{j\Psi_1(\omega)}||H_3(j\omega)|e^{j\Psi_3(\omega)}||H_5(j\omega)|e^{j\Psi_5(\omega)}|}{|H_2(j\omega)|e^{j\Psi_2(\omega)}||H_4(j\omega)|e^{j\Psi_4(\omega)}||H_6(j\omega)|e^{j\Psi_6(\omega)}|} \tag{6.5-144}$$
$$= \frac{|H_1||H_3||H_5|}{|H_2||H_4||H_6|}e^{j\{\Psi_1+\Psi_2+\Psi_3+\Psi_4+\Psi_5+\Psi_6\}} \tag{6.5-145}$$

i.e.

$$|H(j\omega)| = \frac{|H_1(j\omega)||H_3(j\omega)||H_5(j\omega)|}{|H_2(j\omega)||H_4(j\omega)||H_6(j\omega)|} \tag{6.5-146}$$

and

$$\Psi(\omega) = \overbrace{\Psi_1(\omega) + \Psi_3(\omega) + \Psi_5(\omega)}^{\text{zeros}} - \overbrace{\Psi_2(\omega) - \Psi_4(\omega) - \Psi_6(\omega)}^{\text{poles}} \tag{6.5-147}$$
$$A = 20\log|H(j\omega)| = 20\log\frac{|H_1||H_3||H_5|}{|H_2||H_4||H_6|} \tag{6.5-148}$$
$$= 20\log|H_1| + 20\log|H_3| + 20\log|H_5| - 20\log|H_2| - 20\log|H_4| - 20\log|H_6| \tag{6.5-149}$$

i.e.

$$A = \underbrace{A_1 + A_3 + A_5}_{\text{zeros}} - \underbrace{A_2 - A_4 - A_6}_{\text{poles}} \tag{6.5-150}$$

Assume $\omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5 < \omega_6$.

The frequency response $H_1(j\omega)$,$H_2(j\omega)$, and $H_3(j\omega)$ may be implemented using second order low-pass filters by carefully choosing Q (Quality Factor).

6.6 Quality Factor and its importance

For a given RLC circuit shown in Figure 6.12, the transfer function can be obtained as

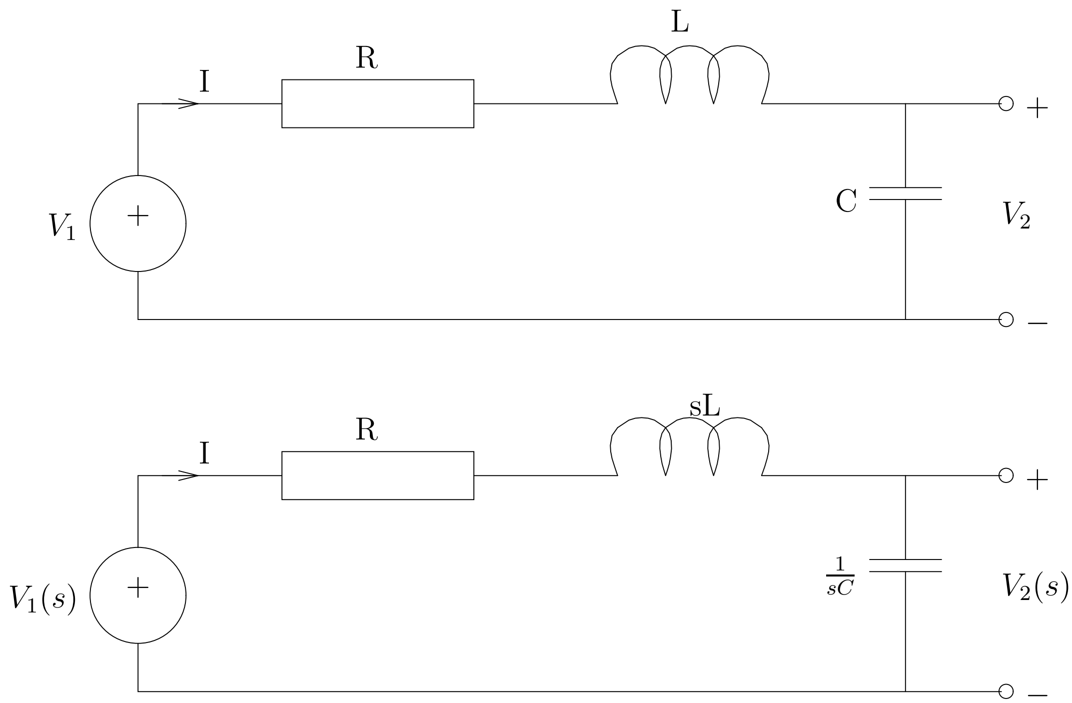


Figure 6.12: RLC circuit

$$H(s) = \frac{\frac{1}{sC}}{Ls + R + \frac{1}{sC}} \tag{6.6-151}$$

$$= \frac{\frac{1}{LC}}{s^2 + \left(\frac{R}{L}\right) s + \frac{1}{LC}} \tag{6.6-152}$$

If $R = 0$ (the circuit is lossless), the pole positions may be determined:

$$s^2 + \frac{1}{LC} = 0 \tag{6.6-153}$$

or

$$s_1, s_2 = \pm j\sqrt{\frac{1}{LC}} = \pm j\omega_0 \tag{6.6-154}$$

where $\omega_0 = \sqrt{\frac{1}{LC}}$, the natural frequency.

If $R \neq 0$ (lossy coils), the Quality factor Q is defines, in this case, as

$$Q = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{6.6-155}$$

which is the ratio of reactance at the frequency ω_0 to resistance.

The transfer function $H(s)$ now may be represented

$$H(s) = \frac{\omega_0^2}{s^2 + \left(\frac{\omega_0}{Q}\right) s + \omega_0^2} = \frac{N(s)}{D(s)} \tag{6.6-156}$$

The pole posistions in the s-plane are

$$s_1, s_2 = -a \pm jb \tag{6.6-157}$$

Thus

$$\begin{aligned} D(s) &= (s + a + jb)(s + a - jb) && (6.6\text{-}158)\\ &= s^2 + 2as + (a^2 + b^2). && (6.6\text{-}159)\end{aligned}$$

The relationship between a, b and ω_0,Q is

$$\begin{cases} a = \frac{\omega_0}{2Q} \\ b = \omega_0\sqrt{1 - \frac{1}{4Q^2}} \end{cases} \tag{6.6-160}$$

or

$$\begin{cases} Q = \frac{\omega_0}{2a} \\ \omega_0^2 = a^2 + b^2 \end{cases} \tag{6.6-161}$$

These relationships are shown in the s-plane in Figure 6.13. The angle Ψ with respect to the negative real axis is

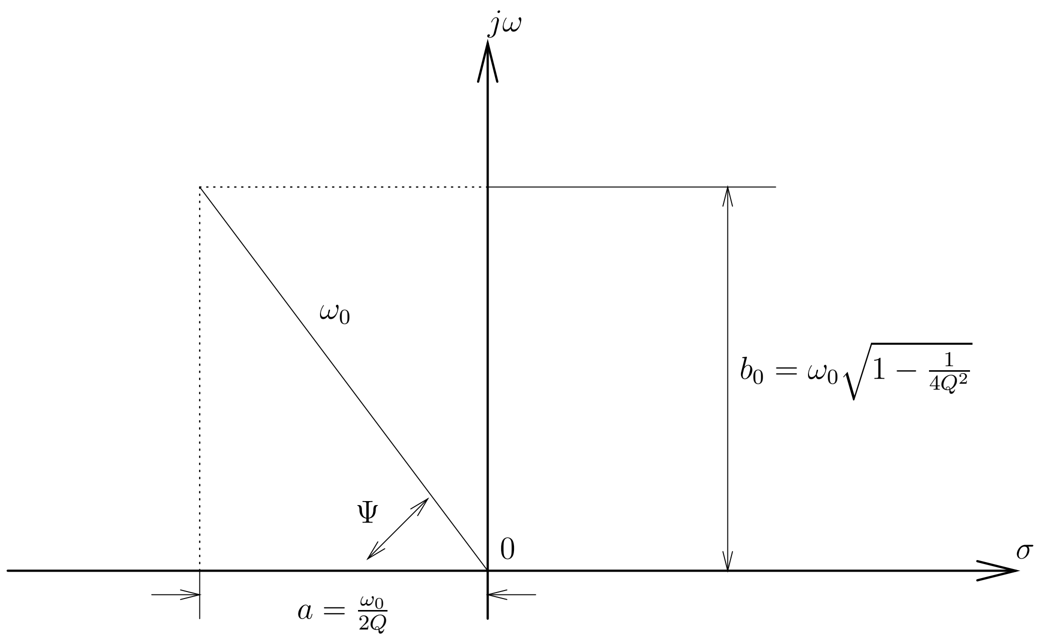


Figure 6.13: Quality factor relationships

$$\Psi = \cos^{-1}\left(\frac{Q}{\omega_0}\right) = \cos^{-1}\left(\frac{1}{2Q}\right) \tag{6.6-162}$$

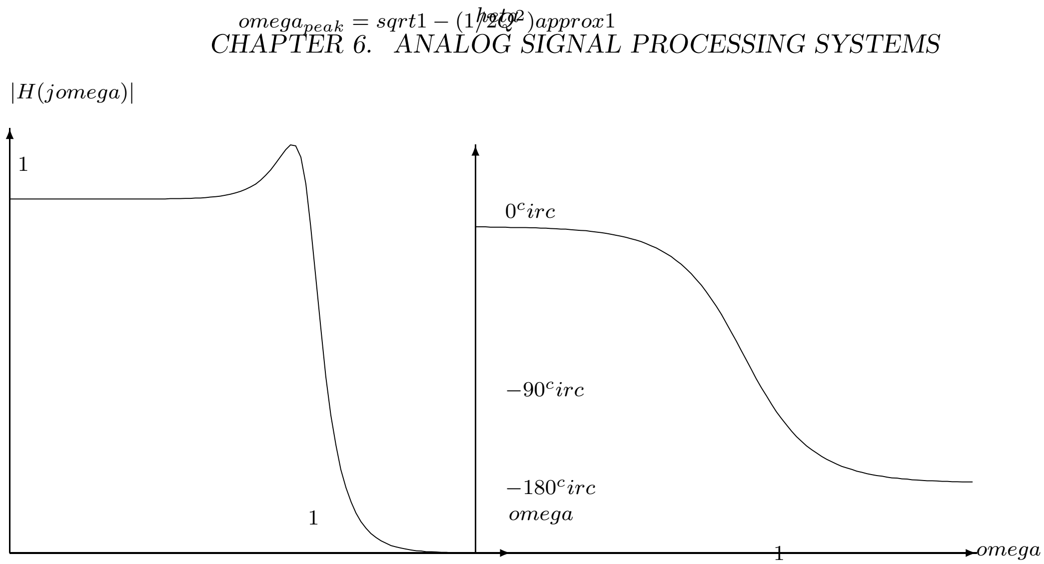


Figure 6.14: Bode plot of quality factor

Assume that $\omega_0 = 1$

$$H(j\omega) = \frac{1}{1 - \omega^2 + j\omega/Q} \quad (6.6\text{-}163)$$

$$|H(j\omega)| = \frac{1}{\sqrt{(1-\omega^2)^2 + (\omega/Q)^2}} \quad (6.6\text{-}164)$$

and

$$\theta = -\tan^{-1}\left(\frac{\omega/Q}{1-\omega^2}\right). \quad (6.6\text{-}165)$$

$$|H(j0)| = 1, \qquad |H(j1)| = Q, \qquad |H(j\infty)| \to 0 \quad (6.6\text{-}166)$$

For large ω,

$$|H(j\omega)| \approx \frac{1}{\omega^2} \quad (6.6\text{-}167)$$

$$\theta(j0) = 0^\circ, \qquad \theta(j1) = -90^\circ, \qquad \theta(j\infty) \to -180^\circ \quad (6.6\text{-}168)$$

The importance of Q may be seen in Figure 6.15

Generally speaking, the higher the order of the transfer funtion, the more accurate the approximation of a filter to the ideal filter.

The amplitude squared function of the filter may be represented as

$$|H(j\omega)|^2 = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1 + \sum_{j=0}^{n} b_j \omega^{2j}} \quad (6.6\text{-}169)$$

6.7 Butterworth Low-Pass filter

The Butterworth response of a low-pass filter is given by

$$|H(j\omega)|^2 = \frac{1}{1 + \omega^{2n}} \quad (6.7\text{-}170)$$

where n is the degree of the filter and the half power frequency occurs at $\omega = 1$ for all n. To determine the poles of the transfer function, we recall that

$$|H(j\omega)|^2 = H(s)H(-s)|_{s=j\omega} \quad (6.7\text{-}171)$$

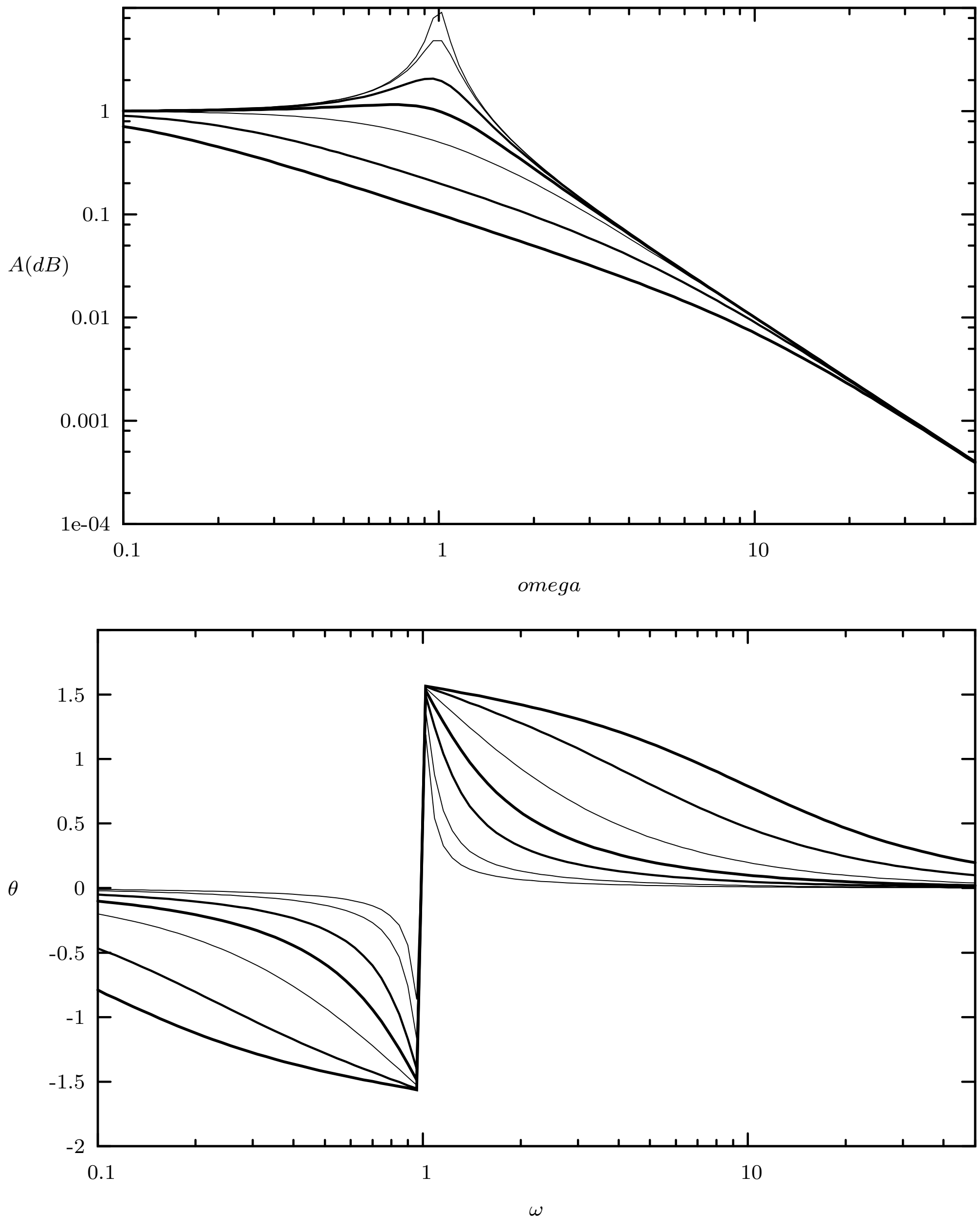


Figure 6.15: Importance of quality factor Q

using $\omega = s/j$

$$H_n(s)H_n(-s) = \frac{1}{1+(s/j)^{2n}} = \frac{1}{1+(-1)^n s^{2n}} \tag{6.7-172}$$

The poles of the above equation are the roots of the equation

$$B_n(s)B_n(-s) = 1 + (-1)^n S^{2n} = 0 \tag{6.7-173}$$

where $B_n(s)$ is the Butterworth polynomial.

Notice also that poles of the Butterworth filter are located on the unit circle, in the s-plane.

6.7.0.6 $n = 1$

$$B_n(s)B_n(-s) = 1 - s^2 = (1+s)(1-s) = 0 \tag{6.7-174}$$

The roots are $s = \pm 1$. However $s = 1$ corresponds to an unstable system which is not desirable. So,

$$B_1(s) = s + 1, \qquad \text{or} \qquad H_1(s) = \frac{1}{s+1} \tag{6.7-175}$$

6.7.0.7 $n = 2$

$$B_n(s)B_n(-s) = 1 + s^4 = 0 \tag{6.7-176}$$

or

$$s^4 = -1 = e^{j(180^\circ + k360^\circ)}, \qquad k \text{ being an integer} \tag{6.7-177}$$

$$s = e^{j(180^\circ + k360^\circ)/4} \tag{6.7-178}$$

It is obvious that there are four roots due to four different angles within 360° (with $k = 0, 1, 2, 3$).

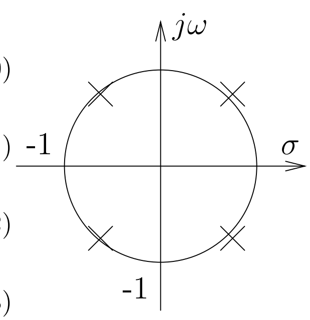
$$\frac{180^\circ + k360^\circ}{4} = 45^\circ, 135^\circ, 225^\circ, 315^\circ \tag{6.7-179}$$

$$s_1 = e^{j45^\circ} = \cos 45^\circ + j \sin 45^\circ = 0.707 + j0.707 \tag{6.7-180}$$

$$s_2 = e^{j135^\circ} = \cos 135^\circ + j \sin 135^\circ = -0.707 + j0.707 \tag{6.7-181}$$

$$s_3 = e^{j225^\circ} = \cos 225^\circ + j \sin 225^\circ = -0.707 - j0.707 \tag{6.7-182}$$

$$s_4 = e^{j315^\circ} = \cos 315^\circ + j \sin 315^\circ = 0.707 - j0.707 \tag{6.7-183}$$



Again, two poles in the right s-plane are not stable. Therefore

$$B_2(s) = (s + 0.707 - j0.707)(s + 0.707 + j0.707) = s^2 + \sqrt{2}s + 1 \tag{6.7-184}$$

or

$$H_s(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \tag{6.7-185}$$

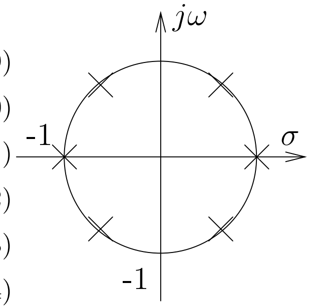
6.7.0.8 $n = 3$

$$1 - s^6 = 0 \tag{6.7-186}$$
$$s^6 = 1 = e^{jk360°} \tag{6.7-187}$$

$$s = e^{j\frac{k360°}{6}}, \qquad (k = 0, 1, 2, 3, 4, 5, 6) \tag{6.7-188}$$

There are six roots:

$$s_1 = e^{j0°} = \cos 0° + j \sin 0° \tag{6.7-189}$$
$$s_2 = e^{j60°} = \cos 60° + j \sin 60° \tag{6.7-190}$$
$$s_3 = e^{j120°} = \cos 120° + j \sin 120° \tag{6.7-191}$$
$$s_4 = e^{j180°} = \cos 180° + j \sin 180° \tag{6.7-192}$$
$$s_5 = e^{j240°} = \cos 240° + j \sin 240° \tag{6.7-193}$$
$$s_6 = e^{j300°} = \cos 300° + j \sin 300° \tag{6.7-194}$$



Leave out the roots in the right half s-plane.

$$B_3(s) = (s - \cos 180° - j \sin 180°)(s - \cos 120° - j \sin 120°)(s - \cos 240° - j \sin 240°) \tag{6.7-195}$$

$$= (s+1)(s + \frac{1}{2} - j\frac{\sqrt{3}}{2})(s + \frac{1}{2} + j\frac{\sqrt{3}}{2}) \tag{6.7-196}$$
$$= (s+1)(s^2 + s + 1) = (s+1)(s^2 + 2\cos 60° s + 1) \tag{6.7-197}$$

or

$$H_3(s) = \frac{1}{(s+1)(s^2 + s + 1)} \tag{6.7-198}$$

To simplify the process of determining the poles for the Butterworth filter, we define an angle Ψ_k with respect to the negative real axis.

If n is odd,

$$s^{2n} = 1 \to s = e^{\frac{jk360°}{2n}} = e^{\frac{jk180°}{n}} \tag{6.7-199}$$

$$\Psi_k = \frac{k180°}{n} - 180° = \frac{(k-n)180°}{n} \tag{6.7-200}$$

when $k = n$, $\Psi_k = 0$.

If n is even

$$s^{2n} = -1 \to s = e^{\frac{j\pm 180° + k360°}{2n}} = e^{j(\frac{90°}{n} + \frac{k180°}{n})} \tag{6.7-201}$$

$$\Psi_k = \frac{\pm 90° + k180°}{n} - 180° = \frac{\pm 90° + (k-n)180°}{n} \tag{6.7-202}$$

when $k = n$, $\Psi_k = \pm 90°/n$. Furthermore, for any n, poles are separated by $180°/n$.

In summary, for n being odd

$$B_n = (s+1)\prod_k (s^2 + 2\cos\Psi_k \cdot s + 1) \tag{6.7-203}$$

Poles of $H_n(s)$ are separated by $180°/n$, equally spaced with respect to $\Psi = 0$, and

$$-90° < \Psi_k < 90° \tag{6.7-204}$$

For n begin even,

$$B_n = \prod_k (s^2 + s\cos\Psi_n \cdot s + 1) \tag{6.7-205}$$

Poles of $H_n(s)$ are separated by $180°/n$, starting frlom $\Psi = \pm 90°/n$.

EXAMPLE 6.7-1 *Design the fifth-order Butterworth low-pass filter with cut-off frequency equal to 1 rad/s.*

Solution:

1. $n = 5$ *(odd), one pole is at* $\Psi = 0$ *and others are separated from it by multiples of* $180°/5 = 36°$.

$$\Psi_k = 0, \pm 36°, \pm 72° \tag{6.7-206}$$

2. *The pole locations are*

$$p_i, \overline{p}_i = -\cos\Psi_k \pm j\sin\Psi_k \tag{6.7-207}$$

or

$$p_o = -\cos 0° = -1 \tag{6.7-208}$$
$$p_1, \overline{p}_1 = -\cos 36° \pm j\sin 36° = -0.8090170 \pm j0.5877852 \tag{6.7-209}$$
$$p_2, \overline{p}_2 = -\cos 72° \pm j\sin 72° = -0.3090170 \pm j0.9510565 \tag{6.7-210}$$

3. *The fifth-order Butterworth function is*

$$B_5(s) = (s+1)(s^2 + 2\cos 36° \cdot s + 1)(s^2 + 2\cos 72° \cdot s + 1) \tag{6.7-211}$$

or the fifth-order Butterworth low-pass filter is given by the transfer function

$$H_5(s) = \frac{1}{(s+1)(s^2 + 2\cos 36° \cdot s + 1)(s^2 + 2\cos 72° \cdot s + 1)} \tag{6.7-212}$$

4. *The quality factor*

$$Q = \frac{1}{2\cos\Psi} \tag{6.7-213}$$

which gives

$$Q_0 = 0.5, \qquad Q_1 = 0.618, \quad \textit{and} \quad Q_2 = 1.618 \tag{6.7-214}$$

Another important issue in the low-pass filter design is to determine the minimum order of the filter under a group of specifications.

Given the nth-order low-pass Butterworth filter

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^{2n}}} \tag{6.7-215}$$

recall that the gain A is defined as

$$A = 20\log|H(j\omega)| dB \tag{6.7-216}$$

and the attenuation α is defined as

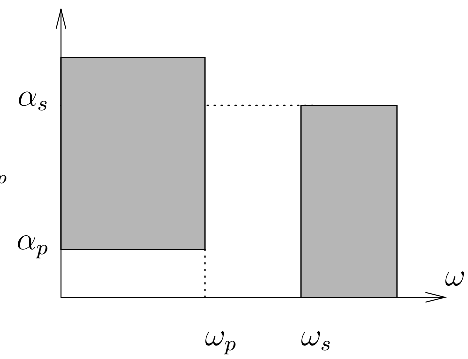
$$\alpha = 20 \log \frac{1}{|H(j\omega)|} \tag{6.7-217}$$

$$= 20 \log \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^{2n}} \tag{6.7-218}$$

$$= 10 \log \left[1 + \left(\frac{\omega}{\omega_0}\right)^{2n}\right] dB \tag{6.7-219}$$

Assume that

Maximum passband attenuation $= \alpha_p$, for $\omega \leqslant \omega_p$
Minimum stopband attenuation $= \alpha_s$, for $\omega > \omega_s$



Dividing both sides of Equation 6.7-219 and finding the antilogarithm gives

$$\left(\frac{\omega}{\omega_0}\right)^{2n} + 1 = 10^{\alpha/10} \tag{6.7-220}$$

To determine n

$$\left(\frac{\omega_p}{\omega_0}\right)^{2n} + 1 = 10^{0.1\alpha_p} - 1 \tag{6.7-221}$$

and

$$\left(\frac{\omega_s}{\omega_0}\right)^{2n} + 1 = 10^{0.1\alpha_s} - 1 \tag{6.7-222}$$

Dividing the above two equations, we obtain

$$\left(\frac{\omega_2}{\omega_p}\right)^{2n} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \tag{6.7-223}$$

Take the log,

$$2n \log \left(\frac{\omega_s}{\omega_p}\right) = \log \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right] \tag{6.7-224}$$

$$n = \frac{\log \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]}{2 \log \left(\frac{\omega_s}{\omega_p}\right)} \tag{6.7-225}$$

However n must be an integer so that the n chosen can be obtained by assigning the next greater intege to n.

Using this n, all specifications cannot be met exactly, the cut-off frequency may be calculated by

$$\omega_0 = \frac{\omega}{[10^{0.1\alpha} - 1]^{\frac{1}{2}n}} \tag{6.7-226}$$

1.

$$\omega_0 = \frac{\omega_p}{[10^{0.1\alpha_p} - 1]^{\frac{1}{2}n}} \tag{6.7-227}$$

2.

$$\omega_0 = \frac{\omega_s}{[10^{0.1\alpha_s} - 1]^{\frac{1}{2}n}} \tag{6.7-228}$$

The results obtained frlom the above equations are compared. The one that offers the advantage will be used.

EXAMPLE 6.7-2 *Realize the following specification with a Butterworth response:*

$$\alpha_p = 0.5dB \qquad \alpha_s = 20dB \tag{6.7-229}$$

$$\omega_p = 1000rad/s \qquad \omega_s = 2000rad/s \tag{6.7-230}$$

Solution:

1.

$$n = \frac{\log\left[\frac{10^{0.1\times 20}-1}{10^{0.1\times 0.5}-1}\right]}{2\log\left(\frac{2000}{1000}\right)} \tag{6.7-231}$$

$$= \frac{\log\left[\frac{10^2-1}{1.1220185-1}\right]}{2\log(2)} \tag{6.7-232}$$

$$= \frac{\log[811.35]}{2\log 2} \tag{6.7-233}$$

$$= 4.83209 \tag{6.7-234}$$

Take $n = 5$.

2. Calculate ω_0:

$$\omega_0 = \frac{\omega_p}{[10^{0.1\times 0.5} - 1]^{\frac{1}{10}}} = 1234rad/s \tag{6.7-235}$$

or

$$\omega_0 = \frac{\omega_s}{[10^{0.1\times 20} - 1]^{\frac{1}{10}}} = 1263rad/s \tag{6.7-236}$$

3. The fifth-order Butterworth low-pass filter with cut-off at $\omega = 1$,

$$H_5(s) = \frac{1}{(s+1)(s^2 + 2\cos 36^\circ \cdot s + 1)(s^2 + s\cos 72^\circ \cdot s + 1)} \tag{6.7-237}$$

4. Transform the cut-off frequency to $\omega_0 = 1263rad/s$, by replacing s by $\frac{s}{\omega_0}$.

$$H_5(s) = \frac{1}{\left(\left(\frac{s}{\omega_0}\right)+1\right)\left(\left(\frac{s}{\omega_0}\right)^2 + 2\cos 36^\circ \cdot \left(\frac{s}{\omega_0}\right) + 1\right)\left(\left(\frac{s}{\omega_0}\right)^2 + \left(\frac{s}{\omega_0}\right)\cos 72^\circ \cdot \left(\frac{s}{\omega_0}\right) + 1\right)} \tag{6.7-238}$$

5.
$$\alpha(1000) = 10\log\left[\left(\frac{1000}{1263}\right)^{10} + 1\right] = 0.4001dB \tag{6.7-239}$$

which is less than the specified $0.5dB$.

6.7.1 Properties of the Butterworth response

Since
$$|H_n(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}} \tag{6.7-240}$$

1.
$$|H_n(j0)| = 1 \qquad \text{for all } n \text{ when } \omega_0 = 1 \tag{6.7-241}$$

2.
$$|H_n(j1)| = \frac{1}{\sqrt{2}} \approx 0.707 \qquad \text{for all } n \tag{6.7-242}$$

3. $|H_n(j\omega)|$ exhibits n-pole roll off, when $\omega \to 0$

4. The derivatives of $|H_n(j\omega)|$ for small ω. Using a Taylor series
$$|H_n(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}} = 1 - \frac{1}{2}\omega^{2n} + \frac{3}{8}\omega^{4n} - \cdots \tag{6.7-243}$$

   it follows that
$$\left.\frac{d^k}{d\omega^k}|H_n(j\omega)|\right|_{\omega=0} = 0 \qquad \text{for } k = 1, 2, \ldots, 2n-1 \tag{6.7-244}$$

   while
$$\left.\frac{d^{2n}}{d\omega^{2n}}|H_n(j\omega)|\right|_{\omega=0} = -\frac{1}{2}(2n)! \tag{6.7-245}$$

   For this reason, the Butterworth response is also known as maximally flat.

6.8 The Chebyshev Response

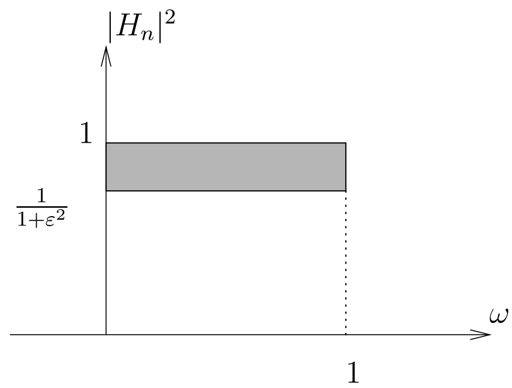
It is noticed that the Butterworth response (of a low-pass filter) has a magnitude-squared form
$$|H_n(j\omega)|^2 = \frac{1}{1+\omega^{2n}} \tag{6.8-246}$$

or
$$= \frac{1}{1+[\omega^n]^2} \tag{6.8-247}$$

A generalization of this response may be written as
$$|H_n(j\omega)|^2 = \frac{1}{1+[F_n(\omega)]^2} \tag{6.8-248}$$

The question now is to find the suitable function $F_n(\omega)$ to provide the required response.

If the required $|H_n(j\omega)|^2$ is defined by



$\varepsilon \leqslant 1$, $\varepsilon T_n(\omega)$ may be used for $F_n(\omega)$ where $T_n(\omega)$ is the Chebyshev polynomial of the first kind.

And

$$|H_n(j\omega)|^2 = \frac{1}{1+\varepsilon^2 T_n^2(\omega)} \tag{6.8-249}$$

where

$$T_n(\omega) = \cos[n \cos^{-1}(\omega)], \qquad \text{for } |\omega| \leqslant 1 \tag{6.8-250}$$

known as the Lissajous function or Chebyshev function. Since the inverse cosine function becomes imaginary for $\omega > 1$

$$\cos^{-1} \omega = jz \tag{6.8-251}$$

or

$$\cos jz = \frac{e^{j(jz)} + e^{-j(jz)}}{2} = \frac{e^{-2} + e^{2}}{2} = \cosh z = \omega \tag{6.8-252}$$

or equally

$$z = \cosh^{-1} \omega \tag{6.8-253}$$

Thus

$$\cos^{-1} = j \cosh^{-1} \omega \tag{6.8-254}$$

Therefore:

$$\begin{aligned} T_n(\omega) &= \cos[n \cos^{-1}(\omega)] && (6.8\text{-}255)\\ &= \cos[nj \cosh^{-1}(\omega)] && (6.8\text{-}256)\\ &= \cosh[n \cosh^{-1}(\omega)] \qquad \text{for } \omega > 1 && (6.8\text{-}257) \end{aligned}$$

In summary

$$|H_n(j\omega)| = \frac{1}{1+\varepsilon^2 T_n^2(\omega)} \tag{6.8-258}$$

where $\varepsilon \leqslant 1$ and

$$T_n(\omega) = \begin{cases} \cos[n \cos^{-1}(\omega)] & \text{for } |\omega| \leqslant 1 \\ \cosh[n \cosh^{-1}(\omega)] & \text{for } |\omega| > 1 \end{cases} \tag{6.8-259}$$

is defined as the Chebyshev response.

6.8.1 The properties of the Chebyshev response

1. Since

$$T_n^2(0) = 0 \qquad \text{for } n \text{ being odd} \tag{6.8-260}$$

$$T_n^2(0) = 1 \qquad \text{for } n \text{ being even} \tag{6.8-261}$$

$$|H_n(j0)| = 1 \qquad \text{for } n \text{ being odd} \tag{6.8-262}$$

and

$$|H_n(j0)| = \frac{1}{\sqrt{1+\varepsilon^2}} \qquad \text{for } n \text{ being even} \tag{6.8-263}$$

2. Since

$$T_n^2(1) = 1 \qquad \text{for all } n \tag{6.8-264}$$

$$|H_n(j1)| = \frac{1}{\sqrt{1+\varepsilon^2}} \qquad \text{for all } n \tag{6.8-265}$$

3. The minima of $|H_n|$ occurs when $T_n^2 = 1$ and the maxima of $|H_n|$ occurs when $T_n^2 = 0$. Since the $|H_n|$ is even, there will be n half-ripples or half-cycles in the range from $\omega = 0$ to $\omega = 1$.

4. The attenuation for the nth order Chebyshev response is

$$\alpha_n = -20\log|H_n(j\omega)| = -20\log\frac{1}{\sqrt{1+\varepsilon^2 T_n^2(\omega)}} \tag{6.8-266}$$

$$= 10\log[1+\varepsilon^2 T_n^2(\omega)]dB \tag{6.8-267}$$

5. In the pass band

$$\alpha_p = 10\log(1+\varepsilon^2) \tag{6.8-268}$$

thus

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} \tag{6.8-269}$$

6. The half power frequency $\omega_{h[}$ may be obtained when $\alpha = 3.01$ or

$$|H_n(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 T_n^2(\omega)}} = \frac{1}{\sqrt{2}} \tag{6.8-270}$$

or

$$\varepsilon^2 T_n^2(\omega) = 1 \tag{6.8-271}$$

or

$$T_n(\omega) = \frac{1}{\varepsilon} = \cosh[n\cosh^{-1}(\omega_{hp})] \tag{6.8-272}$$

$$n\cosh^{-1}(\omega_{hp}) = \cosh^{-1}\left(\frac{1}{\varepsilon}\right) \tag{6.8-273}$$

$$\omega_{hp} = \cosh\left[\frac{1}{n}\cosh^{-1}\left(\frac{1}{\varepsilon}\right)\right] \tag{6.8-274}$$

$$= \cosh\left[\frac{1}{n}\cosh^{-1}\left(10^{0.1\alpha_p} - 1\right)^{-\frac{1}{2}}\right] \tag{6.8-275}$$

since $0 < \varepsilon < 1$, $\omega_{hp} > 1$.

7. If a Chebyshev response is specified by α_p, α_s and ω_s where α_p is the pass-band attenuation ($\alpha(\omega) \leqslant \alpha_p$ for $0 \leqslant \omega \leqslant 1$), α_s is the stop-band attenuation ($\alpha(\omega) \geqslant \alpha_s$ for $\omega \geqslant \omega_s$) the order n of the Chebyshev filter may be obtained by the following equation

$$\alpha_s = \alpha(\omega_s) = 10\log[1 + \varepsilon^2 T_n^2(\omega_s)] \tag{6.8-276}$$

$$\varepsilon^2 \cosh^2(n\cosh^{-1}\omega_s) = 10^{0.1\alpha_s} - 1 \tag{6.8-277}$$

$$\cosh^2(n\cosh^{-1}\omega_s) = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \tag{6.8-278}$$

(Note: $\varepsilon^2 = 10^{0.1\alpha_p} - 1$)

$$n = \frac{\cosh^{-1}\left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\omega_s} \tag{6.8-279}$$

Example 6.8-1 *Given the following specification for a Chebyshev low-pass filter, find ε, half power frequency ω_{hp} and mimum order n of the filter.*

$$\omega_p = 1rad/s, \quad \omega_s = 2.33rad/s, \quad \alpha_p = 0.5dB, \quad \alpha_s = 22dB \tag{6.8-280}$$

Solution:

1.

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} \tag{6.8-281}$$

$$= \sqrt{10^{0.05} - 1} = 0.3493 \tag{6.8-282}$$

2.

$$n = \frac{\cosh^{-1}\left[\frac{10^{0.1\times 22}-1}{10^{0.1\times 0.5}-1}\right]^{\frac{1}{2}}}{\cosh^{-1}[2.33]} \tag{6.8-283}$$

$$= \frac{\cosh^{-1}(35.926)}{\cosh^{-1}(2.33)} \tag{6.8-284}$$

$$= \frac{4.274}{1.489} = 2.87 \tag{6.8-285}$$

$$\approx 3 \qquad \textit{(round up to 3)} \tag{6.8-286}$$

3.

$$\omega_{hp} = \cosh\left[\frac{1}{3}\cosh^{-1}\left(10^{0.1\times 0.5} - 1\right)^{\frac{1}{2}}\right] \tag{6.8-287}$$

$$= \cosh\left[\frac{1}{3}\cosh^{-1}\frac{1}{0.3493}\right] \tag{6.8-288}$$

$$= \cosh\left[\frac{1}{3}1.7129\right] \tag{6.8-289}$$

$$= 1.167rad/s \tag{6.8-290}$$

Comparing with a Butterworth filter that meets the same specifications:

$$n = \frac{\log\left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]}{2\log\left(\frac{\omega_s}{\omega_p}\right)} \tag{6.8-291}$$

$$= \frac{\log\left[\frac{10^{0.1\times 22}-1}{10^{0.1\times 0.5}-1}\right]}{2\log 2.33} \tag{6.8-292}$$

$$= \frac{\log 1290.7008}{2\log 2.33} = 4.234 \tag{6.8-293}$$

$$\approx 5 \qquad \textit{(round upto 5)} \tag{6.8-294}$$

Hence a fifth-order Butterworth response is required to meet the specifications that led to a third-order Chebyshev.

6.8.2 Location of the Chebyshev Poles

From

$$H_n(s)H_n(-s)|_{s=j\omega} = |H_n(j\omega)|^2 \tag{6.8-295}$$

or

$$H_n(s)H_n(-s) = \frac{1}{1+\varepsilon^2 T_n^2(s/j)} \tag{6.8-296}$$

the poles of the $H_n(s)$ can be obtained from

$$1+\varepsilon^2 T_n^2\left(\frac{s}{j}\right) = 0 \tag{6.8-297}$$

$$T_n\left(\frac{s}{j}\right) = 0 \pm j\frac{1}{\varepsilon} \tag{6.8-298}$$

For $\omega < 1$

$$T_n(s/j) = \cos\left[n\cos^{-1}\left(\frac{s}{j}\right)\right] = \cos nw \tag{6.8-299}$$

where w is a complex number:

$$w = \cos^{-1}\left(\frac{s}{j}\right) u + jv \tag{6.8-300}$$

$$\cos(nw) = \cos(nu + jnv) \tag{6.8-301}$$

$$= \cos(nu)\cosh(nv) - j\sin(nu)\sinh(nv) \tag{6.8-302}$$

$$= 0 \pm j\frac{1}{\varepsilon} \tag{6.8-303}$$

That is

$$\cos(nu)\cosh(nv) = 0 \tag{6.8-304}$$

$$\sin(nu)\sinh(nv) = \pm\frac{1}{\varepsilon} \tag{6.8-305}$$

Since $\cosh(nv) \geqslant 1$, to make Equation 6.8-304 stand

$$\cos(nu) = 0 \tag{6.8-306}$$

or

$$u_k = \frac{\pi}{2n}(sk+1) \qquad k = 0, 1, \ldots, 2n-1 \tag{6.8-307}$$

For these values of u_k

$$\sin(nu_n) = \pm 1 \tag{6.8-308}$$

substituted into Equation 6.8-305, we have

$$v_k = \pm\frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) = \pm a \tag{6.8-309}$$

Take u_k and v_k into Equation 6.8-300

$$s_k = j\cos w_k = j\cos\left[\frac{\pi}{2n}(2k+1) + ja\right] \tag{6.8-310}$$

$$= \sigma_k + jw_k \tag{6.8-311}$$

$$\sigma_k = \pm\sinh a \sin\left(\frac{2k+1}{2n}\pi\right) \tag{6.8-312}$$

$$w_k = \cosh a \cos\left(\frac{2k+1}{2n}\pi\right) \tag{6.8-313}$$

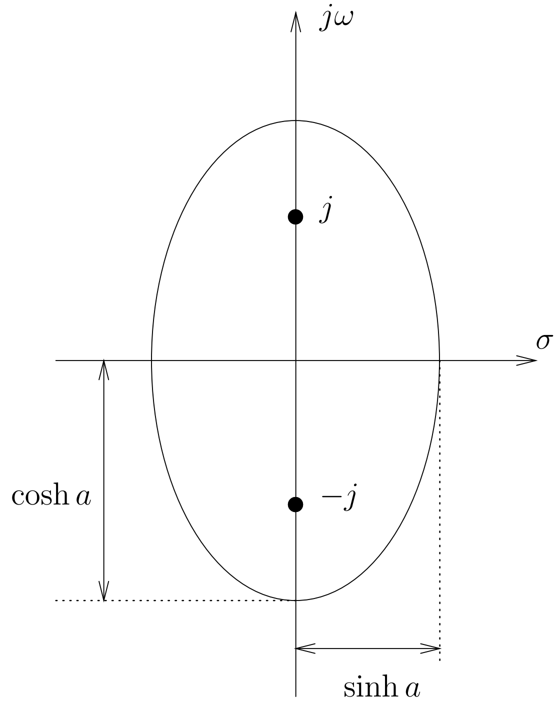
Those s_k which are in the left s-plane will be used as the poles of the Chebyshev filter $H_n(s)$.

From Equations 6.8-312 and 6.8-313, we have

$$\left(\frac{\sigma_k}{\sinh a}\right)^2 + \left(\frac{w_k}{\cosh a}\right)^2 = 1 \tag{6.8-314}$$

which represents an ellipse.

cosh a is the major semiaxis
sinh a is the minor semiaxis
foci are at $\pm j1$.



EXAMPLE 6.8-2 *Given $n = 3$ and $\alpha_p = 1$, determine the location of the Chebyshev poles.*
Solution:

1.
$$a = \frac{1}{3}\sinh^{-1}\left(10^{0.1\times 1} - 1\right)^{-\frac{1}{2}} = 0.476 \tag{6.8-315}$$

2.
$$\cosh a = 1.115 \quad \textit{and} \quad \sinh a = 0.494 \tag{6.8-316}$$

3. *Using Equations 6.8-312 and 6.8-313* $k = 0, 1, 2$

$$k = 0 \qquad s_1 = -0.247 + j0.966 \tag{6.8-317}$$
$$k = 1 \qquad s_2 = -0.494 \tag{6.8-318}$$
$$k = 2 \qquad s_3 = -0.247 - j0.966 \tag{6.8-319}$$

That is:
$$H_n(s) = \frac{1}{(s + 0.494)(s^2 + 0.494s + 0.994)} \tag{6.8-320}$$

6.8.3 Guillemin's Algorithm

If Ψ_k are the Butterworth angles for an nth order filter, the location of the Chebyshev poles (of the same order) is given by

$$s_k = \sigma_k \pm jw_k \tag{6.8-321}$$

$$\begin{cases} -\sigma_k = \sinh a cos \Psi_k \\ \pm w_k = \cosh a \sin \Psi_k \end{cases} \tag{6.8-322}$$

where

$$a = \frac{1}{n}\sinh^{-1}\left(10^{0.1\alpha_p} - 1\right)^{\frac{1}{2}} \tag{6.8-323}$$

EXAMPLE 6.8-3 *Given $n = 5$ and $\alpha_p = 0.5$ dB, find the locations of the Chebyshev poles.*
Solution:

1.

$$a = \frac{1}{5}\sinh^{-1}\left(10^{0.1\times 0.5} - 1\right)^{\frac{1}{2}} = 0.3548 \tag{6.8-324}$$

2.

$$\sinh a = 0.3623 \tag{6.8-325}$$

and

$$\cosh a = 1.0636 \tag{6.8-326}$$

3. The Butterworth angles are

$$\Psi_k = 0^\circ, 36^\circ and 72^\circ \qquad (180^\circ/5 \ apart) \tag{6.8-327}$$

4. The Chebyshev poles

$$for \Psi = 0^\circ, \qquad p_1 = -0.3623 \tag{6.8-328}$$
$$for \Psi = 36^\circ, \qquad p_2, p_3 = -0.2931 \pm j0.6252 \tag{6.8-329}$$
$$for \Psi = 72^\circ, \qquad p_4, p_5 = -1120 \pm j1.0116 \tag{6.8-330}$$

That is

$$H_n(s)\frac{k}{(s+0.3623)(s^2+0.5863s+0.4768)(s^2+0.2240s+1.0359)} \tag{6.8-331}$$

Assume $H_n(0) = 1$, $K = 0.1789$.

6.9 Elliptic Function Response (Caver Response)

In the previous sections, we have found that for a given order n, the Chebyshev response presents a sharper cut-off characteristic that the Butterworth response (smaller $\Delta\omega = \omega_s - \omega_p$); or for the same group of specifications ω_s/ω_p, α_p and α_s, the Chebyshev response requires a smaller value of n than the Butterworth.

This suggests that a new response with equal ripple in both the pass band and the stop band would be superior to even the Chebyshev response. This new response involves Jacobian elliptic integrals and elliptic functions, hence the name elliptic filters. It was dirst demonstrated by Wilhelm Cauer who had studied mathematics under Hilbert at Goettingen and had been famliar with elliptic functions and their applications.

For example, with a given group of specification $\omega_s/\omega_p = 1.5$, $\alpha_p = 0.5$ dB and $\alpha_s = 50$ dB, it is necessary that $n = 17$ for a Butterworth response; that $n = 8$ for a Chebyshev response; and it is necessary only that $n = 5$ for an elliptic response which is a significant improvement.

A typical response of the elliptic response with $n = 4$ is shown as follows.

Unfortunately, the design procedure for the elliptic filter is not as simple as it was for Butterworth and Chebyshev. A commonly used method is the "cut-and-try" method using the extensive tables available in the design handbooks.

1. Select a trivial value for n.
2. Find a group of approximate coefficients for the given specifications α_p, ω_p/ω_s.
3. Check α_s to see if it meets the specifications, if not, take a higher value of n.

   With the filter design software, this design process is made less tedious.

6.10 Frequency Transformation

6.10.1 Low-pass to Low-pass Transformation

If the transfer function for a low-pass filter is designed with the cut-off frequency normalized to $\omega = 1$, the transfer function of a low-pass filter with an arbitrary cut-off frequency ω_0 is obtained by using the following transformation:

$$\omega \rightarrow \omega/\omega_0 \tag{6.10-332}$$

or

$$s \rightarrow s/\omega_0 \tag{6.10-333}$$

6.10.2 Low-pass to High-pass Transformation

Using the following transformation

$$\omega \rightarrow \omega_0/\omega \tag{6.10-334}$$

or

$$s \rightarrow \omega_0/s \tag{6.10-335}$$

in the transfer function of a low-pass filter with the normalized cut-off frequency at $\omega = 1$ may result in a high pass response with a passband edge at ω_0 as shown in Figure 6.16.

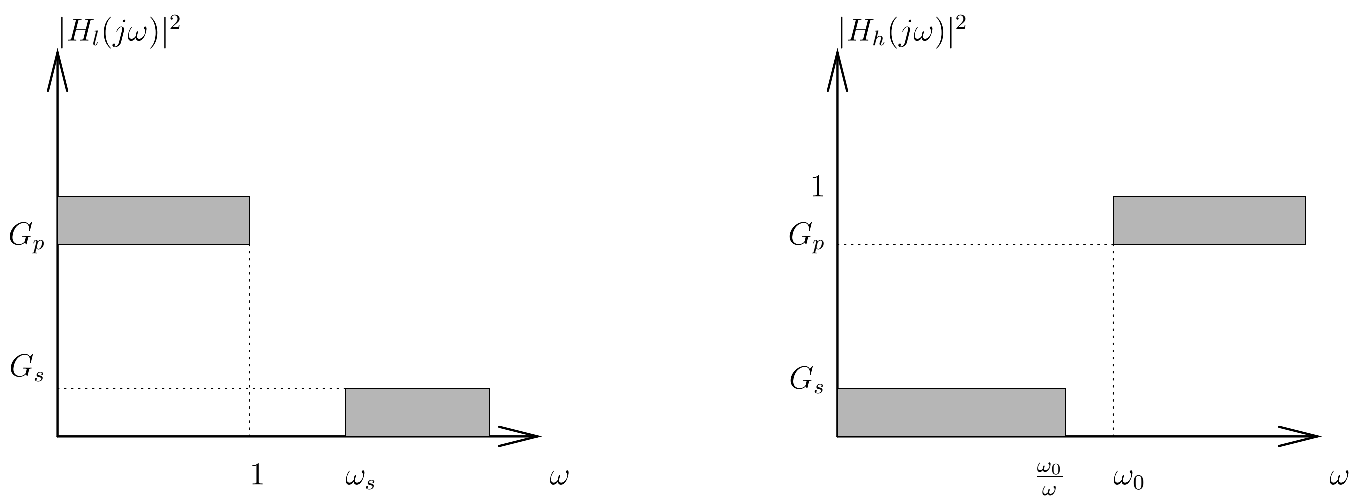


Figure 6.16: Low-pass to High-pass Transformation

6.10.3 Low-pass to Band-pass Transformation

When the negative frequency side is included, the low-pass filter with the normalized cut-off frequency $\omega = 1$ is represented as shown in Figure 6.18.

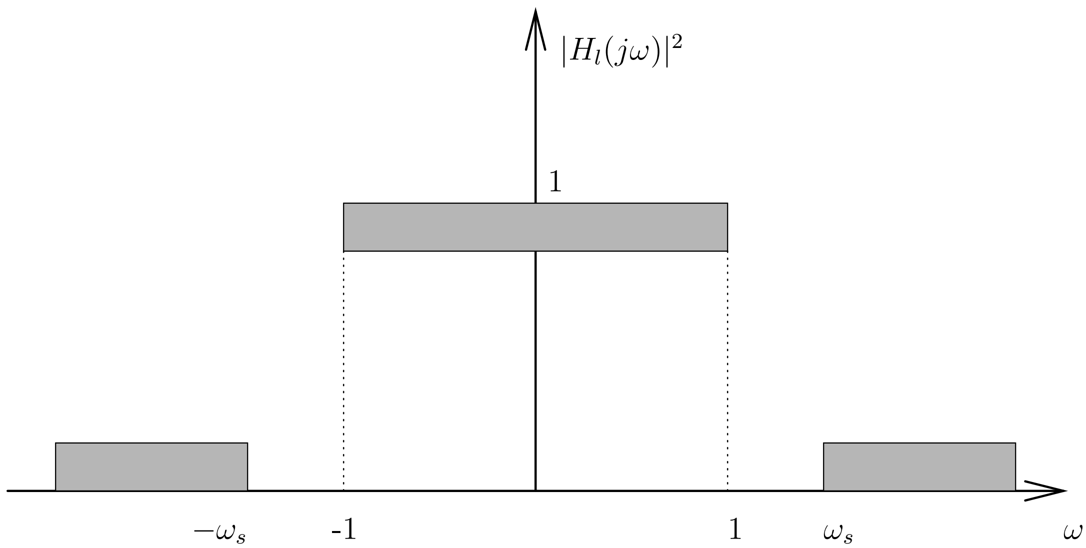


Figure 6.17: Double sided low-pass filter

To find a transformation to a band-pass response with passband extenfing from ω_1 to ω_2 as shown in Figure 6.18.

Let

$$\omega \to \beta\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \tag{6.10-336}$$

or

$$s \to \beta\left(\frac{\omega}{\omega_0} + \frac{\omega_0}{\omega}\right) \tag{6.10-337}$$

where β and ω_0 are to be determined.

Impose the following conditions

$$\begin{cases} -1 = \beta\left(\frac{\omega_1}{\omega_0} = \frac{\omega_0}{\omega_1}\right) \\ 1 = \beta\left(\frac{\omega_2}{\omega_0} = \frac{\omega_0}{\omega_2}\right) \end{cases} \tag{6.10-338}$$

to solve β and ω_0.

We have

$$\begin{cases} -\omega_0\omega_1 = \beta(\omega_1^2 - \omega_0^2) \\ \omega_0\omega_2 = \beta(\omega_2^2 - \omega_0^2) \end{cases} \tag{6.10-339}$$

$$-\frac{\omega_1}{\omega_2} = \frac{\omega_1^2 - \omega_0^2}{\omega_2^2 - \omega_0^2} \tag{6.10-340}$$

$$-\omega_1(\omega_2^2 - \omega_0^2) = \omega_2(\omega_1^2 - \omega_0^2) \tag{6.10-341}$$

$$(\omega_2 + \omega_1)\omega_0^2 = \omega_2\omega_1(\omega_2 + \omega_1) \tag{6.10-342}$$

$$\omega_0^2 = \omega_1\omega_2 \tag{6.10-343}$$

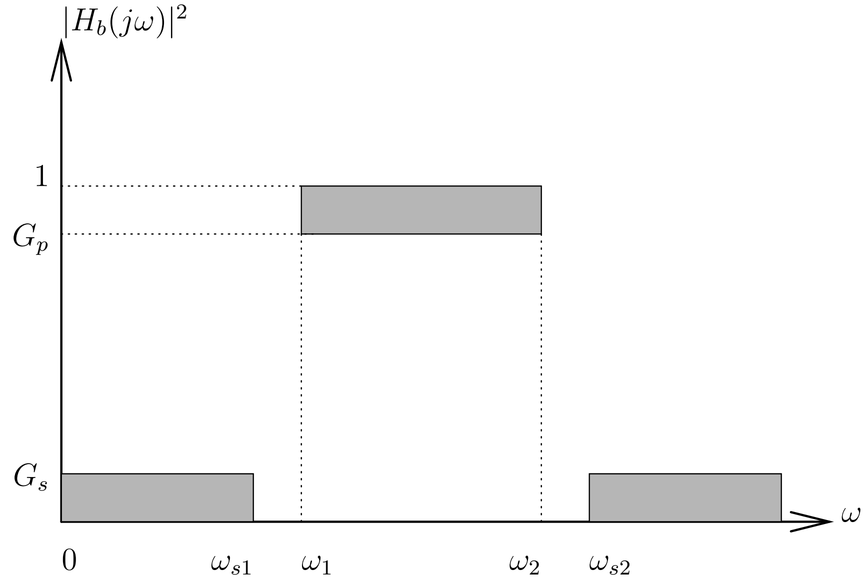


Figure 6.18: Double sided low-pass filter

Therefore

$$\omega_0 = \sqrt{\omega_1\omega_2} \tag{6.10-344}$$
$$\omega_0(\omega_2|\omega_1) = \beta(\omega_2^2 - \omega_1^2) \tag{6.10-345}$$
$$= \beta(\omega_2 + \omega_1)(\omega_2 - \omega_1) \tag{6.10-346}$$
$$\beta(\omega_2 - \omega_1) = \omega_0 \tag{6.10-347}$$
$$\beta = \frac{\omega_0}{\omega_2 - \omega_1} \tag{6.10-348}$$

In summary

$$\begin{cases} \omega + 0 = \sqrt{\omega_1\omega_2} \\ \beta = \frac{\omega_0}{\omega_2 - \omega_1} \end{cases} \tag{6.10-349}$$

where ω_0 is referred to as the band centre. The band-pass response so derived has *geometric symmetry around* ω_0.

For example, if the low-pass Chebyshev response is given by

$$|H_l(j\omega)|^2 = \frac{1}{1 + \varepsilon T_n^2(\omega)} \tag{6.10-350}$$

the band-pass Chebyshev response will be

$$|H_b(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 T_n^2 \left[\beta\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]} \tag{6.10-351}$$

where ω_0 and β are shown in Equation 6.10-349 and ω_1 is the lower stop band edge and ω_2 is the upper stop band edge.

Example 6.10-1 *Find the transfer function of a Butterworth low-pass with the following specifications:*

passband: 0 to 1 kHz, $\alpha_p \leqslant 3$ *dB*
stopband edge : 1.5 kHz, $\alpha_s \geqslant 40$ *dB*

Solution:

1. *Determine the order of required filter*

$$n \geqslant \frac{\log\left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]}{2\log\omega_s/omega_p} \tag{6.10-352}$$

$$= 11.36 \tag{6.10-353}$$

Take $n = 12$, *and work out 12 order Butterworth low-pass filter with the normalized cut-off frequency* $\omega = 1$.

2. *Substitute* ω/ω_0 *into the transfer function obtained, where* $\omega_0 = 1000$ *rad/s.*

6.10.4 Low-pass to Band-stop Transformation

The transformation which converts the low-pass to band-stop response is shown in Figure 6.19 and as follows:

$$\omega \rightarrow \frac{1}{\beta\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \tag{6.10-354}$$

or

$$s \rightarrow \frac{1}{\beta\left(\frac{s}{\omega_0} + \frac{\omega_0}{s}\right)} \tag{6.10-355}$$

6.11 Phase Oriented Design

One of the basic operations in signal processing is time delay which can be approximated by a delay filter.

The ideal delay filter can be depicter as shown in Figure 6.20.

or

$$v_2(t) = v_1(t - D) \tag{6.11-356}$$

The transfer functions of such a system can be written as

$$H(s) = \frac{v_2(s)}{v_1(s)} = e^{-Ds} \tag{6.11-357}$$

Its frequency responses are given as shown in Figure **??**.

As we have proven, $H(s) = e^{-Ds}$ cannot be realized. Therefore the best thing we can do is to approximate.

(Note: Laplace transform pair:

If

$$\mathcal{L}[\{(\sqcup)] = F(s) \tag{6.11-358}$$

$$f(t-\alpha)U(t-\alpha) \overset{\mathcal{L}}{\leftrightarrow} e^{-\alpha s}F(s) \tag{6.11-359}$$

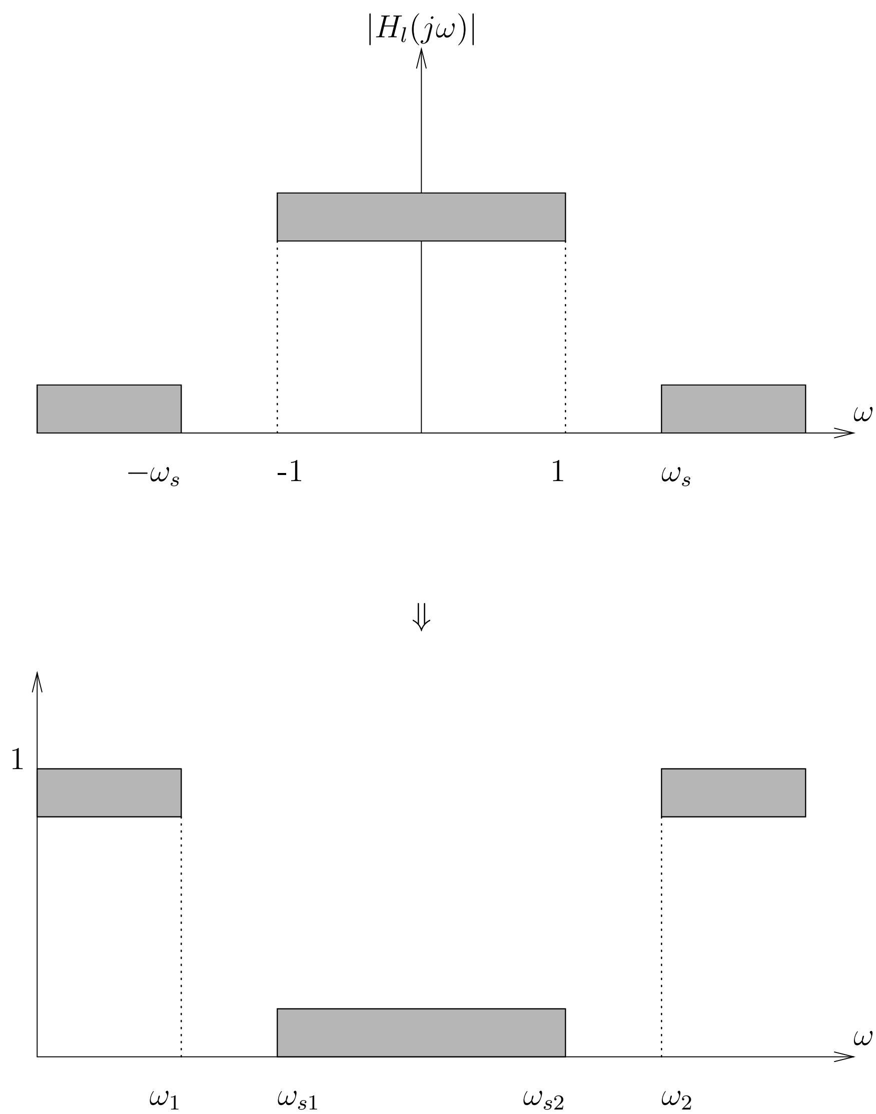


Figure 6.19: Low pass to band stop transformation

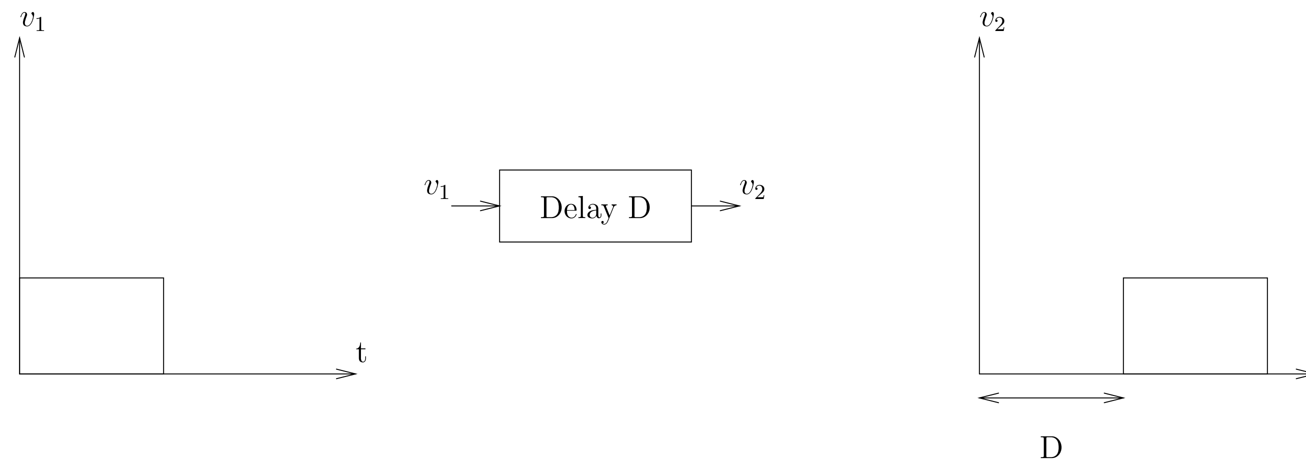


Figure 6.20: Ideal delay filter

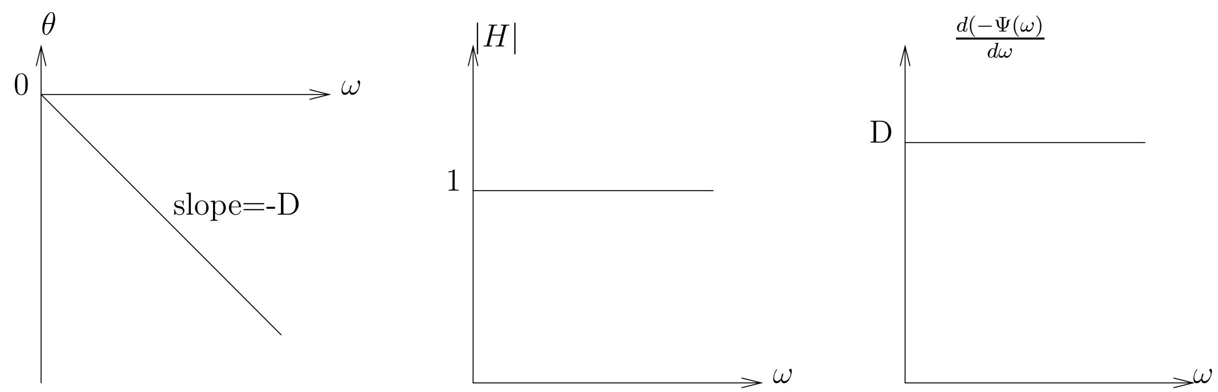


Figure 6.21: Ideal delay filter

As a result

$$\delta(t-D) \overset{\mathcal{L}}{\leftrightarrow} e^{-Ds} \tag{6.11-360}$$

$\delta(t-D)$ function cannot be physically implemented.)

For a given transfer funtion $H(s)$ or

$$H(j\omega) = \Re H(j\omega) + j\Im H(j\omega) = |H(j\omega)|e^{j\Psi(\omega)} \tag{6.11-361}$$

the *phase delay* θ is defined as

$$\theta = \tan^{-1}\frac{\Im[H(j\omega)]}{\Re[H(j\omega)]} = \Psi(\omega) \tag{6.11-362}$$

The *group delay* $T_g(\omega)$ is defined as

$$T_g(\omega) = -\frac{d\theta}{d\omega} = -\frac{d\Psi(\omega)}{d\omega} \tag{6.11-363}$$

$$= \frac{-\Re[H]\frac{d}{d\omega}\Im[H] + \Im[H]\frac{d}{d\omega}\Re[H]}{\{\Re[H(j\omega)]\}^2 + \{\Im[H(j\omega)]\}^2} \tag{6.11-364}$$

The group delay is also known as signal delay or envelope delay. Sometimes it is simply called or referred to as delay.

For instance,

$$H(s) = e^{-DS} \tag{6.11-365}$$

$$e^{-Dj\omega} = \cos(D\omega) - j\sin(D\omega) \tag{6.11-366}$$

$$\theta = \tan^{-1}\left(\frac{-\cos(D\omega)}{\cos(D\omega)}\right) = \tan^{-1}[-\tan(D\omega)] \tag{6.11-367}$$

$$= -D\omega \qquad \text{Phase Delay} \tag{6.11-368}$$

$$T_g(\omega) = -\frac{d\theta}{d\omega} = -(-D\omega)' = D \qquad \text{Group Delay} \tag{6.11-369}$$

A simple approach to approximate the ideal time delay transfer funtion was introduced by L. Storch.

$$H(s) = e^{-s} = \frac{1}{e^s} \tag{6.11-370}$$

Since

$$e^s = \frac{e^s - e^{-s}}{2} + \frac{e^s + e^{-s}}{2} \tag{6.11-371}$$

$$= \sinh s + \cosh s \tag{6.11-372}$$

$$H(s) = \frac{1}{\sinh s + \cosh s} = \frac{\frac{1}{\sinh s}}{1 + \coth s} \tag{6.11-373}$$

The series expansions of the hyperbolic functions of concern are:

$$\cosh s = 1 + \frac{s^2}{2!} + \frac{s^4}{4!} + \frac{s^6}{6!} + \cdots \tag{6.11-374}$$

$$\sinh s = s + \frac{s^3}{3!} + \frac{s^5}{5!} + \frac{s^7}{7!} + \cdots \tag{6.11-375}$$

Using the continued fraction expansion method

$$\frac{\cosh s}{\sinh s} = \coth s = \frac{1}{s} + \cfrac{1}{\frac{3}{s} + \cfrac{1}{\frac{5}{s} + \cfrac{1}{\frac{7}{s} + \ddots \cfrac{1}{\frac{(2n-1)}{s} + \ddots}}}} \tag{6.11-376}$$

In the Storch approach, the continued fraction is simply truncated after n terms to approximate the ideal delay transfer function e^{-s}. Storch showed that this truncation gave maximally flat delay at $\omega = 0$, no matter where the continued fraction is truncated. Further he was the first to associate the denominator polynomial thus obtained with a class of Bessel polynomials. This is the reason why the filter so derived is called the Bessel Filter.

Using the continued fraction expansion $\frac{N(s)}{D(s)}$ to approximate $\frac{\cosh s}{\sinh s}$ is to use $N(s) + D(s)$ to approximate $\sinh s + \cosh s = e^s$. So follow the Storch approach to find the $n = 2$ Bessel function.

We only use two terms from the continued fraction expansion of $\coth s$.

$$\coth s = \frac{1}{s} + \frac{1}{\frac{3}{s}} = \frac{3 + s^2}{3s} = \frac{N_s(s)}{D_2(s)} \tag{6.11-377}$$

To approximate $e^s = \sinh s + \cosh s$ using

$$N_s(s) + D_2(s) = s^2 + 3s + 3 = \mathcal{B}_{\in}(\int) \tag{6.11-378}$$

Similarly, for $n = 3$

$$\coth s = \frac{1}{s} + \frac{1}{\frac{3}{s} + \frac{1}{\frac{5}{s}}} \tag{6.11-379}$$

$$= \frac{6s^2 + 15}{s^3 + 15s} = \frac{N_3(s)}{D_3(s)} \tag{6.11-380}$$

$$\mathcal{B}_{\ni}(\int) = N_3(s) + D_3(s) = s^3 + 6s^2 + 15s + 15 \tag{6.11-381}$$

A recursive formula may be used to generate the Bessel polynomial for any value of n.

$$\mathcal{B}_{\backslash}(\int) = (\int\backslash - \infty)\mathcal{B}_{\backslash-\infty}(\int) + \int^{\in}\mathcal{B}_{\backslash-\in}(\int) \tag{6.11-382}$$

To form the transfer function of the Bessel-Thomson response

$$H_n(s) = \frac{\mathcal{B}_{\backslash}(\prime)}{\mathcal{B}_{\backslash}(\int)} \tag{6.11-383}$$

in which $H_n(j0) = 1$, and n is the order of the filter.

It is called the Bessel-Thomson response because W. E. Thomson was one of the first to use this approach.

The Bessel polynomials that can be computed using Equation 6.11-382 are listed below upto $n = 4$.

$$\mathcal{B}_\prime = \infty \qquad \mathcal{B}_\prime(\prime) = \infty \tag{6.11-384}$$

$$\mathcal{B}_\infty = \int + \infty \qquad \mathcal{B}_\infty(\prime) = \infty \tag{6.11-385}$$

$$\mathcal{B}_\in = \int^\in + \ni\int + \ni \qquad \mathcal{B}_\in(\prime) = \ni \tag{6.11-386}$$

$$B_3 = s^3 + 6s^2 + 15s + 15 \qquad \mathcal{B}_\ni(\prime) = \infty\bigtriangledown \tag{6.11-387}$$

$$B_4 = s^4 + 10s^3 + 45s^2 + 105s + 105 \qquad \mathcal{B}_\triangle(\prime) = \infty\prime\bigtriangledown \tag{6.11-388}$$

Unlike for the Butterworth and Chebyshev responses, there is no simple rule to determine the roots of $\mathcal{B}_\backslash(\int) = \prime$, which are the poles of the Bessel-Thomson response. However, they can be ound by computer methods.

$$\mathcal{B}_\infty(\int) = s + 1 \tag{6.11-389}$$

$$\mathcal{B}_\in(\int) = s^2 + 3s + 3 \tag{6.11-390}$$

$$\mathcal{B}_\ni(\int) = (s^2 + 3.67782s + 6.45944)(s + 2.32219) \tag{6.11-391}$$

$$\mathcal{B}_\triangle(\int) = (s^2 + 5.79242s + 9.14013)(s^2 + 4.207585 + 11.4878) \tag{6.11-392}$$

A cut-and-try method has to be used for the Bessel-Thomson filter design, based on, normally, the specifications of the group delay.

REMARK 6.11-1 *1. From the formation procedure of the Bessel polynomial (hence the filter) using the Storch method, namely, the continued fraction expansion, the Bessel polynomial is strictly Hurwitz. Thus the filter is stable.*

2. Since the Bessel filter is designed based on the delay response, the magnitude response of the filter will be poor compared with other types of filter.

3. The low-pass to low-pass transformation can be used to adjust the cut-off to arbitrary value. However, none of the other transformations; to high-pass, band-pass or band-stop are valid since these transformations will distort the delay characteristics.

Chapter 7

Digitization of Analog Signals

Contents

The conversion of an analog signal (continuous-time, continuous magnitude) to a digital signal involves

sampling — Approximation in time,

quantization — Approximation in magnitude, and

coding — Representation.

Although sampling of an analog signal does not have to be at regular time intervals, the periodic sampling is commonly used in most of the cases. T is called the sampling period.

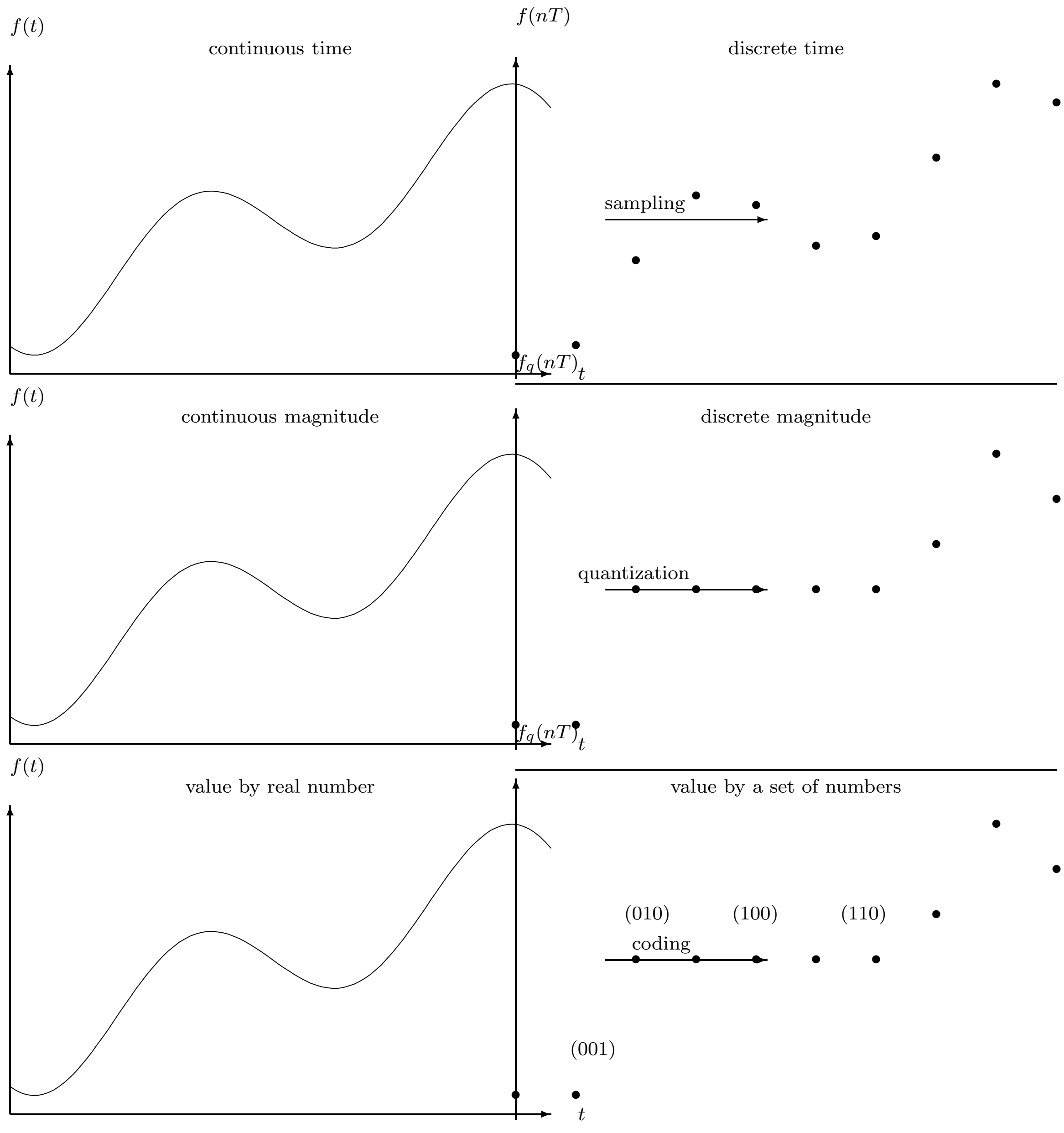


Figure 7.1: Sampling, quantization and coding of analog signals

Uniform and non-uniform quantization may be used at the quantization operation and linear or non-linear coding may also be used at the encoding stage.

In practice, the entire process of sampling, quantization and encoding is usually called analog-to-digital (A/D) conversion.

7.1 Sampling

7.1.1 Sampling using the ideal impulse

Given an analog signal $f(t)$, it may be sampled using a periodic train of impulses

$$\delta_\infty(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{7.1-1}$$

and the sampled function $f_s(t)$ is presented as follows

$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{7.1-2}$$

Recalling the property of the unit impulse

$$f(t)\delta(t - \alpha) = f(\alpha)\delta(t - \alpha) \tag{7.1-3}$$

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT) \tag{7.1-4}$$

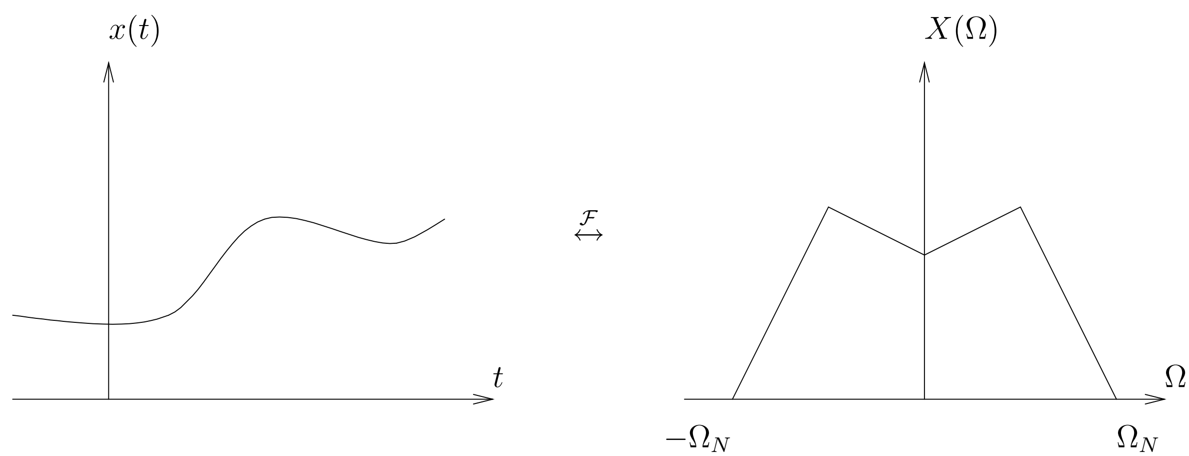
The operation is called the impulse modulation and it is depicted in Figure 7.2.

Assume the Fourier spectrum of the function $f(t)$ is $F(\omega)$ which is band limited to ω_m,

$$|F(\omega)| = 0 \qquad \text{for } |\omega| > \omega_m \tag{7.1-5}$$

we shall find out what effects that the sampling operation will have on the spectrum of the given signal

$$f(t) \overset{\mathcal{F}}{\leftrightarrow} F(\omega) \tag{7.1-6}$$



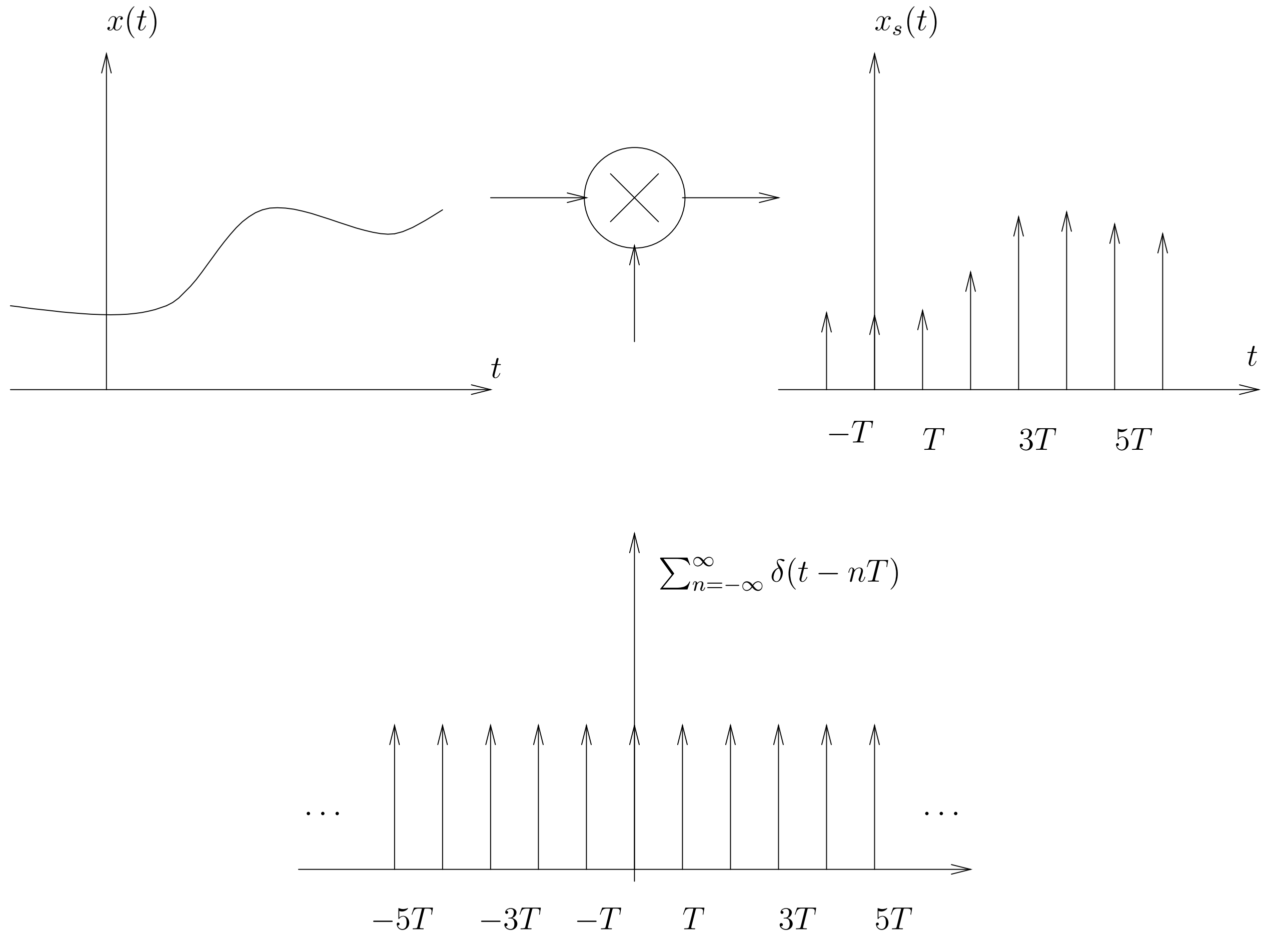


Figure 7.2: Impulse modulation

Find the Fourier transform of $f_s(t)$. Using the property of the Fourier transform (convolution-in-frequency domain)

$$f_1(t)f_s(t) \leftrightarrow \frac{1}{2\pi}F_1(\omega) * F_2(\omega) \tag{7.1-7}$$

$$\mathcal{F}[\{_f(\sqcup)] = \mathcal{F}[\{(\sqcup)\delta_\infty(\sqcup)] \tag{7.1-8}$$

$$= \frac{1}{2\pi}\mathcal{F}[\{(\sqcup)] * \mathcal{F}\left[\sum_{\backslash=-\infty}^{\infty} \delta(\sqcup - \backslash\mathcal{T})\right] \tag{7.1-9}$$

where

$$\mathcal{F}[\{(\sqcup)] = F(\omega) \tag{7.1-10}$$

and

$$\mathcal{F}\left[\sum_{\backslash=-\infty}^{\infty} \delta(\sqcup - \backslash\mathcal{T})\right] = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \tag{7.1-11}$$

$$\omega_0 = \frac{2\pi}{T} \tag{7.1-12}$$

ω_0 is the sampling frequency and T is the sampling period.

Thus:

$$F_s(\omega) = \mathcal{F}[\{_f(\sqcup)] = \frac{1}{2\pi}F(\omega) * \left\{\frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)\right\} \tag{7.1-13}$$

$$= \frac{1}{T}\left\{F(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)\right\} \tag{7.1-14}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_0) \tag{7.1-15}$$

(Noticing:

$$F(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega - v)\delta(v - n\omega_0)dv \tag{7.1-16}$$

$$= \sum_{n=-\infty}^{\infty} F(\omega - n\omega_0). \tag{7.1-17}$$

)

From Equation (7.1-15) the spectrum of the sampled signal consists of the periodic extension of the spectrum $F(\omega)$ of the analog signal $f(t)$.

From Figure 7.3, when the sampling frequency $\omega_0 \geqslant 2\omega_m$ where ω_m is the highest frequency component of the band-limited spectrum $F(\omega)$ of the $f(t)$, $F(\omega)$ can be retrieved from $F_s(\omega)$, if an ideal low-pass filter $H(\omega)$ is applied. Thus $f(t)$ can be fully recovered, by inverse Fourier tranforming $F(\omega) = H(\omega)F_s(\omega)$.

Definition 7.1-1 (Nyquist frequency) *The minimum sampling rate required to prevent aliasing is twice the highest frequency component ω_m in the spectrum $F(\omega)$ of $f(t)$. This minimum sampling rate is defined as the Nyquist frequency ω_N, i.e.*

$$\omega_N = 2\omega_m \tag{7.1-18}$$

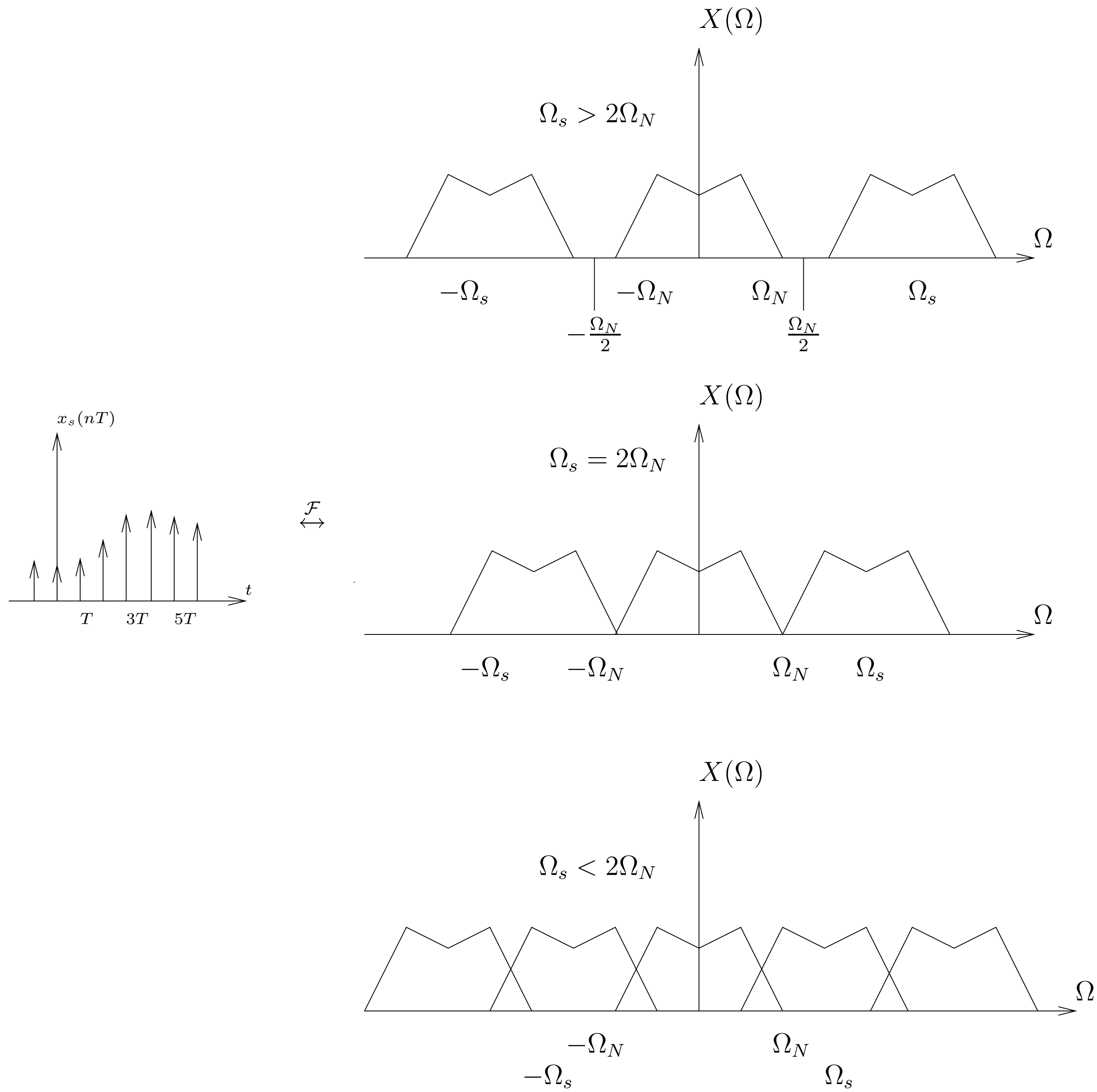


Figure 7.3: Effects of changing sampling rate

7.1.1.1 The sampling theorem

A signal $f(t)$ whose spectrum is band-limited to below a frequency ω_m, can be completely recovered from its samples $\{f(nT)\}$ taken at a rate

$$f_N = \frac{\omega_N}{2\pi} = \frac{1}{T_N} \qquad \text{where } \omega_N = 2\omega_m \tag{7.1-19}$$

The signal $f(t)$ is determined from its sample values $\{f(nT)\}$ by

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin \omega_m (t - nT)}{\omega_m (t - nT)} \tag{7.1-20}$$

where

$$T = \frac{\pi}{\omega_m} = \frac{2\pi}{\omega_N} = \frac{1}{f_N} \tag{7.1-21}$$

Proof:

If the sampling rate is greater than the Nyquist frequency $\omega_N = 2\omega_m$, $F(\omega)$ can be recovered from $F_s(\omega)$ by using an ideal low-pass filter of magnitude T and cut-off at ω_m as shown in Figure 7.4.

$$H(j\omega) = \begin{cases} T & |\omega| \leqslant \omega_m \\ 0 & |\omega| > \omega_m \end{cases} \tag{7.1-22}$$

$$F(\omega) = F_s(\omega) H(\omega) \tag{7.1-23}$$

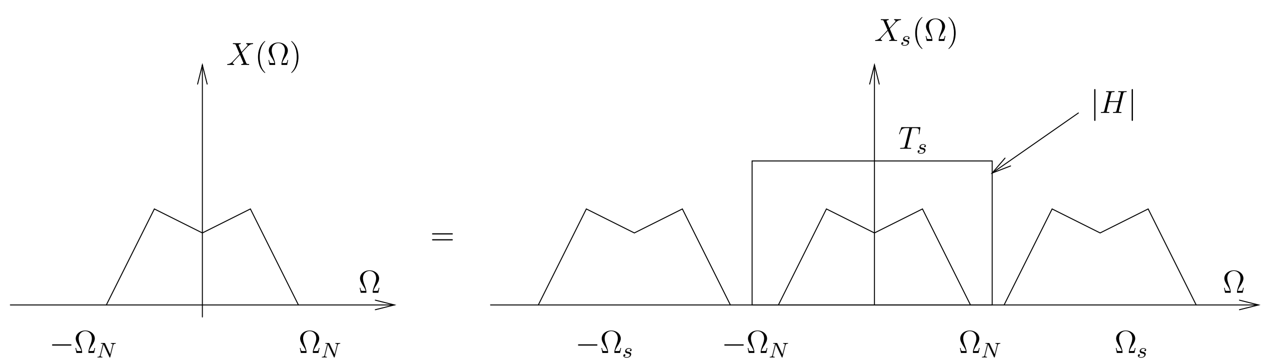


Figure 7.4: Low-pass reconstruction filter

That is

$$f(t) = \mathcal{F}^{-\infty}[\mathcal{F}_f(\omega)] * \mathcal{F}^{-\infty}[\mathcal{H}(\omega)] \tag{7.1-24}$$

$$= f_s(t) * h(t) \tag{7.1-25}$$

where $h(t)$ is the impulse response of the ideal low-pass filter $H(\omega)$.

$$H(\omega) = Te^{-jk\omega}|_{k=0} \tag{7.1-26}$$

$$\hat{h}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{jt\omega}d\omega \tag{7.1-27}$$

$$= \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} Te^{-jk\omega}e^{jt\omega}d\omega \tag{7.1-28}$$

$$= \frac{T}{2\pi}\int_{-\omega_m}^{\omega_m} e^{j(t-k)\omega}d\omega \tag{7.1-29}$$

$$= \frac{T}{2\pi}\frac{1}{j(t-k)}e^{j(t-k)\omega}\Big|_{-\omega_m}^{\omega_m} \tag{7.1-30}$$

$$= \frac{T}{2\pi}\frac{1}{j(t-k)}\left\{e^{j(t-k)\omega_m} - e^{-j(t-k)\omega_m}\right\} \tag{7.1-31}$$

$$\begin{aligned} &= \frac{T}{2\pi}\frac{1}{j(t-k)}\{\cos[(t-k)\omega_m] + j\sin[(t-k)\omega_m] - \\ &\qquad \cos[(t-k)\omega_m] + j\sin[(t-k)\omega_m]\} \end{aligned} \tag{7.1-32}$$

$$= \frac{T}{2\pi}\frac{1}{j(t-k)}2j\sin[(t-k)\omega_m] \tag{7.1-33}$$

$$= \frac{T}{\pi(t-k)}\sin[(t-k)\omega_m] \tag{7.1-34}$$

$$h(t) = \hat{h}(t)\Big|_{k=0} = T\frac{\sin(\omega_m t)}{\pi t} \tag{7.1-35}$$

$$= \frac{\sin(\omega_m t)}{\omega_m t} \tag{7.1-36}$$

$$f(t) = f_s(t) * h(t) \tag{7.1-37}$$

$$= \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} f(nT)\delta(\tau - nT)\right)\frac{\sin[\omega_m(t-\tau)]}{\omega_m(t-\tau)}d\tau \tag{7.1-38}$$

$$= \sum_{n=-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(nT)\frac{\sin[\omega_m(t-\tau)]}{\omega_m(t-\tau)}\delta(\tau - nT)d\tau\right\} \tag{7.1-39}$$

$$= \sum_{n=-\infty}^{\infty} f(nT)\frac{\sin[\omega_m(t-nT)]}{\omega_m(t-nT)} \tag{7.1-40}$$

Equation (7.1-20) or (7.1-40) is essentially a formula for the interpolation of the signal values by its values at the sampling points.

Remark 7.1-1

1. *$f(t)$ is recoverable from its sample values if and only if the sampling theorem is satisfied.*

2. *In practice, we cannot expect all signals to have a limited band spectrum. Therefore the spectrum of the signal is made band-limited to a frequency $\omega_N/2$ prior to sampling at ω_N. The prefiltering is required to accomplish this task.*

3. *Because the ideal filter is physically unrealizable, in practice the sampling frequency is chosen to be higher than the Nyquist rate. (Speech signals are band limited to 3.4 kHz and sampled at 8 kHz. PCM uses 7 bits plus 1 bit parity → 64 kbits/s → 64 kHz which is the channel bandwidth allocated for telephone lines).*

4. *For a band-pass signal the spectrum of which lies in the range*

$$\omega_1 < |\omega| < \omega_2 \tag{7.1-41}$$

the Nyquist frequency ω_N is given by

$$\omega_N = 2(\omega_2 - \omega_1) \tag{7.1-42}$$

and the band-pass filter must be used for the reconstruction of the signal.

7.1.2 Practical sampling functions

Since the ideal impulse train is not feasible in practice, a periodic train of rectangular pulse is used in its place.

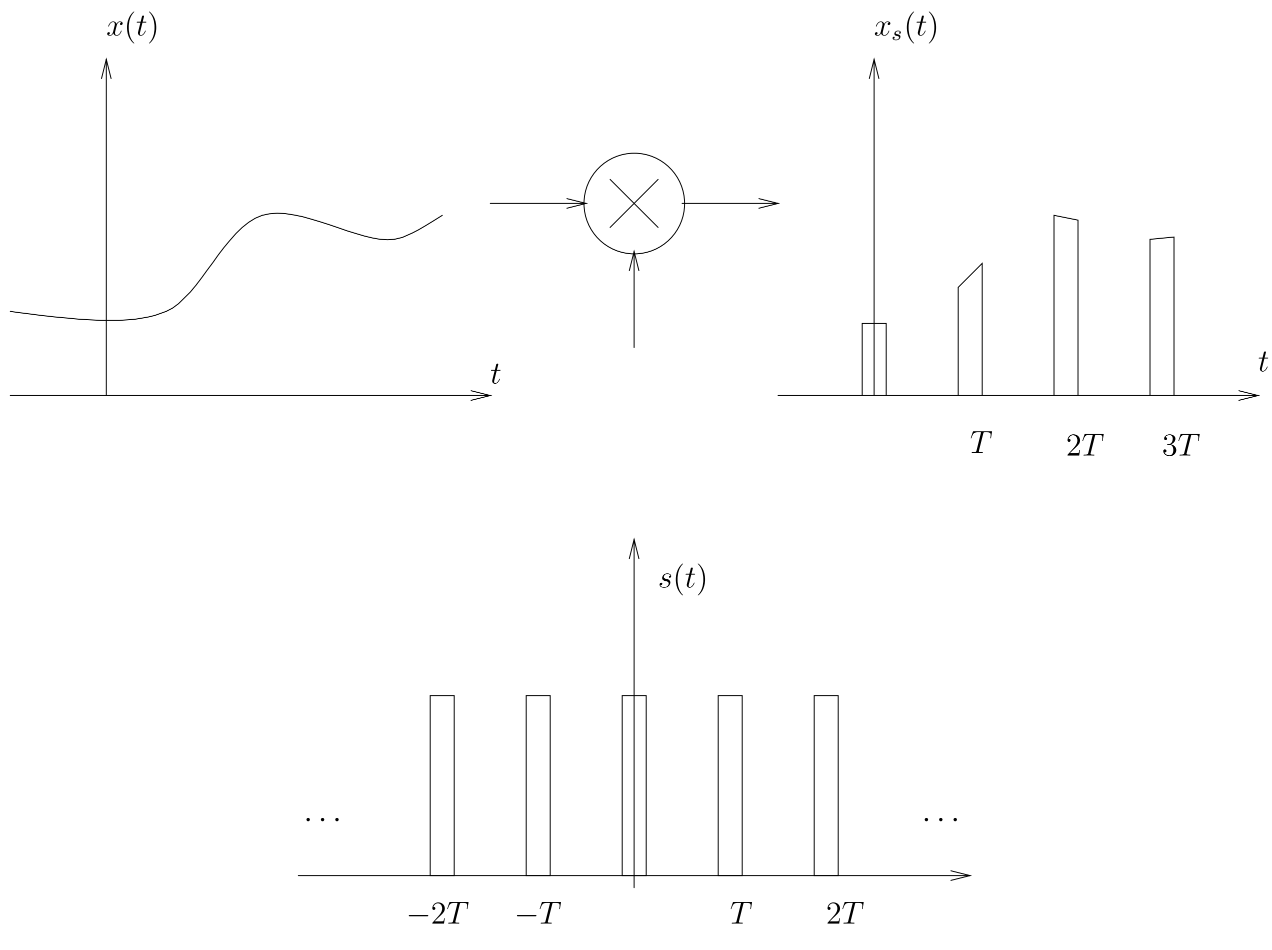


Figure 7.5: Rectangular pulse train

In Figure 7.5, τ is the pulse width and $T = 2\pi/\omega_0$ is the sampling period of the function $s(t)$.

$$f_s(t) = f(t)s(t) \tag{7.1-43}$$

The Fourier series of $s(t)$ is given by

$$s(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0} \tag{7.1-44}$$

where

$$C_k = \frac{\tau}{T} \frac{\sin(k\pi\tau/T)}{k\pi\tau/T} \tag{7.1-45}$$

Therefore

$$f_s(t) = f(t)s(t) \tag{7.1-46}$$

$$= \sum_{k=-\infty}^{\infty} C_k f(t) e^{jk\omega_0 t} \tag{7.1-47}$$

The Fourier transform of $f_s(t)$ is given by

$$F_s(\omega) = \mathcal{F}[\{_f(\sqcup)] \tag{7.1-48}$$

$$= \int_{-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} C_k f(t) e^{jk\omega_0 t} \right\} e^{-j\omega t} dt \tag{7.1-49}$$

$$= \sum_{k=-\infty}^{\infty} C_k \underbrace{\int_{-\infty}^{\infty} f(t) e^{-j(\omega - k\omega_0)t} dt}_{F(\omega - k\omega_0)} \tag{7.1-50}$$

$$= \sum_{k=-\infty}^{\infty} C_k F(\omega - k\omega_0) \tag{7.1-51}$$

From Equation (7.1-51), apart from a coefficient C_k, it has the same form as Equation (7.1-15). That is, the sampling theorem still holds.

$$(\qquad C_0 = \frac{\tau}{T} \left. \frac{\frac{\pi\tau}{T} \cos \frac{k\pi\tau}{T}}{\frac{\pi\tau}{T}} \right|_{k=0} \tag{7.1-52}$$

$$= \frac{\tau}{T} \qquad) \tag{7.1-53}$$

As well, from Equation (7.1-51), it is obvious the actual shape of the sampling pulses is immaterial, the theorem is always valid.

7.2 Windowed Signals

It is often the case in digital signal processing that only part of a waveform can be captured or stored for processing.

Definition 7.2-1 (Windowing) *The operation of multiplying the true signal by a rectangular function of appropriate duration is called windowing.*

Remark 7.2-1

1. *The wider the window, the more closely the windowed spectrum approaches that of the original continuous signal.*

2. *The effect of windowing is to smear out the spectrum.*

Definition 7.2-2 (Leakage) *Leakage is used to refer to the spectral spreading caused by time domain windowing.*

Different windows may be applied to reduce the leakage caused by time domain windowing. Examples of windowing can be seen in Figure 7.6 and Figure 7.7.

7.3 Quantization

Definition 7.3-1 *Quantization is defined as an approximation of each sample value $f(nT)$ by an integer multiple of a basic quantity q. q is called the quantizing step. The system which does the quantization is called a quantizer as shown in Figure 7.8.*

Two quantization methods are rounding and truncation.

The sampled signal $f(nT)$ can be represented by a sum of the quantizer output $f_q(nT)$ and an error signal $\mathcal{E}(\backslash\mathcal{T})$, i.e.

$$f(nT) = f_q(nT) + \mathcal{E}(\backslash\mathcal{T}) \tag{7.3-54}$$

7.3.1 Rounding

When rounding is used in the quantization, each sample value $f(nT)$ is rounded to nq where $(n-\frac{1}{2})q \leqslant f(nT) < (n+\frac{1}{2})q$.

The maximum error $|\mathcal{E}(\backslash\mathcal{T})| \leqslant \frac{\text{II}}{\in}$.

7.3.2 Truncation

When truncating is used in the quantization, each sample value $f(nT)$ is truncated to nq where $nq \leqslant f(nT) < (n+1)q$.

The maximum error $|\mathcal{E}(\backslash\mathcal{T})| < \text{II}$.

7.4 Encoding and Binary Number Representation

The sampled and quantized signal can be represented by a set of numbers which are almost always in binary form.

A given number N can be represented with finite precision in the following general form

$$N = \sum_{i=-m}^{n} C_i r^2 \tag{7.4-55}$$

where C_i is the ith coefficient and r is the radix and $0 \leqslant C_i \leqslant (r-1)$.

It can also be written as

$$N = (C_n C_{n-1} \cdots C_1 C_0 . C_{-1} C_{-2} \cdots C_{-m})_r \tag{7.4-56}$$

If $r = 10$, it is the decimal system. If $r = 2$ it is the binary system.

Multiplication and division methods can be used for the conversion.

To implement arithmetic operations on binary numbers using the finite word length, two number representations may be used, i.e. the fixed-point numbers and floating-point numbers.

Thus the names fixed-point arithmetic and floating-point arithmetic.

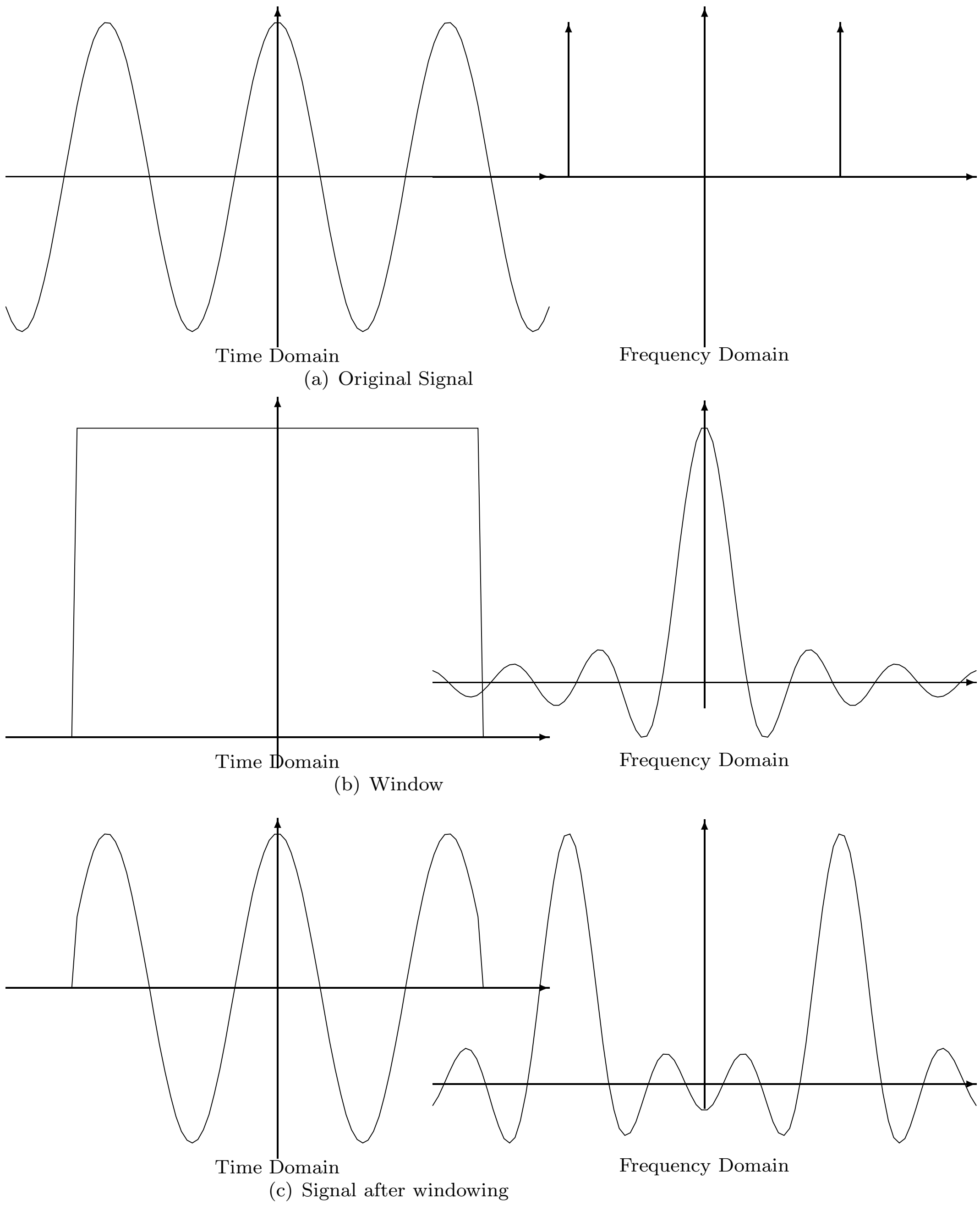
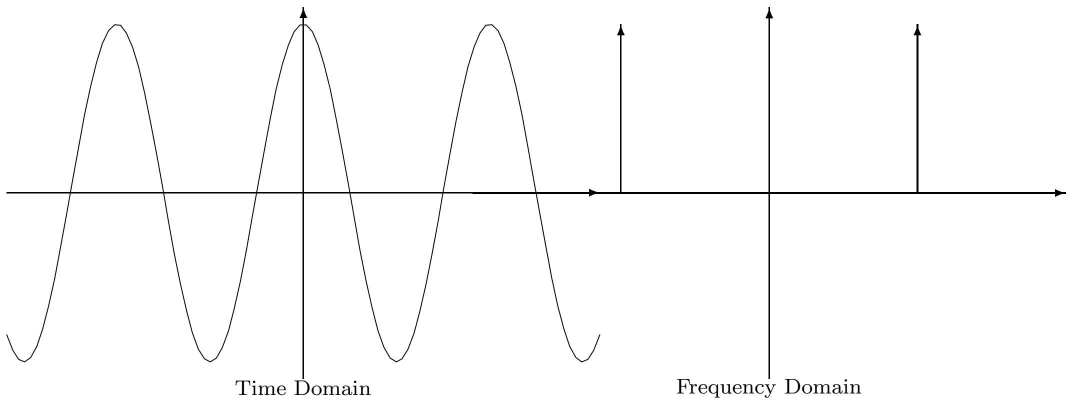
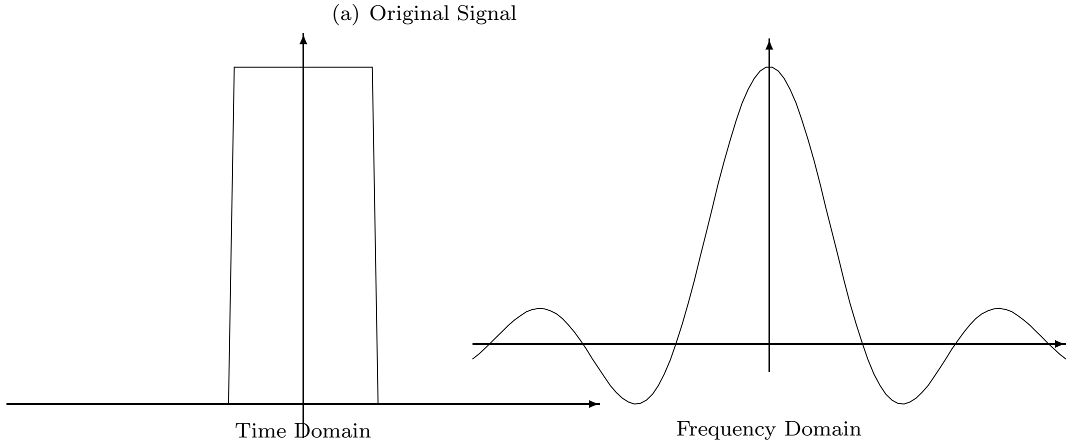


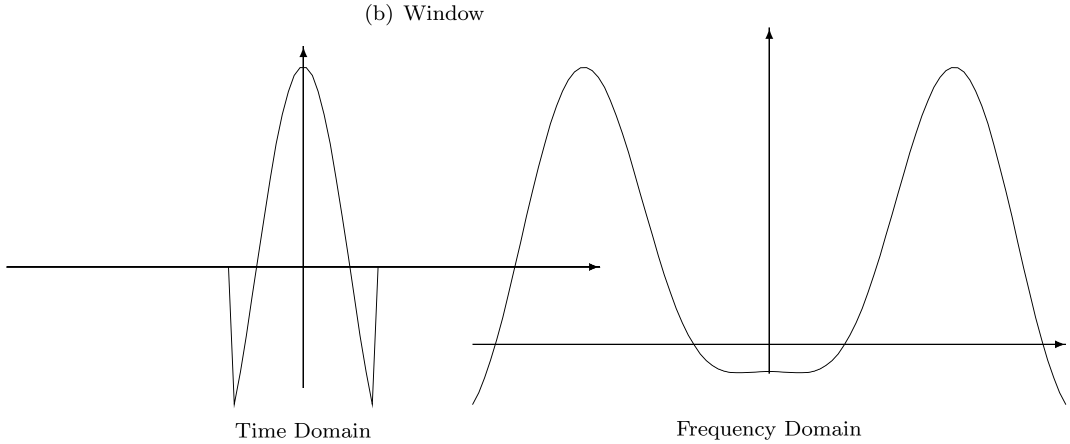
Figure 7.6: Windowing with wide rectangular window



(a) Original Signal



(b) Window



(c) Signal after windowing

Figure 7.7: Windowing with narrow rectangular window

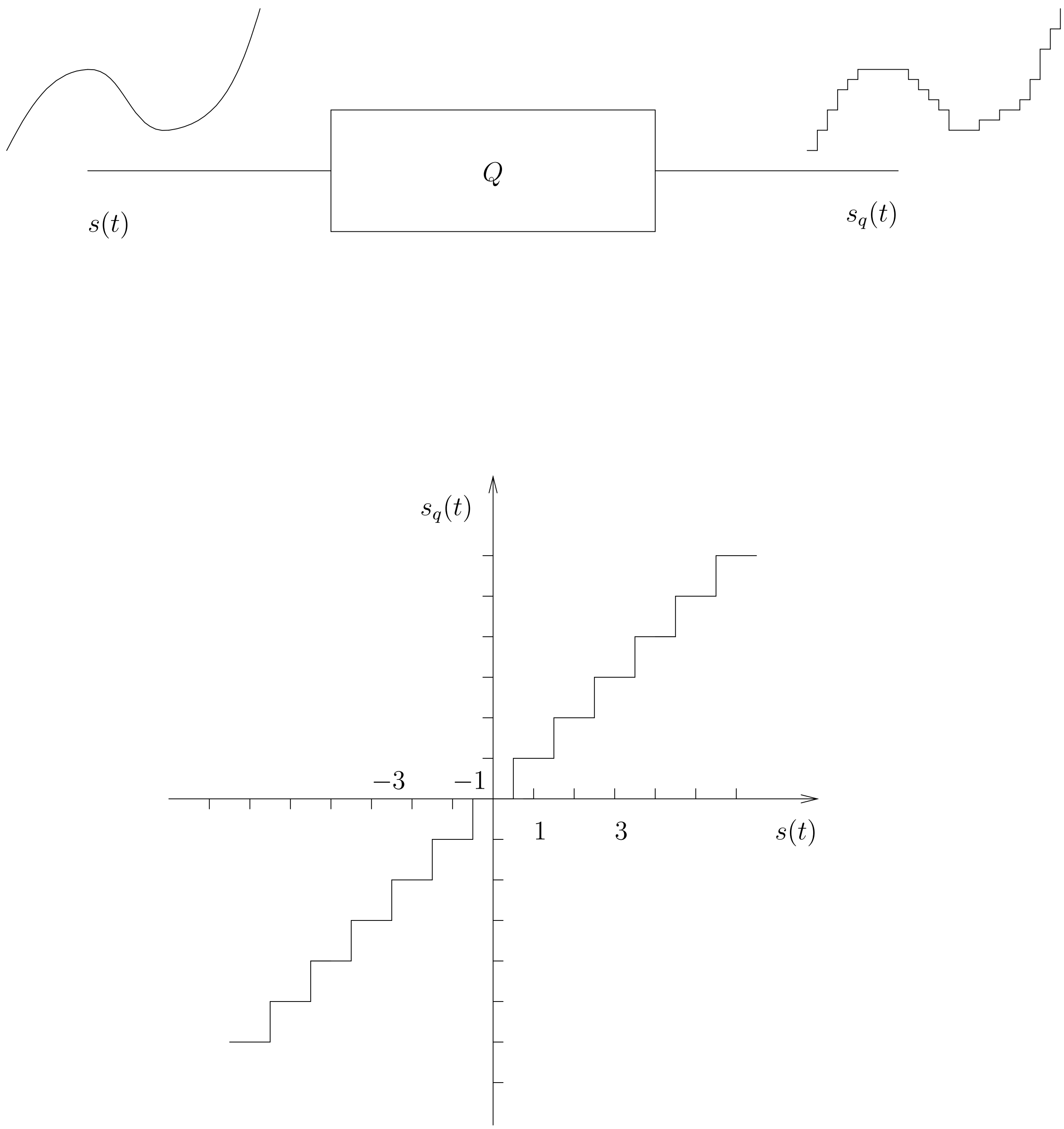


Figure 7.8: Quantization operation

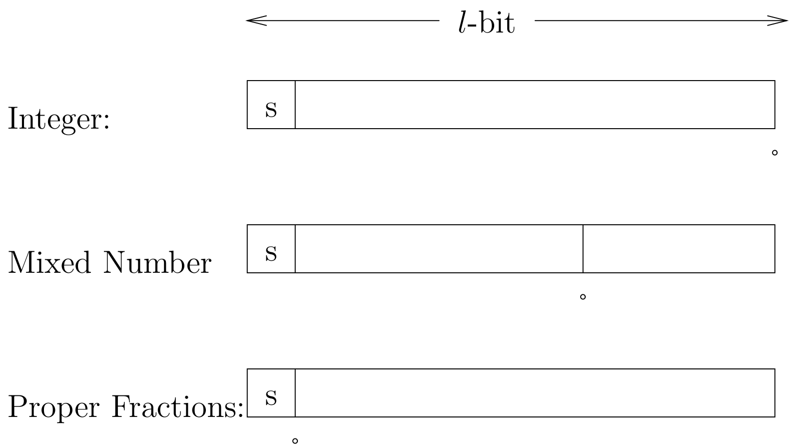


Figure 7.9: Fixed-point Number Representation

7.4.1 Fixed-point arithmetic

Using l-bit word, a fixed-point number may be represented as shown in Figure 7.9.

In all cases, the most significant bit (MSB) is used as the sign bit.

Multiplication of two integers results in an increased word-length which cannot be reduced by rounding or quantization.

Mixed numbers are difficult to multiply.

For one's complement representation,

$$N_{OC} = \begin{cases} N & \text{for } N \geqslant 0 \\ 2 - 2^{-l} - |N| & \text{for } N < 0 \end{cases} \tag{7.4-57}$$

where N is a proper fraction within the range $-(1-2^{-l}) \leqslant N \leqslant 1-2^{-l}$ and l is the word-length with sign-bit excluded.

For two's complement representation

$$N_{TC} = \begin{cases} N & \text{for } N \geqslant 0 \\ 2 - |N| & \text{for } N < 0 \end{cases} \tag{7.4-58}$$

where N is a proper fraction within the range $-1 \leqslant N \leqslant 1-2^{-l}$ and l is the word-length with the sign-bit excluded.

Remarks on the three types of fixed-point arithmetic:

1. Sign-magnitude addition requires sign checks, complementing and end-around carry.

2. Special algorithms are required for one's and two's complement arithmetic multiplication.

As a result, fixed-point arithmetic uses a format which represents proper fractions only. (Multiplication of two proper fractions is still a proper fraction, $|N| < 1$ and the product can be rounded off to maintain a pre-set word length.) The rage of the numbers under this arrangement is:

$$-1 \leqslant N \leqslant 1 - 2^{-l} \tag{7.4-59}$$

where l is the word-length (sign-bit excluded).

Example 7.4-1

One's Complement	Two's Complement	Numeric Value N
0 . 1 1	0.1 1	3/4
0 . 1 0	0.1 0	1/2
0 . 0 1	0.0 1	1/4
0 . 0 0 / 1 . 1 1	0.0 0	0
1 . 1 0	1.1 1	-1/4
1 . 0 1	1.1 0	-1/2
1 . 0 0	1.0 1	-3/4
	1.0 0	-1

Example 7.4-2 *Given $l = 2$, find $(-\frac{1}{4} - \frac{1}{2})$.*

One's complement arithmetic (end-around carry)

```
      1.10   -1/4
  +   1.01   -1/2
  ----------------
  1   0.11
         . 1
  ----------------
      1.00   -3/4
```

End-around carry: A carry bit at most significant position is added at the least significant position

Two's complement arithmetic

```
      1.11   -1/4
  +   1.10   -1/2
  ----------------
  1   1.01
  ----------------
      1.01   -3/4
```

A carry bit at the most significant position is discarded.

7.4.2 Number Quantization

In the finite-word-length computation, there are two problems.

1. Overflow occurs when the result of an addition operation cannot be accommodated by the given word-length.
2. Round off operation is required if a pre-set word-length is to be maintained after multiplication.

Again, rounding or truncation may be used for the round off.

In truncation, all bits after lth are simply discarded.

In rounding, if the $(l + 1)$th bit is 0, discard $(l + 1)$th as well as any bits after it, where l is the word-length, excluding the sign bit. If the $(l + 1)$th bit is 1, add 1 to the lth bit and discard any bits after it.

This is a simple rounding towards $+\infty$:

$$\begin{array}{c} 1.6 \rightarrow 2 \\ 1.5 \rightarrow 2 \\ 1.4 \rightarrow 1 \\ -1.6 \rightarrow -2 \\ -1.5 \rightarrow -1 \\ -1.4 \rightarrow -1 \end{array} \tag{7.4-60}$$

7.4.3 Floating point numbers

7.4.3.1 Representation

A number N can be represented in the floating-point form

$$N = 2^c M \tag{7.4-61}$$

where c is the exponent, M is the mantissa. c is an integer, M is usually normalized to the range

$$\frac{1}{2} \leqslant M \leqslant 1 \tag{7.4-62}$$

Example 7.4-3

$$\begin{aligned}(2.5)_{10} &= 2^{10.0}0.101 &(7.4\text{-}63)\\ &= 2^2(2^{-1}+2^{-3}) &(7.4\text{-}64)\\ &= 2+2^{-1} &(7.4\text{-}65)\\ &= 2.5 &(7.4\text{-}66)\\ (1.25)_{10} &= 2^{1.0}0.101 &(7.4\text{-}67)\end{aligned}$$

7.4.3.2 Multiplication and division

Given $N_1 = 2^{c_1}M_1$ and $N_2 = 2^{c_2}M_2$.

$$\begin{aligned}N_1 \times N_2 &= 2^{c_1}M_1 \times 2^{c_2}M_2 &(7.4\text{-}68)\\ &= 2^{c_1+c_2}(M_1 \times M_2) &(7.4\text{-}69)\end{aligned}$$

and

$$\frac{N_1}{N_2} = \frac{2^{c_1}M_1}{2^{c_2}M_2} = 2^{c_1-c_2}\frac{M_1}{M_2} \tag{7.4-70}$$

7.4.3.3 Addition

The floating point addition requires the adjustment of the mantissa of the smaller number until the exponents c_1 and c_2 are equal. The result is usually normalized.

Example 7.4-4 (Floating-point binary representation) *Given*

$$N_1 = 2^{10.0}0.101 = (2.5)_{10} \tag{7.4-71}$$

and

$$N_2 = 2^{1.0}0.101 = (1.25)_{10} \tag{7.4-72}$$

1.

$$\begin{aligned}N_1 \times N_2 &= 2^{10.0+1.0}\{0.101 \times 0.101\} &(7.4\text{-}73)\\ &= 2^{11.0}0.011001 &(7.4\text{-}74)\\ &= 2^{11.0}2^{-1}0.11001 &(7.4\text{-}75)\\ &= 2^{10.0}0.11001 = (3.125)_{10} &(7.4\text{-}76)\end{aligned}$$

2.

$$\begin{aligned}N_1 + N_2 &= 2^{10.0}0.101 + 2^{1.0}0.101 &(7.4\text{-}77)\\ &= 2^{10.0}0.101 + 2^{10.0}2^{-1}0.101 &(7.4\text{-}78)\\ &= 2^{10.0}\{0.101 + 0.0101\} &(7.4\text{-}79)\\ &= 2^{10.0}0.1111 = (3.75)_{10} &(7.4\text{-}80)\end{aligned}$$

3. Rounding

$$2^{10.0}1.000 = 2^{11.0}0.100 = (4)_{10} \tag{7.4-81}$$

4. Truncation

$$2^{10.0}0.111 = (3.5)_{10} \tag{7.4-82}$$

7.4.4 Comparison between the finite-word-length-arithmetics

1. In the fixed-point arithmetic multiplication results in round off errors, while addition will introduce overflow.

2. In the floating-point arithmetic, round off errors exist in both multiplication and addition.

3. The valid dynamic range in the fixed-point arithmetic is small

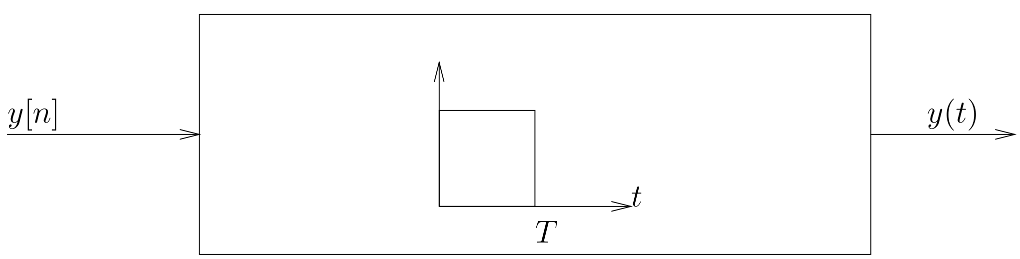
$$-1 \leqslant N \leqslant 1 - 2^{-l} \tag{7.4-83}$$

while the dynamic range in the floating-point arithmetic is large. If the exponent is 6 signed bit integer

$$2^{-64} \leqslant |N| < 2^{63} \tag{7.4-84}$$

7.5 Reconstruction Filter

To reconstruct an analog signal after digital processing, a conversion is required to change the signal into its analog for, A zero-order hold D/A converter may be used



$$g(t) = \sum_n g(n)h(t - nT) \tag{7.5-85}$$

The Fourier spectrum of the zero-order hold D/A converter is shown in Figure 7.10 while the input and output of the D/A converter is shown in Figure 7.11.

Obviously, a post-filter is required to help the reconstruction of the analog signal.

Since the Fourier transform of the zero-order hold D/A is given by

$$H(j\omega)\mathcal{F}[\langle(\sqcup)] = \int_{,}^{\mathcal{T}} \rceil^{-|\omega\sqcup} \lceil\sqcup = -\frac{\infty}{|\omega} \rceil^{-|\omega\sqcup} \Big|_{,}^{\mathcal{T}} \tag{7.5-87}$$

$$= \frac{j}{\omega}\left(e^{-j\omega T} - 1\right) \tag{7.5-88}$$

$$= \frac{j}{\omega} e^{-j\frac{\omega T}{2}} \left(e^{-\frac{j\omega T}{2}} - e^{\frac{j\omega T}{2}}\right) \tag{7.5-89}$$

$$= \frac{j}{\omega} e^{-\frac{j\omega T}{2}} \left(-2j \sin\frac{\omega T}{2}\right) \tag{7.5-90}$$

$$= \frac{2\sin\left(\frac{\omega T}{2}\right)}{\omega} e^{-\frac{j\omega T}{2}} \tag{7.5-91}$$

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \tag{7.5-86}$$

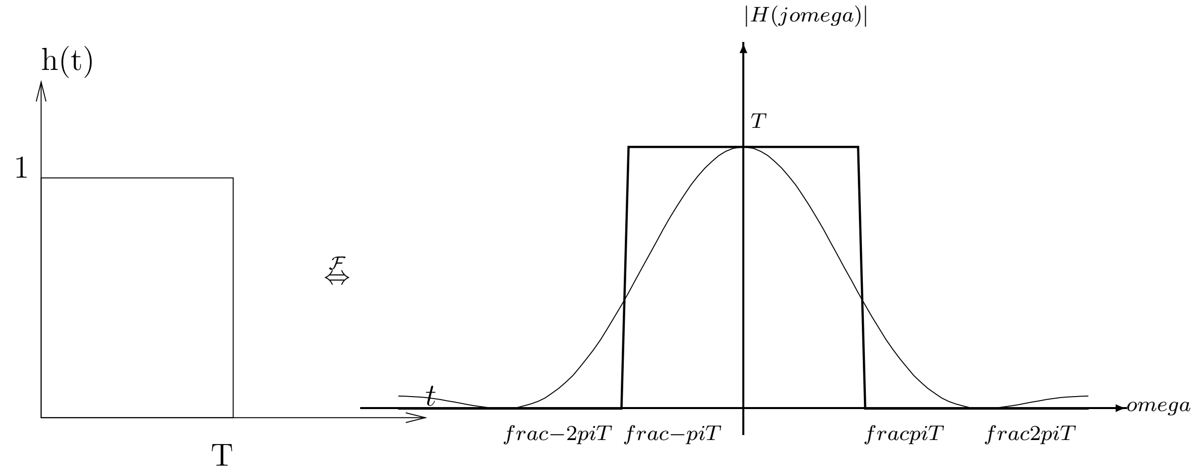


Figure 7.10: Zero-order-hold D/A converter

If we are going to implement an ideal low-pass interpolating filter $H_r(j\omega)$, as shown in Figure 7.12 we need a post filter after the D/A converted which makes

$$\underbrace{H_r(j\omega)}_{\text{Reconstruction filter}} = \overbrace{H(j\omega)}^{\text{zero-order hold D/A converter}} \cdot \underbrace{\tilde{H}_r(j\omega)}_{\text{post filter}} \tag{7.5-92}$$

where

$$\tilde{H}_r(j\omega) = \begin{cases} \frac{(\omega T/2)}{\sin\left(\frac{\omega T}{2}\right)} e^{\frac{j\omega T}{2}} & |\omega| < \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases} \tag{7.5-93}$$

For more information/discussion, refer to "Discrete-Time Signal Processing", by A. V. Oppenheim and R. W. Schafer.

7.6 The Concept of Frequency in Continuous-Time and Discrete-Time Signals

7.6.1 Continuous-time Sinusoidal Signals

A continuous-time sinusoidal signal $x_a(t)$ can be used to describe a simple harmonic oscillation

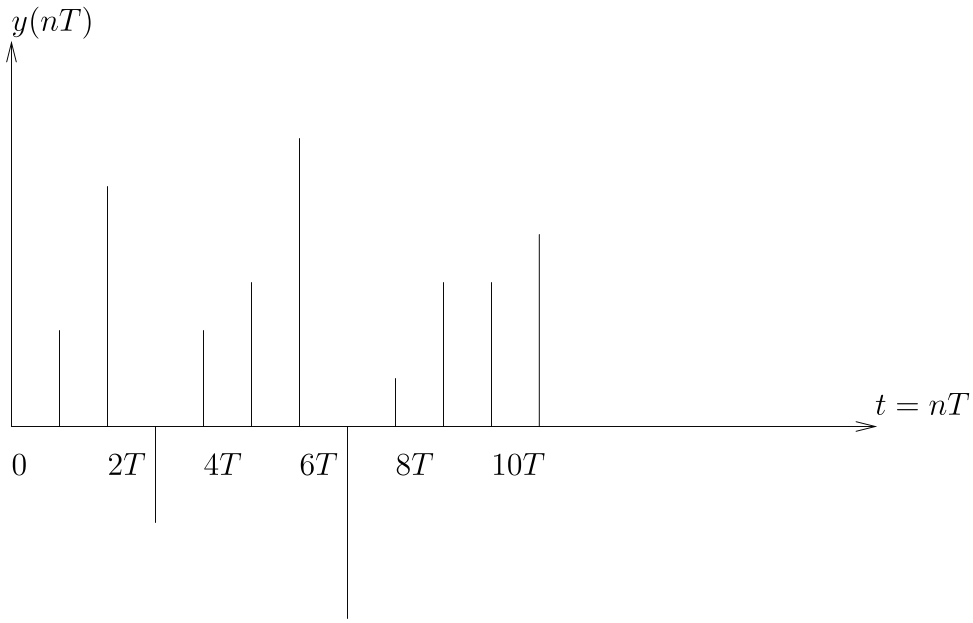
$$x_a = A\cos(\Omega t + \theta) \tag{7.6-94}$$

where $-\infty < t < \infty$, A is the amplitude, Ω is the frequency in rad/s and θ is the phase in radians.

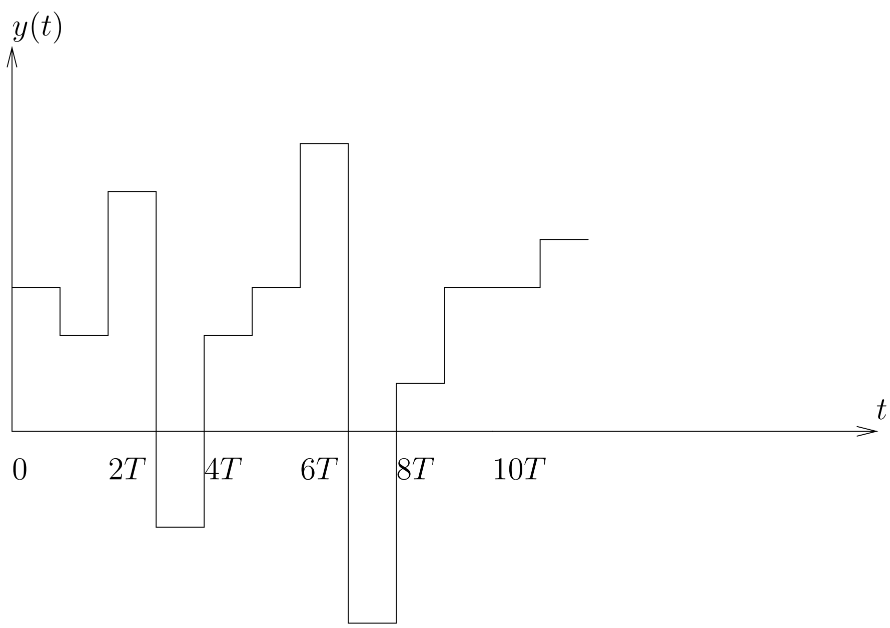
$$\Omega = 2\pi F \tag{7.6-95}$$

where F is the frequency in cycles per second or Hertz (Hz). Hence

$$x_a(t) = A\cos(2\pi F t + \theta) \tag{7.6-96}$$

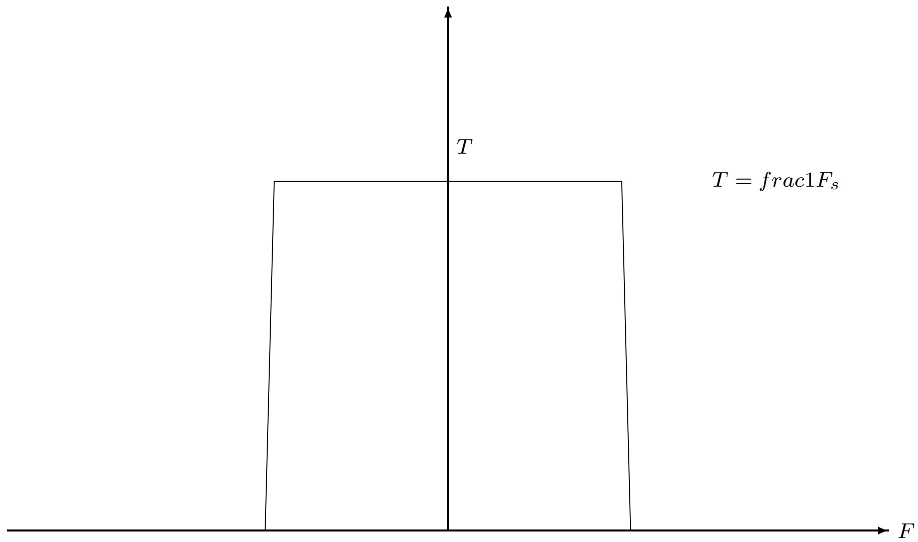


(a) Input to D/A Converter

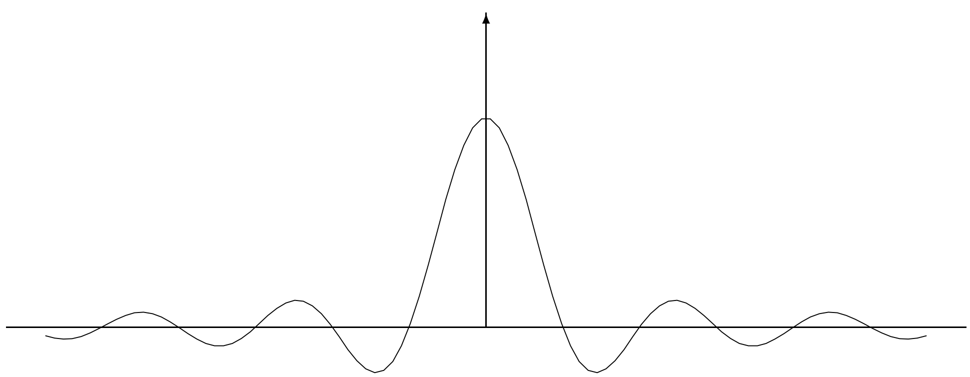


(b) Resulting Output from D/A Converter

Figure 7.11: D/A Converter



(a) Frequency response



(b) Impulse response

Figure 7.12: Ideal low pass filter charactersitics

The waveform is shown in Figure 7.13.

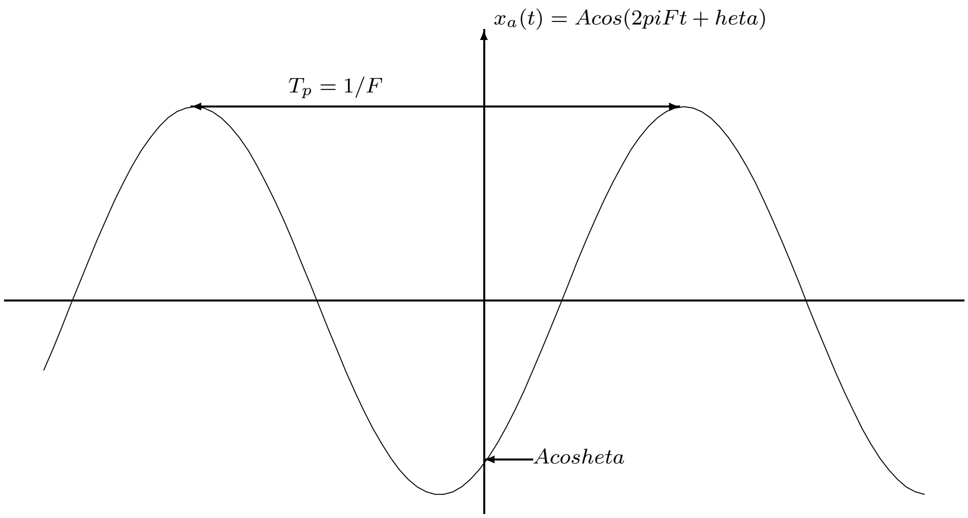


Figure 7.13: Waveform of continuous-time sinusoidal signal

7.6.1.1 Properties of the sinusoidal signal $x_a(t)$

i. $x_a(t)$ is periodic for every fixed value of F, i.e.

$$x_a(t + T_p) = x_a(t) \tag{7.6-97}$$

where $T_p = \frac{1}{F}$ is the fundamental period of the $x_a(t)$.

ii. If $F_1 \neq F_2$

$$x_a(t) = A\cos(2\pi F_1 t + \theta) \tag{7.6-98}$$
$$\neq A\cos(2\pi F_2 t + \theta) = x'_a(t) \tag{7.6-99}$$

or

Continuous-time sinusoids signals with distinct (different) frequencies are themselves distinct.

iii. Increasing F results in an increase in the rate of oscillation of the signal or more periods are included in a given time interval.

Remark 7.6-1 *Due to continuity of the time variable t, $F = \frac{1}{T_p}$ can be increased without limit. i.e. $F \to \infty$.*

The frequency F of the continuous time sinusoidal signal is defined in $(-\infty, \infty)$.
The negative frequencies come about for mathematical convenience.
Since (the Euler identity)

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \tag{7.6-100}$$
$$X_a(t) = A\cos(2\pi Ft + \theta) \tag{7.6-101}$$
$$= \underbrace{\frac{A}{2}e^{j(2\pi Ft+\theta)} + \frac{A}{2}e^{-j(2\pi Ft+\theta)}}_{\text{Phasor expression}} \tag{7.6-102}$$

7.6.2 Discrete-time Sinusoidal Signals

A discrete-time sinusoidal signal may be expressed as

$$x[n] = A\cos(\omega n + \theta) \tag{7.6-103}$$

where n is an integer variable ($-\infty < n < \infty$) which is the sample number, A is again the amplitude of the sinusoid, ω is the frequency in radians per sample, and θ is the phase in radians. See Figure 7.14.

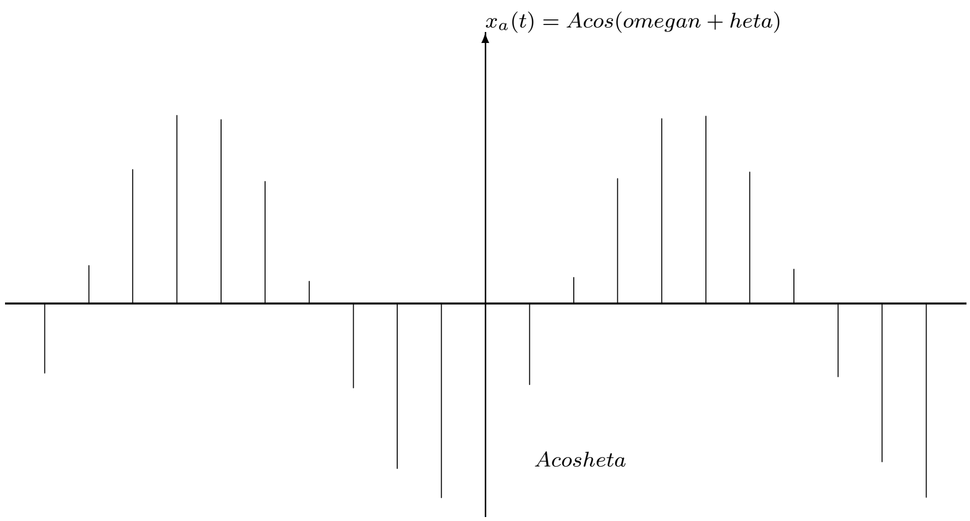


Figure 7.14: Waveform of discrete-time sinusoidal signal

If we define

$$\omega \triangleq 2\pi f \tag{7.6-104}$$
$$x[n] = A\cos(2\pi fn + \theta) \tag{7.6-105}$$

where f is the frequency in cycles per second or Hz.

7.6.2.1 Properties of the discrete sinusoidal signals

i. A discrete-time sinusoid is periodic only if its frequency f is a rational number.
   A discrete-time signal $x[n]$ is periodic with period N ($N > 0$), if and only if

$$x[n+N] = x[n] \qquad \text{for all } n \tag{7.6-106}$$

The smallest value of N for which Equation (7.6-106) is true is called the fundamental period.

Proof:

For $x[n]$ with a given f_0 to be periodic, the following must be held according to Equation (7.6-106):

$$\cos[2\pi f_0(N+n)+\theta] = \cos(2\pi f_0 n + \theta) \tag{7.6-107}$$

For Equation (7.6-107) to be true, it requires

$$2\pi f_0(N+n) = 2\pi f_0 n + 2k\pi \tag{7.6-108}$$

where k is an integer. Or

$$2\pi f_0 N = 2k\pi \tag{7.6-109}$$
$$f_0 = \frac{k}{N} \tag{7.6-110}$$

This means that a discrete-time sinusoidal signal $x[n]$ is periodic only is f_0 is rational (expressed as the ratio of two integers).

When k and N is Equation (7.6-110) are prime numbers (i.e. they have no common factors), N is called the fundamental period.

Remark:
A small change in the frequency can result in a large change in the fundamental period.

Example 7.6-1

$$f_1 = \frac{31}{60} \rightarrow N = 60 \tag{7.6-111}$$
$$f_1 = \frac{30}{60} \rightarrow N = 2 \tag{7.6-112}$$

ii. Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.(Compare with the continuous time case!)

Proof:

For a given ω_0,

$$\cos[(\omega_0+2\pi)n+\theta] = \cos(\omega_0 n + \theta + 2\pi n) \tag{7.6-113}$$
$$= \cos(\omega_0 n + \theta) \tag{7.6-114}$$

Or in general

$$x_k = \cos[\overbrace{(\omega_0 + 2k\pi)}^{\omega_k} n + \theta] \tag{7.6-115}$$
$$= \cos[\omega_0 n + \theta + 2k\pi n] \tag{7.6-116}$$
$$= \cos[\omega_0 n + \theta] \tag{7.6-117}$$

where $k = 0, \pm 1, \pm 2, \ldots$

As a result, all $x_k(n)$ are indistinguishable or identical.

However, the discrete-time sinusoidals with different frequencies in the range of $|\omega| \leqslant \pi$ or $|f| \leqslant \frac{1}{2}$ are unique or distinct.

Definition 7.6-1 (Alias) *The sinusoid with frequency $|\omega| > \pi$ is called an alias of a corresponding sinusoid with frequency $|\omega| < \pi$.*

iii. The highest rate of oscillation in a discrete-time sinusoid is attained when $|\omega| = \pi$ or $|f| = \frac{1}{2}$.

Assume that

$$x[n] = \cos(\omega_0 n) \qquad (7.6\text{-}118)$$

when $\omega_0 = 0$, there is no oscillation.

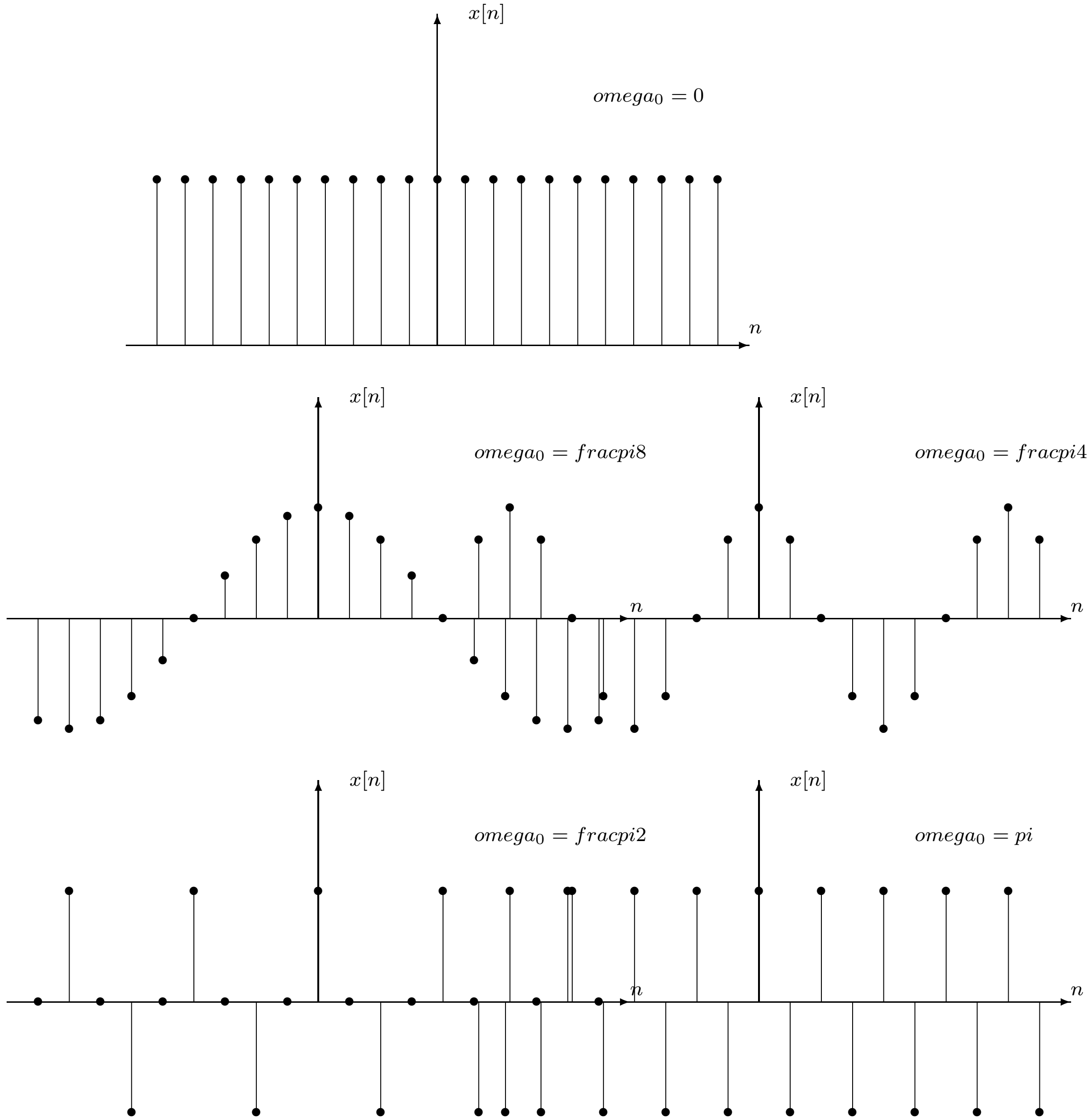


Figure 7.15: Change in oscillation of discrete-time sinusoid

If $\pi < \omega_0' < 2\pi$, $\omega_0' = 2\pi - \omega_0$ where $|\omega_0| \leqslant \pi$. As a result

$$\begin{aligned} x[n] &= \cos(\omega_0' n) && (7.6\text{-}119)\\ &= \cos[(2\pi - \omega_0)n] && (7.6\text{-}120)\\ &= \cos(-\omega_0 n) && (7.6\text{-}121)\\ &= \cos(\omega_0 n) && (7.6\text{-}122)\end{aligned}$$

or ω_0' is an alias of ω_0.

Note: when $\pi < \omega_0' < 2\pi$ and increases, corresponding ω_0 ($\omega_0 = 2\pi - \omega_0'$) is decreasing.

Again, negative frequencies can be introduced for discrete-time sinusoids.

$$x[n] = A\cos(\omega n + \theta) = \frac{A}{2}e^{j(\omega n+\theta)} + \frac{A}{2}e^{-j(\omega n+\theta)} \qquad (7.6\text{-}123)$$

Final remarks:

The frequency range for discrete-time sinusoidal signals is finite with duration 2π. The fundamental rage is defined as

$$-\pi \leqslant \omega \leqslant \pi \qquad (\text{or } 0 \leqslant \omega < 2\pi) \qquad (7.6\text{-}124)$$

or

$$-\frac{1}{2} \leqslant f \leqslant \frac{1}{2} \qquad (\text{or } 0 \leqslant f \leqslant 1) \qquad (7.6\text{-}125)$$

7.7 Analog to Digital Conversion and Digital to Analog Conversion

A digital signal processing system can be illustrated in Figure 7.16, provided both input and output signals are in analog form.

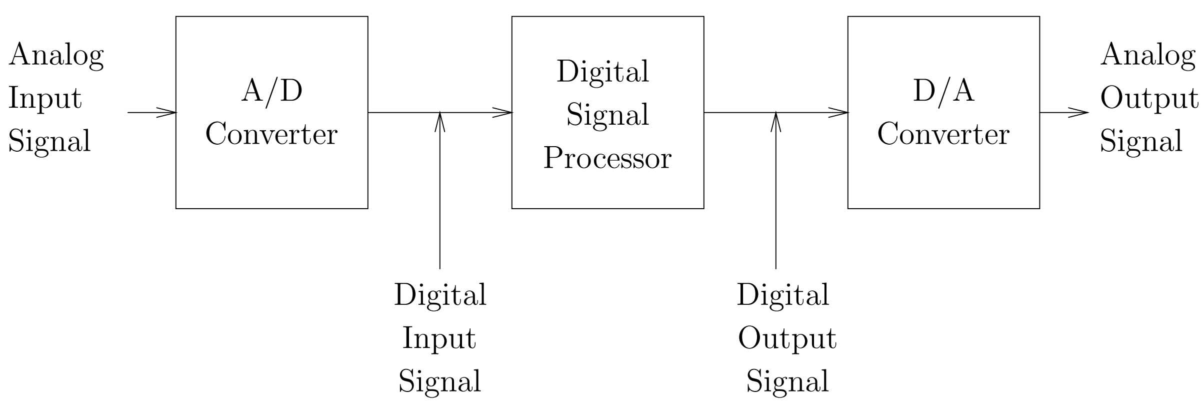


Figure 7.16: Digital signal processing system

The A/D converter can be further decomposed into two major parts as shown in Figure 7.17

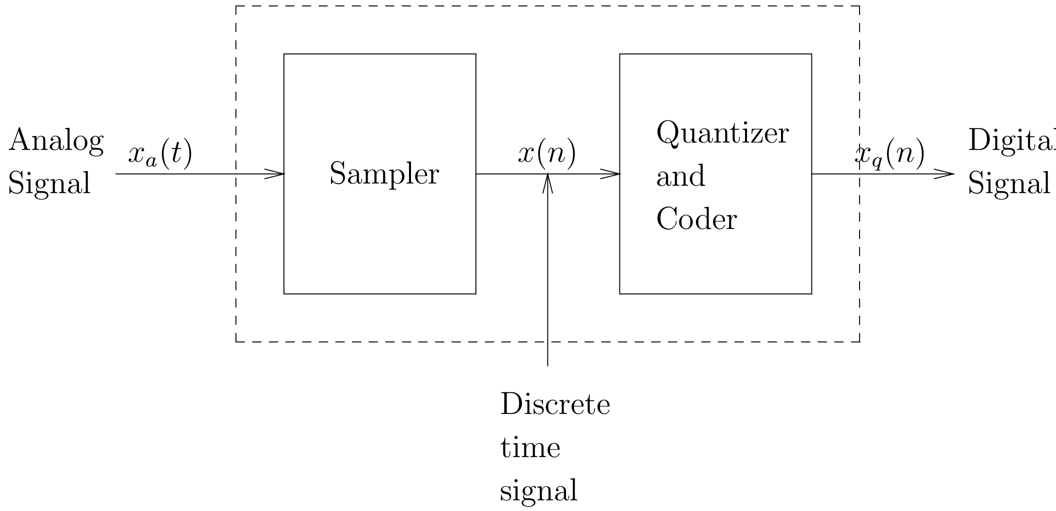


Figure 7.17: A/D converter

7.7.1 Sampling of Analog Sinusoidal Signals

7.7.1.1 Periodic sampling

Periodic or uniform sampling is used most often in practice.

In the relation

$$x[n] = x_a(nT) \qquad \text{for } -\infty < n < \infty \tag{7.7-126}$$

$x[n]$ is the discrete time signal obtained by taking samples of the analog signal every T seconds.

T sampling period (or sample interval)

F_s $\frac{1}{T}$ is the sampling rate in samples per second (or sampling frequency in Hz).

Obviously,

$$x_a(nT) = x_a(t)|_{t=nT=n\frac{1}{F_s}} \tag{7.7-127}$$

7.7.1.2 Relationship between analog and discrete time sinusoids

The relationship between the variable F (or Ω) for analog sinusoids and the frequency variable f (or ω) for discrete time sinusoids is discussed in this subsection.

Considering an analog sinusoidal signal given by

$$x_a(t) = A\cos(2\pi Ft + \theta) \tag{7.7-128}$$

Use a sampling rate $F_s = \frac{1}{T}$ to obtain a discrete-time sinusoid $x[n]$, which yields

$$\begin{aligned} x[n] \triangleq x_a(nT) &= A\cos(2\pi FnT + \theta) && (7.7\text{-}129)\\ &= A\cos\left(2\pi n\frac{F}{F_s} + \theta\right) && (7.7\text{-}130)\\ &= A\cos(2\pi fn + \theta) && (7.7\text{-}131)\end{aligned}$$

where $f = \frac{F}{F_s}$ is called relative frequency or normalized frequency.

Notice that

$$\omega = 2\pi f = 2\pi \frac{F}{F_s} \tag{7.7-132}$$
$$= 2\pi FT \tag{7.7-133}$$
$$= \Omega T \tag{7.7-134}$$

Although the range of F or Ω for $x_a(t)$ is infinite:

$$-\infty < F < \infty \tag{7.7-135}$$

or

$$-\infty < \Omega < \infty \tag{7.7-136}$$

the range of f or ω for $x[n]$ is finite with fundamental range

$$-\frac{1}{2} \leqslant f \leqslant \frac{1}{2} \tag{7.7-137}$$

or

$$-\pi \leqslant \omega \leqslant \pi \tag{7.7-138}$$

($x[n]$ is obtained using a sampling rate F_s.)

From Equation (7.7-134) and Equation (7.7-138), it yields

$$-\frac{1}{2} \leqslant \frac{F}{F_s} \leqslant \frac{1}{2} \tag{7.7-139}$$

or

$$-\pi \leqslant \Omega T \leqslant \pi \tag{7.7-140}$$

Thus

$$-\frac{F_s}{2} \leqslant F \leqslant \frac{F_s}{2} \tag{7.7-141}$$

or

$$-\frac{\pi}{T} \leqslant \Omega \leqslant \frac{\pi}{T} \tag{7.7-142}$$

Using a sampling rate F_s to convert $x_a(t)$ into $x[n]$, Equation (7.7-142) conveys the following messages:

i. If F satisfies Equation (7.7-142), the sampling will yield a unique $x[n]$ with distinct f ($|f| \leqslant \frac{1}{2}$) from $x_a(t)$; otherwise, it will produce $x'[n]$ which is an alias of $x[n]$.

ii. The highest frequency in a continuous time signal that can be uniquely distinguished is $F_{max} = F_s/2$ or $\Omega_{max} = \pi F_s$ when such a signal is sampled a rate $F_s = \frac{1}{T}$. ($F_s/2$ is also called the folding frequency).

EXAMPLE 7.7-1 *Given two analog sinusoids:*

$$x_1(t) = \cos[2\pi(10)t] \tag{7.7-143}$$

Analog Signals		Discrete-Time Signals
$\Omega = 2\pi F$		$\omega = 2\pi f$
$\frac{\text{radians}}{\text{sec}}$ hertz		$\frac{\text{radians}}{\text{sample}}$ $\frac{\text{cycles}}{\text{sample}}$
$-\infty < \Omega < \infty$	$\omega = \Omega T$	$-\pi \leqslant \omega \leqslant \pi$
$-\infty < F < \infty$	$f = F/F_s$	$-\frac{1}{2} \leqslant f \leqslant \frac{1}{2}$
	$\Leftrightarrow$	
$-\pi/T \leqslant \Omega \leqslant \pi/T$	$\Omega = \omega/T$	
$-F_s/2 \leqslant F \leqslant F_s/2$	$F = f \cdot F_s$	

Table 7.1: Relations between frequency and relative frequency

and

$$x_2(t) = \cos[2\pi(50)t] \tag{7.7-144}$$

use a sampling rate $F_s = 40$ Hz to produce discrete-time sinusoids:

$$x_1[n] = \cos\left[2\pi\left(\frac{10}{40}\right)n\right] = \cos\frac{\pi}{2}n \tag{7.7-145}$$

$$= \cos\left[2\pi\left(\frac{1}{4}\right)n\right] \qquad |f_1| \leqslant \frac{1}{2} \tag{7.7-146}$$

and

$$x_2[n] = \cos\left[2\pi\left(\frac{50}{40}\right)n\right] \tag{7.7-147}$$

$$= \cos\frac{5\pi}{2}n \tag{7.7-148}$$

$$= \cos\left[2\pi\left(\frac{5}{4}\right)n\right] \qquad |f_2| > \frac{1}{2} \tag{7.7-149}$$

$$= \cos\left(\frac{\pi}{2}n + 2\pi n\right) \tag{7.7-150}$$

$$= \cos\frac{\pi}{2}n \tag{7.7-151}$$

$$= x_1(n) \tag{7.7-152}$$

$$\triangleq x[n] \tag{7.7-153}$$

Remark 7.7-1

1. *$x_2[n]$ is an alias of $x_1[n]$*
2. *F_2 is an alias of F_1 at the sampling frequency F_s*
3. *All of the sinusoids $\cos[2\pi(f_1 + 40k)t]$ for $k = 1, 2, \ldots$ sampled at $F_s = 40$ Hz yields $x[n]$.*

Example 7.7-2 (Aliasing) *Given*

$$y_0(t) = \cos(2\pi F_0 t) \qquad \text{with } F_0 = \tfrac{1}{8} \text{ Hz} \tag{7.7-154}$$

and

$$y_1(t) = \cos(2\pi F_1 t) \qquad \text{with } F_1 = -\tfrac{7}{8} \text{ Hz} \tag{7.7-155}$$

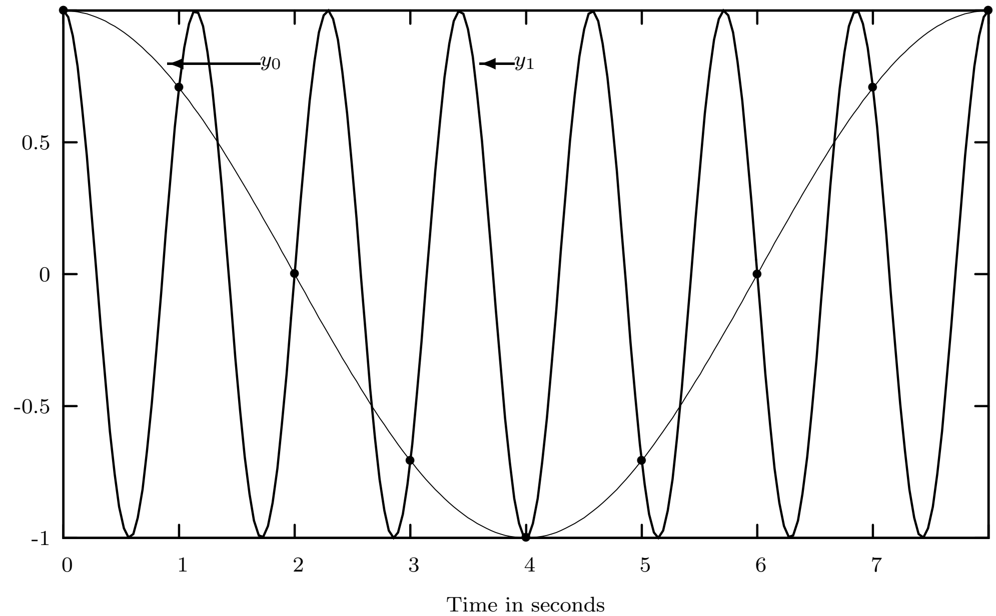


Figure 7.18: Example of aliasing

when $y_0(t)$ and $y_1(t)$ are sampled at $F_s = 1$ Hz (or $T = 1$ seconds), they produce identical samples as shown in Figure 7.18.

As a result, given $y[n]$, there is an ambiguity as to which analog signal $y_i(t)$ these samples represent.

Given $y_0(t)$ and $y_1(t)$ as before and

$$y_2(t) = \cos(2\pi F_2 t) \qquad \textit{with } F_2 = \tfrac{9}{8} \textit{ Hz} \tag{7.7-156}$$

Using $F_s = 1$ Hz, we obtain the same group of samples in the form of $y[n]$ as shown in Figure 7.19.

Note:

$$-\frac{F_s}{2} \leqslant F_0 \leqslant \frac{F_s}{2} \tag{7.7-157}$$

while

$$F_1 < -\frac{F_s}{2} \tag{7.7-158}$$

and

$$F_2 > \frac{F_s}{2} \tag{7.7-159}$$

In fact any

$$y_k(t) = \cos(2\pi F_k t) \tag{7.7-160}$$

where

$$F_k = F_0 + kF_s \qquad \textit{for } k = \pm 1, \pm 2, \ldots \tag{7.7-161}$$

F_k is outside the fundamental range $(-\frac{F_s}{2} \leqslant F \leqslant \frac{F_s}{2})$, will produce the same $y[n]$ when sampled at F_s.

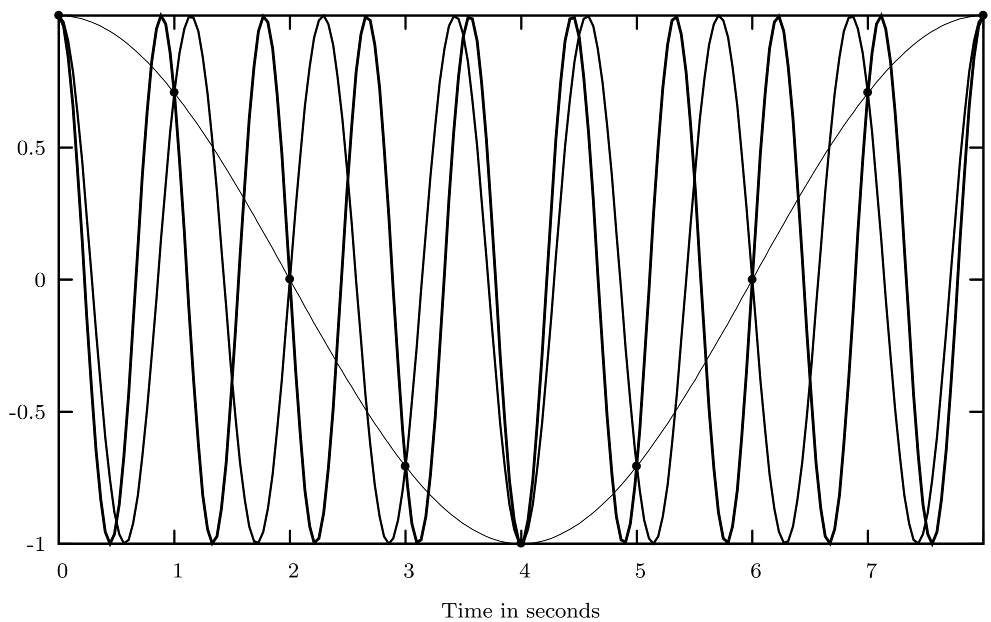


Figure 7.19: Example of aliasing

7.7.2 Reconstruction of Analog Signals

From our previous discussions, if F_{max} is the bandwidth (or the highest spectral component) of the analog signal $x_a(t)$, and the sampling frequency $F_s = 2F_{max}$ (Nyquist rate) is used to obtain the discrete-time signal $x(nT)$. ($T = \frac{1}{F_s}$), the analog signal $x_a(t)$ can be fully recovered using the following ideal reconstruction formula (or ideal interpolation formula)

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin\left(\frac{\pi}{T}\right)(t - nT)}{\left(\frac{\pi}{T}\right)(t - nT)} \tag{7.7-162}$$

However, the ideal interpolation function

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T} \tag{7.7-163}$$

is non causal and cannot be physically realized.

(Note:

$$\mathcal{F}[\langle(\sqcup)] = \begin{cases} T & |F| = \frac{F_s}{2} \\ 0 & \text{otherwise} \end{cases} \tag{7.7-164}$$

).

7.7.2.1 Zero-order hold

A zero-order hold approximates the analog signal $x[t]$ by a series of rectangular pulses whose height is equal to the corresponding value of the signal pulse ($x(nT)$) as shown in Figure 7.20.

The impulse response of the zero-order hold is given by

$$h_0(t) = \begin{cases} 1 & 0 \leqslant t \leqslant T \\ 0 & \text{otherwise} \end{cases} \tag{7.7-165}$$

Its frequency response is

$$H_0(F) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi Ft}dt \tag{7.7-166}$$

$$= \int_{0}^{T} e^{-j2\pi Ft}dt \tag{7.7-167}$$

$$= T\left(\frac{\sin(\pi FT)}{\pi FT}\right)e^{-j\pi FT} \tag{7.7-168}$$

as shown in Figure 7.21.

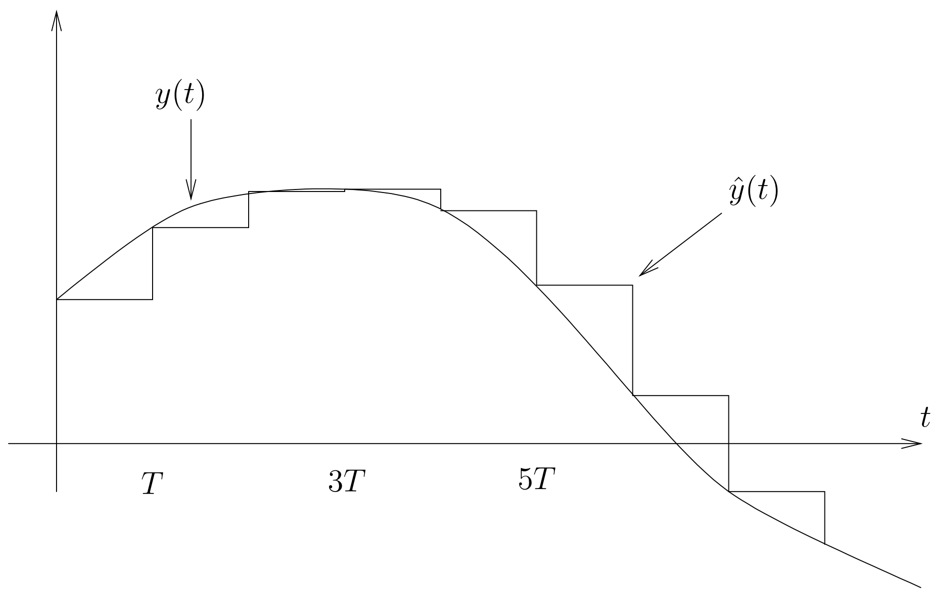


Figure 7.20: Approximation of an analog signal by a zero-order hold

From the frequency response of a zero-order hold, it is seen that the zero-order hold does not possess a sharp cutoff frequency response characteristic.

As a result, the zero-order hold passes undesirable aliased frequency components which are above $F_s/2$ to its output. A low-pass analog filter is normally used to suppress frequency components above $F_s/2$.

7.7.2.2 First-order hold

A first-order hold approximates analog $x[t]$ by straight-line segments which have a slope that is determined by the current sample $x(nT)$ and the previous sample $x(nT-T)$ as shown in Figure 7.24.

It is expressed by a piece-wise linear function:

$$\hat{x}(t) = x(nT) + \frac{x(nT) - x(nT-T)}{T}(t - nT) \tag{7.7-169}$$

where

$$nT \leqslant t < (n+1)T \tag{7.7-170}$$

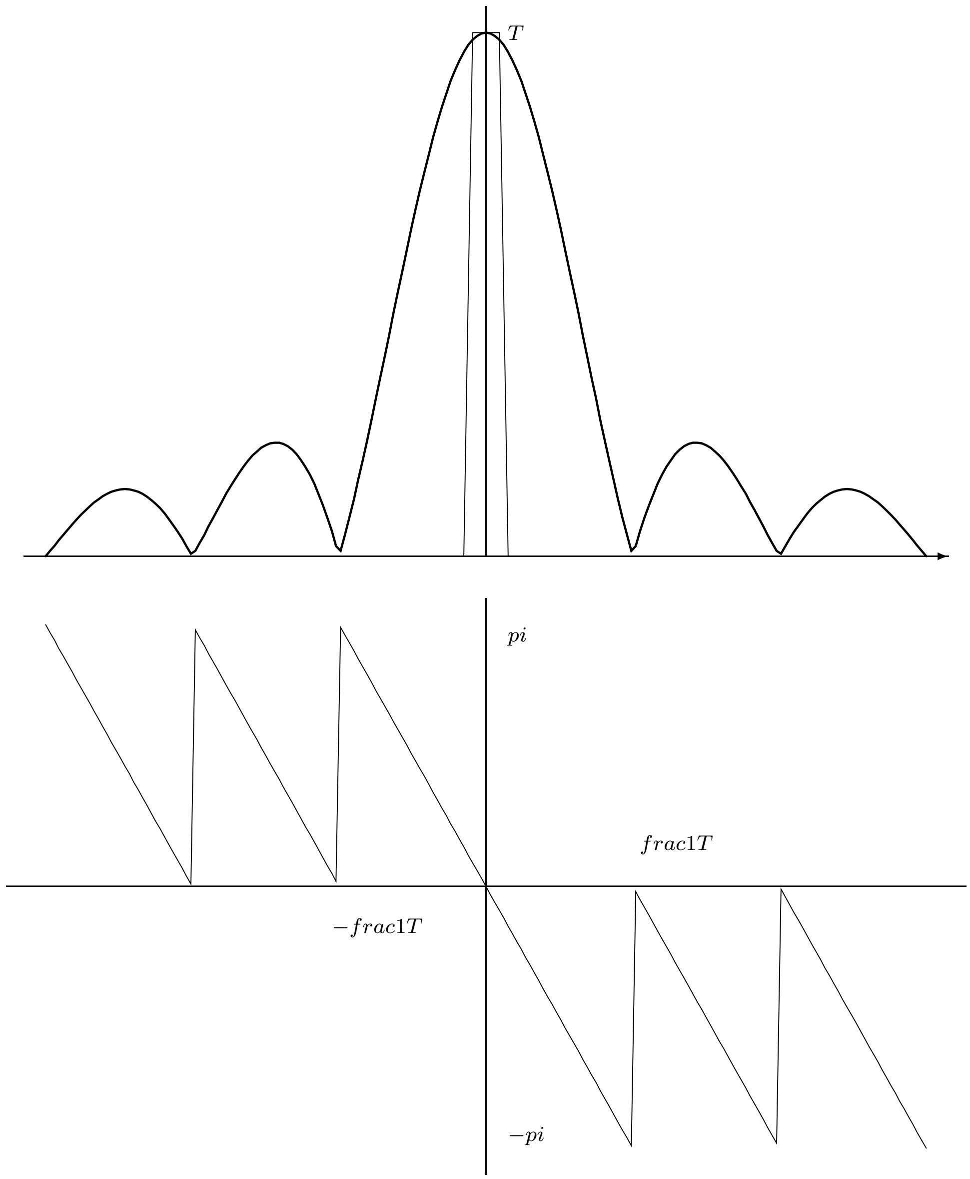


Figure 7.21: Frequency response characteristics of a zero-order hold

The impulse response of the first-order hold is given by

$$h(t) = \begin{cases} 1 + \frac{t}{T} & \text{for } 0 \leqslant t \leqslant T \\ 1 - \frac{t}{T} & \text{for } T \leqslant t < 2T \\ 0 & \text{otherwise} \end{cases} \tag{7.7-171}$$

as shown in Figure 7.22.

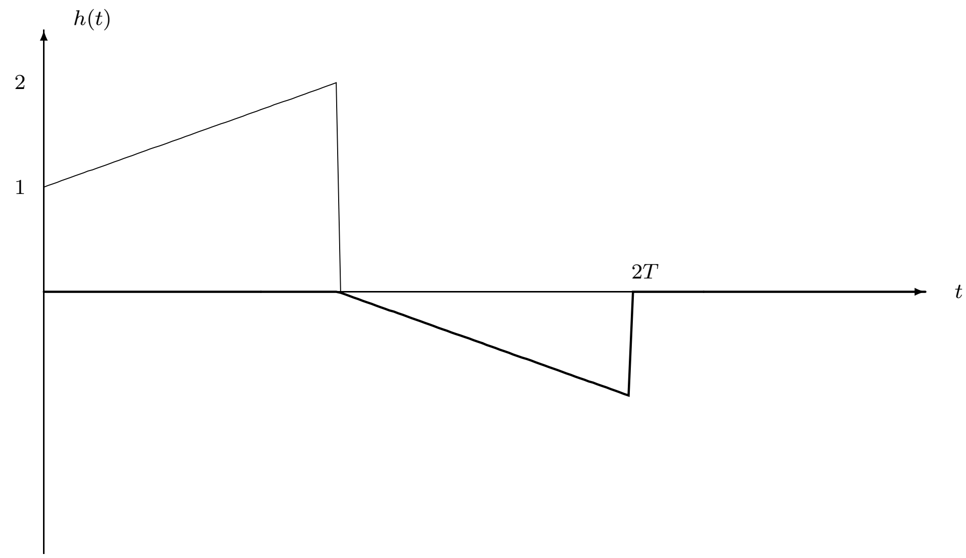


Figure 7.22: Impulse response of first-order hold

The frequency response of the first-order hold is obtained by Fourier transforming $h(t)$

$$H(F) = \mathcal{F}[\langle(\sqcup)] \tag{7.7-172}$$

$$= T(1 + 4\pi F^2 T^2)^{\frac{1}{2}} \left(\frac{\sin \pi F T}{\pi F T} \right)^2 e^{j\theta(F)} \tag{7.7-173}$$

where

$$\theta(F) = -\pi F T + \tan^{-1} 2\pi F T \tag{7.7-174}$$

as shown in Figure 7.23.

Since this reconstruction technique also suffers from distortion due to its passing frequency components above $F_s/2$, it is followed by an analog filter that significantly attenuates frequencies above the folding frequency $F_s/2$.

7.7.2.3 Linear interpolation with delay

The first-order hold with delay performs signal reconstruction by computing the slope of the straight line based on the current sample $x(nT)$ and the past sample $x(nT - T)$ of the signal as shown in Figure 7.25.

$$\hat{x}(t) = x(nT - T) + \frac{x(nT) - x(nT - T)}{T}(t - nT) \tag{7.7-175}$$

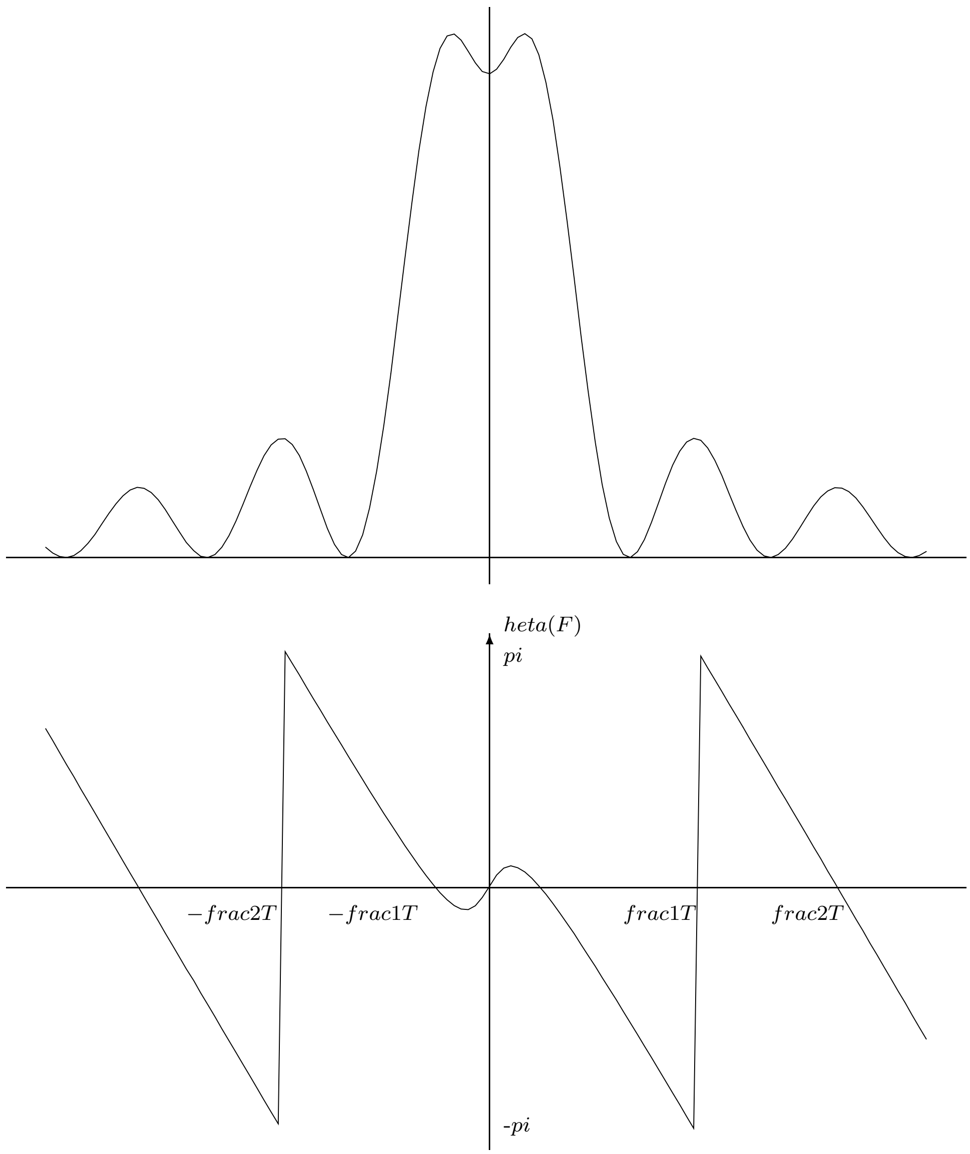


Figure 7.23: Frequency response characterstics of first-order hold

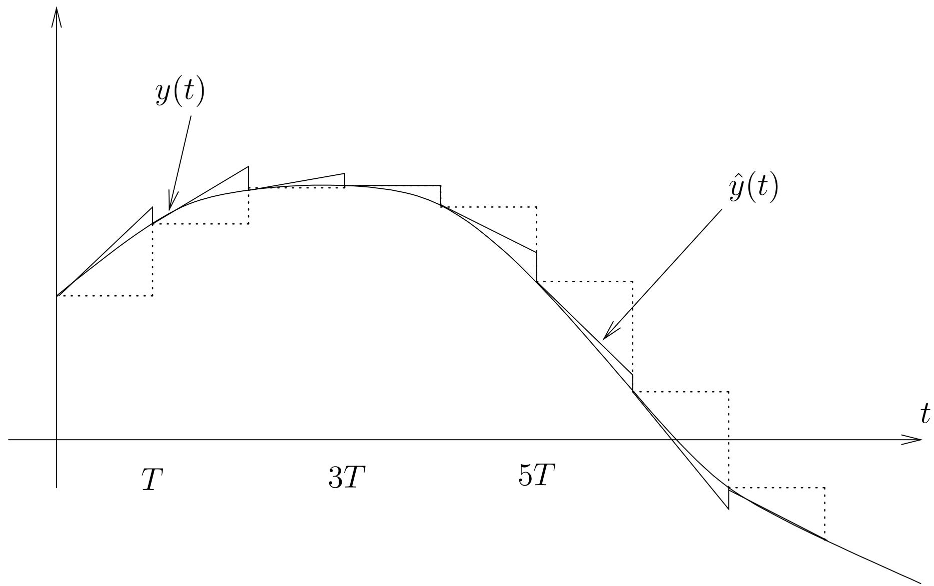


Figure 7.24: Approximation of an analog signal by a first-order hold

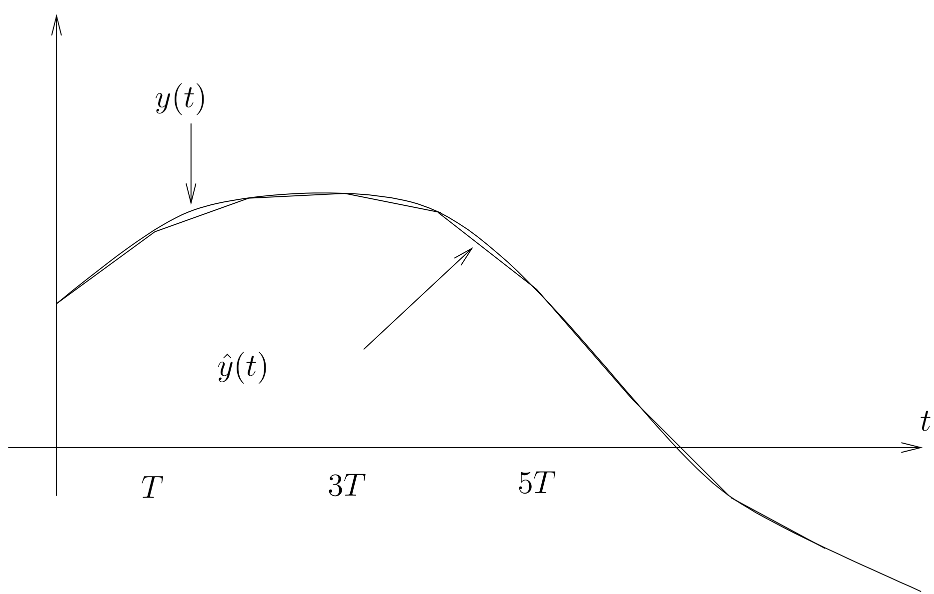


Figure 7.25: Linear interpolation of $x[t]$ with a T-second delay

where

$$nT \leqslant t < (n+1)T \tag{7.7-176}$$

Its impulse response, shown in Figure 7.26 is given by

$$h(t) = \begin{cases} t/T & 0 \leqslant t < T \\ 2 - t/T & T \leqslant t < 2T \\ 0 & \text{otherwise} \end{cases} \tag{7.7-177}$$

Its frequency response is

$$H(F) = T\left(\frac{\sin \pi FT}{\pi FT}\right)^2 e^{-j2\pi Ft} \tag{7.7-178}$$

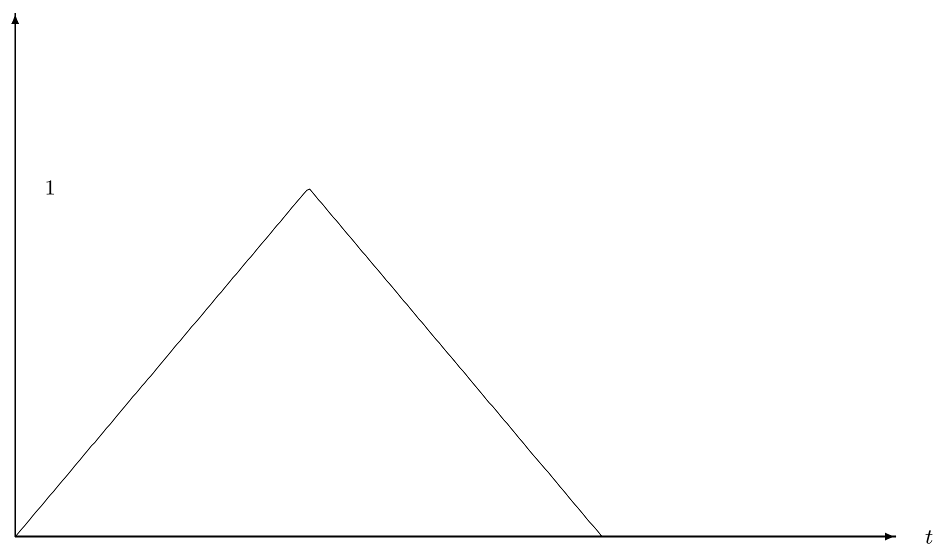


Figure 7.26: Impulse response for the linear interpolator with delay

It can be see in Figure 7.27 that its magnitude response falls off rapidly and has small side lobes beyond F_s, and its phase response is linear due to the delay T.

EXAMPLE 7.7-3 *Consider the analog signal*

$$x_a(t) = 3\cos(100\pi t) \tag{7.7-179}$$

(a). Determine the minimum required sampling rate to avoid aliasing.

(b). Suppose that the signal is sampled at $F_s = 200$ Hz. What is the discrete-time signal obtained after sampling?

(c). Suppose that the signal is sampled at $F_s = 75$ Hz. What is the discrete-time signal obtained after sampling?

(d). What is the frequency $F < F_s/2$ of a sinusoid that yields samples identical to those obtained in part (c)?

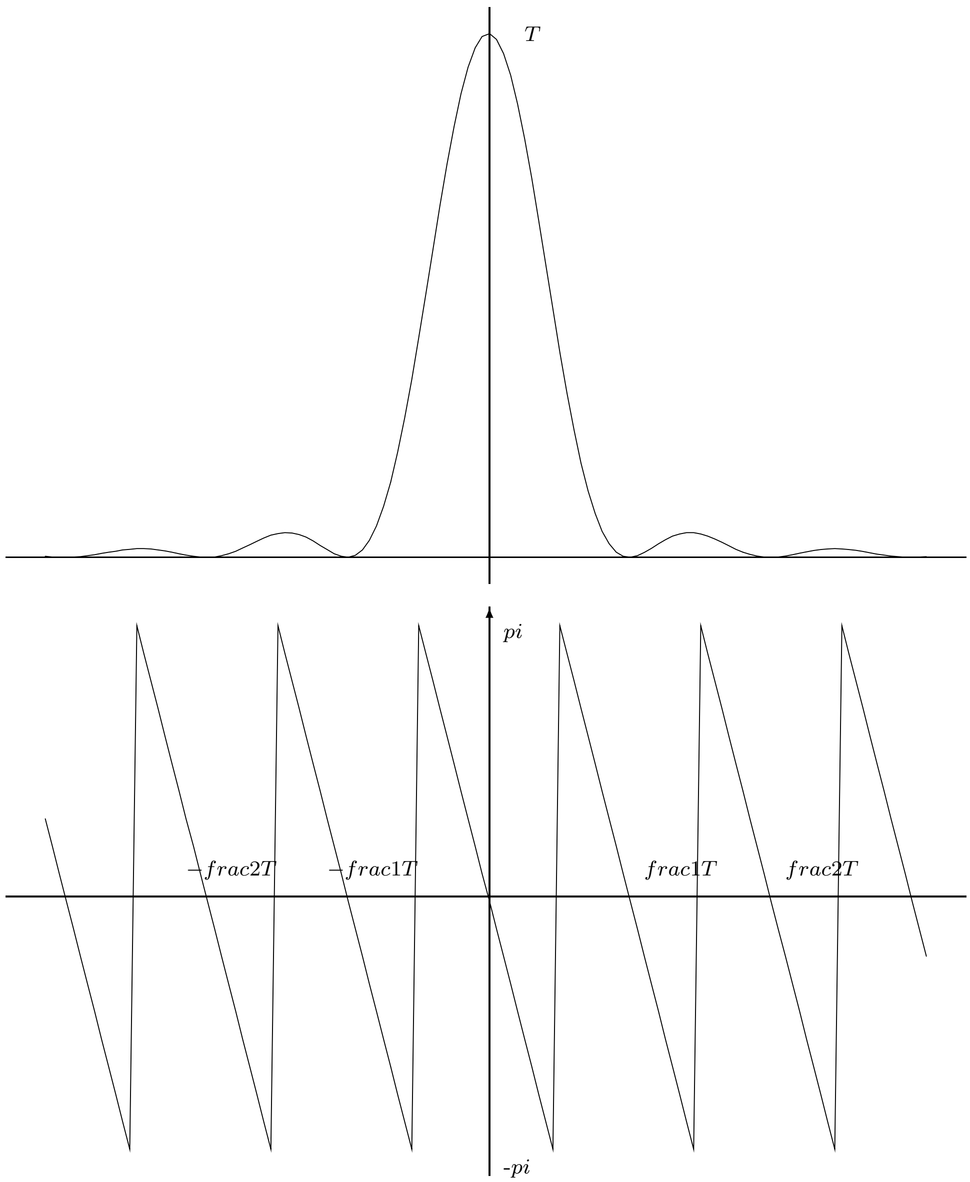


Figure 7.27: Frequency response characteristics for the linear interpolator with delay

Solution:

(a).

$$x_a(t) = 3\cos(2\pi 50t) \tag{7.7-180}$$

i.e. the frequency of the analog signal is $F = 50$ Hz. Hence the minimum sampling rate required to avoid aliasing is $F_s = 100$ Hz. (=$2 \times F$)

(b). If $x_a(t)$ is sampled at $F_s = 200$ Hz the discrete-time signal is

$$x[n] = 3\cos(100\pi nT) \tag{7.7-181}$$

$$= 3\cos\left(100\pi n\frac{1}{F_s}\right) \tag{7.7-182}$$

$$= 3\cos\left(\frac{100\pi n}{200}\right) \tag{7.7-183}$$

$$= 3\cos\left(\frac{\pi}{2}n\right) \tag{7.7-184}$$

Note: $T = \frac{1}{F_s}$, $f = \frac{1}{4} < \frac{1}{2}$

(c). If $x_a(t)$ is sampled at $F_s = 75$ Hz,

$$x[n] = 300\cos\left(\frac{100\pi}{75}n\right) \tag{7.7-185}$$

$$= 3\cos\left(\frac{4}{3}\pi n\right) \tag{7.7-186}$$

$$= 3\cos\left(2\pi - \frac{2\pi}{3}\right)n \tag{7.7-187}$$

$$= 3\cos\left(\frac{2}{3}\pi n\right) \tag{7.7-188}$$

(d). For $F_s = 75$,

$$F = F_s f = 75f \tag{7.7-189}$$

since $f_0 = \frac{1}{3}$ as shown in (c)

$$F = 75 \times \frac{1}{3} = 25Hz \tag{7.7-190}$$

where F is the frequency in $(-\frac{F_s}{2}, \frac{F_s}{2})$ that yields identical samples to those obtained in (c).

It can be seen that $F = 50$ Hz is an alias of $F = 25$ Hz for the sampling rate $F_s = 75$ Hz.

$$\left(\quad 3\cos\left(2\pi \times 50 \times \frac{n}{75}\right) = 3\cos\left(2\pi \times 25 \times \frac{n}{75}\right) \quad\right) \tag{7.7-191}$$

EXAMPLE 7.7-4 *Given the analog signal*

$$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t \tag{7.7-192}$$

Find the Nyquist rate for this signal.

Solution:

Since $F_1 = 25$ Hz, $F_2 = 150$ Hz and $F_3 = 50$ Hz, the highest frequency component is $F_{max} = 150$ Hz. The Nyquist rate F_N is given by

$$F_N = 2F_{max} = 300Hz \tag{7.7-193}$$

However, if we choose the sampling rate $F_s = 300$ Hz

$$10\sin 300\pi\frac{n}{300} = 10\sin(n\pi) = 0 \tag{7.7-194}$$

That is $10\sin(300\pi t)$ sampled at its zero-crossing points an, as a result, this component will be missed completely. F_s should be higher than F_N to avoid this.

EXAMPLE 7.7-5 *Given*

$$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t \tag{7.7-195}$$

(a). What is the Nyquist rate for this signal?

(b). If $F_s = 5000$ samples/s, what is the discrete-time signal obtained after sampling?

(c). What is the analog signal $y_a(t)$ we can construct from the samples of we use the ideal interpolation?

Solution:

(a). $F_1 = 1$ kHz, $F_2 = 3$ kHz and $F_3 = 6$ kHz. Thus $F_{max} = 6$ kHz, and the Nyquist rate is $F_N = 12$ kHz. Therefore, to avoid aliasing the sampling rate should be

$$F_s > F_N = 12kHz \tag{7.7-196}$$

(b). If $F_s = 5$ kHz, the folding frequency is

$$\frac{F_s}{2} = 2.5kHz \tag{7.7-197}$$

and this is the maximum frequency that can be represented uniquely by the sampled signal.

$$x[n] = x_a(nT) = x_a\left(\frac{n}{F_s}\right) \tag{7.7-198}$$

$$= 3\cos\left(\frac{1000}{5000}2\pi n\right) + 5\sin\left(2\pi\frac{3000}{5000}n\right) + 10\cos\left(2\pi\frac{6000}{5000}n\right) \tag{7.7-199}$$

$$= 3\cos\left(2\pi\frac{1}{5}n\right) + 5\sin\left(2\pi\left(1-\frac{2}{5}\right)n\right) + 10\cos\left(2\pi\left(1+\frac{1}{5}\right)n\right) \tag{7.7-200}$$

$$= 13\cos\left(2\pi\frac{1}{5}n\right) - 5\sin\left(\frac{2}{5}n\pi\right) \tag{7.7-201}$$

(c). Since only the frequency components at 1 kHz and 2 kHz are present in the $x[n]$, the analog signal that can be recovered is

$$y_a(t) = 13\cos 2000\pi t + 5\sin 4000\pi t \tag{7.7-202}$$

7.8 Changing the Sampling Rate Using Discrete Time Processes

7.8.1 Introduction

In many practical applications of digital signal processing, it is often required to change the sampling rate of a signal, either increasing it or decreasing it by some amount. Such applications include, for example, teletype, facsimile speech and video, etc.

Definition 7.8-1 (Sampling Rate Conversion) *The process of converting a discrete-time signal from a given rate to a different rate is called sampling rate conversion.*

Definition 7.8-2 (Multirate DSP Systems) *Systems which employ multiple sampling rates in the processing of rates are called multirate digital signal processing systems.*

Definition 7.8-3 (Downsampling) *The operation of reducing the sampling rate (including any filtering) is called down sampling.*

Definition 7.8-4 (Upsampling) *The operation of increasing the sample rate is called upsampling.*

Two general methods can be used for sampling rate conversion of a discrete-time signal:

- One approach is to reconstruct the continuous signal $x_c(t)$ from the sampled discrete-time sequence $x[n]$. Then obtain the new discrete-time sequence $x'[n]$ by sampling $x_c(t)$ at the desired rate. Using this approach, the new sampling rate can be selected arbitrarily and need not have any special relationship to the old sampling rate. However, signal distortion will be introduced by the D/A (as the ideal reconstruction cannot be realized) in the signal reconstruction and by the quantization effects in the A/D conversion.

- The second approach is to change the sampling rate that involves only discrete-time operations or is entirely in the discrete-time domain.

  Avoiding problems which the former approach faces, the only apparent problem in performing the sampling rate conversion in the discrete-time domain is that the ratio of new to old sampling rate is constrained to be rational.

  However, this constraint does not pose a limitation in most practical applications.

  Assume that

  $$x[n] = x_c(nT) \tag{7.8-203}$$

  where $x_c(t)$ is a continuous-time signal and $x[n]$ is the discrete-time sequence obtained by sampling $x_x[t]$ with a rate $\frac{1}{T}$. Even if $x[n]$ was not obtained originally by sampling, an analog band limited signal $x_r(t)$ can be found whose samples are $x[n] = x_c(nT)$ where

  $$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left[\frac{\pi(t-nT)}{T}\right]}{\frac{\pi(t-nT)}{T}} \tag{7.8-204}$$

  Using a different sampling rate, a new discrete-time sequence can be obtained

  $$x'[n] = x_c(nT') \tag{7.8-205}$$

  where $T' \neq T$.

7.8.2 Sampling rate reduction by an integer factor

The sampling rate of a sequence $x[n]$ can be reduced by “sampling” $x[n]$, to form a new sequence

$$x_d[n] = x[nM] = x_c(nMT) \tag{7.8-206}$$

The system defined by Equation (7.8-206) is called a sampling rate compressor. Obviously, the compressor performs downsampling.

It can be seen that $x_d[n]$ can be obtained directly from the analog signal $x_c(t)$ by sampling with period $T' = MT$.

If $x_c(t)$ is band limited to Ω_{max} i.e.

$$|X_c(j\Omega)| = 0 \qquad \text{for } |\Omega| > \Omega_{max} \tag{7.8-207}$$

$x_d[n]$ is an exact representation of $x_c(t)$ or $x_c(t)$ can be reconstructed from $x_d[n]$ if the sampling rate Ω_s' is equal to or greater than $2\Omega_{max}$, i.e.

$$\Omega_s' \geqslant 2\Omega_{max} \tag{7.8-208}$$

It follows

$$\Omega_s' = 2\pi F_s' = \frac{2\pi}{T'} > 2\Omega_{max} \tag{7.8-209}$$

$$\frac{\pi}{T'} = \frac{\pi}{MT} > \Omega_{max} \tag{7.8-210}$$

or:

$$\frac{\pi}{T} > M\Omega_{max} \tag{7.8-211}$$

$$\Omega_s = \frac{2\pi}{T} > M(\underbrace{2\Omega_{max}}_{\text{Nyquist rate}}) \tag{7.8-212}$$

That is the sampling rate can be reduced by a factor of M without aliasing if the original sampling rate was at least M times the Nyquist rate or if the bandwidth of the sequence $x[n]$ is first reduced by a factor of M by discrete-time filtering.

The downsampler or discrete-time sampler can be represented by the diagram in Figure 7.28.

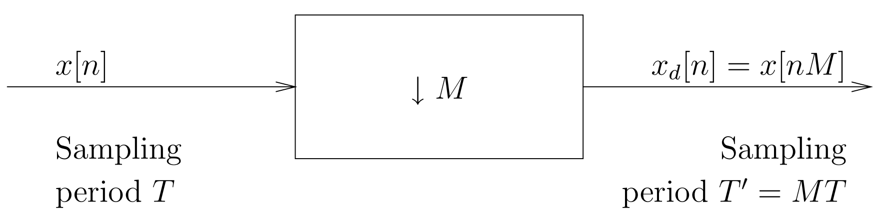


Figure 7.28: Representation of downsampler

The frequency-domain relation between the input and output of the compressor in terms of the discrete-time Fourier transform:

Assume that the Fourier transform of $x_c(t)$ is $X_c(j\Omega)$, that of the periodic impulse train $s(t)$ is $S(j\Omega)$, where $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

Recall that the product of $x_c(t)$ and $s(t)$, and its Fourier transform are given as follows:

$$x_s(t) = x_c(t)s(t) \tag{7.8-213}$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{7.8-214}$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \tag{7.8-215}$$

and

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \tag{7.8-216}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega - kj\Omega_s) \tag{7.8-217}$$

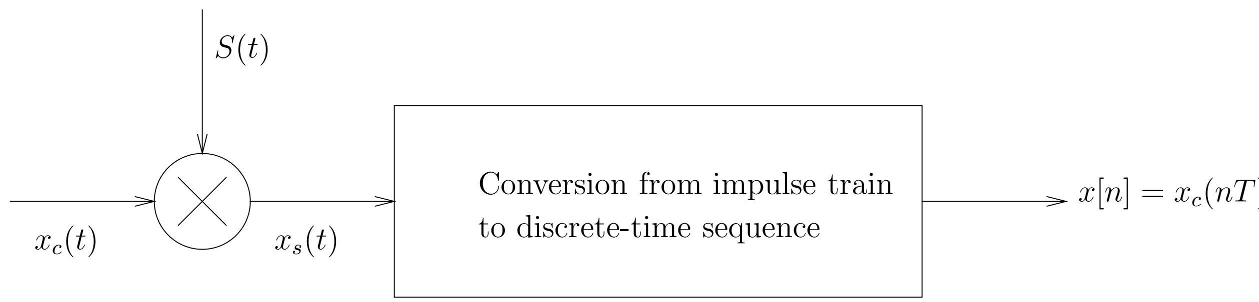
(Note:

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) \tag{7.8-218}$$

where $\Omega_s = 2\pi/T$.)

Recall that the discrete-time sequence $x[n]$ is defined as $x_c(nT)$.

(Note:



where

$x_s(t)$: continuous time, areas of impulses;

$x[n]$: indexed on the integer n, finite number.

)

Now, to express the discrete-time Fourier transform $X(e^{j\omega})$ of the sequence $x[n]$, in terms of $X_s(j\Omega)$ and $X_c(j\Omega)$.

Step 1: Find the Fourier transform of $x_s(t)$:

$$X_s(j\Omega) = \mathcal{F}[\S_{\int}(\sqcup)] \tag{7.8-219}$$

$$= \mathcal{F}\left[\sum_{\backslash=-\infty}^{\infty} \S_{\rfloor}(\backslash\mathcal{T})\delta(\sqcup - \backslash\mathcal{T})\right] \tag{7.8-220}$$

$$= \int_{-\infty}^{\infty} \left\{\sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)\right\} e^{-j\Omega t} dt \tag{7.8-221}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(t - nT) e^{-j\Omega t} dt \tag{7.8-222}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) e^{-j\Omega T n} \tag{7.8-223}$$

Step 2: Since

$$x[n] = x_c(nT) \tag{7.8-224}$$

and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{7.8-225}$$

it follows that

$$X_s(j\Omega) = X\left(e^{j\omega}\right)\Big|_{\omega=\Omega T} = X\left(e^{j\Omega T}\right) \tag{7.8-226}$$

Similarly, the discrete-time Fourier transform of $x_d[n] = x[nM] = x_c(nT')$ with $T' = MT$ is

$$X_d\left(e^{j\omega}\right) = \frac{1}{T'}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T'} - j\frac{2\pi r}{T'}\right) \tag{7.8-227}$$

Substitute MT for T' in Equation (7.8-227)

$$X_d\left(e^{j\omega}\right) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi r}{MT}\right) \tag{7.8-228}$$

Define

$$r = i + kM \tag{7.8-229}$$

where k and i are integers such that $-\infty < k < \infty$ and $0 \leqslant i \leqslant M-1$.

It follows that

$$X_d\left(e^{j\omega}\right) = \frac{1}{MT}\sum_{i=0}^{M-1}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi(i+kM)}{MT}\right) \tag{7.8-230}$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{MT} - j\frac{2\pi i}{MT} - j\frac{2\pi k}{T}\right)\right\} \tag{7.8-231}$$

From Equation (7.8-226) and Equation (7.8-217),

$$X\left(e^{j\Omega T}\right) = X_s(j\Omega) \tag{7.8-232}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\Omega - jk\Omega_s) \tag{7.8-233}$$

and

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \tag{7.8-234}$$

Thus, the discrete-time Fourier transform $X(e^{j\omega})$ of the sequence $x[n]$ is simply a frequency scaled version of $X_s(j\Omega)$ with the frequence scaling specified by $\omega = \Omega T$, and therefore, can also be expressed using the Fourier transform $X_c(j\Omega)$ of the original analog signal $x_c(t)$.

Notice that:

$$X\left(e^{j(\omega-2\pi i)/M}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega-2\pi i}{MT} - j\frac{2\pi k}{T}\right) \tag{7.8-235}$$

Thus

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\omega-2\pi i)/M}\right) \tag{7.8-236}$$

Equation (7.8-236) expressed the Fourier transform of the discrete-time sampled sequence $x_d[n]$ (sampling period MT) in terms of the Fourier transform of the sequence $x[n]$ (sampling period T).

From Equation (7.8-228), $X_d(e^{j\omega})$ can be thought of as being composed of an infinite set of copies of $X_c(j\Omega)$, with frequency scaled through $\omega = \Omega T'$ and shifted by integer multiples of $2\pi/T'$.

From Equation (7.8-236), $X_d(e^{j\omega})$ can be seen as M copies of the periodic Fourier transform $X(e^{j\omega})$, with frequency scaled by M and shifted by integer multiples of $2\pi/M$.

Remark 7.8-1

1. *$X_d(e^{j\omega})$ is periodic with period 2π (as are all discrete-time Fourier Transforms).*
2. *Aliasing can be avoided if $X(e^{j\omega})$ is band limited,*

$$X(e^{j\omega}) = 0 \qquad \text{for } \omega_N \leqslant |\omega| \leqslant \pi \tag{7.8-237}$$

and

$$2\pi/M \geqslant 2\omega_N \tag{7.8-238}$$

In the example shown in Figure 7.29, $2\pi/T = 4\Omega_{max}$, i.e. the original sampling rate is exactly twice the minimum rate to avoid aliasing. Thus, when the original sampled sequence $x[n]$ is downsampled by a factor of $M = 2$, no aliasing occurs. However, if $M > 2$ in this case, aliasing will result.

Aliasing due to downsampling is shown in Figure 7.30 and Figure 7.31. In this case $2\pi/T = 4\Omega_{max}$. Thus, $\omega_N = \Omega_{max}T = \pi/2$. If we downsample by a factor of $M = 3$, it is obtained $x_d[n] = x[3n] = x_c(3nT)$.

Because $\omega'_N = M\omega_N = 3\pi/2$, which is greater than π aliasing occurs.

Remark 7.8-2

(a). *In general, to avoid aliasing in downsampling $X[n]$ by a factor of M requires that*

$$M\omega_N < \pi \quad \text{or} \quad \omega_N < \pi/M \tag{7.8-239}$$

where $\omega_N = \Omega_{max}T$, Ω_{max} is the bandwidth of $x_c(t)$, T is the sampling period to obtain $x[n] = x_c(nT)$.

(b). *To avoid aliasing due to downsampling $x[n]$ can be filtered by an ideal low-pass filter with cutoff frequency π/M, then the output $\tilde{x}[n]$ can be downsampled without aliasing. However, the sequence $\tilde{x}_d[n] = \tilde{x}[nM]$ no longer represents the original continuous-time signal $x_c(t)$.*

(c). *General system for downsampling by a factor of M is shown in Figure 7.32. The system is called a* decimator. *The operation is called* decimation *which performs downsampling by low-pass filter followed by compression.*

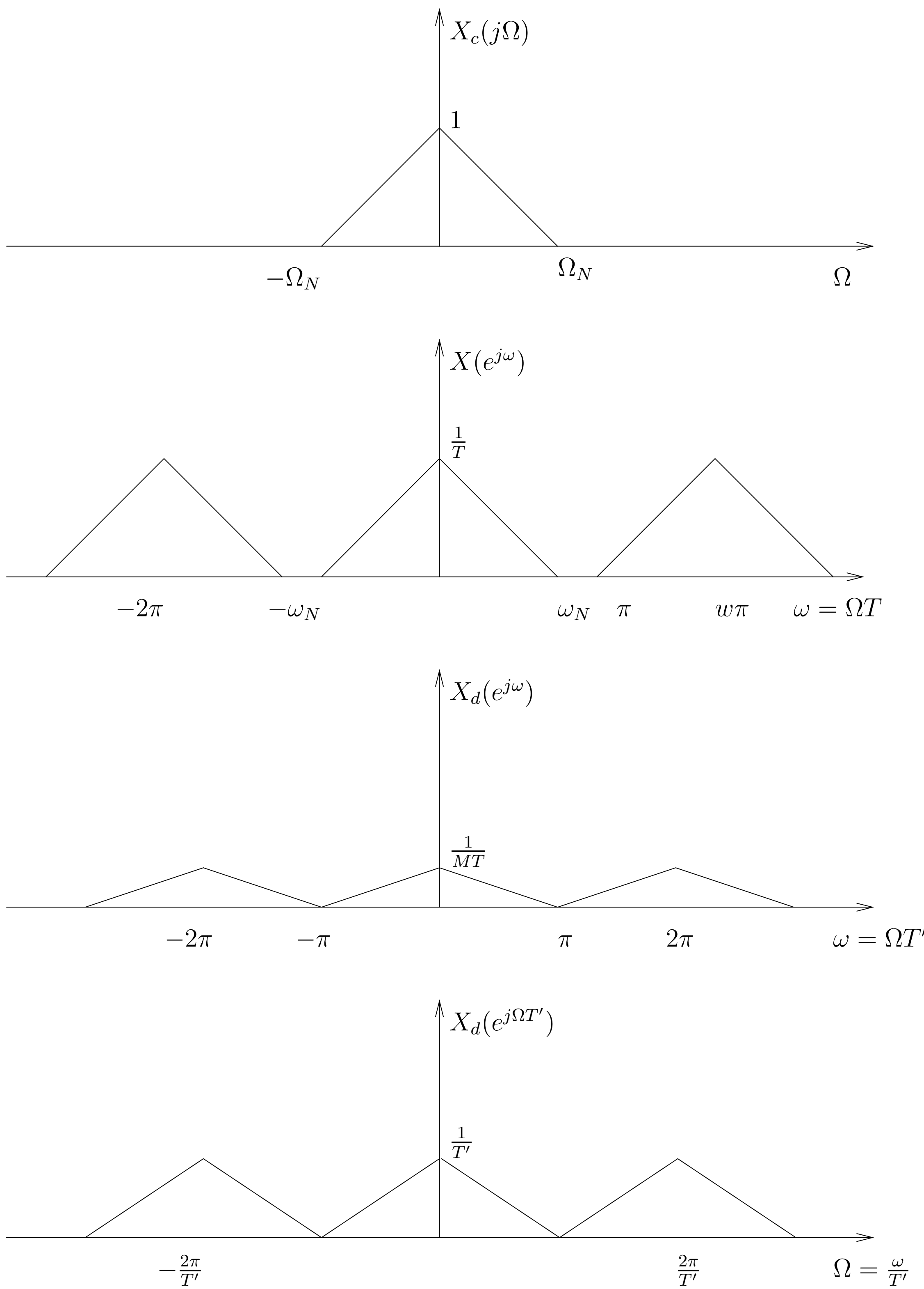


Figure 7.29: Frequency domain illustration of downsampling

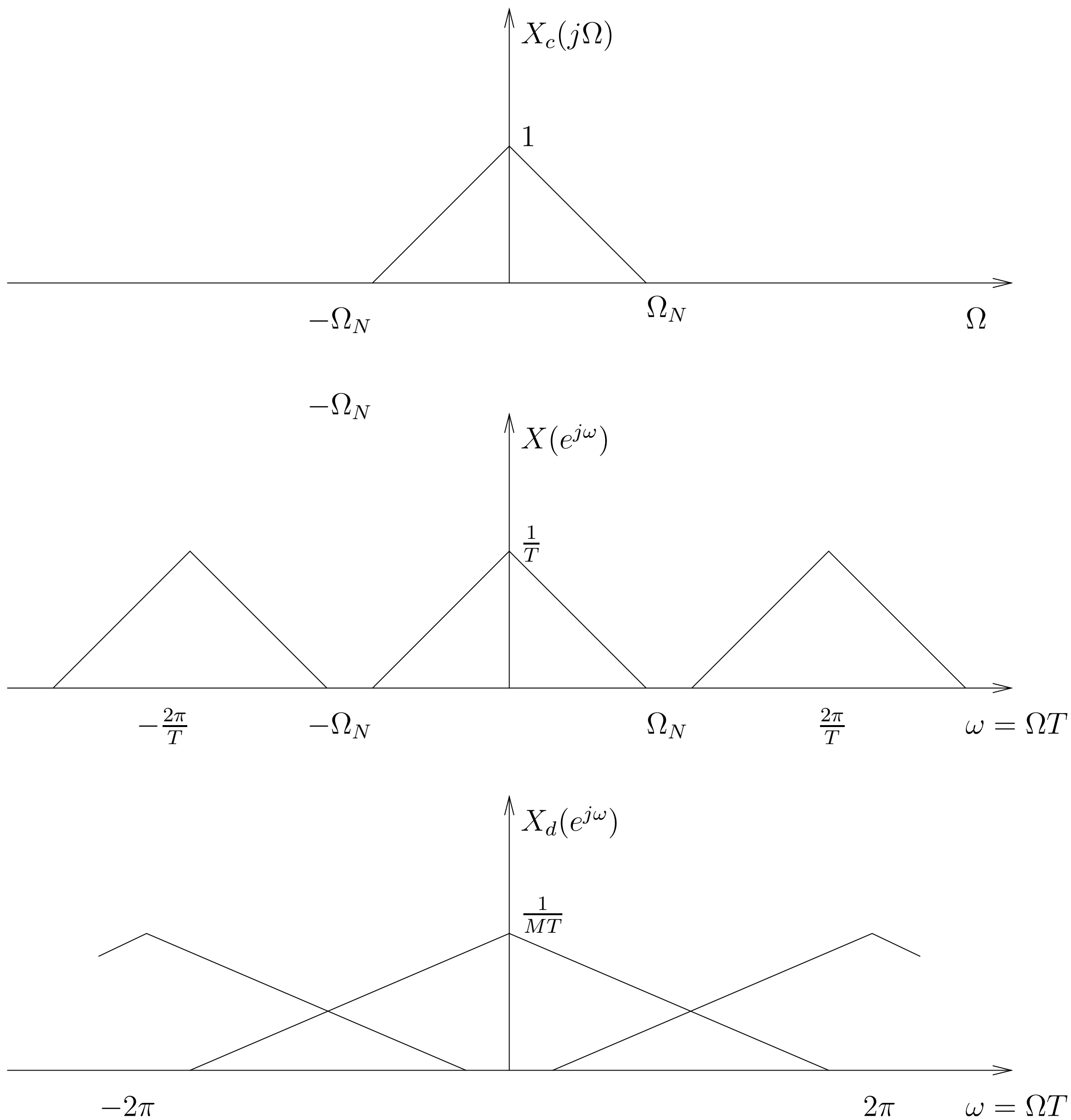


Figure 7.30: Frequency domain illustration of downsampling with aliasing

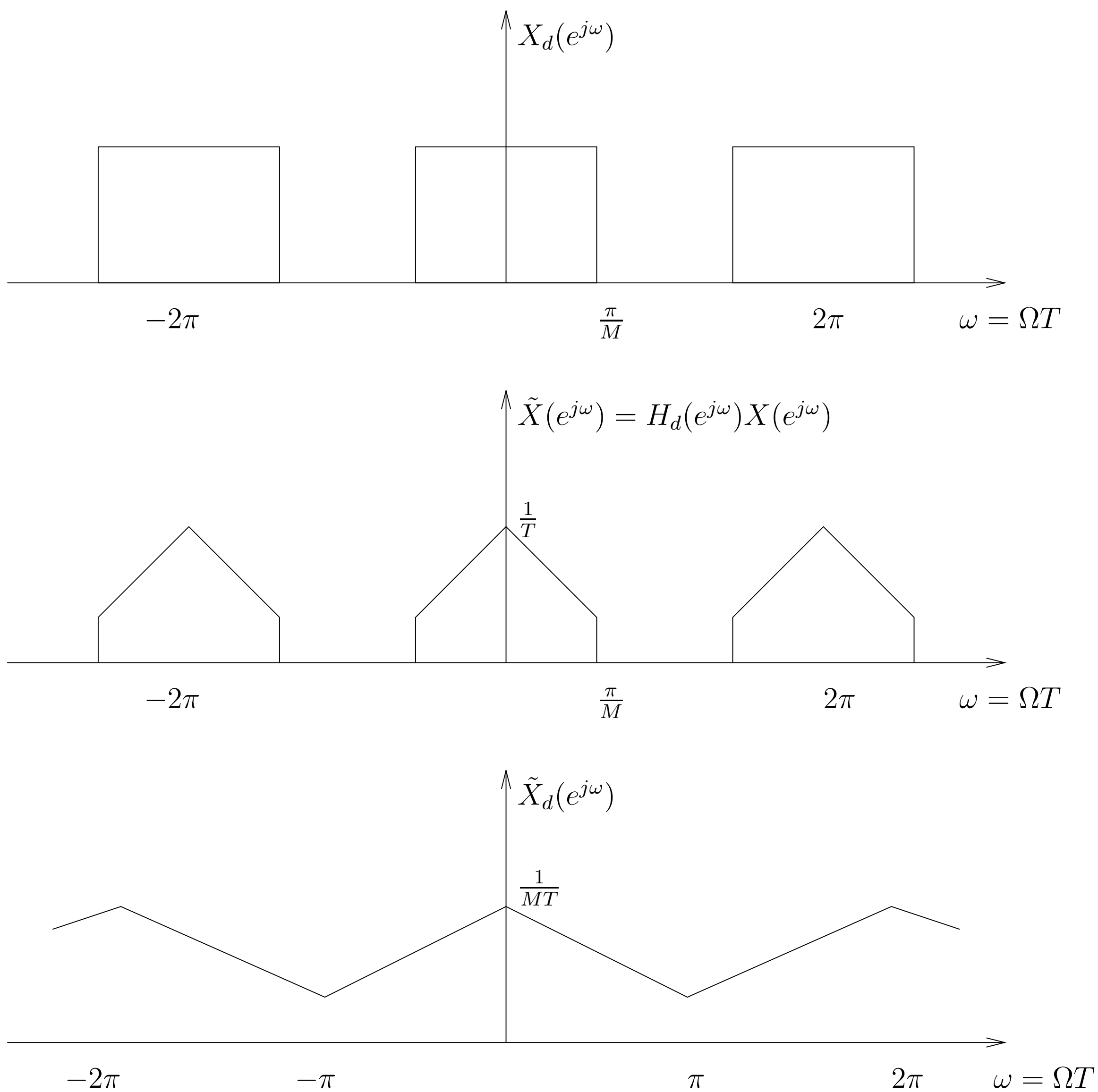


Figure 7.31: Downsampling with prefiltering to avoid aliasing

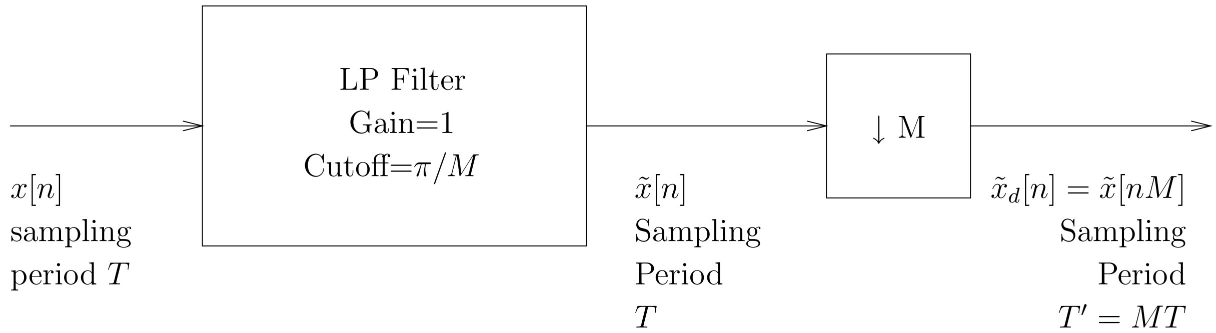


Figure 7.32: General system for downsampling

7.8.3 Sampling rate increase by an integer factor

The reduction of the sampling rate of a discrete-time signal by an integer factor involves sampling the sequence in a manner analagous to sampling a continuous-time signal (A/D). Not surprisingly, increasing the sampling rate involves operations analagous to D/A conversion.

Assume that we wish to increase the sampling rate, by a factor of L, of a signal sequence $x[n]$ whose underlying continuous-time signal is $x_c(t)$, i.e. to obtain the sample sequence

$$x_i[n] = x_c(nT') \tag{7.8-240}$$

where $T' = T/L$, from the sequence of samples

$$x[n] = x_c(nT) \tag{7.8-241}$$

This operation of increasing the sampling rate is referred to as upsampling.

It can be seen that

$$x_i[n] = x[n/L] = x_c(nT/L) \qquad \text{for } n = 0, \pm L, \pm 2L, \ldots \tag{7.8-242}$$

a system can be constructed to obtain $x_i[n]$ from $x[n]$ using only discrete-time processing as shown in Figure 7.33.

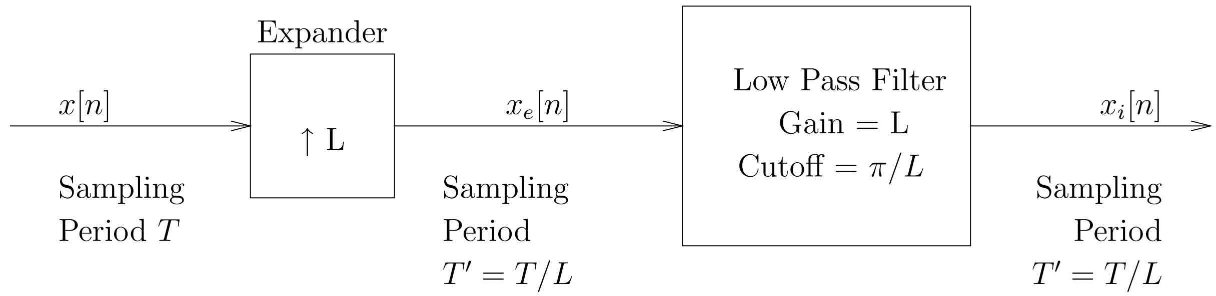


Figure 7.33: Expander

The sampling rate expander is defined as

$$x_e[n] = \begin{cases} x[n/L] & \text{for } n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{7.8-243}$$

or equivalently

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL] \tag{7.8-244}$$

The lowpass filter with gain $= L$ and cutoff frequency $= \pi/L$ works in a similar way as the reconstruction filter does in D/A conversion.

On the whole, the system works in a similar manner to the ideal D/A converter.

The operation of the upsampling system is most easily understood in the frequency domain. (What is the relationship between the spectrum of $x_e[n]$ and that of $x[n]$? How can we obtain an interpolated sequence $x_i[n]$ to represent $x_c(t)$ from $x[n]$?)

The Fourier transform of $x_e[n]$ is given as:

$$\begin{aligned} X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL] \right\} e^{-j\omega n} && (7.8\text{-}245) \\ &= \sum_{k=-\infty}^{\infty} x[k] \left\{ \sum_{n=-\infty}^{\infty} \delta[n - kL] e^{-j\omega n} \right\} && (7.8\text{-}246) \\ &= \sum_{k=-\infty}^{\infty} x[k] e^{-jk(\omega L)} && (7.8\text{-}247) \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega L)n} && (7.8\text{-}248) \\ &= X(e^{j\omega L}) && (7.8\text{-}249) \end{aligned}$$

Please notice that $\omega = \Omega T$ where Ω is the frequency of $x_c(t)$ and T is the sampling period to obtain $x[n]$.

From Equation (7.8-249) the Fourier transform of the output of the expander, $x_e[n]$, is a frequency-scaled version of the Fourier transform of the input, $x[n]$, i.e. replacing ω by ωL in $X(e^{j\omega})$ will result in $X_e(e^{j\omega})$.

Examine Equation (7.8-249) once again

$$X_e(e^{j\omega}) = X(e^{j\omega L}) \tag{7.8-250}$$

If we define ω',

$$\omega' = \Omega T' = \Omega \frac{T}{L} = \frac{\omega}{L} \tag{7.8-251}$$

Equation (7.8-249) gives

$$X_e\left(e^{j\frac{\omega'}{L}}\right) = X\left(e^{j\omega'}\right) \tag{7.8-252}$$

Since for the discrete-time signal $x[n]$, $-\pi \leqslant \omega \leqslant \pi$, (fundamental range of ω),

$$-\frac{\pi}{L} \leqslant \omega' \leqslant \frac{\pi}{L} \tag{7.8-253}$$

($\omega' = \frac{\omega}{L}$) in which the frequency components of $x_e[n]$ are unique.

Assume the spectrum of $x_i[n]$ is $X_i(e^{j\omega})$ where $-\frac{\pi}{L} \leqslant \omega_i \leqslant \frac{\pi}{L}$. We would like to obtain $x_i[n] = x[n/L] = x_c(nT/L)$ for $n = 0, \pm L, \pm 2L, \ldots$ by finding the inverse Fourier transform of $X_i(e^{j\omega_i})$ which is obtained by band limiting $X_e(e^{j\omega_i})$ to $-\frac{\pi}{L} \leqslant \omega_i \leqslant \frac{\pi}{L}$.

$$x_i[n] = \frac{1}{2\pi}\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} X_i(e^{j\omega_i})e^{j\omega_i n}d\omega_i \tag{7.8-254}$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} X_e(e^{j\omega_i})e^{j\frac{\omega}{L}n}d\left(\frac{\omega}{L}\right) \tag{7.8-255}$$

$$= \frac{1}{L}\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega_i L})e^{j\omega\left(\frac{n}{L}\right)}d\omega \tag{7.8-256}$$

$$= \frac{1}{L}\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega\left(\frac{n}{L}\right)}d\omega \tag{7.8-257}$$

$$= \frac{1}{L}x\left[\frac{n}{L}\right] \tag{7.8-258}$$

$$\left(\text{Note:} \qquad \left[\frac{n}{L}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega\left(\frac{n}{L}\right)}d\omega. \qquad\right) \tag{7.8-259}$$

From Equation (7.8-258), it can be seen that if a band limited lowpass ideal filter with unit gain was used to obtain $x_i[n]$, the value of each sample is only $\frac{1}{L}$ of that of the $x\left[\frac{n}{L}\right]$, therefore, $x_c\left(n\frac{T}{L}\right)$, meaning the result would not be correct.

To correct that, an ideal lowpass filter with gain of L and cutoff frequency $|\omega| \leqslant \frac{\pi}{L}$ should be used after the expander.

This shows that the general upsampling system depicted previously in Figure 7.33 does indeed give an output satisfying $x_i[n] = x_c(nT')$ if the input sequence $x[n] = x_c(nT)$ was obtained by sampling without aliasing, where $T' = \frac{T}{L}$.

The system is called an *interpolater* and the operation of upsampling is considered to be synonymous with *interpolation*.

The impulse response of the lowpass filter in the ideal "interpolator" is

$$h_i[n] = \frac{\sin\left(\frac{\pi n}{L}\right)}{\pi\left(\frac{n}{L}\right)} \tag{7.8-260}$$

Thus

$$x_i[n] = \sum_{m=-\infty}^{\infty} x_e[m]h[n-m] \tag{7.8-261}$$

$$= \sum_{m=-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[m-kL]\right\}\frac{\sin\left[\frac{\pi(n-m)}{L}\right]}{\frac{\pi(n-m)}{L}} \tag{7.8-262}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\left\{\sum_{m=-\infty}^{\infty}\delta[m-kL]\frac{\sin\left[\frac{\pi(n-m)}{L}\right]}{\frac{\pi(n-m)}{L}}\right\} \tag{7.8-263}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\frac{\sin\left[\frac{\pi(n-m)}{L}\right]}{\frac{\pi(n-m)}{L}} \tag{7.8-264}$$

From Equation (7.8-260), the impulse response has the properties

$$\begin{cases} h_i[0] = 1 \\ h_i[n] = 0, \text{for } n = \pm L, \pm 2L, \ldots \end{cases} \tag{7.8-265}$$

Thus using the ideal lowpass interpolation filter defined by Equation (7.8-260) and Equation (7.8-265) produces

$$x_i[n] = x[n/L] = x_c(nT/L) = x_c(nT') \qquad \text{for } n = 0, \pm L, \pm 2L, \ldots \tag{7.8-266}$$

The fact that $x_i[n] = x_c(nT')$ for all n follows from our frequency domain argument.

7.8.3.1 Linear interpolator

In practice, ideal lowpass filters cannot be implemented exactly, filters of other types are used to approximate the characteristics of the ideal filter.

Although linear interpolation is not very accurate, it is often used in practice.

Linear interpolation can be implemented using an expander followed by a filter which has an impulse response

$$h_{lin}[n] = \begin{cases} 1 - |n|/L & \text{for } |n| \leqslant L \\ 0 & \text{otherwise} \end{cases} \tag{7.8-267}$$

Figure 7.34 shows $h_{lin}[n]$ for $L = 5$.

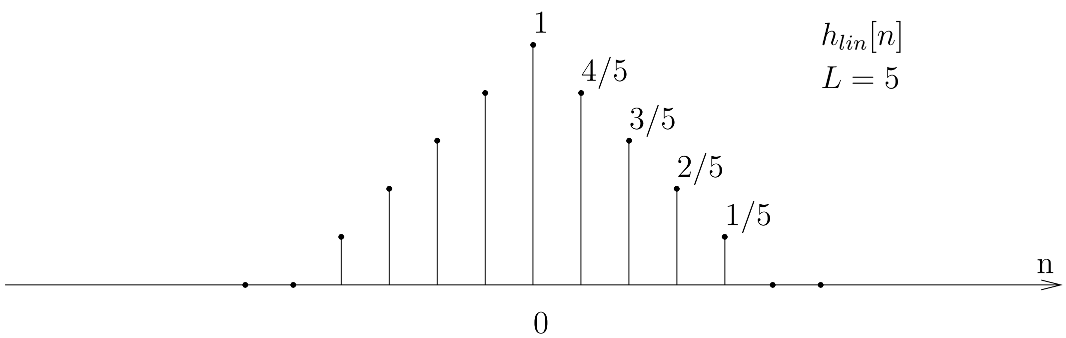


Figure 7.34: Impulse response for linear interpolation

Therefore, the interpolated output is

$$\begin{aligned}
x_{lin}[n] &= \sum_{k=-\infty}^{\infty} x_e[k] h_{lin}[n-k] &(7.8\text{-}268)\\
&= \sum_{k=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} x[m]\delta[k-mL] \right\} h_{lin}[n-k] &(7.8\text{-}269)\\
&= \sum_{m=-\infty}^{\infty} x[m] \left\{ \sum_{k=-\infty}^{\infty} \delta[k-mL] h_{lin}[n-k] \right\} &(7.8\text{-}270)\\
&= \sum_{m=-\infty}^{\infty} x[m] h_{lin}[n-mL] &(7.8\text{-}271)\\
&= \sum_{k=-\infty}^{\infty} x[k] h_{lin}[n-kL] &(7.8\text{-}272)
\end{aligned}$$

Note that

$$h_{lin}[0] = 1 \tag{7.8-273}$$
$$h_{lin}[n] = 0 \qquad \text{for } n = \pm L, \pm 2L, \ldots \tag{7.8-274}$$

so that

$$x_{lin}[n] = x[n/L] \qquad \text{for } n = 0, \pm L, \pm 2L, \ldots \tag{7.8-275}$$

The frequency response of $h_{lin}[n]$ is

$$H_{lin}(e^{j\omega}) = \frac{1}{L}\left\{\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right\}^2 \tag{7.8-276}$$

Comparing $H_{lin}(e^{j\omega})$ with the frequency response of the ideal lowpass interpolator filter, when the original signal is sampled at the Nyquist rate, linear interpolation will not be very good since *the output of the filter will contain considerable energy in the band $\pi/L < |\omega| \leqslant \pi$.*

However, if the original sampling rate is much higher than the Nyquist rate, then the linear interpolator will be more successful in removing the frequency-scaled image of $X_c(j\Omega)$ at multiples of $2\pi/L$.

(When the original sampling rate is higher than the Nyquist rate, the spectrum of the signal $X_c(t)$ will be close to $\Omega = 0$, and $X(e^{j\omega})$ close to $\omega = \Omega T = 0, \pm 2\pi, \pm 4\pi, \ldots$ and $X_e(e^{j\omega})$ close to $\omega_i = \Omega T' = 0, \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \ldots$. At these areas, the $H_{lin}(e^{j\omega})$ attenuates the most.)

7.8.4 Changing the Sampling Rate by a non-integer factor

By combining decimation and interpolation, it is possible to change the sampling rate by a non-integer factor.

It is shown in Figure 7.35 a system which consists of an interpolator and a decimator. The interpolator decreases the sampling period from T to T/L. The decimator increases the sampling period by M. The overall system produces an output sequence $\tilde{x}_d[n]$ that has an effective sampling period of $T' = \left(\frac{M}{L}\right) T$.

EXAMPLE 7.8-1 *If $M = 101$ and $L = 100$,*

$$T' = \frac{M}{L} T = 1.01T \tag{7.8-277}$$

By choosing L and M arbitrarily, we can approach arbitrarily close to any desired ratio of sampling periods.

If $M > L$, there is a net increase in the sampling period (decrease in sampling rate) and π/M is the dominant cutoff frequency. If $x[n]$ was obtained by sampling at the Nyquist rate, the sequence $\tilde{x}_d[n]$ will represent a lowpass-filtered version of the original underlying band limited signal $x_c(t)$ if we are to avoid aliasing.

If $M < L$, the opposite is true, and π/L is the dominant cutoff frequency and there will be no need to further limit the bandwidth of the signal below the original Nyquist frequency.

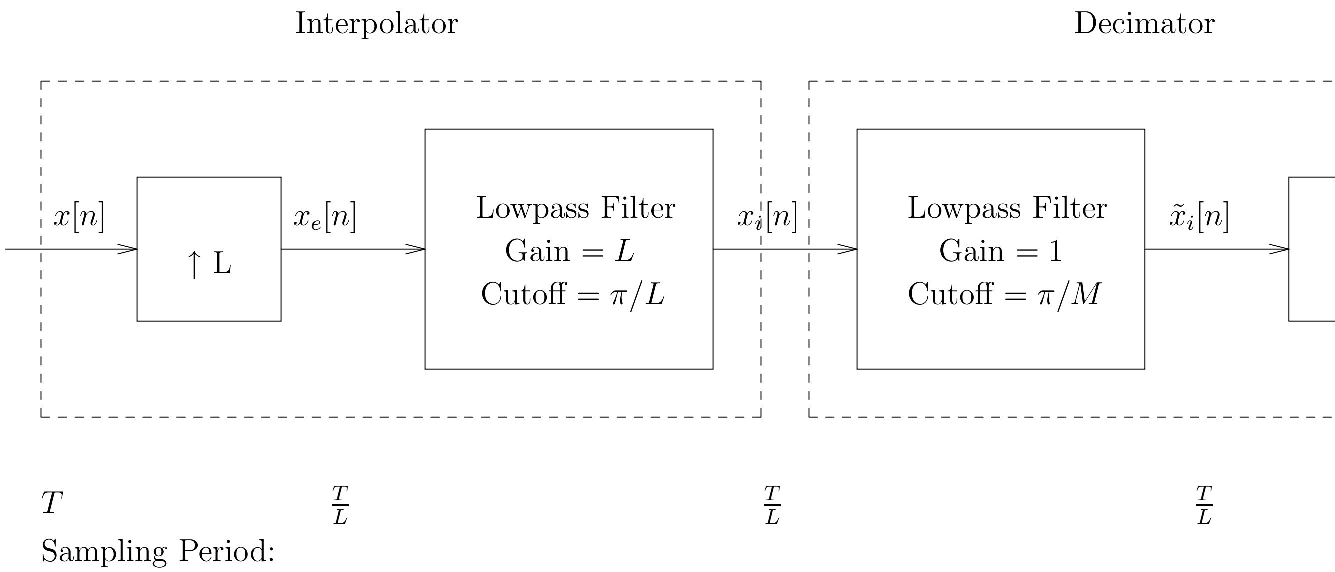


Figure 7.35: System for changing sampling rate by a non-integer factor

Chapter 8

Discrete Signals and Systems

Contents

8.1 Introduction

Definition 8.1-1 (Discrete System) *A discrete system is one which has discrete input and output signals.*

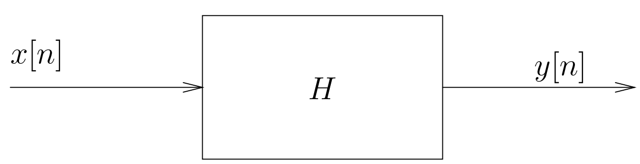
As a majority part of discrete systems involves finite-word-length operations, whether it is using the fixed-point or the floating-point arithmetic, digital systems are our major concern.

The advantages of the digital signal processing systems over analog ones are as follows:

- allowing flexibility in reconfiguring;
- providing better control of accuracy requirements;
- storing digital signals on magnetic media without deterioration or loss of signal fidelity beyond that introduced in the A/D conversion;
- allowing for the implementation of more sophisticated signal processing algorithms;
- allowing multidimensional signal processing which has no equivalent in the analog form.

8.2 Properties of Discrete Systems

Given a discrete system



where $f[n]$ is the input and $g[n]$ is the output.

$$g[n] = H(f[n]) \tag{8.2-1}$$

8.2.1 Linearity

If two output signals $g_1[n]$ and $g_2[n]$ are response signals of the system to the input signals $f_1[n]$ and $f_2[n]$ respectively, the system is linear if

$$H(a_1 f_1[n] + a_2 f_2[n]) = a_1 g_1[n] + a_2 g_2[n] \tag{8.2-2}$$

where a_1 and a_2 are constants.

In general, the system is linear if

$$g_i[n] = H(f_i[n]) \qquad \text{for } i = 0, 1, \ldots, m \tag{8.2-3}$$

and

$$H\left(\sum_{i=0}^{m} a_i f_i[n]\right) = \sum_{i=0}^{m} a_i g_i[n] \tag{8.2-4}$$

where a_i $(i = 0, 1, \ldots, m)$ are constants.

8.2.2 Time-Invariant or Shift-Invariant

If $g[n] = H(f[n])$,

$$H(f[n - n_0]) = g[n - n_0] \tag{8.2-5}$$

where n_0 is a constant time delay, the system (H) is time-invariant.

8.2.3 Causality

A system is causal if the output of the system at any time n, $g[n]$ depends ONLY on present and past inputs, $f[n], f[n-1], f[n-2], \ldots$ but does not depend on the future inputs $f[n+1], f[n+2], \ldots$.

$$g[n] = H(f[n], f[n-1], \ldots) \tag{8.2-6}$$

where $H(\cdot)$ is some arbitrary function.

8.2.4 Static or Dynamic Systems

A discrete-system is called static or memory-less if its output at any instant depends at most on the input sample at the same time, but not on past or future samples of the input. Otherwise the system is said to be dynamic or to have memory.

8.2.5 Stability

A system is stable in the Bounded-Input Bounded-Output (BIBO) sense if and only if *every* bounded input sequence produces a bounded output sequence.

Namely, for every input sequence $f[n]$ such that

$$|f[n]| \leqslant M_f < \infty \qquad \text{for all } n \tag{8.2-7}$$

where M_f is a fixed positive finite value, there exists a finite fixed positive value M_g such that

$$|g[n]| \leqslant M_g < \infty \qquad \text{for all } n \tag{8.2-8}$$

where $g[n]$ is the output sequence or the system response to the input $f[n]$.

Example 8.2-1 *Determine the properties of a given system, assuming $g[n]$ is the output sequence and $f[n]$ is the input sequence.*

(a). Static or Dynamic?

- *(i)* $g[n] = af[n]$, *where a is a constant.*
- *(ii)* $g[n] = nx[n] + bx^3[n]$ *where b is a constant.*
- *(iii)* $g[n] = x[n] + 3x[n-1]$
- *(iv)* $g[n] = \sum_{k=0}^{N} x[n-k]$ *where $N = 0$ or N is a finite number but not zero*

(b). Time-variant or time-invariant

(i) $g[n] = H(f[n]) = f[n] - f[n-1]$

(ii) $g[n] = H(f[n]) = nf[n]$

(iii) $g[n] = H(f[n]) = f[-n]$

(iv) $g[n] = H(f[n]) = f[n]\cos(\omega_0 n)$

(c). Linear or nonlinear?

(i) $g[n] = H(f[n]) = nf[n]$

(ii) $g[n] = H(f[n]) = f[n^2]$

(iii) $g[n] = H(f[n]) = f^2[n]$

(iv) $g[n] = H(f[n]) = Af[n] + B$, *($A$, B constants)*

(v) $g[n] = H(f[n]) = e^{f[n]}$

(d). Causal or noncausal?

(i) $g[n] = f[n] - f[n-1]$

(ii) $g[n] = \sum_{k=-\infty}^{n} f[k]$

(iii) $g[n] = af[n]$

(iv) $g[n] = f[n] + 3f[n+4]$

(v) $g[n] = f[n^2]$

(vi) $g[n] = f[2n]$

(vii) $g[n] = f[-n]$

Solutions:

(a). Static or Dynamic?

(i) *Static*

(ii) *Static*

(iii) *Dynamic*

(iv) $N = 0$ *Static*
$N \neq 0$ *Dynamic*

(b). Time-variant or time-invariant

(i) $g[n] = H(f[n]) = f[n] - f[n-1]$. *If the input is delayed by n_0 units in time and applied to the system, the output is*

$$g[n, n_0] = f[n - n_0] - f[n - n_0 - 1]. \tag{8.2-9}$$

On the other hand, from the given condition

$$g[n - n_0] = f[n - n_0] - f[(n - n_0) - 1] \tag{8.2-10}$$

i.e.

$$g[n, n_0] = g[n - n_0]. \tag{8.2-11}$$

The system is time-invariant.

(ii) $g[n] = H(f[n]) = nf[n]$. *If the input is delayed by* n_0 *units and applied to the system, the response to* $f[n-n_0]$ *is*

$$g[n, n_0] = nf[n-n_0] \tag{8.2-12}$$

On the other hand, from the given condition

$$g[n-n_0] = (n-n_0)f[n-n_0] \tag{8.2-13}$$
$$= nf[n-n_0] - n_0 f[n-n_0] \tag{8.2-14}$$

i.e.

$$g[n, n_0] \neq g[n-n_0] \tag{8.2-15}$$

The system is time-variant.

(iii) $g[n] = H(f[n]) = f[-n]$. *The system response to* $f[n-n_0]$ *is*

$$g[n, n_0] = f[-n-n_0] \tag{8.2-16}$$

(which is $f[-n]$ *delayed by* n_0 *units.) However,*

$$g[n-n_0] = f[-(n-n_0)] \tag{8.2-17}$$
$$= f[-n+n_0] \tag{8.2-18}$$

The system is time-variant, since

$$g[n, n_0] \neq g[n-n_0] \tag{8.2-19}$$

(iv) $g[n] = H(f[n]) = f[n]\cos(\omega_0 n)$. *The system response to* $f[n-n_0]$ *is*

$$g[n, n_0] = f[n-n_0]\cos(\omega_0 n) \tag{8.2-20}$$

On the other hand,

$$g[n-n_0] = f[n-n_0]\cos(\omega_0[n-n_0]) \tag{8.2-21}$$

The system is time-variant.

(c). Linear or nonlinear?

(i) $g[n] = H(f[n]) = nf[n]$

$$g_1[n] = nf_1[n] \tag{8.2-22}$$
$$g_2[n] = nf_2[n] \tag{8.2-23}$$
$$g_3[n] = H\left(a_1 f_1[n] + a_2 f_2[n]\right) \tag{8.2-24}$$
$$= n\left\{a_1 f_1[n] + a_2 f_2[n]\right\} \tag{8.2-25}$$
$$= a_1 n f_1[n] + a_2 n f_2[n] \tag{8.2-26}$$
$$= a_1 g_1[n] + a_2 g_2[n] \tag{8.2-27}$$

Linear.

(ii) $g[n] = H(f[n]) = f[n^2]$

$$g_1[n] = f_1[n^2] \tag{8.2-28}$$
$$g_2[n] = f_2[n^2] \tag{8.2-29}$$
$$g_3[n] = H\left(a_1 f_1[n] + a_2 f_2[n]\right) \tag{8.2-30}$$
$$= a_1 f_1[n^2] + a_2 f_2[n^2] \tag{8.2-31}$$
$$= a_1 g_1[n] + a_2 g_2[n] \tag{8.2-32}$$

Linear.

(iii) $g[n] = H(f[n]) = f^2[n]$

$$g_1[n] = f_1^2[n] \tag{8.2-33}$$
$$g_2[n] = f_2^2[n] \tag{8.2-34}$$
$$g_3[n] = H\left(a_1 f_1[n] + a_2 f_2[n]\right) \tag{8.2-35}$$
$$= \left(a_1 f_1[n] + a_2 f_2[n]\right)^2 \tag{8.2-36}$$
$$\neq a_1 f_a^2[n] + a_2 f_2^2[n] \tag{8.2-37}$$

Nonlinear.

(iv) $g[n] = H(f[n]) = Af[n] + B$, *($A$, B constants)*

$$g_1[n] = Af_1[n] + B \tag{8.2-38}$$
$$g_2[n] = Af_2[n] + B \tag{8.2-39}$$
$$g_3[n] = H(a_1 f_1[n] + a_2 f_2[n]) \tag{8.2-40}$$
$$= A\left\{a_1 f_1[n] + a_2 f_2[n]\right\} + B \tag{8.2-41}$$
$$= a_1 A f_1[n] + a_2 A f_2[n] + B \tag{8.2-42}$$
$$\begin{cases} B = 0 & \textit{Linear} \\ B \neq 0 & \textit{Nonlinear} \end{cases} \tag{8.2-43}$$

(v) $g[n] = H(f[n]) = e^{f[n]}$ *Nonlinear.*

(d). Causal or noncausal?

(i) *Causal*

(ii) *Causal*

(iii) *Causal*

(iv) *Noncausal*

(v) *Noncausal*

(vi) *Noncausal*

(vii) *Noncausal Since* $n = -1$, $g[-1] = f[1]$, *with respect to* -1, 1 *is two steps into the future.*

EXAMPLE 8.2-2 *Consider the system referred to as an accumulator and defined by the relation*

$$g[n] = \sum_{k=-\infty}^{n} f[k] \tag{8.2-44}$$

where $f[k]$ *is the input sequence and* $g[n]$ *is the output sequence.*
Is this system stable? (BIBO?)
Solution:
Choose $f[n] = u[n]$, *where* $u[n]$ *is the unit step sequence defined as*

$$u[n] = \begin{cases} 1 & n \geqslant 0 \\ 0 & n < 0 \end{cases} \tag{8.2-45}$$

Obviously, $u[n]$ *is a bounded input as*

$$|u[n]| = 1 < \infty \qquad \textit{for all } n \tag{8.2-46}$$

However, for this input to the accumulator, the output is

$$g[n] = \sum_{k=-\infty}^{n} u[n] = \begin{cases} 0 & n < 0 \\ (n+1) & n \geqslant 0 \end{cases} \tag{8.2-47}$$

Although this response is finite if n is finite, it is nevertheless unbounded, i.e., there is no fixed positive value M_g such that $(n+1) \leqslant M_g < \infty$ for ALL n.

8.3 Discrete Unit Impulse and Impulse Response of the System

DEFINITION 8.3-1 *The discrete unit impulse is defined as the sequence*

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{8.3-48}$$

$\delta[n]$ is also known as the unit sample sequence. In contrast to the analog unit impulse function $\delta[t]$ which is infinite at $t = 0$, zero everywhere else and has unit area. $\delta[n]$ is one at $n = 0$ and zero elsewhere.

Using the unit sample sequence, a sequence $f[n]$ can be expressed as a sum of shifted impulses

$$f[n] = \sum_{m=-\infty}^{\infty} f[m]\delta[n-m] \tag{8.3-49}$$

Assume that the impulse response of the system (H) to a unit impulse is the sequence $h[n]$:

$$h[n] = H(\delta[n]) \tag{8.3-50}$$

If the system is *time-invariant* (or shift invariant) it follows that

$$h[n-m] = H(\delta[n-m]) \tag{8.3-51}$$

Furthermore, if the system is also *linear*, the system response to the input signal $f[n]$ is

$$g[n] = H(f[n]) \tag{8.3-52}$$

$$= H\left(\sum_{m=-\infty}^{\infty} f[m]\delta[n-m]\right) \tag{8.3-53}$$

$$= \sum_{m=-\infty}^{\infty} f[m]H(\delta[n-m]) \tag{8.3-54}$$

$$= \sum_{m=-\infty}^{\infty} f[m]h[n-m] \tag{8.3-55}$$

$$\triangleq h[n] * f[n] \tag{8.3-56}$$

where $*$ stands for the convolution of two sequences.

From Equation (8.3-56), it can be seen that a linear time-invariant system is completely defined by its impulse response.

If the system is *causal*, i.e.

$$h[n] = 0 \qquad \text{for } n < 0 \tag{8.3-57}$$

Equation (8.3-56) becomes

$$g[n] = \sum_{m=0}^{\infty} h[m]f[n-m] \tag{8.3-58}$$

$$\left(\text{Note:} \quad \sum_{m=-\infty}^{\infty} h[m]f[n-m] = \sum_{m=-\infty}^{\infty} f[m]h[n-m]\right) \tag{8.3-59}$$

REMARK 8.3-1 *All linear time-invariant (or LTI) systems are described by the convolution sum of Equation (8.3-56).*

8.3.1 Sufficient and necessary condition for BIBO stability of LTI systems

Linear time-invariant systems are stable if and only if (iff) the impulse response is absolutely summable, i.e., if

$$S = \sum_{m=-\infty}^{\infty} |h[m]| < \infty \tag{8.3-60}$$

Proof:

1. To prove that Equation (8.3-60) is a sufficient condition for BIBO stability, we assume that $g[n]$ and $f[n]$ are the output and input sequences of the LTI system respectively. It follows that, from Equation (8.3-58)

$$|g[n]| = \left| \sum_{m=-\infty}^{\infty} h[m]f[n-m] \right| \tag{8.3-61}$$

$$\leqslant \sum_{m=-\infty}^{\infty} |h[m]||f[n-m]| \tag{8.3-62}$$

   If $f[n]$ is bounded such that

$$|f[n]| \leqslant M_f < \infty \tag{8.3-63}$$

   then

$$|g[n]| \leqslant M_f \sum_{m=-\infty}^{\infty} |h[m]| \tag{8.3-64}$$

   Thus $g[n]$ is bounded if Equation (8.3-60) holds, i.e., Equation (8.3-60) is a sufficient condition for BIBO stability.

2. To show that Equation (8.3-60) is also a necessary condition, we must show that if $S = \infty$, a bounded input can be found that will cause an unbounded output. Consider the input sequence with values

$$f[n] = \begin{cases} \frac{h^*[-n]}{|h[-n]|} & h[n] \neq 0 \\ 0 & h[n] = 0 \end{cases} \tag{8.3-65}$$

where $h^*[n]$ is the complex conjugate of $h[n]$. $f[n]$ is clearly bounded by unity. However, the value of the output sequence at $n = 0$ is

$$g[0] = \sum_{m=-\infty}^{\infty} h[m]f[-m] \tag{8.3-66}$$

$$= \sum_{m=-\infty}^{\infty} h[m]\frac{h^*[m]}{|h[m]|} \tag{8.3-67}$$

$$= \sum_{m=-\infty}^{\infty} \frac{|h[m]|^2}{|h[m]|} = S \tag{8.3-68}$$

Thus, if $S = \infty$, it is possible for a bounded input sequence to produce an unbounded output sequence. Therefore Equation (8.3-60) is a necessary condition for BIBO stability.

Example 8.3-1 (Stability test for LTI systems)

(a). Given the ideal delay system defined by

$$g[n] = f[n - n_d] \qquad -\infty < n < \infty \tag{8.3-69}$$

where n_d is a fixed positive integer called the delay of the system.

(b). Given the general moving average system defined by

$$g[n] = \frac{1}{M_1 + M_2 + 1} \sum_{m=-M_1}^{M_2} f[n-m] \tag{8.3-70}$$

$$= \frac{1}{M_1 + M_2 + 1} \{f[n + M_1] + f[n + M_1 - 1] + \cdots$$
$$+ f[n] + f[n-1] + \cdots + f[n - M_2]\} \tag{8.3-71}$$

This system computes the n^{th} sample of the output sequence as the average of $(M_1 + M_2 + 1)$ samples of the input sequence around the n^{th} sample.

(c). Given the accumulator system defined by

$$g[n] = \sum_{k=-\infty}^{n} f[k] \tag{8.3-72}$$

(d). Given the forward difference system defined by

$$g[n] = f[n+1] - f[n] \tag{8.3-73}$$

(e). Given the backward difference system defined by

$$g[n] = f[n] - f[n-1] \tag{8.3-74}$$

Using the iff condition for BIBO stability determine whether the above systems are stable.

Solution:

Follow these two steps to solve the problem:

1. *Find the impulse response of the system by computing the response of the system to $\delta[n]$; using the defining relationship for the system*

2. *Compute the sum*

$$S = \sum_{n=-\infty}^{\infty} |h[n]| \tag{8.3-75}$$

and see if $S < \infty$.

(a). *The impulse response of the ideal delay system is*

$$h[n] = \delta[n - n_d] \tag{8.3-76}$$

where n_d *is a positive fixed integer.*

$$S = \sum_{n=-\infty}^{\infty} \delta[n - n_d] = 1 < \infty \tag{8.3-77}$$

since

$$\delta[n - n_d] = \begin{cases} 1 & \textit{for } n = n_d \\ 0 & \textit{otherwise} \end{cases} \tag{8.3-78}$$

It is BIBO stable.

(b). *The impulse response of the moving average system*

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{m=-M_1}^{M_2} \delta[n - m] \tag{8.3-79}$$

$$= \begin{cases} \frac{1}{M_1+M_2+1} & \textit{for } -M_1 \leqslant n \leqslant M_2 \\ 0 & \textit{otherwise} \end{cases} \tag{8.3-80}$$

Then

$$S = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-M_1}^{M_2} \frac{1}{M_1 + M_2 + 1} \tag{8.3-81}$$

$$= 1 < \infty \tag{8.3-82}$$

It is stable.

(c). *The impulse response of the accumulator:*

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1 & n \geqslant 0 \\ 0 & n < 0 \end{cases} \tag{8.3-83}$$

$$= u[n] \tag{8.3-84}$$

$$\left(\textit{Note:} \quad \delta[k] = \begin{cases} 1 & \textit{for } k = 0 \\ 0 & \textit{otherwise} \end{cases} \right) \tag{8.3-85}$$

$$S = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} u[n] \to \infty \tag{8.3-86}$$

It is unstable.

(d). The impulse response of the forward difference system:

$$h[n] = \delta[n+1] - \delta[n] \tag{8.3-87}$$

$$S = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n+1] - \delta[n]| \tag{8.3-88}$$

$$= 1 + 1 = 2 < \infty \tag{8.3-89}$$

$$\left(\textit{Note:} \quad |\delta[n+1] - \delta[n]|_{n=-1} + |\delta[n+1] - \delta[n]|_{n=0} \qquad \right) \tag{8.3-90}$$

It is stable.

(e). The impulse response of the backward difference system:

$$h[n] = \delta[n] - \delta[n-1] \tag{8.3-91}$$

Then

$$S = \sum_{n=-\infty}^{\infty} |\delta[n] - \delta[n-1]| \tag{8.3-92}$$

$$= |\delta[0] - \delta[-1]| + |\delta[1] - \delta[0]| \tag{8.3-93}$$

$$= 1 + 1 = 2 < \infty \tag{8.3-94}$$

The system is stable.

REMARK 8.3-2 *The impulse response of the ideal delay, moving average, forward difference system has only a finite number of non zero samples. Such systems are called Finite-duration Impulse Response (FIR) systems. FIR systems will always be stable as long as each of the impulse value is finite in magnitude.*

In comparison, the impulse response of the accumulator is infinite in duration. Such a system is called an Infinite-duration Impulse Response (IIR) system.

EXAMPLE 8.3-2 *Given an IIR system*

$$h[n] = a^n u[n] \tag{8.3-95}$$

where $u[n]$ is the unit step sequence, test the stability of the system.

Solution:

$$S = \sum_{n=-\infty}^{\infty} |h[n]| \tag{8.3-96}$$

$$= \sum_{n=-\infty}^{\infty} |a^n u[n]| \tag{8.3-97}$$

$$= \sum_{n=0}^{\infty} |a|^n \tag{8.3-98}$$

$$= \begin{cases} \frac{1}{1-|a|} & \textit{for } |a| < 1 \\ \infty & \textit{for } |a| \geqslant 1 \end{cases} \tag{8.3-99}$$

Therefore if $|a| < 1$, the system is stable, if $|a| \geqslant 1$, it is unstable.

8.4 Properties of Convolution and the Interconnection of LTI Systems

The convolution operation, denoted by $*$, of two discrete sequences is defined as

$$g[n] = h[n] * f[n] = \sum_{m=-\infty}^{\infty} h[m]f[n-m] \tag{8.4-100}$$

Define $m' = n - m$, and substitute m' into Equation (8.4-100).

$$g[n] = \sum_{m'=-\infty}^{\infty} h[n-m']f[m'] \tag{8.4-101}$$

$$= \sum_{m=-\infty}^{\infty} h[n-m]f[m] \tag{8.4-102}$$

$$= f[n] * h[n] \tag{8.4-103}$$

Thus the convolution operation is commutative.

Commutative Law

$$h[n] * f[n] = f[n] * h[n] \tag{8.4-104}$$

Associative Law

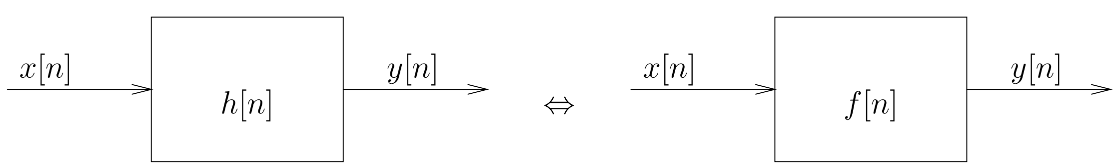
$$(h_1[n] * h_2[n]) * f[n] = h_a[n] * (h_2[n] * f[n]) \tag{8.4-105}$$

Distributive Law

$$(h_1[n] + h_2[n]) * f[n] = h_1[n] * f[n] + h_2[n] * f[n] \tag{8.4-106}$$

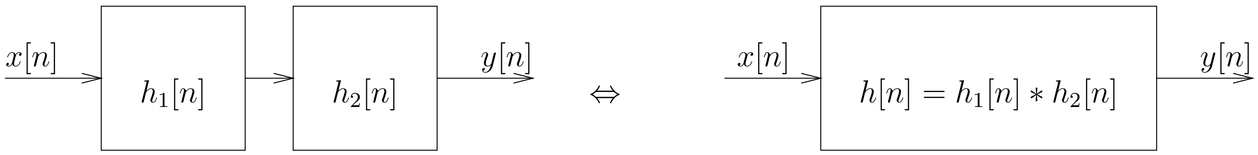
8.4.1 Implications and interpretation of properties of convolution

8.4.1.1 Commutative Property



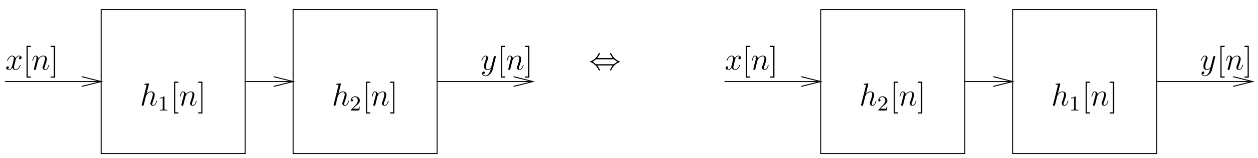
$y[n] = h[n] * x[n]$ $\qquad$ $y[n] = f[n] * x[n]$

8.4.1.2 Associative Property



$$y[n] = h_1[n] * h_2[n] * x[n]$$

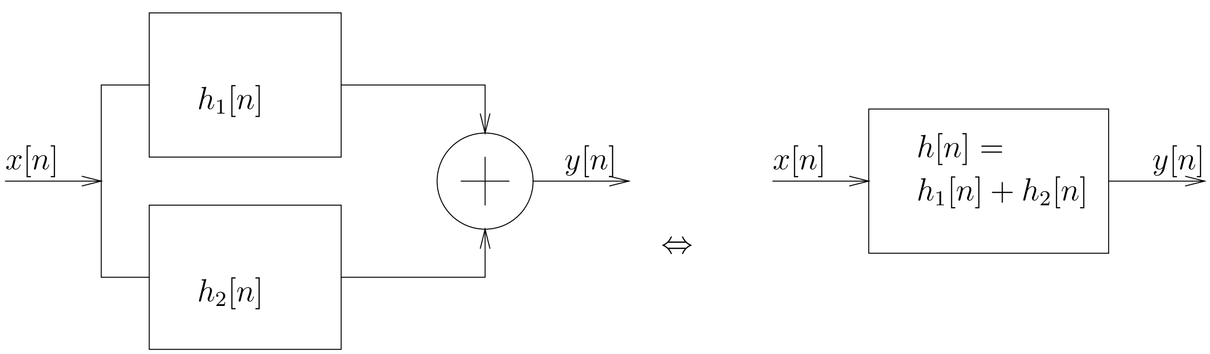
$$y[n] = h[n] * x[n]$$
where $h[n] = h_1[n] * h_2[n]$



$$y[n] = (h_1[n] * h_2[n]) * x[n]$$

$$y[n] = (h_2[n] * h_1[n]) * x[n]$$

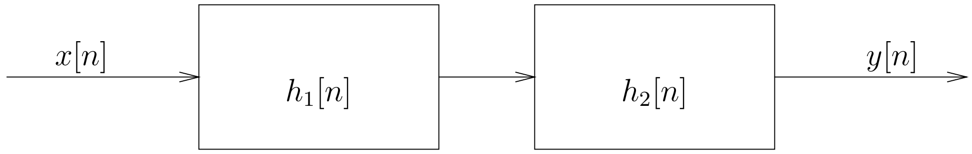
8.4.1.3 Distributive Property



$$y[n] = h_1[n] * x[n] + h_2[n] * x[n]$$

$$y[n] = (h_1[n] + h_2[n]) * x[n]$$

Example 8.4-1 *Consider a system that consists of a forward difference system cascaded with an ideal delay of one sample*



where $h_1[n] = \delta[n+1] - \delta[n]$ *forward difference, and* $h_2[n] = \delta[n-1]$ *one sample ideal delay.*

Find the impulse response of the overall system.

Solution:

$$\begin{aligned}
h[n] &= h_1[n] * h_2[n] && (8.4\text{-}107)\\
&= (\delta[n+1] - \delta[n]) * \delta[n-1] && (8.4\text{-}108)\\
&= \delta[n-1] * (\delta[n+1] - \delta[n]) && (8.4\text{-}109)\\
&= \delta[n-1] * \delta[n+1] - \delta[n-1] * \delta[n] && (8.4\text{-}110)\\
&= \sum_{k=-\infty}^{\infty} \delta[k-1]\delta[n-k+1] - \sum_{k=-\infty}^{\infty} \delta[k-1]\delta[n-k] && (8.4\text{-}111)\\
&= \delta[n] - \delta[n-1] && (8.4\text{-}112)
\end{aligned}$$

The impulse response of the cascade system is

$$\begin{aligned}
h[n] &= u[n] * (\delta[n] - \delta[n-1]) && (8.4\text{-}113)\\
&= \sum_{k=-\infty}^{\infty} u[k]\delta[n-k] - \sum_{k=-\infty}^{\infty} u[k]\delta[n-k-1] && (8.4\text{-}114)\\
&= u[n] - u[n-1] && (8.4\text{-}115)\\
&= \delta[n] && (8.4\text{-}116)
\end{aligned}$$

Thus the overall impulse response is an impulse. As a result,

$$\begin{aligned}
g[n] &= h[n] * f[n] && (8.4\text{-}117)\\
&= \sum_{k=-\infty}^{\infty} h[k]f[n-k] && (8.4\text{-}118)\\
&= \sum_{k=-\infty}^{\infty} \delta[k]f[n-k] && (8.4\text{-}119)\\
&= f[n] && (8.4\text{-}120)
\end{aligned}$$

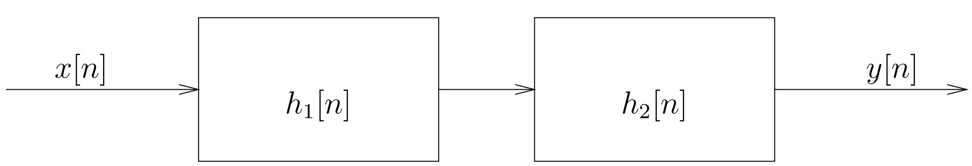
In this case, the backward difference system compensates exactly for the effect of the accumulator, i.e., the backward difference system is the inverse system for the accumulator, and vice versa.

The overall system is identical to the backward difference system.

REMARK 8.4-1 *The noncausal forward difference system has been converted to a causal system by cascading with a delay.*

In general, any noncausal FIR system can be made causal by cascading it with a sufficiently long delay.

EXAMPLE 8.4-2 *Consider a system which consists of an accumulator system and a backward difference system*



where $h_1[n] = u[n]$*, the impulse response of the accumulator,* $h_2[n] = \delta[n] - \delta[n-1]$*, the impulse response of the backward difference system.*

DEFINITION 8.4-1 (INVERSE SYSTEMS) *If a linear time-invariant system has impulse response* $h[n]$*, then its inverse system, if it exists, has impulse response* $h_i[n]$ *defined by the relation*

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n] \tag{8.4-121}$$

8.5 Linear Constant Coefficient Difference Equations

An important subclass of LTI systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an Nth-order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{8.5-122}$$

The z-transform can be used to solve the difference equations.

EXAMPLE 8.5-1 *The accumulator can be represented by a linear constant-coefficient difference equation*

$$y[n] = \sum_{k=-\infty}^{n} x[k] \tag{8.5-123}$$

$$= \sum_{k=-\infty}^{n-1} x[k] + x[n] \tag{8.5-124}$$

$$= y[n-1] + x[n] \tag{8.5-125}$$

or

$$y[n] - y[n-1] = x[n] \tag{8.5-126}$$

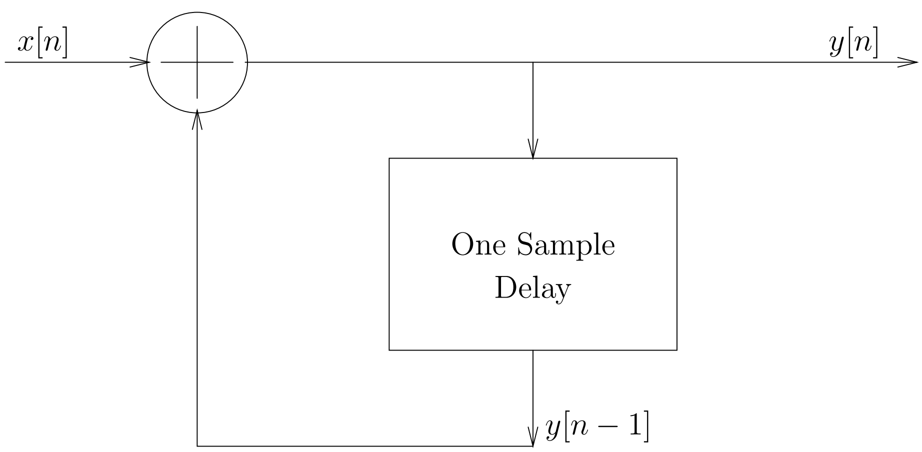


Figure 8.1: Recursion representation of $y[n] - y[n-1] = x[n]$

Figure 8.1 is often referred to as a recursion representation *of the system since each value* $y[n]$ *is computed using previous computed values* $y[n-1]$*.*

EXAMPLE 8.5-2 *The moving average system can also be expressed by a linear constant coefficient difference equation.*

Assume $M_1 = 0$ so that the system is causal. The impulse response of the system is

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k] \tag{8.5-127}$$

$$= \frac{1}{M_1 + M_2 + 1} \{u[n + M_1] - u[n - M_2 - 1]\} \tag{8.5-128}$$

$$= \frac{1}{M_2 + 1} \{u[n] - u[n - M_2 - 1]\} \tag{8.5-129}$$

It follows that

$$y[n] = \sum_{m=-\infty}^{\infty} \left\{ \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} \delta[m-k] x[n-m] \right\} \tag{8.5-130}$$

$$= \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} \left\{ \sum_{m=-\infty}^{\infty} \delta[m-k] x[n-m] \right\} \tag{8.5-131}$$

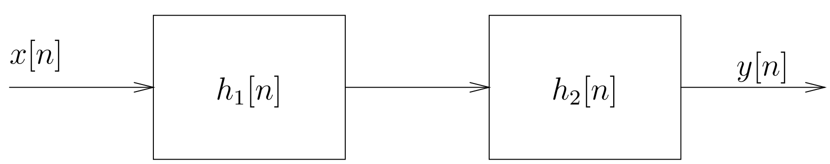
$$= \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k] \tag{8.5-132}$$

Equation (8.5-132) is a special case of Equation (8.5-122) with $N = 0$, $a_0 = 1$ and $b_k = \frac{1}{M_2+1}$ for $0 \leqslant k \leqslant M_2$.

From Equation (8.5-129)

$$h[n] = \underbrace{\frac{1}{M_2 + 1} \{\delta[n] - \delta[n - M_2 - 1]\}}_{h_1[n]} * \underbrace{u[n]}_{h_2[n]} \tag{8.5-133}$$

Equation (8.5-133) suggests that the moving average system can be represented as a cascade system as shown below



and noticing $h_2[n] = u[n]$ is the impulse response of the accumulator.

From Figure 8.2

$$y[n] - y[n-1] = x_1[n] \qquad \textit{(Accumulator)} \tag{8.5-134}$$

and

$$x_1[n] = \frac{1}{M_2 + 1} \{x[n] - x[n - M_2 - 1]\} \tag{8.5-135}$$

$$= h_1[n] * x[n] \tag{8.5-136}$$

Therefore

$$y[n] - y[n-1] = \frac{1}{M_2 + 1} \{x[n] - x[n - M_2 - 1]\} \tag{8.5-137}$$

Equation (8.5-137) is also a special case of Equation (8.5-122) with $N = 1$, $a_0 = 1$, $a_1 = -1$ and $b_0 = -b_{M_2+1} = \frac{1}{M_2+1}$, and $b_k = 0$ otherwise.

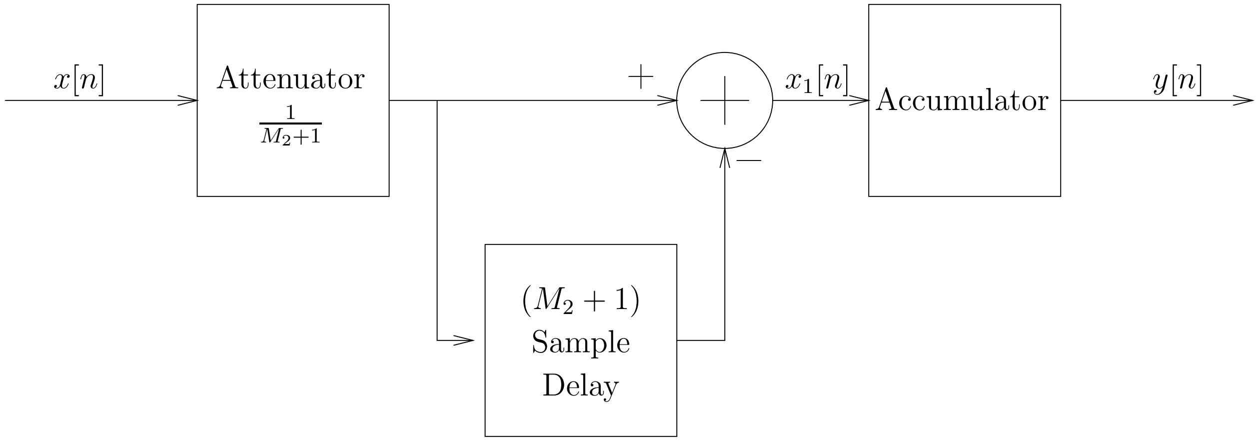


Figure 8.2: Recursive representation of moving average system

8.6 Frequency Domain Representation of Discrete Time Signals and Systems

Assume an LTI system with the impulse response $h[n]$, its response to an input sequence $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$, with a complex exponential of radian frequency ω, is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{8.6-138}$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega[n-k]} \tag{8.6-139}$$

$$= e^{j\omega n}\left\{\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right\} \tag{8.6-140}$$

Define

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \tag{8.6-141}$$

It follows that

$$y[n] = H(e^{j\omega})e^{j\omega n} \tag{8.6-142}$$

Consequently, $e^{j\omega n}$ is an eigenfunction of the system, and $H(e^{j\omega})$ is its associated eigenvalue.

$H(e^{j\omega})$ is called the frequency response of the system and it is complex in general

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) \tag{8.6-143}$$

or

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\measuredangle H(e^{j\omega})} \tag{8.6-144}$$

Example 8.6-1 *Find the frequency response of the ideal delay system (LTI)*

$$y[n] = x[n - n_d] \tag{8.6-145}$$

or

$$h[n] = \delta[n - n_d] \tag{8.6-146}$$

where n_d is a fixed integer.
Use $x[n] = e^{j\omega n}$ as the input,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} \delta[k - n_d] e^{j\omega(n-k)} && (8.6\text{-}147) \\ &= e^{j\omega(n-n_d)} && (8.6\text{-}148) \\ &= e^{-j\omega n_d} e^{j\omega n} && (8.6\text{-}149) \\ &= H(e^{j\omega}) e^{j\omega n} && (8.6\text{-}150) \\ |H(e^{j\omega})| &= \left| e^{-j\omega n_d} \right| = 1 && (8.6\text{-}151) \end{aligned}$$

and

$$\measuredangle H(e^{j\omega}) = -\omega n_d \tag{8.6-152}$$

Example 8.6-2 *Given an LTI system with the frequency response $H(e^{j\omega})$, find its response to a sinusoid*

$$x[n] = A\cos(\omega_0 n + \phi) \tag{8.6-153}$$

Solution:
Express $x[n]$ as

$$x[n] = \underbrace{\frac{A}{2} e^{j\phi} e^{j\omega_0 n}}_{x_1[n]} + \underbrace{\frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}}_{x_2[n]} \tag{8.6-154}$$

The system response to $x_1[n]$

$$y_1[n] = H(e^{j\omega_0}) \frac{A}{2} e^{j\phi} e^{j\omega_0 n} \tag{8.6-155}$$

and to $x_2[n]$

$$y_2[n] = H(e^{-j\omega_0}) \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n} \tag{8.6-156}$$

Thus

$$\begin{aligned} y[n] &= y_1[n] + y_2[n] && (8.6\text{-}157) \\ &= \frac{A}{2} \left\{ H(e^{j\omega_0}) e^{j\phi} e^{j\omega_0 n} + H(e^{-j\omega_0}) e^{-j\phi} e^{-j\omega_0 n} \right\} && (8.6\text{-}158) \end{aligned}$$

If $h[n]$ is real, then

$$H(e^{-j\omega_0}) = H^*(e^{j\omega_0}) \tag{8.6-159}$$

From Equation (8.6-158)

$$y[n] = \frac{A}{2}\left\{\left|H(e^{j\omega_0})\right| e^{j\measuredangle H(e^{j\omega_0})} e^{j\phi} e^{j\omega_0 n} + \left|H(e^{j\omega_0})\right| e^{-j\measuredangle H(e^{j\omega_0})} e^{-j\phi} e^{-j\omega_0 n}\right\} \tag{8.6-160}$$

$$= A\left|H(e^{j\omega_0})\right| \cos\left(\omega_0 n + \phi + \theta\right) \tag{8.6-161}$$

where $\theta = \measuredangle H(e^{j\omega_0})$ is the phase of the system at frequency ω_0.
Sinusoid in $\rightarrow$ sinusoid out.

8.7 Representation of Sequences by Fourier Transforms

Definition 8.7-1 (Discrete-time Fourier Transform and Its Inverse) *The discrete-time Fourier transform and its inverse are defined as follows:*

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8.7-162}$$

and

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{8.7-163}$$

where $x[n]$ is the discrete-time sequence and $X(e^{j\omega})$ is the Fourier transform of $x[n]$. Equation (8.7-162) gives the Fourier transform of $x[n]$, which is used for analyzing the sequence $x[n]$ to determine how much of each frequency component is required to synthesize $x[n]$ using Equation (8.7-163). Equation (8.7-163) is the inverse Fourier transform, which is a synthesis formula.

$-\pi < \measuredangle X(e^{j\omega}) < \pi$ is denoted as $ARG(X(e^{j\omega}))$ which is the principal range.

Comparing Equation (8.6-141) and Equation (8.7-162) it can be seen that the frequency response of an LTI system is simply the Fourier transform of the impulse response $h[n]$ and therefore the impulse response can be obtained from the frequency response $H(e^{j\omega})$ using the inverse Fourier transform integral, i.e.,

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \tag{8.7-164}$$

Remark 8.7-1 *It is useful to be aware of the equivalence between the Fourier series representation of continuous variable periodic functions and the Fourier transform representation of discrete-time signals, since all the familiar properties of the Fourier series can be applied, with appropriate interpretation of variables, to the Fourier transform representation of a sequence.*

In general, the discrete-time Fourier transform is a complex-valued function of ω, which can either be expressed in rectangular form as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \tag{8.7-165}$$

or in polar form as

$$X(e^{j\omega}) = \underbrace{\left|X(e^{j\omega})\right|}_{\text{magnitude}} \underbrace{e^{j\measuredangle X(e^{j\omega})}}_{\text{phase}} \tag{8.7-166}$$

The Fourier transform is also referred to as the Fourier transform spectrum or simply the spectrum.

$$\begin{array}{ll} \left|X(e^{j\omega})\right| & \text{magnitude spectrum} \\ \measuredangle X(e^{j\omega}) & \text{phase spectrum} \end{array}$$

Note: The phase $\measuredangle X(e^{j\omega})$ is not uniquely defined by Equation (8.7-166) since any integer multiple of 2π may be added to $\measuredangle X(e^{j\omega})$ at any value of ω without affecting the result of complex exponentiation.

Absolute summability of $x[n]$ is a sufficient condition for existence of a Fourier transform representation, and it also guarantees uniform convergence. In other words, if $x[n]$ is absolutely summable $X(e^{j\omega})$ exists.

Since any finite-length sequence is absolutely summable, it will have a Fourier transform representation.

EXAMPLE 8.7-1 *Given $x[n] = a^n u[n]$, its Fourier Transform is*

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} \tag{8.7-167}$$

$$= \sum_{n=0}^{\infty} \left[ae^{-j\omega}\right]^n \tag{8.7-168}$$

$$= \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |a| < 1 \tag{8.7-169}$$

Clearly the condition $|a| < 1$ is also the condition for absolute summability of $x[n]$. i.e.,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1 - |a|} \qquad \text{if } |a| < 1 \tag{8.7-170}$$

	Sequence	Transform	ROC
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4	$\delta[n-m]$	z^{-m}	All z except 0 [if $m > 0$] or ∞ [if $m < 0$]
5	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7	$na^n u[n]$	$\frac{az^{-1}}{[1-az^{-1}]^2}$	$\lvert z\rvert > \lvert a\rvert$
8	$-na^n u[-n-1]$	$\frac{az^{-1}}{[1-az^{-1}]^2}$	$\lvert z\rvert < \lvert a\rvert$
9	$(\cos\omega_0 n)u[n]$	$\frac{1-(\cos\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10	$(\sin\omega_0 n)u[n]$	$\frac{(\sin\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11	$(r^n \cos\omega_0 n)u[n]$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12	$(r^n \sin\omega_0 n)u[n]$	$\frac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13	$\begin{cases} a^n & 0 \leqslant n \leqslant N-1 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Table 8.1: Some common z-transform pairs

8.8 The z-Transform and Its Properties

Definition 8.8-1 (Two sided z-transform) *Given a sequence of real numbers $f[n]$, its two-sided z-transform is defined by*

$$\hat{F}(z) = \mathcal{Z}\left\{\{[\backslash]\right\} \tag{8.8-171}$$

$$\triangleq \sum_{n=-\infty}^{\infty} f[n]z^{-n} \tag{8.8-172}$$

where z is a complex variable.

Definition 8.8-2 (One sided z-transform) *The one sided z transform of the sequence is defined by*

$$F(z) = \mathcal{Z}\left\{\{[\backslash]\right\} \tag{8.8-173}$$

$$\triangleq \sum_{n=0}^{\infty} f[n]z^{-n} \tag{8.8-174}$$

where z is a complex variable. If the sequence is causal, i.e.,

$$f[n] = 0 \qquad \textit{for } n < 0 \tag{8.8-175}$$

$$\hat{F}(z) = F(z) \tag{8.8-176}$$

i.e., the two-sided and one-sided z-transforms are identical.

Example 8.8-1 *Given a sequence $f[n]$*

$$f[n] = \begin{cases} 1 & n = 0 \\ -1.5 & n = 1 \\ 2 & n = 2 \\ 0 & n = 3 \\ 3 & n = 4 \\ 0 & \textit{for } n > 4 \textit{ and } n < 0 \end{cases} \tag{8.8-177}$$

the z-transform of $f[n]$ is

$$F(z) = \sum_{n=0}^{\infty} f[n]z^{-n} \tag{8.8-178}$$

$$= 1 - 1.5z^{-1} + 2z^{-2} + 3z^{-4} \tag{8.8-179}$$

Example 8.8-2 *The z-transform of the discrete impulse sequence is*

$$\mathcal{Z}\left\{\delta[\backslash]\right\} = \sum_{n=0}^{\infty} z^{-n} \tag{8.8-180}$$

$$= \delta[0]z^0 \tag{8.8-181}$$

$$= 1 \tag{8.8-182}$$

Example 8.8-3 *The z-transform of the discrete unit step sequence.*

$$\mathcal{Z}\left\{\sqcap[\backslash]\right\} = \sum_{n=0}^{\infty} u[n]z^{-n} \tag{8.8-183}$$

$$= \sum_{n=0}^{\infty} z^{-n} \tag{8.8-184}$$

$$= \frac{1}{1 - z^{-1}} \qquad \textit{for } |z| > 1 \tag{8.8-185}$$

EXAMPLE 8.8-4 *The z-transform of the shifted discrete impulse sequence $\delta[n-m]$ is*

$$\mathcal{Z}\{\delta[\backslash - \Updownarrow]\} = \sum_{n=0}^{\infty} \delta[n-m] z^{-n} \tag{8.8-186}$$

$$= z^{-m} \tag{8.8-187}$$

EXAMPLE 8.8-5 *The z-transform of the shifted unit step sequence $u[n-m]$ is*

$$\mathcal{Z}\{\sqcap[\backslash - \Updownarrow]\} = \sum_{n=0}^{\infty} u[n-m] z^{-n} \tag{8.8-188}$$

$$= \sum_{n=m}^{\infty} z^{-n} \tag{8.8-189}$$

$$= z^{-m} + z^{-(m+1)} + z^{-(m+2)} + \cdots \tag{8.8-190}$$

$$= z^{-m} \left\{1 + z^{-1} + z^{-2} + \cdots\right\} \tag{8.8-191}$$

$$= \frac{z^{-m}}{1 - z^{-1}} \qquad \textit{for } |z| > 1 \tag{8.8-192}$$

EXAMPLE 8.8-6 *Given an exponential sequence*

$$f[n] = e^{-\alpha n} \qquad \alpha > 0 \tag{8.8-193}$$

the z-transform of $f[n]$ is

$$\mathcal{Z}\left\{\rceil^{-\alpha\backslash}\right\} = \sum_{n=0}^{\infty} e^{-\alpha n} z^{-n} \tag{8.8-194}$$

$$= \sum_{n=0}^{\infty} \left[e^{-\alpha} z^{-1}\right]^{n} \tag{8.8-195}$$

$$= \frac{1}{1 - e^{-\alpha} z^{-1}} \qquad \textit{for } |e^{-\alpha} z^{-1}| < 1 \tag{8.8-196}$$

8.8.1 Linearity of the z-transform

Assume that

$$F_1(z) = \mathcal{Z}\{\{_\infty[\backslash]\} \tag{8.8-197}$$

and

$$F_2(z) = \mathcal{Z}\{\{_\in[\backslash]\} \tag{8.8-198}$$

$$\mathcal{Z}\left\{\dashv\{_\infty[\backslash] + \lfloor\{_\in[\backslash]\right\} = aF_1(z) + bF_2(z) \tag{8.8-199}$$

where a and b are constants. That is the z-transform is linear.

Proof:

$$\mathcal{Z}\left\{\dashv\{_\infty[\backslash] + \lfloor\{_\in[\backslash]\right\} = \sum_{n=0}^{\infty} \left\{a f_1[n] + b f_2[n]\right\} z^{-n} \tag{8.8-200}$$

$$= a \sum_{n=0}^{\infty} f_1[n] z^{-n} + b \sum_{n=0}^{\infty} f_2[n] z^{-n} \tag{8.8-201}$$

$$= aF_1(z) + bF_2(z) \tag{8.8-202}$$

8.8.2 Shifting

If $\mathcal{Z}\{\{[\backslash]\} = \mathcal{F}(\ddagger)$

$$\mathcal{Z}\left\{\{[\backslash - \Updownarrow]\right\} = \ddagger^{-\Updownarrow}\mathcal{F}(\ddagger) \tag{8.8-203}$$

with $m > 0$ and $f[n]$ assumed causal.

Proof:

$$\mathcal{Z}\{\{[\backslash - \Updownarrow]\} = \sum_{n=0}^{\infty} f[n-m]z^{-n} \tag{8.8-204}$$

$$= \sum_{k=-m}^{\infty} f[k]z^{-[k+m]} \tag{8.8-205}$$

$$= z^{-m} \sum_{k=-m}^{\infty} f[k]z^{-k} \tag{8.8-206}$$

$$= z^{-m} \sum_{k=-m}^{-1} f[k]z^{-k} + z^{-m} \sum_{k=0}^{\infty} f[k]z^{-k} \tag{8.8-207}$$

$$= z^{-m} \sum_{k=0}^{\infty} f[k]z^{-k} + z^{-m} \sum_{k=1}^{m} f[-k]z^{k} \tag{8.8-208}$$

$$= z^{-m}F(z) + z^{-m} \sum_{k=1}^{m} f[-k]z^{k} \tag{8.8-209}$$

where $F(z)$ is the z-transform of $f[n]$. If the sequence $f[n]$ is causal, i.e., $f[-k] = 0$ for $k > 0$,

$$\mathcal{Z}\{\{[\backslash - \Updownarrow]\} = \ddagger^{-\Updownarrow}\mathcal{F}(\ddagger) \tag{8.8-210}$$

The term

$$z^{-m} \sum_{k=1}^{m} f[-k]z^{k} \tag{8.8-211}$$

represents the initial conditions.

From the shifting property, delaying a sequence by 1 unit (or 1 sample period) is equivalent to multiplication by z^{-1} in the z-domain. Therefore, z^{-1} is called the unit delay operator.

$$\mathcal{Z}\{\{[\backslash + \Updownarrow]\} = \sum_{n=0}^{\infty} f[n+m]z^{-n} \tag{8.8-212}$$

$$= \sum_{k=m}^{\infty} f[k]z^{-[k-m]} \tag{8.8-213}$$

$$= \sum_{k=0}^{\infty} f[k]z^{-[k-m]} - \sum_{k=0}^{m-1} f[k]z^{-[k-m]} \tag{8.8-214}$$

$$= z^{m} \sum_{k=0}^{\infty} f[k]z^{-k} - z^{-m} \sum_{k=0}^{m-1} f[k]z^{-k} \tag{8.8-215}$$

8.8.3 Convolution of sequences

Recalling that the convolution of two sequences $f_1[n]$ and $f_2[n]$ is deifned as

$$f_1[n] * f_2[n] \triangleq \sum_{m=0}^{\infty} f_1[m] f_2[n-m] \tag{8.8-216}$$

$$= \sum_{m=0}^{\infty} f_2[m] f_1[n-m] \tag{8.8-217}$$

it follws that

$$f_1[n] * f_2[n] = \sum_{m=0}^{n} f_1[m] f_2[n-m] \tag{8.8-218}$$

if $f_1[n]$ and $f_2[n]$ are causal.

The z-transform of the convolution of two sequences $f_1[n]$ and $f_2[n]$ is

$$\mathcal{Z}\{\{_\infty[\backslash] * \{_\in[\backslash]\} = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty} f_1[m] f_2[n-m] \right\} z^{-n} \tag{8.8-219}$$

$$= \sum_{m=0}^{\infty} f_1[m] \sum_{n=0}^{\infty} f_2[n-m,] z^{-n} \tag{8.8-220}$$

$$= \sum_{m=0}^{\infty} f_1[m] \sum_{k=-m}^{\infty} f_2[k] z^{-[k+m]} \tag{8.8-221}$$

$$= \sum_{m=0}^{\infty} f_1[m] z^{-m} \left\{ \sum_{k=0}^{\infty} f_2[k] z^{-k} + \sum_{k=-m}^{-1} f_2[k] z^{-k} \right\} \tag{8.8-222}$$

$$= \left\{ \sum_{m=0}^{\infty} f_1[m] z^{-m} \right\} \left\{ \sum_{k=0}^{\infty} f_2[k] z^{-k} \right\} \tag{8.8-223}$$

$$= F_1(z) F_2(z) \tag{8.8-224}$$

where $F_1(z) = \mathcal{Z}\{\{_\infty[\backslash]\}$ and $F_2(z) = \mathcal{Z}\{\{_\in[\backslash]\}$. i.e.,

$$\mathcal{Z}\{\{_\infty[\backslash] * \{_\in[\backslash]\} = \mathcal{Z}\{\{_\infty[\backslash]\}\mathcal{Z}\{\{_\in[\backslash]\} \tag{8.8-225}$$

If $f_1[n]$ and $f_2[n]$ are causal, the z-transform of the convolution of two sequences is equivalent to the product of z-transforms of each sequence.

8.8.4 Convergence

The one-sided z-transform of the causal sequence $f[n]$:

$$F(z) = \sum_{n=0}^{\infty} f[n] z^{-n} \tag{8.8-226}$$

is only defined at all points in the z-plane where the Equation (8.8-226) converges.

8.9 Relationship between the z-Transform and the Laplace Transform

Assume that the sequence $f[n]$ is obtained by sampling a causal continuous signal $f(t)$. i.e.,

$$f[n] = f(nT) \qquad n = 0, 1, \ldots \tag{8.9-227}$$

Property Number	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R_x
	$x_1[n]$	$X_1(z)$	R_{x1}
	$x_2[n]$	$X_2(z)$	R_{x2}
1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x1} \cap R_{x2}$
2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x except for the possible addition or deletion of the origin or ∞
3	$z_0^n x[n]$	$X(z/Z_0)$	$\|z_0\|R_x$
4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x except for the possible addition or deletion of the origin or ∞
5	x^*	$X^*(z^*)$	R_x
6	$\Re\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7	$\Im\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	$x[-n]$	$X(1/z)$	$1/R_x$
9	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x1} \cap R_{x2}$
10	Initial Value Theorem: $x[n] = 0, n < 0$ $\quad \lim_{z\to\infty} X(z) = x[0]$		
11	$x_1[n]x_2[n]$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2(z/v)v^{-1}dv$	Contains $R_{x1}R_{x2}$
12	Parseval's Relation: $\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$		

Table 8.2: Some z-transform properties

the z-transform of the sequence is

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n} \tag{8.9-228}$$

Recalling that the sampled signal $f_s(t)$ of a causal continuous-time $f(t)$ is

$$f_s(t) = \sum_{n=0}^{\infty} f(nT)\delta(t - nT) \tag{8.9-229}$$

the Laplace transform of $f_s(t)$ is

$$\begin{aligned}
\mathcal{L}\{\{_f(\sqcup)\} &= F_s(s) && (8.9\text{-}230)\\
&= \int_0^{\infty} f_s[t]e^{-ts}dt && (8.9\text{-}231)\\
&= \int_0^{\infty} \sum_{n=0}^{\infty} f(nT)\delta(t - nT)e^{-ts}dt && (8.9\text{-}232)\\
&= \sum_{n=0}^{\infty} f(nT)\int_0^{\infty} \delta(t - nT)e^{-ts}dt && (8.9\text{-}233)\\
&= \sum_{n=0}^{\infty} f(nT)e^{-nTs} && (8.9\text{-}234)
\end{aligned}$$

Comparing Equation (8.9-228) and Equation (8.9-234)

$$\begin{aligned}
F(z) &= \mathcal{Z}\{\{[\backslash]\}|_{\ddagger=\rceil^{\mathcal{T}f}} && (8.9\text{-}235)\\
&= \mathcal{L}\{\{_f(\sqcup)\} = \mathcal{F}_f(f) && (8.9\text{-}236)
\end{aligned}$$

i.e., the z-transform and the Laplace transform of the causal sampled signal are identical if we define $z = e^{Ts}$.

Noticing that:

$$s = \sigma + j\omega \tag{8.9-237}$$
$$z = e^{\sigma T} e^{j\omega T} \tag{8.9-238}$$

$$\sigma < 0 \quad \text{maps into} \quad |z| < 1 \tag{8.9-239}$$
$$\sigma > 0 \quad \text{maps into} \quad |z| > 1 \tag{8.9-240}$$
$$j\omega \text{ axis} \quad \text{maps into} \quad |z| = 1 \tag{8.9-241}$$

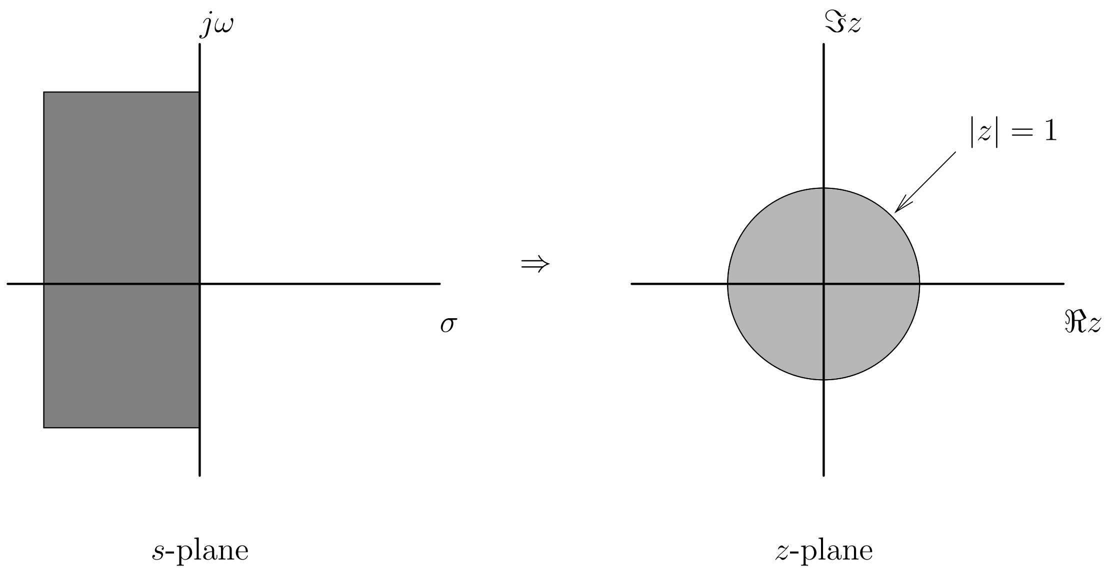


Figure 8.3: Relationship between s and z-planes

The Fourier transform of the sequence $f[n]$ is obtained by substituting $j\omega$ for s in Equation (8.9-234).

$$F(j\omega) = \sum_{n=0}^{\infty} f(nT)e^{-jnT\omega} \tag{8.9-242}$$

8.10 The Inverse z-Transform

There are several ways to calculate the inverse z-trasnform to obtain the sequence $f[n]$ from its z-transform.

- the inversion integral
- long division (series expansion)
- partial fractions

8.10.1 Use of the Inversion Integral

Given the z-transform of the sequence $f[n]$

$$F(z) = \sum_{n=0}^{\infty} z^{-n} \tag{8.10-243}$$

multiply Equation (8.10-243) by z^{k-1} and integrate over a closed contour C within the region of convergence of $F(z)$, which gives

$$\oint_C F(z) z^{k-1} dz = \oint_C \sum_{n=0}^{\infty} f[n] z^{-n+k-1} dz \tag{8.10-244}$$

Providing that $f[n]$ is absolutely summable, i.e.,

$$\sum_{n=0}^{\infty} |f[n]| < \infty \tag{8.10-245}$$

the order of integration and summation in Equation (8.10-244) can be reversed. That is

$$\oint_C F(z) z^{k-1} dz = \sum_{n=0}^{\infty} f[n] \oint_C z^{-n+k-1} dz \tag{8.10-246}$$

Using the Cauchy integral theorem

$$\frac{1}{2\pi j} \oint_C z^{-n+k-1} dz = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} \tag{8.10-247}$$

$$\sum_{n=0}^{\infty} f[n] \oint_C z^{-n+k-1} dz = 2\pi j f[k] \qquad \text{for } n = k \tag{8.10-248}$$

Thus

$$2\pi j f[n] = \oint_C F(z) z^{n-1} dz \tag{8.10-249}$$

or

$$f[n] = \frac{1}{2\pi j} \oint_C F(z) z^{n-1} dz \tag{8.10-250}$$

$$= \mathcal{Z}^{-\infty}[\mathcal{F}(\ddagger)] \tag{8.10-251}$$

where C is a contour (counter clockwise) enclosing all the singularities of poles of $F(z)$.

Using Cauchy's residue theorem,

$$f[n] = \frac{1}{2\pi j} \oint_C F(z) z^{n-1} dz \tag{8.10-252}$$

$$= \sum \Big(\text{residues of } F(z) z^{n-1} \text{ at the poles inside } C\Big) \tag{8.10-253}$$

If $F(z)$ is a rational function

$$F(z) = \frac{\sum_{r=0}^{M} a_r z^{-r}}{\sum_{r=0}^{N} b_r z^{-r}} \tag{8.10-254}$$

define

$$F_0(z) = F(z)z^{n-1} \tag{8.10-255}$$
$$= \frac{z^{n-1}P_M(z)}{\prod_{i=0}^{l}(z-p_i)^{m_i}} \tag{8.10-256}$$

where p_i are the poles of $F_0(z)$ and m_i are their multiplicities.

From Equation (8.10-253),

$$f[n] = \sum_i k_i \tag{8.10-257}$$

where k_i are the residues of $F_0(z)$ at its poles

At a simple pole

$$k_i = \lim_{z\to p_i}[(z-p_i)F_0(z)] \tag{8.10-258}$$

while at an mth order pole

$$k_i = \frac{1}{[m_i-1]!}\lim_{z\to p_i}\frac{d^{m_i-1}}{dz^{m_i-1}}(z-p_i)^{m_i}F_0(z) \tag{8.10-259}$$

EXAMPLE 8.10-1 *Given*

$$F(z) = \frac{z(2z-1)}{(z-1)(z+0.5)} \tag{8.10-260}$$
$$F_0(z) = z^{n-1}F(z) \tag{8.10-261}$$
$$= z^n\frac{(2z-1)}{(z-1)(z+0.5)} \tag{8.10-262}$$
$$k_1 = \left.\frac{(2z-1)}{(z+0.5)}z^n\right|_{z=1} = \frac{2}{3} \tag{8.10-263}$$
$$k_{0.5} = \left.\frac{(2z-1)}{(z-1)}z^n\right|_{z=-0.5} \tag{8.10-264}$$
$$= \frac{2[-\frac{1}{2}]-1}{-\frac{1}{2}-1}\left[-\frac{1}{2}\right]^n \tag{8.10-265}$$
$$= \frac{4}{3}\left[-\frac{1}{2}\right]^n \tag{8.10-266}$$

Hence

$$f[n] = \frac{2}{3} + \frac{4}{3}\left[-\frac{1}{2}\right]^n \tag{8.10-267}$$

8.10.2 Use of Long Division (Power Series Expansion)

If the z-traznsform is given as a power series in the form

$$F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n} \tag{8.10-268}$$
$$= \cdots + f[-2]z^2 + f[-1]z + f[0] + f[1]z^{-1} + f[2]z^{-2} + \cdots \tag{8.10-269}$$

any particular value of the sequence can be determined by finding the coefficient of the appropriate power of z^{-1}.

For a given rational function

$$F(z) = \frac{\sum_{r=0}^{M} a_r z^{-r}}{\sum_{r=0}^{N} b_r z^{-r}} \tag{8.10-270}$$

the power series expansion can be obtained by dividing the numerator by the denominator.

EXAMPLE 8.10-2 *Consider the function*

$$F(z) = \frac{1}{1 - 0.8z^{-1}} \tag{8.10-271}$$

where $|0.8z^{-1}|$ *or* $|z| > 0.8$.

$$\begin{array}{r|l}
 & 1 + 0.8z^{-1} + 0.64z^{-2} + \cdots \\
\hline
1 - 0.8z^{-1} & 1 \\
 & \underline{1 - 0.8z^{-1}} \\
 & \quad 0.8z^{-1} \\
 & \quad \underline{0.8z^{-1} - 0.64z^{-2}} \\
 & \qquad\quad 0.64z^{-2} \\
 & \qquad\quad \underline{0.64z^{-2} - 0.512z^{-3}} \\
 & \qquad\qquad\quad 0.512z^{-3} \\
 & \qquad\qquad\qquad \vdots
\end{array}$$

Thus

$$F(z) = 1 + 0.8z^{-1} + 0.64z^{-2} + \cdots \tag{8.10-272}$$

$$= 1 + 0.8z^{-1} + [0.8]^2 z^{-2} + \cdots + (0.8)^n z^{-n} + \cdots \tag{8.10-273}$$

and

$$f[n] = (0.8)^n u[n] \tag{8.10-274}$$

where $u[n]$ *is the unit step sequence.*

8.10.3 Use of Partial Fractions

Consider $F(z)$ to be a rational function

$$F(z) = \frac{P_M(z^{-1})}{D_N(z^{-1})} \tag{8.10-275}$$

$$= \frac{P_M(z^{-1})}{\sum_{r=1}^{N}(1 - P_r z^{-1})} \tag{8.10-276}$$

Express Equation (8.10-276) using the partial fraction expansion,

$$F(z) = \sum_{r=1}^{N} \frac{a_r}{(1 - P_r z^{-1})} \tag{8.10-277}$$

Notice that the z-transform pair

$$a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}} \qquad \text{for } |z| > |a| \tag{8.10-278}$$

From Equation (8.10-277) and Equation (8.10-278)

$$f_n[n] = \sum_{r=1}^{N} a_r P_r^n u[n] \tag{8.10-279}$$

$$= \begin{cases} \sum_{r=1}^{N} a_r P_r^n & \text{for } n \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8.10-280}$$

Example 8.10-3 *Given*

$$F(z) = \frac{1}{(1 - z^{-1})(1 + 0.8z^{-1})} \tag{8.10-281}$$

the partial fraction expansion of $F(z)$,

$$F(z) = \frac{a_1}{1 - z^{-1}} + \frac{a_2}{1 + 0.8z^{-1}} \tag{8.10-282}$$

where

$$a_1 = (1 - z^{-1})F(z)|_{z=1} = \frac{1}{1 + 0.8} = \frac{1}{1.8} = \frac{5}{9} \tag{8.10-283}$$

$$a_2 = (1 + 0.8z^{-1})F(z)|_{z=-0.8} = \frac{1}{1 + \frac{1}{0.8}} = \frac{1}{1 + \frac{5}{4}} = \frac{4}{9} \tag{8.10-284}$$

Thus

$$F(z) = \frac{\frac{5}{9}}{1 - z^{-1}} + \frac{\frac{4}{9}}{1 + 0.8z^{-1}} \tag{8.10-285}$$

The inverse trasnform is the sequence

$$f[n] = \frac{5}{9} + \frac{4}{9}[-0.8]^n \qquad \textit{for } n \geqslant 0 \tag{8.10-286}$$

8.11 Complex Convolution Theorem

Given two discrete-time sequences $f[n]$ and $g[n]$, assuming that they have z-transforms

$$F(z) = \mathcal{Z}\{\{[\backslash]\} = \sum_{\backslash=\prime}^{\infty} \{[\backslash]\ddagger^{-\backslash} \tag{8.11-287}$$

and

$$G(z) = \mathcal{Z}\{\}[\backslash]\} = \sum_{\backslash=\prime}^{\infty} \}[\backslash]\ddagger^{-\backslash} \tag{8.11-288}$$

define the sequence

$$q[n] = f[n]g[n] \tag{8.11-289}$$

The complex convolution theorem states that the z-transform of $q[n]$ can be obtained using the z-transforms of $f[n]$ and $g[n]$ in the following manner:

$$Q(z) = \mathcal{Z}\{\text{II}[\backslash]\} \tag{8.11-290}$$

$$= \frac{1}{2\pi j}\oint_{C1} F(\nu)G\left(\frac{z}{\nu}\right)\frac{d\nu}{\nu} \tag{8.11-291}$$

$$= \frac{1}{2\pi j}\oint_{C2} F\left(\frac{z}{\nu}\right)G(z)\frac{d\nu}{\nu} \tag{8.11-292}$$

where $C1$ (or $C2$) is a contour in the common region of convergence of $F[\nu]$ and $G\left[\frac{z}{\nu}\right]$ (or $F\left[\frac{z}{\nu}\right]$ and $G[\nu]$).

Proof:

The z-transform of $q[n]$ is given by

$$Q(z) = \mathcal{Z}\{\text{II}[\backslash]\} \tag{8.11-293}$$

$$= \sum_{n=0}^{\infty}\{f[n]g[n]\}z^{-n} \tag{8.11-294}$$

Notice that $g[n]$ can be expressed using the inverse z-transform:

$$g[n] = \frac{1}{2\pi j}\oint_{C} G(z)z^{n-1}dz \tag{8.11-295}$$

Thus

$$Q(z) = \sum_{n=0}^{\infty}\left\{f[n]\frac{1}{2\pi j}\oint_{C_2} G(\nu)\nu^{n-1}d\nu\right\}z^{-n} \tag{8.11-296}$$

$$= \frac{1}{2\pi j}\oint_{C_2}\sum_{n=0}^{\infty} f[n]\left[\frac{z}{\nu}\right]^{-n} G(\nu)\frac{d\nu}{\nu} \tag{8.11-297}$$

$$= \frac{1}{2\pi j}\oint_{C_2} F\left(\frac{z}{\nu}\right)G(\nu)\frac{d\nu}{\nu} \tag{8.11-298}$$

Note:

$$F(z) = \sum_{n=0}^{\infty} f[n]z^{-n} \tag{8.11-299}$$

and

$$F\left[\frac{z}{\nu}\right] = \sum_{n=0}^{\infty} f[n]\left[\frac{z}{\nu}\right]^{-n} \tag{8.11-300}$$

8.11.1 Parseval's Relation

From Equation (8.11-294) and Equation (8.11-298),

$$\sum_{n=0}^{\infty}\{f[n]g[n]\}z^{-n} = \frac{1}{2\pi j}\oint_{C} F\left[\frac{z}{\nu}\right]G[\nu]\frac{d\nu}{\nu} \tag{8.11-301}$$

If $f[n] = g[n]$ and $z = 1$, Equation (8.11-301) becomes

$$\sum_{n=0}^{\infty} f^2[n] = \frac{1}{2\pi j}\oint_{C} F[\nu^{-1}]F[\nu]\frac{d\nu}{\nu} \tag{8.11-302}$$

or

$$\sum_{n=0}^{\infty} f^2[n] = \frac{1}{2\pi j} \oint_C F(z)F(z^{-1}) \frac{dz}{z} \tag{8.11-303}$$

Equation (8.11-303) is the discrete form of Parseval's relation for causal sequences and the one-sided z-transform.

8.12 Discription of Discrete Systems

A very important subclass of linear time-invariant discrete systems consists of those systems for which the input $f[n]$ and the output $g[n]$ are related by a linear difference equation with constant coefficients in the following form.

$$g[n] = \sum_{r=0}^{M} a_r f[n-r] - \sum_{r=1}^{N} b_r g[n-r] \tag{8.12-304}$$

where $M \leqslant N$, providing that the system is causal, and a_r and b_r are real constants.

Applying z-transform to Equation (8.12-304),

$$G(z) = \sum_{r=0}^{M} a_r F(z) z^{-r} - \sum_{r=1}^{N} b_r G(z) z^{-r} \tag{8.12-305}$$

$$= \left\{ \sum_{r=0}^{M} a_r z^{-r} \right\} F(z) - \left(\sum_{r=1}^{N} b_r z^{-r} \right) G(z) \tag{8.12-306}$$

Thus, the transfer function of the system can be given as

$$H(z) = \frac{G(z)}{F(z)} = \frac{\sum_{r=0}^{M} a_r z^{-r}}{1 + \sum_{r=1}^{N} b_r z^{-r}} \tag{8.12-307}$$

This means that the relationship between the input and output of the above system is represented by a real rational function in the z-domain.

In the discussion of the relationship between the z-transform and the Laplace transform, it is defined that

$$z^{-1} \triangleq e^{-Ts} \tag{8.12-308}$$

Substitute Equation (8.12-308) into Equation (8.12-307) to obtain the system frequency response as

$$H(e^{j\omega T}) = \frac{\sum_{r=0}^{M} a_r e^{-jr\omega T}}{1 + \sum_{r=1}^{N} b_r e^{-jr\omega T}} \tag{8.12-309}$$

or

$$H(e^{j\omega T}) = \left| H(e^{j\omega T}) \right| e^{j\Psi[\omega]} \tag{8.12-310}$$

where $|H(e^{j\omega T})|$ is the amplitude response of the system and $\Psi[\omega]$ is the phase response.

Since $H(z)$ is a rational function of z and the exponential function is periodic with a period of 2π, it follows that $|H(e^{j\omega T})|$ and $\Psi[\omega]$ are also periodic with respect to ω, the period being $\omega_N = 2\pi f_N$, where $f_N = \frac{1}{T}$ is the sampling frequency.

Therefore, the system frequency response is obtained by considering the points on the $j\omega$-axis of the s-plane, which is equaivalent to considering the points on the unit circle $|z| = 1$ in the z-plane.

8.12.1 Stability and causality

Using the transfer function $H(z)$, the system response in the z-domain can be expressed

$$G(z) = H(z)F(z) \tag{8.12-311}$$

where $G(z)$ and $F(z)$ are z-transforms of the output and input of the system respectively.

Using the convolution theorem, the system response is given

$$g[n] = \sum_{m=0}^{n} h[m]f[n-m] \tag{8.12-312}$$

i.e.,

$$g[n] = h[n] * f[n] \tag{8.12-313}$$

where $h[n]$ is the impulse response of the causal system given by

$$h[n] = \mathcal{Z}^{-\infty}\{\mathcal{H}(\ddagger)\} \tag{8.12-314}$$

If the system is strictly stable (BIBO), the system impulse response $h[n]$ is absolutely summable, i.e.,

$$\sum_{n=0}^{\infty} |h[n]| < \infty \tag{8.12-315}$$

Assume that the transfer function is expressed as

$$H(z) = \frac{P[z^{-1}]}{\prod_{r=1}^{N}[1 - P_r z^{-1}]} \tag{8.12-316}$$

$$= \sum_{r=1}^{N} \frac{a_r}{[1 - P_r z^{-1}]} \tag{8.12-317}$$

Therefore,

$$h[n] = \sum_{r=1}^{N} a_r P_r^n \tag{8.12-318}$$

Note that:

$$\mathcal{Z}\left\{\lfloor^{\backslash}\right\} = \frac{1}{1 - bz^{-1}} \qquad \text{for } |z| > b \tag{8.12-319}$$

Hence, the impulse response $h[n]$ is absolutely summable, if and only if

$$|P_r| < 1 \qquad \text{for all } r \tag{8.12-320}$$

which means that the poles of the trasnfer function defined by $z = P_r$ must lie inside the unit circle in the z-plane.

Notice that in general

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{8.12-321}$$

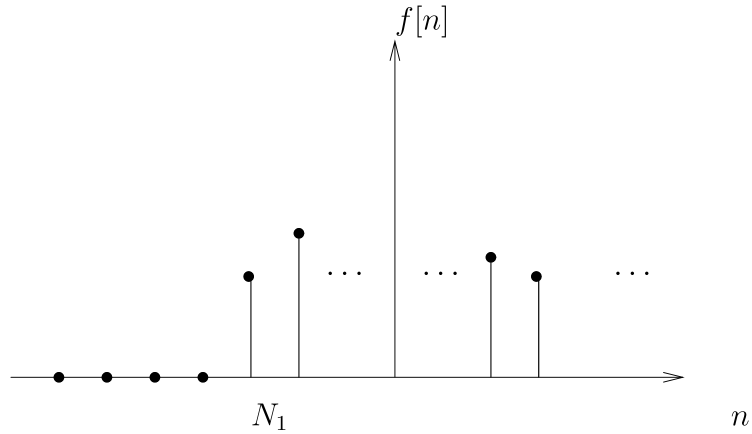
is identical to

$$\sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty \tag{8.12-322}$$

for $|z| = 1$, the condition for stability is equivalent to the condition that the ROC of $H(z)$ includes the unit circle.

Now let's consider the relation between the causality and the ROC of a sequence.

Definition 8.12-1 (Right-Sided sequence) *A sequence $x[n]$ is said to be a right-sided sequence if $x[n]$ is zero for a given $N_1 < \infty$ and $n \leqslant N_1$.*



A causal sequence is a right-sided sequence.

Property of the ROC for the z-transform of a Right-Sided Sequence

If $x[n]$ is a right-sided sequence, the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

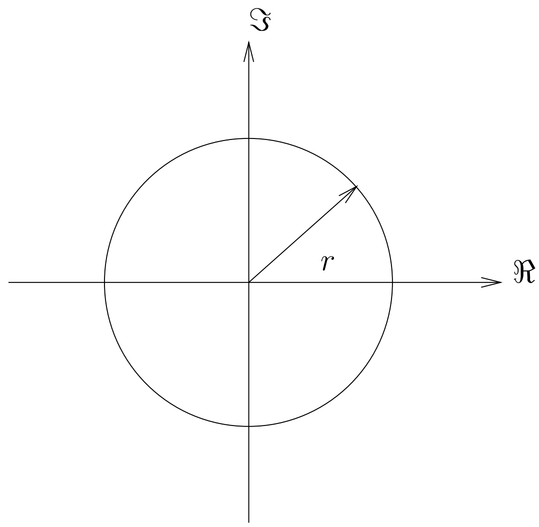
To understand the above property and its implication, the following facts have to be understood first.

The region of convergence (ROC) of the z-transform cannot contain any poles by definition because $X(z)$ is infinite at a pole and therefore does not converge.

Since the complex variable z can be expressed in polar form as

$$z = re^{j\omega} \tag{8.12-323}$$

where $r = |z|$, the convergence of the z-transform requires absolute summabilty of the sequence $x[n]|z|^{-n}$ or equivalently $x[n]r^{-n}$. It is clear that the ROC is concentric with the origin.



For a given right-sided sequence, if the circle $|z| = r_0$ is in the ROC, $x[n]r_0^{-n}$ is absolutely summable, or equivalently, the Fourier transform of $x[n]r_0^{-n}$ converges. Since the sequence $x[n]$ is right-sided, the term $x[n]$ multiplied by any real exponential sequence r_1^{-n} $[r_1 > r_0]$ which, with increasing n decays faster than r_0^{-n} will also be absolutely summable.

As a result, for a right-sided sequence, the ROC extends outward from some circle in the z-plane, concentric with the origin.

What is left is to verify that this circle is in fact at the outermost pole in $X(z)$.

Without losing generality, assume that $X(z)$ has simple poles at $z = p_1, p_2, \ldots, p_N$ with p_1 having the smallest magnitude corresponding to the innermost pole, and p_N having the largest magnitude corresponding to the outermost pole.

As we discussed previously, $X(z)$ can be expressed in partial fraction expansion, and

$$x[n] = \sum_{k=1}^{N} a_k p_k^n \qquad n \geqslant N_1 \tag{8.12-324}$$

referring to Equation (8.12-317) and Equation (8.12-324)

Consider

$$x[n]r^{-n} = r^{-n} \sum_{k=1}^{N} a_k p_k^n \tag{8.12-325}$$

$$= \sum_{k=1}^{N} a_k [p_k r^{-1}]^n \qquad n \geqslant N_1 \tag{8.12-326}$$

Absolutle summability of $x[n]r^{-n}$ requires that

$$\sum_{n=-\infty}^{\infty} \left| x[n]r^{-n} \right| < \infty \tag{8.12-327}$$

or

$$\sum_{n=N_1}^{\infty} \left| x[n]r^{-n} \right| < \infty \tag{8.12-328}$$

as $x[n]$ is right-sided

$$\sum_{n=N_1}^{\infty} \left| \sum_{k=1}^{N} a_k [p_k r^{-1}]^n \right| < \infty \tag{8.12-329}$$

If each exponential $a_k[p_k r^{-1}]^n$ is absolutely summable, Equation (8.12-329) and therefore Equation (8.12-327) holds. i.e.,

$$\sum_{n=N_1}^{\infty} \left| a_k [p_k r^{-1}]^n \right| < \infty \tag{8.12-330}$$

or equivalently

$$\sum_{n=N_1}^{\infty} |p_k r^{-1}|^n < \infty \qquad \text{where } k = 1, \ldots, N \tag{8.12-331}$$

For Equation (8.12-331) to be true, the following must hold

$$|p_k r^{-1}| < 1 \tag{8.12-332}$$

i.e.,

$$|r| > |p_k| \qquad \text{for all } k = 1, \ldots, N \tag{8.12-333}$$

Since p_N has the largest magnitude, the absolute summability of $x[n]r^{-n}$ requires that

$$|r| > |p_N| \tag{8.12-334}$$

which means that the ROC is outside the outermost pole, extending to infinity.

If $N_1 < 0$ [noncausal sequence], the ROC will not include $|z| = r = \infty$ since r^{-n} is infinite for r infinite and n negative.

In the following example, we shall see that causality and stability are not necessarily compatible requirements.

EXAMPLE 8.12-1 *Consider the LTI system with input and output related through the difference equation*

$$g[n] - \frac{5}{2} g[n-1] + g[n-2] = f[n] \tag{8.12-335}$$

The transfer function $H(z)$ is obtained using the z-transform.

$$H(z) = \frac{G(z)}{F(z)} = \frac{1}{1 - \frac{5}{2} z^{-1} + z^{-2}} \tag{8.12-336}$$

$$= \frac{1}{[1 - \frac{1}{2} z^{-1}][1 - 2z^{-1}]} \tag{8.12-337}$$

The pole-zero plot for $H(z)$ is shown in Figure 8.4.

There are two poles $(\frac{1}{2}, 2)$ and two zeros (both at origin). Note that $\frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$ and $\frac{1}{1-2z^{-1}} = \frac{z}{z-2}$.

There are three possible choices for the ROC.

1. *If the system is assumed to be causal, the ROC is outside the outermost pole*

$$|z| = r > 2 \tag{8.12-338}$$

In this case the system will not be stable since the ROC does not include the unit circle.

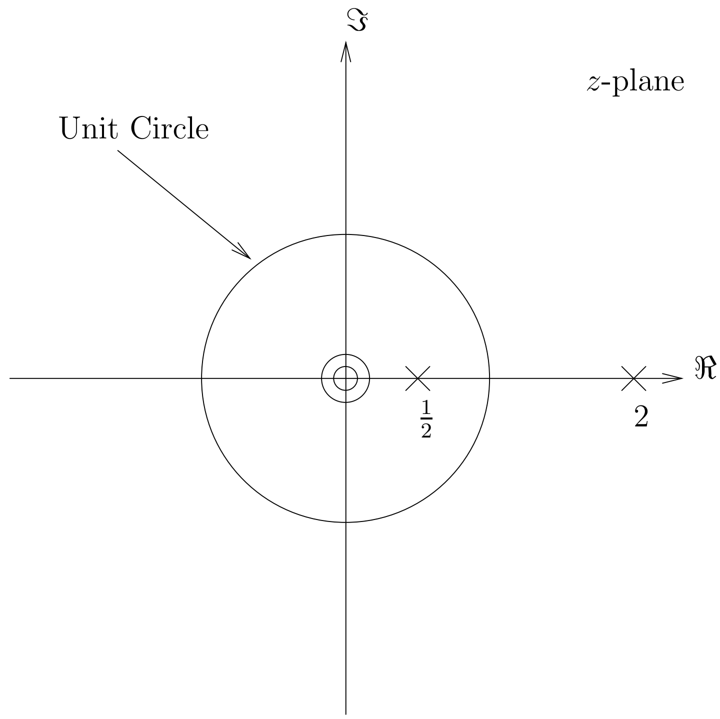


Figure 8.4: Pole-zero plot for $H(z)$

2. *If the system is assumed to be stable, the ROC will be* $\frac{1}{2} < |z| < 2$. *The system is not causal though.*

3. *For* $|z| < \frac{1}{2}$ *as the ROC, the system will be neither stable nor causal.*

REMARK 8.12-1 *In order for an LTI system whose input and output satisfy a difference equation in the form*

$$\sum_{k=0}^{N} a_k g[n-k] = \sum_{k=0}^{M} b_k f[n-k] \tag{8.12-339}$$

to be both causal and stable, the ROC of the corresponding system function $H(z)$ *must be outside the outermost pole and include the unit circle. Clearly, this requires that all the poles of the system function be inside the unit circle.*

8.12.2 The Bilinear variable

To facilitate testing the stability of the sampled data system without resorting to factorization of polynomials and to assist the digital filter design the bilinear variable λ is introduced as

$$\lambda = \frac{1-z^{-1}}{1+z^{-1}} = \frac{z-1}{z+1} \tag{8.12-340}$$

or

$$z = \frac{1+\lambda}{1-\lambda} \tag{8.12-341}$$

λ is a bilinear function of z. The relationship between the λ-plane and the z-plane is explained as follows.

Both λ and z are complex and can be given as

$$z = x + jy \tag{8.12-342}$$

and

$$\lambda = \Sigma + j\Omega \tag{8.12-343}$$

From Equation (8.12-340)

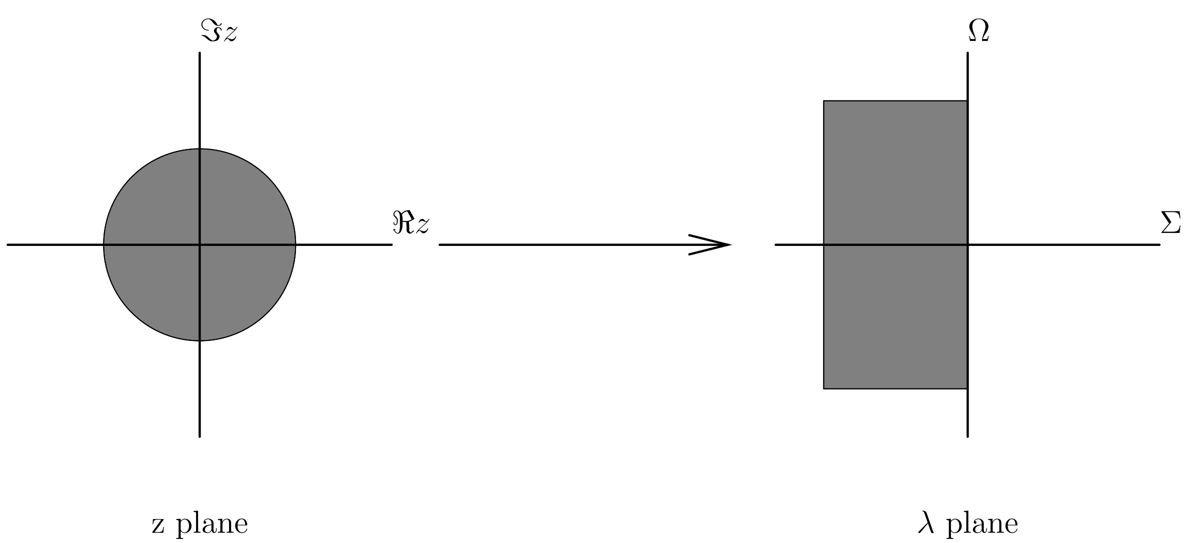
$$\Sigma + j\Omega = \frac{[x + jy] - 1}{[x + jy] + 1} \tag{8.12-344}$$

$$= \frac{[x^2 + y^2] - 1}{[1 + x]^2 + y^2} + j\frac{2y}{[1 + x]^2 + y^2} \tag{8.12-345}$$

or equivalently

$$\Sigma + j\Omega = \frac{|z|^2 - 1}{[1 + x]^2 + y^2} + j\frac{2y}{[1 + x]^2 + y^2} \tag{8.12-346}$$

1.	$\lvert z\rvert < 1$ point in the unit circle in the z-plane.	$\underset{\leftrightarrow}{\text{maps onto}}$	$\Sigma < 0$ points in the left half of the λ-plane.
2.	$\lvert z\rvert > 1$ points outside the unit circle in the z-plane.	$\underset{\leftrightarrow}{\text{maps onto}}$	$\Sigma > 0$ points in the right half of the λ-plane.
3.	$\lvert z\rvert = 1$ points on the unit circle in the z-plane.	$\underset{\leftrightarrow}{\text{maps onto}}$	$\Sigma = 0$ points on the $j\Omega$-axis.
	$z = 1$	$\underset{\leftrightarrow}{\text{maps onto}}$	$\Omega = 0$
	$z = -1$	$\underset{\leftrightarrow}{\text{maps onto}}$	$\Omega = \infty$



For a given transfer function

$$H(z) = \frac{N[z^{-1}]}{D[z^{-1}]} \tag{8.12-347}$$

where $N[z^{-1}]$ and $D[z^{-1}]$ are polynomials with real coefficients.

Substitute Equation (8.12-341) into Equation (8.12-347) resulting in

$$H(\lambda) = \frac{P_M[\lambda]}{Q_N[\lambda]} \tag{8.12-348}$$

where $P_M[\lambda]$ and $Q_N[\lambda]$ are real polynomials of λ.

As we discussed previously, the system is strictly stable if all the poles of $H(z)$ are inside the unit circle in the z-plane. As a result, $H[\lambda]$ is stable if and only if

$$M \leqslant N \tag{8.12-349}$$

and

$$Q_N[\lambda] \neq 0 \qquad \text{for } \Re[\lambda] > 0 \tag{8.12-350}$$

i.e. $H(\lambda)$ has all its poles in the open left half plane. Condition (8.12-349) excludes poles at $\lambda = \pm\infty$ corresponding to the point $z = -1$ on the unit circle in the z-plane. Therefore, the stability of the system requires that the polynomial $Q_N[\lambda]$ be strictly Hurwitz in the λ-plane.

Example 8.12-2 *Given a transfer function*

$$H(z) = \frac{[1+z^{-1}]^3}{37 + 51z^{-1} + 27z^{-2} + 5z^{-3}} \tag{8.12-351}$$

test for stability.

Solution:

$$z^{-1} = \frac{1-\lambda}{1+\lambda} \tag{8.12-352}$$

into $H(z)$ to obtain

$$\begin{aligned} H(\lambda) &= \frac{\left[1 + \frac{1-\lambda}{1+\lambda}\right]^3}{37 + 51\left[\frac{1-\lambda}{1+\lambda}\right] + 27\left[\frac{1-\lambda}{1+\lambda}\right]^2 + 5\left[\frac{1-\lambda}{1+\lambda}\right]^3} && (8.12\text{-}353)\\ &= \frac{8}{37[1+\lambda]^3 + 51[1-\lambda][1+\lambda]^2 + 27[1-\lambda]^2[1+\lambda] + 5[1-\lambda]^3} && (8.12\text{-}354)\\ &= \frac{8}{120 + 120\lambda + 48\lambda^2 + 8\lambda^3} && (8.12\text{-}355)\\ &= \frac{1}{15 + 15\lambda + 6\lambda^2 + \lambda^3} && (8.12\text{-}356) \end{aligned}$$

Using the Hurwitz polynomial testing method separate $Q_N[\lambda] = \lambda^3 + 6\lambda^2 + 15\lambda + 15$,

$$N[\lambda] = \lambda^3 + 15\lambda \tag{8.12-357}$$

$$M[\lambda] = 6\lambda^2 + 15 \tag{8.12-358}$$

$$
\begin{array}{r|l}
 & \frac{1}{6}\lambda \\ \hline
6\lambda^2+15 & \lambda^3+15\lambda \\
 & \underline{\lambda^3+\frac{5}{2}\lambda} \\
 & \quad \frac{25}{2}\lambda
\end{array}
\qquad
\begin{array}{|l}
\frac{12}{25}\lambda \\ \hline
6\lambda^2+15 \\
\underline{6\lambda^2} \\
\qquad 15
\end{array}
\qquad
\begin{array}{|l}
\frac{25}{30}\lambda \\ \hline
\frac{25}{2}\lambda \\
\underline{\frac{25}{2}\lambda} \\
\qquad 0
\end{array}
$$

Since all the quotients are positive, $H(\lambda)$ is strictly Hurwitz. Therefore, all the poles of $H(\lambda)$ are in the open left half λ-plane which corresponds to the inside of the unit circle in the z-plane. The $H(z)$ is stable.

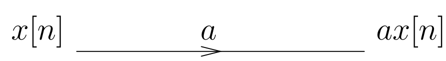
8.13 Digital Filters

Digital filters described by the linear time-invariance equation

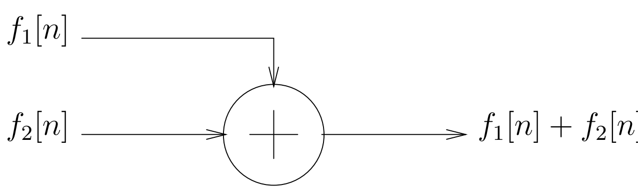
$$g[n] = \sum_{r=0}^{M} a_r f[n-r] - \sum_{r=1}^{N} b_r g[n-r] \tag{8.13-359}$$

involve essentially three types of operations

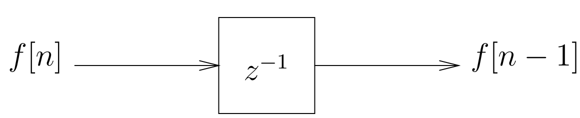
1. multiplications



2. additions



3. delay or shifting



A block diagram using these operations can be used to realize a system represented by Equation (8.13-359).

EXAMPLE 8.13-1 *Given a system defined by*

$$g[n] = af[n] - b_1 g[n-1] - b_2 g[n-2] \tag{8.13-360}$$

where $g[n]$ and $f[n]$ are output and input sequences of the system, respectively.
The corresponding system function is

$$H(z) = \frac{G(z)}{F(z)} = \frac{a}{1 + b_1 z^{-1} + b_2 z^{-2}} \tag{8.13-361}$$

The block diagram representation of the system realization is given as shown in Figure 8.5. This representation aids in many ways.

- *representation of the algorithm.*
- *structure of the hardware architecture.*
- *storage needs for z^{-1}.*
- *number of additions.*
- *number of multiplications.*
- *steps or computation sequence.*

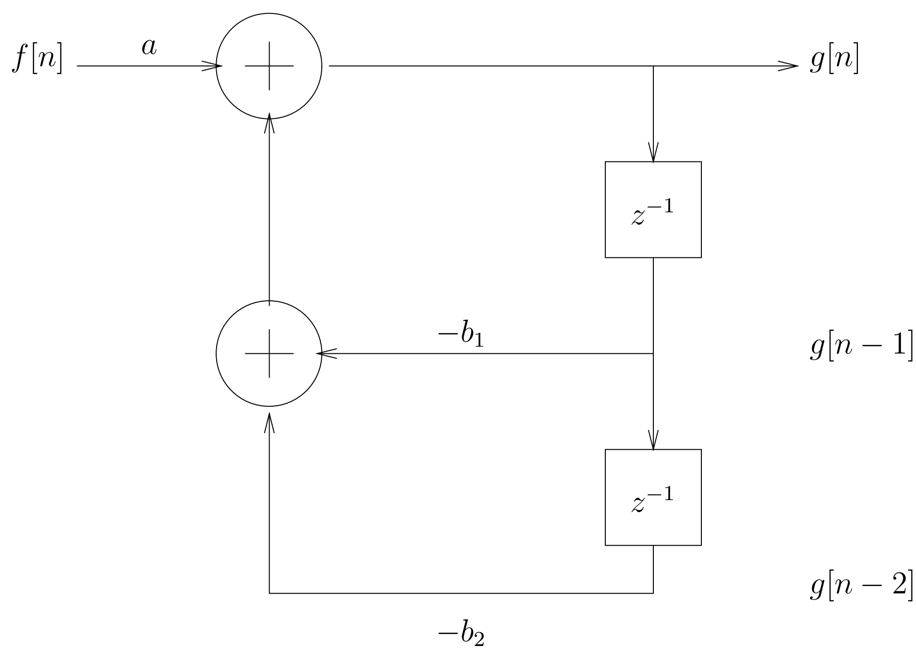


Figure 8.5: Block diagram representation

Figure 8.5 can also be represented in the form of the Signal Flow Graph as shown in Figure 8.6.

Formally, a signal flow graph is a network of directed branches that connect at nodes.

1. Associated with each node is a variable or node value which migh be designated as w_k or $w_k[n]$ for the node k since node variables for digital filters are generally sequences.

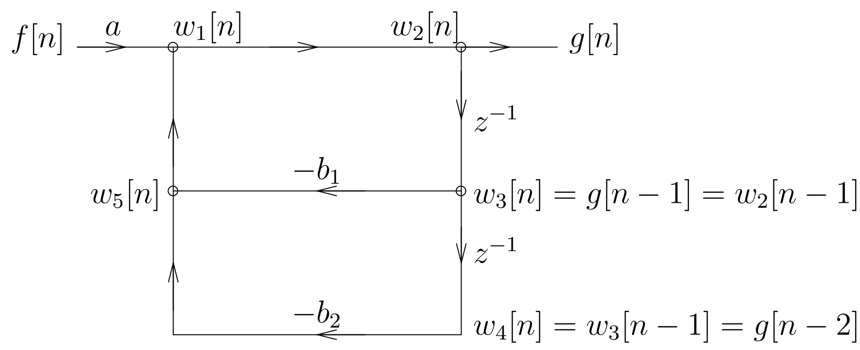


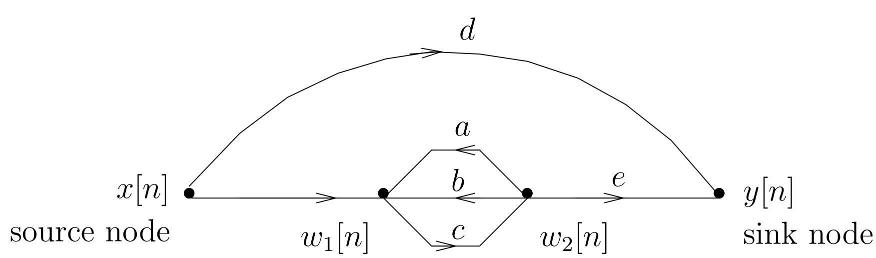
Figure 8.6: Signal flow graph

2. Branch $[j, k]$ denotes a branch originating at node j and terminating at node k being indicated by an arrowhead on the branch. Each branch has an input signal and an output signal. The input signal from node j to branch $[j, k]$ is the node value $w_j[n]$. In a linear SFG, the output of a branch is a linear transformation of the input to the branch, In the simplest cast, the transform is a constant gain when the output of the branch is simply a constant multiple of the input to the branch. When an explicit indication of the branch operation is omitted, the branch transmittance is unity.

3. The node value at each node is the sum of the outputs of all the branches entering the node.

4. There are two special types of nodes — source nodes and sink nodes. Source nodes are nodes that have no entering bracnhes. They are used to represent the injection of external inputs or signal sources into an SFG. Sink nodes are nodes tha have only entering branches. They are used to extract outputs from an SFG.

Example 8.13-2 *Given a signal flow graph*



Solution:

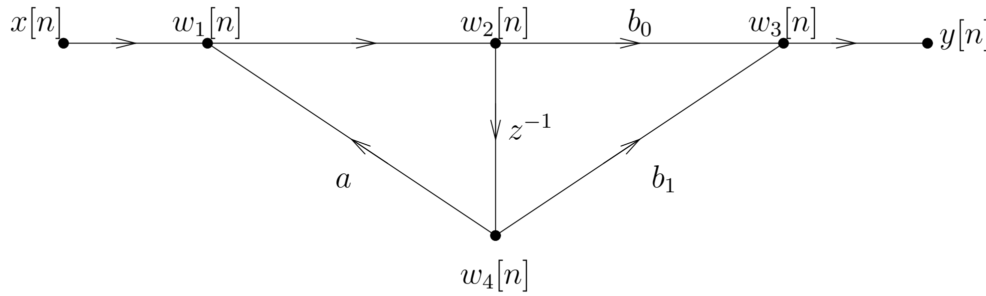
$$w_1[n] = x[n] + aw_2[n] + bw_2[n] \tag{8.13-362}$$
$$w_2[n] = cw_1[n] \tag{8.13-363}$$
$$y[n] = dx[n] + ew_2[n] \tag{8.13-364}$$
$$w_1[n] = x[n] + acw_1[n] + bcw_1[n] \tag{8.13-365}$$
$$y[n] = dx[n] + ecw_1[n] \tag{8.13-366}$$
$$w_1[n] = \frac{1}{1 - ac - bc} x[n] \tag{8.13-367}$$

Substitute Equation (8.13-367) into Equation (8.13-366)

$$y[n] = dx[n] + ec\frac{1}{1-ac-bc}x[n] \tag{8.13-368}$$
$$= \frac{d-acd-bcd+ec}{1-ac-bc}x[n] \tag{8.13-369}$$
$$= \alpha x[n] \tag{8.13-370}$$

Assume that $[1-ac-bc] \neq 0$.

EXAMPLE 8.13-3 *Given a SFG*



find the realtion between $y[n]$ *and* $x[n]$.
Solution:

$$w_1[n] = aw_4[n] + x[n] \tag{8.13-371}$$
$$w_2[n] = w_1[n] \tag{8.13-372}$$
$$w_3[n] = b_0w_2[n] + b_1w_4[n] \tag{8.13-373}$$
$$w_4[n] = w_2[n-1] \tag{8.13-374}$$
$$y[n] = w_3[n] \tag{8.13-375}$$

Substitute Equation (8.13-374) into Equation (8.13-371)

$$w_1[n] = aw_2[n-1] + x[n] \tag{8.13-376}$$

Then, substitute Equation (8.13-376) into Equation (8.13-372)

$$w_2[n] = aw_2[n-1] + x[n] \tag{8.13-377}$$

By the same token, substitute Equation (8.13-374) into Equation (8.13-373)

$$w_3[n] = b_0w_2[n] + b_1w_2[n-1] \tag{8.13-378}$$

and therefore

$$y[n] = b_0w_2[n] + b_1w_2[n-1] \tag{8.13-379}$$

or

$$\begin{cases} w_2[n] & = aw_2[n-1] + x[n] \\ y[n] & = b_0w_2[n] + b_1w_2[n-1] \end{cases} \tag{8.13-380}$$

Often, the manupulation of the difference equations of a flow graph is difficult when dealing with the time-domain variables due to feedback of delayed variables.

To obtain the corresponding linear difference equation which represents the input output relation, use z-transform on Equation (8.13-380) to obtain

$$\begin{cases} W_2(z) & = az^{-1}W_2(z) + X(z) \\ Y(z) & = b_0W_2(z) + b_1z^{-1}W_2(z) \end{cases} \tag{8.13-381}$$

From Equation (8.13-381)

$$W_2(z) = \frac{X(z)}{1 - az^{-1}} \tag{8.13-382}$$

*Substitute Equation (8.13-382) into Equation (**??**)*

$$Y(z) = \frac{b_0 + b_1z^{-1}}{1 - az^{-1}}X(z) \tag{8.13-383}$$

Thus

$$[1 - az^{-1}]Y(z) = [b_0 + b_1z^{-1}]X(z) \tag{8.13-384}$$

Take the inverse z-transform of Equation (8.13-384)

$$y[n] - ay[n-1] = b_0x[n] + b_1x[n-1] \tag{8.13-385}$$

or

$$y[n] = ay[n-1] + b_0x[n] + b_1x[n-1] \tag{8.13-386}$$

8.13.1 Classification of digital filters

In terms of the realization methods used for digital filters, they can be classified into two types:

- recursive realization
- non-recursive realization

The relationship between the output and input sequence of a filter for a recursive realization is given by

$$g[n] \triangleq \text{function of}\{g[n-1], g[n-2], \ldots; f[n], f[n-1], \ldots\} \tag{8.13-387}$$

The present output sample $g[n]$ is a funtion of past outputs as well as present and past input samples.

For a non-recursive realization, the relationship between the input sequence and output sequence is given by

$$g[n] \triangleq \text{function of}\{f[n], f[n-1], \ldots\} \tag{8.13-388}$$

In this case, the present output sample is a function only of past and present inputs.

Digital filters can be also classified in terms of their impulse responses. If the input and output sequences of the system are related as

$$g[n] = \sum_{r=0}^{M} a_r f[n-r] \tag{8.13-389}$$

where $g[n]$ is the output and $f[n]$ is the input sequence. The system function is given by

$$\frac{G(z)}{F(z)} = H(z) = \sum_{r=0}^{M} a_r z^{-r} \tag{8.13-390}$$

Therefore, the impulse response is

$$h[n] = \sum_{r=0}^{M} a_r \delta[n-r] = \begin{cases} a_n & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{8.13-391}$$

Since the impulse response $h[n]$ is finite in length, i.e., it is zero outside a finite interval, the system is called Finite Impulse Response (FIR) system. It can be seen that Equation (8.13-389) is identical to the convolution sum. An important property of the FIR filter is that it is inherently stable since all poles of its transfer funtions are at $z = 0$ which is inside the unit circle.

If a system's input-output relation is defined as

$$g[n] = \sum_{r=0}^{M} a_r f[n-r] - \sum_{r=1}^{N} b_r g[n-r] \tag{8.13-392}$$

where $g[n]$ and $f[n]$ are the output and input sequences of the system respectively, the system function will be given as

$$H(z) = \frac{\sum_{r=0}^{M} a_r z^{-r}}{1 + \sum_{r=1}^{N} b_r z^{-r}} \tag{8.13-393}$$

which is a rational function of z^{-1}. If $H(z)$ only possess first-order poles, it can be expressed in the form

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - P_k z^{-1}} \tag{8.13-394}$$

where the first summation term is present only if $M \geqslant N$ and obtained by long division of the denominator into the numerator.

For a causal system, the ROC is outside all of the poles in Equation (8.13-394) and it follows that

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k P_k^n u[n] \tag{8.13-395}$$

where the first term is present only if $M \geqslant N$. When there is at least one nonzero pole of $H(z)$ which is not cancelled by a zero, there will be a term in the form of $A_k P_k^n u[n]$ and $h[n]$ will not be of finite length, i.e., $h[n]$ will not be zero outside a finite interval. Therefore, this system is called Infinite Impulse Response (IIR) system.

Remark 8.13-1

- *The terms FIR and IIR refer to the type of the filter*

- *The terms non-recursive and recursive refer to the method of realization or implementation.*

8.13.2 Structures for discrete-time systems

A generalized Nth-order difference equation can be given as

$$g[n] - \sum_{r=1}^{N} b_r g[n-r] = \sum_{r=0}^{M} a_r f[n-r] \tag{8.13-396}$$

with the corresponding system function

$$H(z) = \frac{\sum_{r=0}^{M} a_r z^{-r}}{1 - \sum_{r=1}^{N} b_r z^{-r}} \tag{8.13-397}$$

Rewriting Equation (8.13-396) as a recursive realization leads to

$$g[n] = \sum_{r=1}^{N} b_r g[n-r] + \sum_{r=0}^{M} a_r f[n-r] \tag{8.13-398}$$

and further to a pair of difference equations

$$v[n] = \sum_{r=0}^{M} a_r f[n-r] \tag{8.13-399}$$

$$g[n] = \sum_{r=1}^{N} b_r g[n-r] + v[n] \tag{8.13-400}$$

In block diagram form, Equations (8.13-399) and 8.13-400 can be represented as shown in Figure 8.7

In the z-domain

$$V(z) = \left(\sum_{r=0}^{M} a_r z^{-r} \right) F(z) \tag{8.13-401}$$

$$= H_1(z) F(z) \tag{8.13-402}$$

and

$$G(z) = \frac{1}{1 - \sum_{r=1}^{N} b_r z^{-r}} \tag{8.13-403}$$

$$= H_2(z) V(z) \tag{8.13-404}$$

or

$$G(z) = H_2(z) V(z) \tag{8.13-405}$$

$$= H_2(z) H_1(z) F(z) \tag{8.13-406}$$

$$= H(z) F(z) \tag{8.13-407}$$

where

$$H(z) = H_2(z) H_1(z) \tag{8.13-408}$$

$$= \left[\frac{1}{1 - \sum_{r=1}^{N} b_r z^{-r}} \right] \left[\sum_{r=0}^{M} a_r z^{-r} \right] \tag{8.13-409}$$

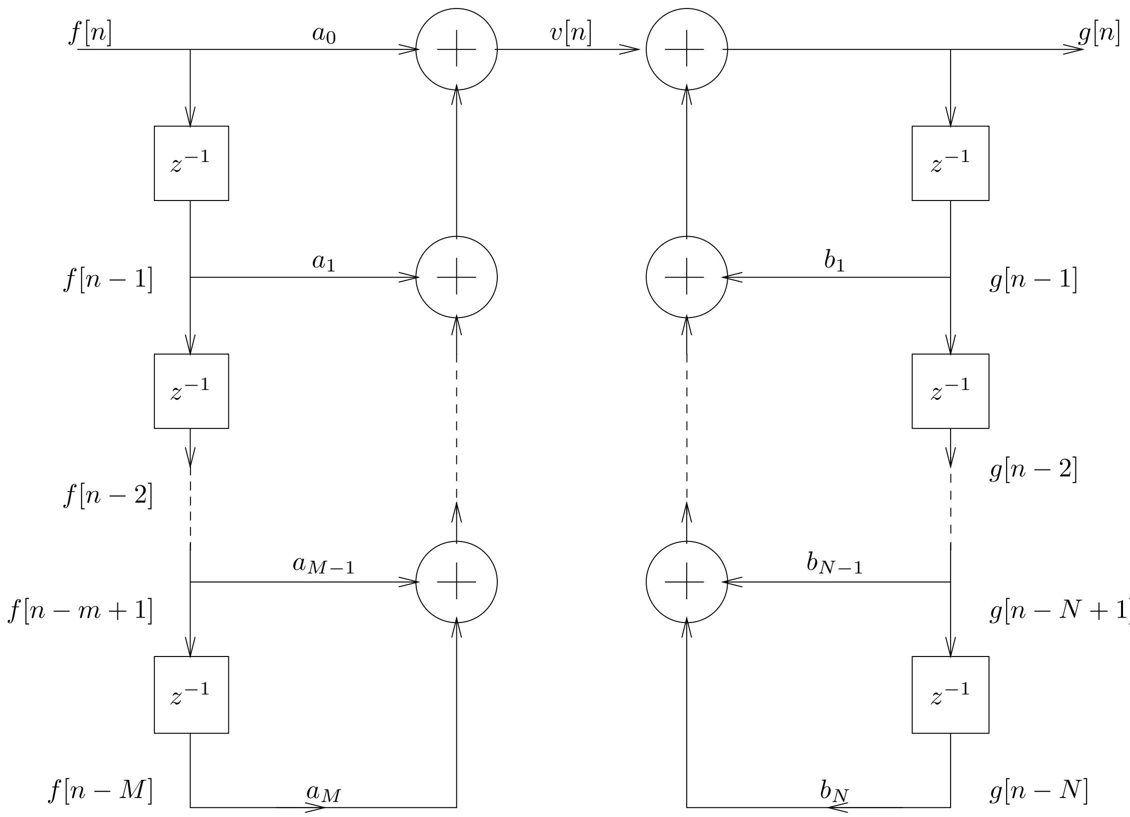


Figure 8.7: Block diagram: Direct Form I

Since $H(z)$ can be also decomposed into

$$H(z) = H_1(z)H_2(z) \tag{8.13-410}$$

$$= \left[\sum_{r=0}^{M} a_r z^{-r}\right]\left[\frac{1}{1-\sum_{r=1}^{N} b_r z^{-r}}\right] \tag{8.13-411}$$

A new pair of difference equations are obtained rendering the same $f[n]$ and $g[n]$ relation

$$w[n] = \sum_{r=1}^{N} b_r w[n-r] + f[n] \tag{8.13-412}$$

$$g[n] = \sum_{r=0}^{M} a_r w[n-r] \tag{8.13-413}$$

Depending on whether zeros or poles of the transfer function $H(z)$ are implemented first the block diagrams are categorized as the direct form I [zero first] or direct form II [poles first]. The direct form I requires $[N+M]$ delay elements in its realization, whilst the direct form II only requires $[MAX[N,M]]$.

8.13.3 Basic structures for IIR filters

For any given rational system function, there exists a wide variety of equivalent sets of difference equations or network structures. The choice between various structures is usually determined by practical considerations such as computational complexity [mults., adds. and delay elements], modularity and simplicity of data transfer and effects of finite register length and finite-precision arithmetic.

An LTI IIR system, as we have discussed can be represented by the difference equation resulting from the z-domain expression.

$$W(z) = H_2(z)F(z) \tag{8.13-414}$$

$$= \left[\frac{1}{1-\sum_{r=1}^{N} b_r z^{-r}}\right] F(z) \tag{8.13-415}$$

and

$$G(z) = H_1(z)W(z) \tag{8.13-416}$$

$$= \left[\sum_{r=0}^{M} a_r z^{-r}\right] W(z) \tag{8.13-417}$$

Their block diagram representation will have the form shown in Figure 8.8.

If $M \neq N$, some of the coefficients will be zero. The above diagram can be further modified to obtain Figure 8.9, which is also known as the Canonic Direct Form.

$$g[n] - \sum_{r=1}^{N} b_r g[n-r] = \sum_{r=0}^{M} a_r f[n-r] \tag{8.13-418}$$

with the corresponding rational transfer function

$$H(z) = \frac{\sum_{r=0}^{M} a_r z^{-r}}{1-\sum_{r=1}^{N} b_r z^{-r}} \tag{8.13-419}$$

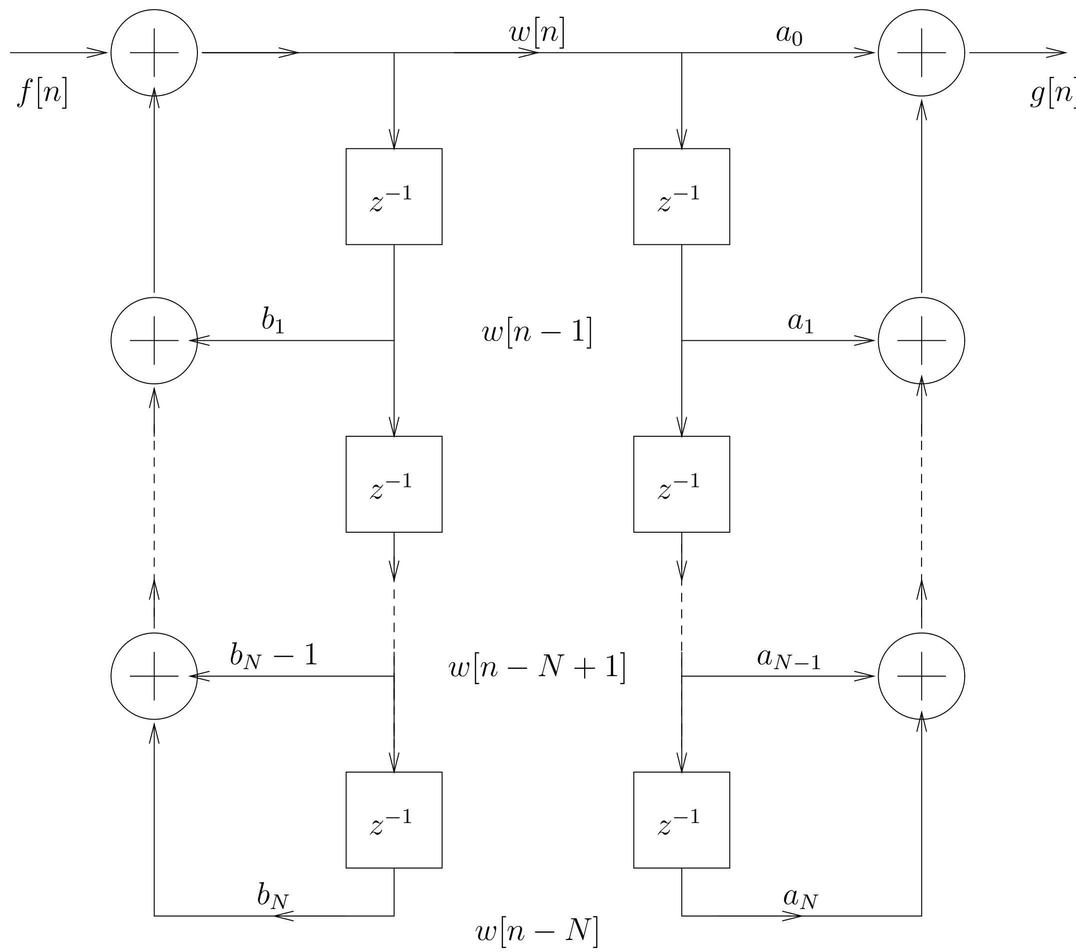


Figure 8.8: Block diagram of LTI IIR system

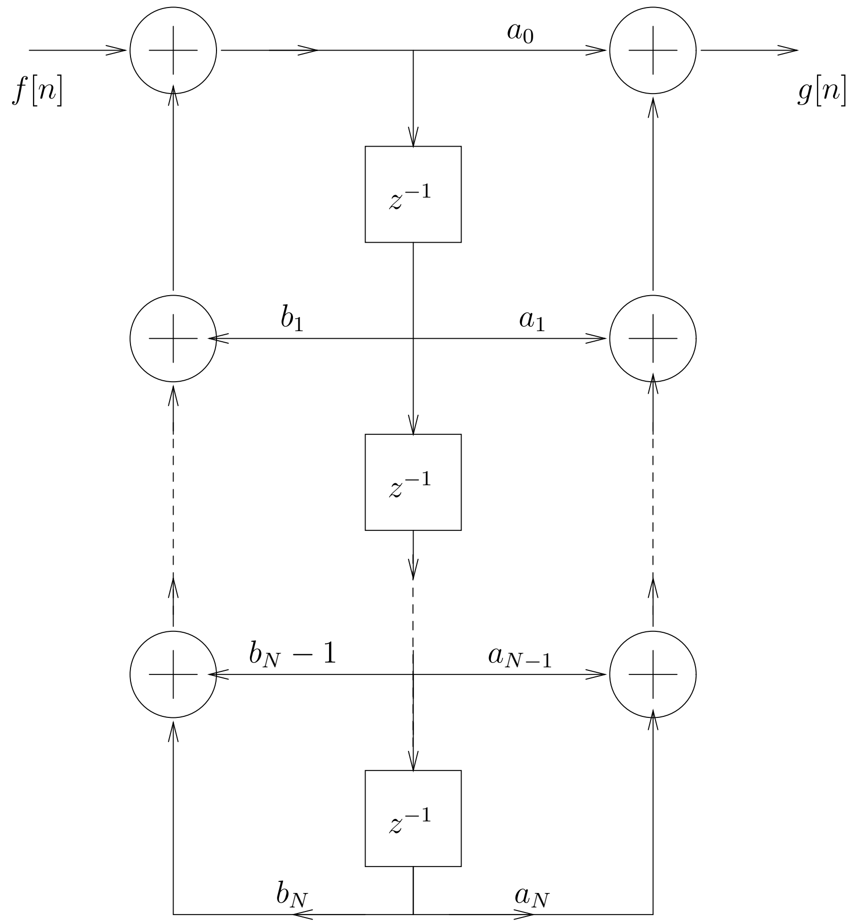


Figure 8.9: Block diagram of LTI IIR system: Direct Form II

The direct form I and II realizations of Equation (8.13-418) are given in signal flow graphs for $M = N$ in Figure 8.10 and Figure 8.11. (For $M \neq N$, some of the coefficients will be zero.)

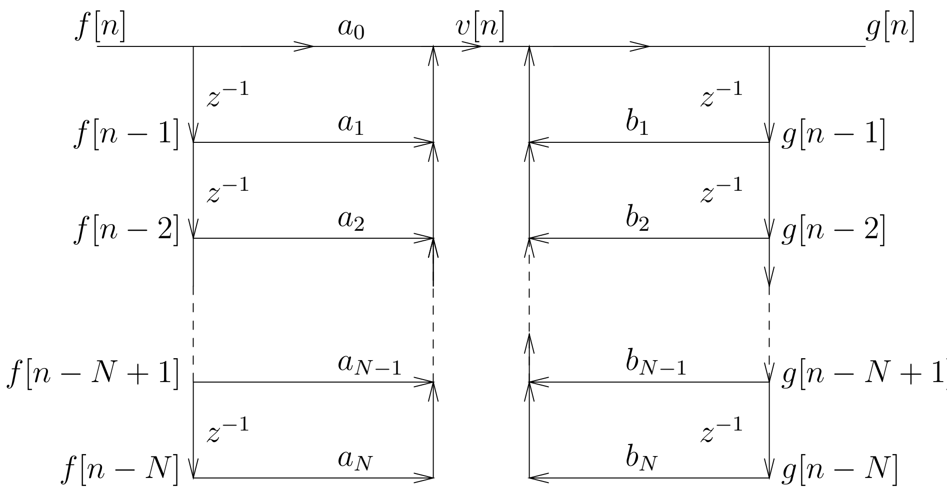


Figure 8.10: Direct Form I for an Nth order system

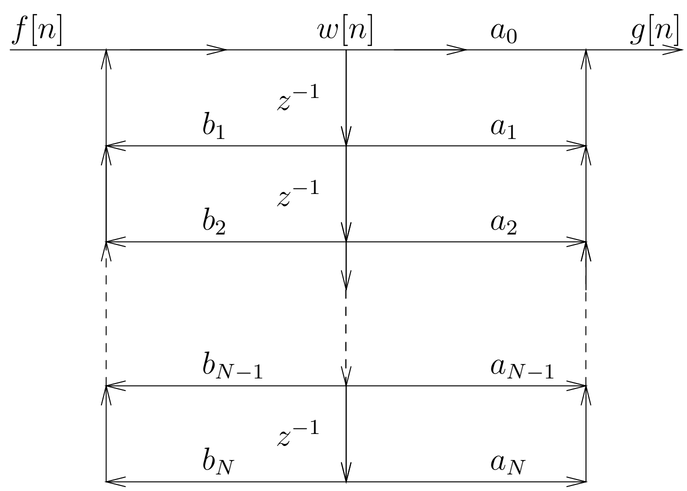


Figure 8.11: Direct Form II for an Nth order system

Example 8.13-4 *Given the transfer function*

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} \tag{8.13-420}$$

the direct form I structure is given as shown in Figure 8.12, while the direct form II structure is shown in Figure 8.13.

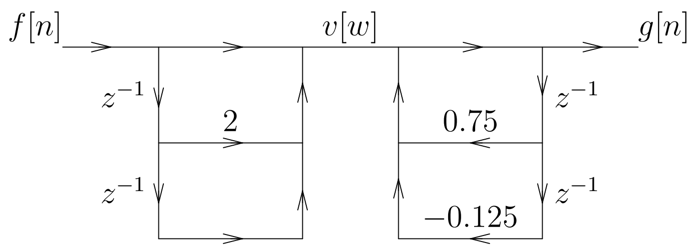


Figure 8.12: Direct Form I for $H(z)$

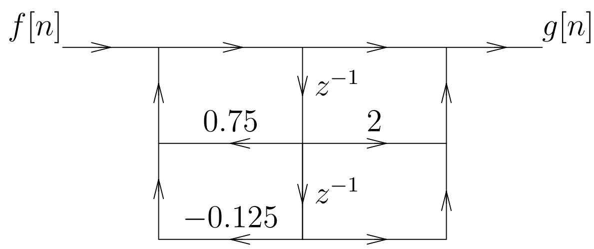


Figure 8.13: Direct Form II for $H(z)$

8.13.3.1 Cascade form

If the system function $H(z)$ is a rational function with the numerator and denominator polynomials being real, $H(z)$ can be expressed by

$$H(z) = A\frac{\prod_{k=1}^{M_1}\left[1 - g_k z^{-1}\right]\prod_{k=1}^{M_2}\left[1 - h_k z^{-1}\right]\left[1 - h_k^* z^{-1}\right]}{\prod_{k=1}^{N_1}\left[1 - c_k z^{-1}\right]\prod_{k=1}^{N_2}\left[1 - d_k z^{-1}\right]\left[1 - d_k^* z^{-1}\right]} \tag{8.13-421}$$

where $M = M_1 + 2M_2$ and $N = N_1 + 2N_2$. The first order factors represent real zeros at g_k and real poles at c_k, and the second-order factors represent complex conjugate pairs of zeros at h_k and h_k^* and complex conjugate pairs of poles at d_k and d_k^*.

Combining pairs of real factors and complex conjugate pairs into second-order factors results in a modular structure provided that $M \leqslant N$.

$$H(z) = \prod_{k=1}^{N_s}\frac{a_{0k} + a_{1k}z^{-1} + a_{2k}z^{-2}}{1 - b_{1k}z^{-1} - b_{2k}z^{-2}} \tag{8.13-422}$$

where $N_s = [[N+1]/2]$ is the largest integer contained in $[N+1]/2$.

If there are an odd number of real zeros, one of the coefficients b_{2k} will be zero. Likewise, if there are an odd number of real poles, one of the coefficients a_{2k} will be zero.

The second order factor

$$H_k(z) = \frac{a_{0k} + a_{1k}z^{-1} + a_{2k}z^{-2}}{1 - b_{1k}z^{-1} - b_{2k}z^{-2}} \tag{8.13-423}$$

can be represented by the structure shown in Figure 8.14.

The difference equations represented by a general cascade of direct form II second-

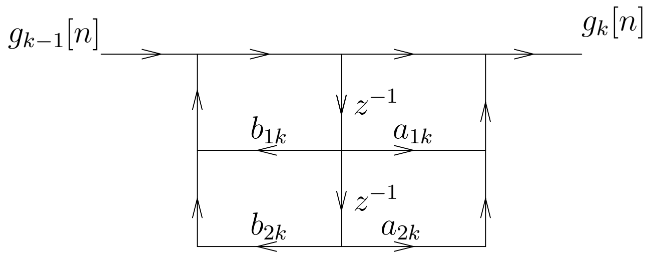


Figure 8.14: Second order factor in direct form II

order sections are of the form

$$g_0[n] = f[n] \tag{8.13-424}$$
$$w_k[n] = b_{1k}w_k[n-1] + b_{2k}w_k[n-2] + g_{k-1}[n] \tag{8.13-425}$$
$$g_k[n] = a_{0k}w_k[n] + a_{1k}w_k[n-1] + a_{2k}w[n-2] \tag{8.13-426}$$
$$g[n] = g_{N_s}[n] \qquad \text{where } k = 1, 2, \ldots, N_s \tag{8.13-427}$$

Example 8.13-5 *A cascade structure for a sixth-order system is given by a direct-form II realization of each second order system as shown in Figure 8.15.*

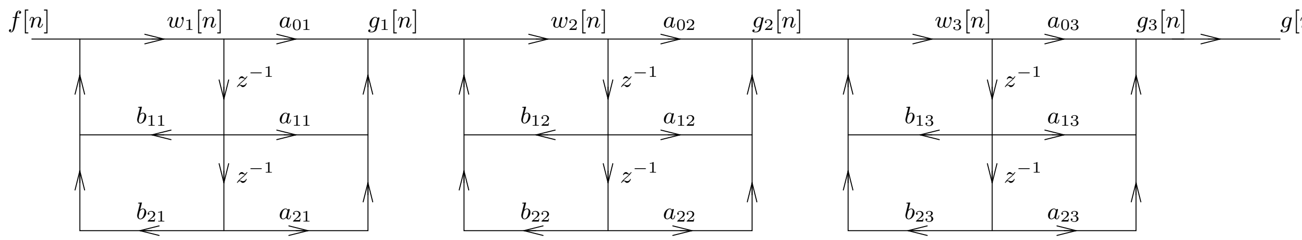


Figure 8.15: Cascade structure of sixth order system

Example 8.13-6 *Given a second order system*

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} \tag{8.13-428}$$
$$= \frac{[1 + z^{-1}][1 + z^{-1}]}{[1 - 0.5z^{-1}][1 - 0.25z^{-1}]} \tag{8.13-429}$$

it can be implemented in different forms as shown in Figure 8.16 and Figure 8.17.

8.13.3.2 Parallel form

The system function $H(z)$ can be expressed as a partial fraction expansion.

$$H(z) = \sum_{k=0}^{N_p} c_k z^{-k} + \sum_{k=1}^{N_1} \frac{A_k}{1 - c_k z^{-1}} + \sum_{k=1}^{N_2} \frac{B_k[1 - e_k z^{-1}]}{(1 - d_k z^{-1})(1 - d_k^* z^{-1})} \tag{8.13-430}$$

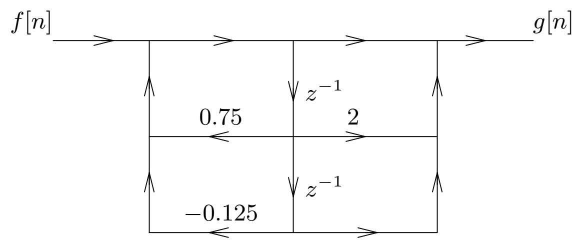


Figure 8.16: Direct Form II second-order section

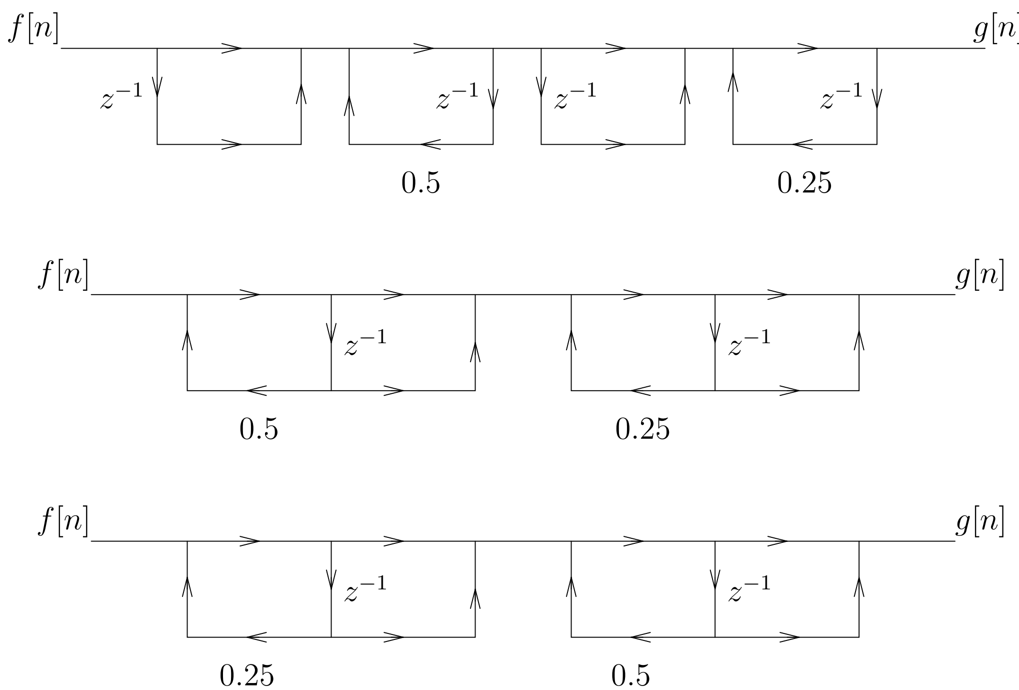


Figure 8.17: Cascade structures with first-order sections

where $N = N_1 + 2N_2$. If $M \geqslant N$, $N_p = M - N$; otherwise $c_k = 0$. If a_k and b_k for all k are real in $H(z)$, A_k, B_k, C_k c_k and e_k are all real.

If the real poles are grouped in pairs

$$H(z) = \sum_{k=0}^{N_p} c_k z^{-1} + \sum_{k=1}^{N_s} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}} \tag{8.13-431}$$

where $N_s = ((N+1)/2)$ is the largest integer contained in $(N+1)/2$.

The signal flow graph for the second-order term in Equation (8.13-431) is given in Figure 8.18.

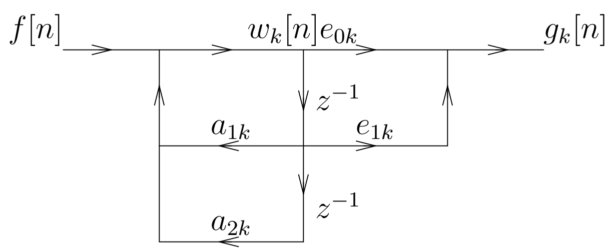


Figure 8.18: Signal flow graph for second order term

The general difference equations for the parallel form with second-order direct form II sections are

$$w_k[n] = 1_{ak} w_k[n-1] + a_{2k} w_k[n-2] + f[n] \tag{8.13-432}$$
$$g_k[n] = e_{0k} w_k[n] + e_{1k} w_k[n-1] \tag{8.13-433}$$
$$g[n] = \sum_{k=0}^{N_p} c_k f[n-k] + \sum_{k=1}^{N_s} g_s[n] \tag{8.13-434}$$

EXAMPLE 8.13-7 *Given a system function*

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} \tag{8.13-435}$$
$$= 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}} \tag{8.13-436}$$

the paralled realization is presented as shown in Figure 8.19.

The given $H(z)$ can also be written as

$$H(z) = 8 + \frac{18}{1 - 0.5z^{-1}} - \frac{25}{1 - 0.25z^{-1}} \tag{8.13-437}$$

Therefore, another form of the Signal Flow Graph is shown in Figure 8.20.

8.13.4 Feedback in IIR systems

The flow graphs of IIR systems have feedback loops which are closed paths that begin at a node and return to that node by traversing branches only in the direction of their arrowheads. For example, the SFG

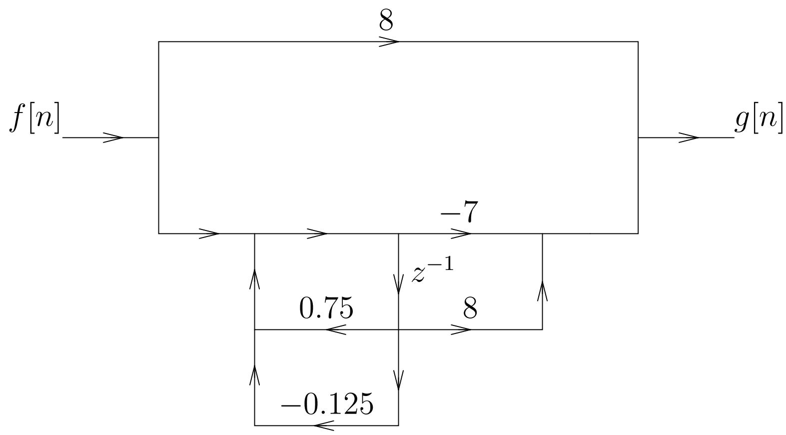


Figure 8.19: Parallel realization

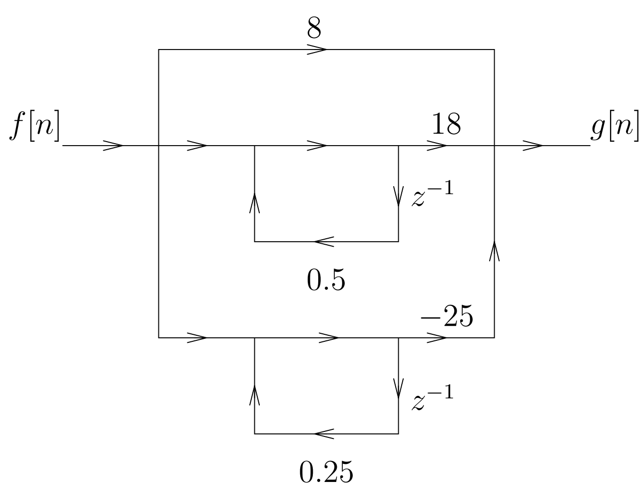
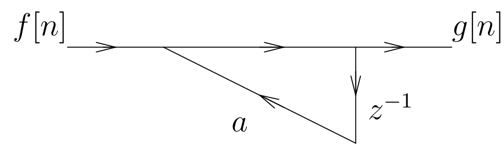


Figure 8.20: Alternative realization



represents the difference equation

$$g[n] = ag[n-1] + f[n] \tag{8.13-438}$$

which has an infinite impulse response

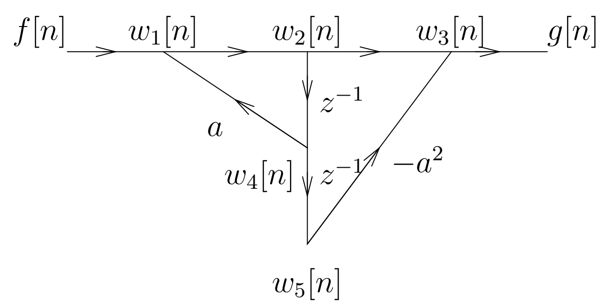
$$h[n] = a^n u[n] \tag{8.13-439}$$

The system function is

$$H(z) = \frac{1}{1 - az^{-1}} \tag{8.13-440}$$

However, neither poles in the system function nor loops in the network are sufficient for the impulse response to be infinitely long.

Example 8.13-8 *Given an SFG with feedback loop*



the difference equations representing the input output relation is obtianed by

$$
\begin{aligned}
w_1[n] &= f[n] + aw_4[n] && (8.13\text{-}441)\\
w_2[n] &= w_1[n] && (8.13\text{-}442)\\
w_4[n] &= w_2[n-1] && (8.13\text{-}443)\\
w_5[n] &= w_4[n-1] = w_2[n-2] && (8.13\text{-}444)\\
w_3[n] &= w_2[n] - a^2 w_5[n] && (8.13\text{-}445)\\
&= w_2[n] - a^2 w_2[n-2] && (8.13\text{-}446)\\
g[n] &= w_3[n] && (8.13\text{-}447)\\
w_2[n] &= f[n] + aw_2[n-1] && (8.13\text{-}448)\\
g[n] &= w_2[n] - a^2 w_2[n-2] && (8.13\text{-}449)
\end{aligned}
$$

using the z-transform

$$
\begin{aligned}
W_2(z)[1 - az^{-1}] &= F(z) && (8.13\text{-}450)\\
G(z) &= [1 - a^2 z^{-2}]W(z) && (8.13\text{-}451)\\
G(z) &= \frac{1 - a^2 z^{-2}}{1 - az^{-1}} F(z) && (8.13\text{-}452)
\end{aligned}
$$

That is

$$H(z) = \frac{1 - a^2 z^{-2}}{1 - az^{-1}} \tag{8.13-453}$$
$$= 1 + az^{-1} \tag{8.13-454}$$

Therefore

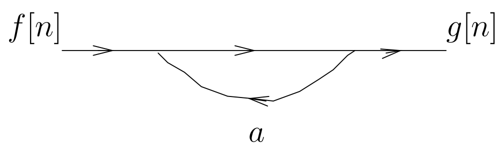
$$g[n] = f[n] + af[n-1] \tag{8.13-455}$$

with the system impulse response

$$h[n] = \delta[n] + a\delta[n-1] \tag{8.13-456}$$

The system is an FIR system (frequency sampling system.)

EXAMPLE 8.13-9 *The followinf SFG is noncomputable.*



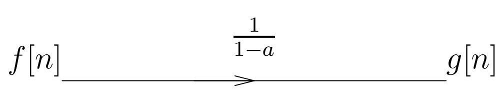
since

$$g[n] = ag[n] + f[n] \tag{8.13-457}$$

However, the solution to the problem can be found if the SFG is modified according to

$$g[n] = \frac{1}{1-a} f[n] \tag{8.13-458}$$

provided that $a \neq 1$.



8.13.5 Signal flow graph transposition

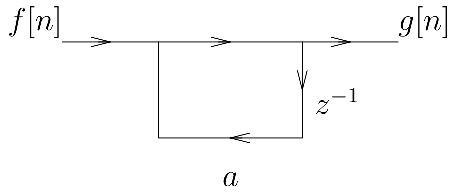
Transposition of a flow graph is accomplished by reversing the directiopns of all the branches in the network while keeping the branch trasnmittance unchanged and changing the source nodes to sink nodes and vice versa.

For single-input/single-output systems, resulting flow graphs have the same system function as the original graph if the input and output nodes are interchanged.

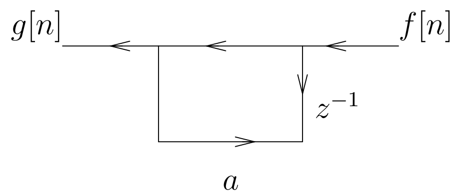
EXAMPLE 8.13-10 *Given a first order system presented by*

$$H(z) = \frac{1}{1 - az^{-1}} \tag{8.13-459}$$

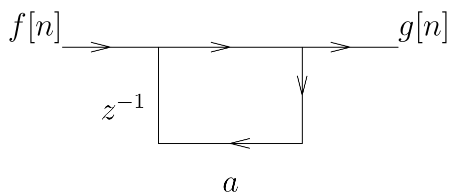
with the SFG



the transposed form of the given SFG is shown



and redrawing the above SFG with input on the left



By inspection of the original system and the corresponding transposed system, the only difference between the two is the order of operations by the coefficient a and the units delay (z^{-1}). Therefore they have the same system function.

8.13.6 Basic structures for FIR systems

FIR system may be considered as a special case of the IIR systems when the coefficients b_k of the denominator polynomial are all zero resulting in

$$g[n] = \sum_{r=0}^{M} a_k f[n-r] \tag{8.13-460}$$

with the impulse response

$$h[n] = \begin{cases} a_k & \text{for} n = 0, 1, \ldots, M \\ 0 & \text{otehrwise} \end{cases} \tag{8.13-461}$$

The direct form I and II structures of the IIR system reduces to the same direct form FIR structure shown in Figure 8.21.

This structure is also referred to as a *tapped delay line* structure or a transversal filter structure.

Using the transposition theorem, the transposed direct form for the FIR systems is obtained as is shown in Figure 8.22.

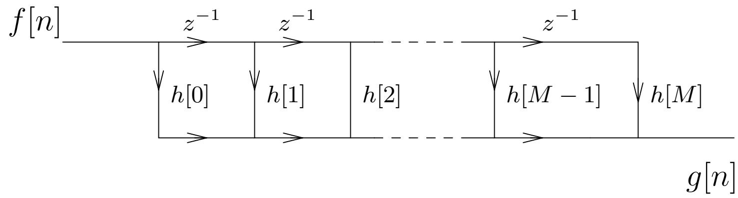


Figure 8.21: Direct form FIR structure

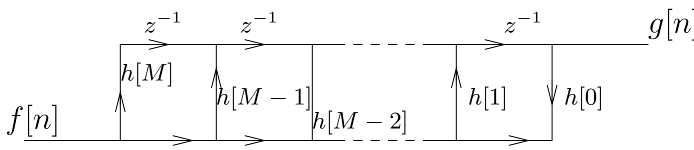


Figure 8.22: Transposed direct form FIR structure

8.13.6.1 Cascade form for FIR systems

The cascade form for FIR systems is obtained by factorizing the system function $H(z)$,

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} = \prod_{k=1}^{M_s}(a_{0k} + a_{1k}z^{-1} + a_{2k}z^{-2}) \tag{8.13-462}$$

where $M_s = ((M+1)/2)$ is the largest integer contained in $(M+1)/2$. When M is odd, one of the b_{2k} will be zero. The corresponding SFG is shown in Figure 8.23.

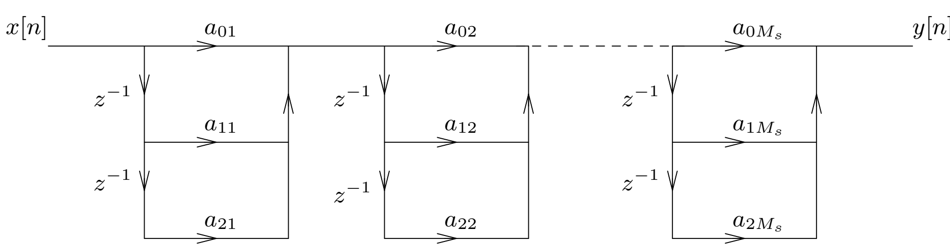


Figure 8.23: Cascaded form for FIR structure

Chapter 9

Filter Design Techniques

Contents

Definition 9.0-1 *Any system which modifies certain frequencies relative to others is called a filter.*

Our main emphasis is on the design of frequency-selective filters which are linear, time-invariant, causal and stable systems.

Since discrete-time signals to be processed are generally represented in digital forms and discrete-time systems (filters) are often implemented with digital computation, discrete-time filters are commonly referred to as digital filters.

Digital filter design involves the following three stages:

1. the specification of the desired properties of the system — typically given in the frequency domain.
2. the approximation of the specifications using a causal discrete-time system.
3. the realization of the system — different forms and structures.

9.1 Design of Discrete Time IIR Filters from Continuous-Time Filters

The design procedure for the discrete-time filter by transforming a prototype continuous-time analog filter begins with a set of discrete-time specifications and includes the following steps:

1. Obtain the specifications on the continuous-time filter by a transformation of the specifications for the desired discrete-time filter.

2. Acquire the system function $H_c(s)$ or the impulse response $h_c(t)$ of the continuous-time filter using the established approximation methods.

3. Transform the system function $H_c(s)$ is the s-plane or impulse response $h_c(t)$ in the continuous-time domain into the system function $H(z)$ in the z-plane, or impulse response $h[n]$ in the discrete-time domain.

REMARK 9.1-1 *In the above transformation, the essential properties of the continuous-time frequency response must be preserved in the frequency response of the resulting discrete-time filter. (eg. stability)*

Three design methods for IIR will be discussed:

1. design by impulse invariance.
2. design using bilinear transformation.
3. algorithmic technique — computer aided design.

9.1.1 Filter design by impulse invariance

An equivalent discrete-time system to a continuous-time linear time-invariant system can be described in Figure 9.1 for bandlimited inputs.

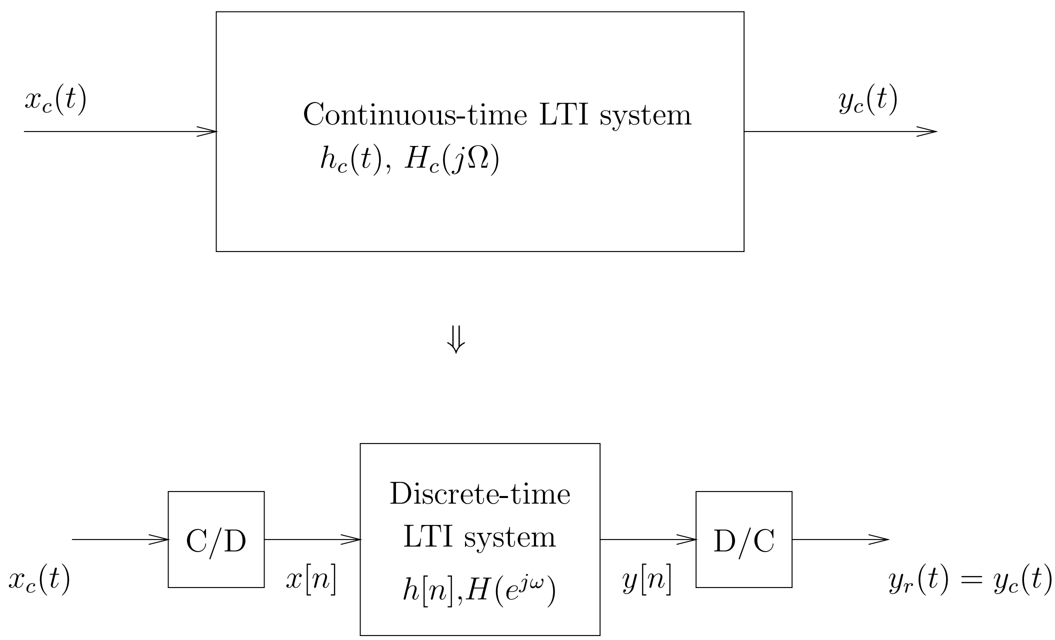


Figure 9.1: Equivalent systems for bandlimited inputs

Our objective is to establish the relationship between $h_c(t)$ and $h[n]$ or $H_c(j\Omega)$ and $H(e^{j\omega})$.

Since

$$x[n] = x_c(nT) \tag{9.1-1}$$

the Fourier transform of $x[n]$ is related to the Fourier transform, $X_c(j\Omega)$ of $x_c(t)$ as

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \tag{9.1-2}$$

where $\omega = \Omega T$ according to previous discussion.

Since the discrete-time system in Figure 9.1 is LTI

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \tag{9.1-3}$$

where $H(e^{j\omega})$ is the frequency response of the system, or equivalently, the Fourier transform of the unit sample response, and $X(e^{j\omega})$ and $Y(e^{j\omega})$ are Fourier transforms of the input $x[n]$ and output, $y[n]$, respectively.

Note that the frequency response of the ideal reconstruction filter in an ideal reconstruction system (D/C) is given by

$$H_r(j\Omega) = \begin{cases} T & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases} \tag{9.1-4}$$

Therefore

$$Y_r(j\Omega) = H_r(j\Omega)Y(j\Omega) \tag{9.1-5}$$

$$= \begin{cases} TY(j\Omega) & |\Omega| < \pi/T \\ 0 & \text{otherwise} \end{cases} \tag{9.1-6}$$

Combining Equations 9.1-2, 9.1-3 and 9.1-6 we obtain

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}) \tag{9.1-7}$$

$$= H_r(j\Omega)H(e^{j\Omega T})\left\{\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\Omega - j\frac{2\pi k}{T}\right)\right\} \tag{9.1-8}$$

If $X_c(j\Omega) = 0$ for $|\Omega| \geqslant \pi/T$ (the input is bandlimited), then the ideal lowpass reconstruction filter $H_r(j\Omega)$ cancels the factor $1/T$ and selects only the term in Equation 9.1-8 for $k = 0$, i.e.,

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega) & \text{for} |\Omega| < \frac{\pi}{T} \\ 0 & \text{for} |\Omega| \geqslant \frac{\pi}{T} \end{cases} \tag{9.1-9}$$

The Equation 9.1-9 states that if $X_c(j\Omega)$ is bandlimited and the sampling rate is above the Nyquist rate, the overall continuous-time system is equivalent to a linear time-invariant system whose effective frequency response is defined as

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \frac{\pi}{T} \\ 0 & |\Omega| \geqslant \frac{\pi}{T} \end{cases} \tag{9.1-10}$$

and the input-output relation is of the form

$$Y_r(j\Omega) = H_{eff}(j\Omega)X_c(j\Omega) \tag{9.1-11}$$

To make two systems equivalent for bandlimited inputs is to make

$$H_{eff}(j\Omega) = H_c(j\Omega) \tag{9.1-12}$$

or equivalently, according to Equation 9.1-10

$$H(e^{j\omega}) + H_c\left(\frac{j\omega}{T}\right) \qquad |\omega| < \pi \tag{9.1-13}$$

where T is chosen such that

$$H_c(j\Omega) = 0 \qquad \text{for} |\Omega| \geqslant \frac{\pi}{T} \tag{9.1-14}$$

Recall that if we choose

$$h[n] = h_c(nT) \tag{9.1-15}$$

then the relation between the Fourier transform of $h[n]$ and that of $h_c(nT)$ is

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right) \tag{9.1-16}$$

Which is the frequency domain representation of the sampling. Since $H_c(j\frac{\omega}{T})$ is bandlimited

$$H(e^{j\omega}) = \frac{1}{T} H_c\left(j\frac{\omega}{T}\right) \tag{9.1-17}$$

where $|\omega| < \pi$.

To make two systems equivalent as shown by Equation 9.1-13 we have to require that the relationship between the discrete-time impulse response $h[n]$ and the continuous-time impulse response $h_c(t)$ satisfy the following equation

$$h[n] = T h_c(nT) \tag{9.1-18}$$

In summary, for the continuous-time LTI system and the system consisting of a discrete-time LTI system to be equivalent, the following equations must be satisfied

$$h[n] = T h_c(nT) \tag{9.1-19}$$

and

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right) \tag{9.1-20}$$

where $h[n]$ and $H(e^{j\omega})$ are the discrete-time impulse response and the frequency response of the discrete system respectively; $h_c(t)$ and $H_c(j\Omega)$ are the continuous-time impulse response and the frequency response of the continuous time system respectively.

When $h[n]$ and $h_c(t)$ are related through Equation 9.1-19, the discrete-time system is called an impulse-invariant version of the continuous-time system.

In the impulse invariance design procedure for transforming continuous-time filters into discrete-time filters.

$$h[n] = T_d h_c(nT_d) \tag{9.1-21}$$

where T_d represents a sampling interval.

Since the design begins with the discrete-time filter specifications, T_d cancels in the impulse invariance procedure. The design sampling period T_d need not be the same as the sampling period T which is associated with the C/D and D/C conversion. T_d is usually chosen to be 1 so that $\omega = \Omega T_d = \Omega$ or any other convenient value.

As a result, in the frequency domain the frequency response of the discrete-time filter obtained from Equation 9.1-21 is related to the frequency response of the continuous-time filter by

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right) \tag{9.1-22}$$

If the continuous-time filter is bandlimited so that

$$H_c(j\Omega) = 0 \qquad \text{for} |\Omega| \geqslant \frac{\pi}{T_d} \tag{9.1-23}$$

recalling that the sampling rate $\Omega_s = \frac{2\pi}{T_d}$ rad/s, no aliasing will occur, then

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right) \qquad \text{for } |\omega| \leqslant \pi \tag{9.1-24}$$

Assume that Ω_c is the cutoff frequency for $H_c(j\Omega)$, as shown in Figure 9.2.

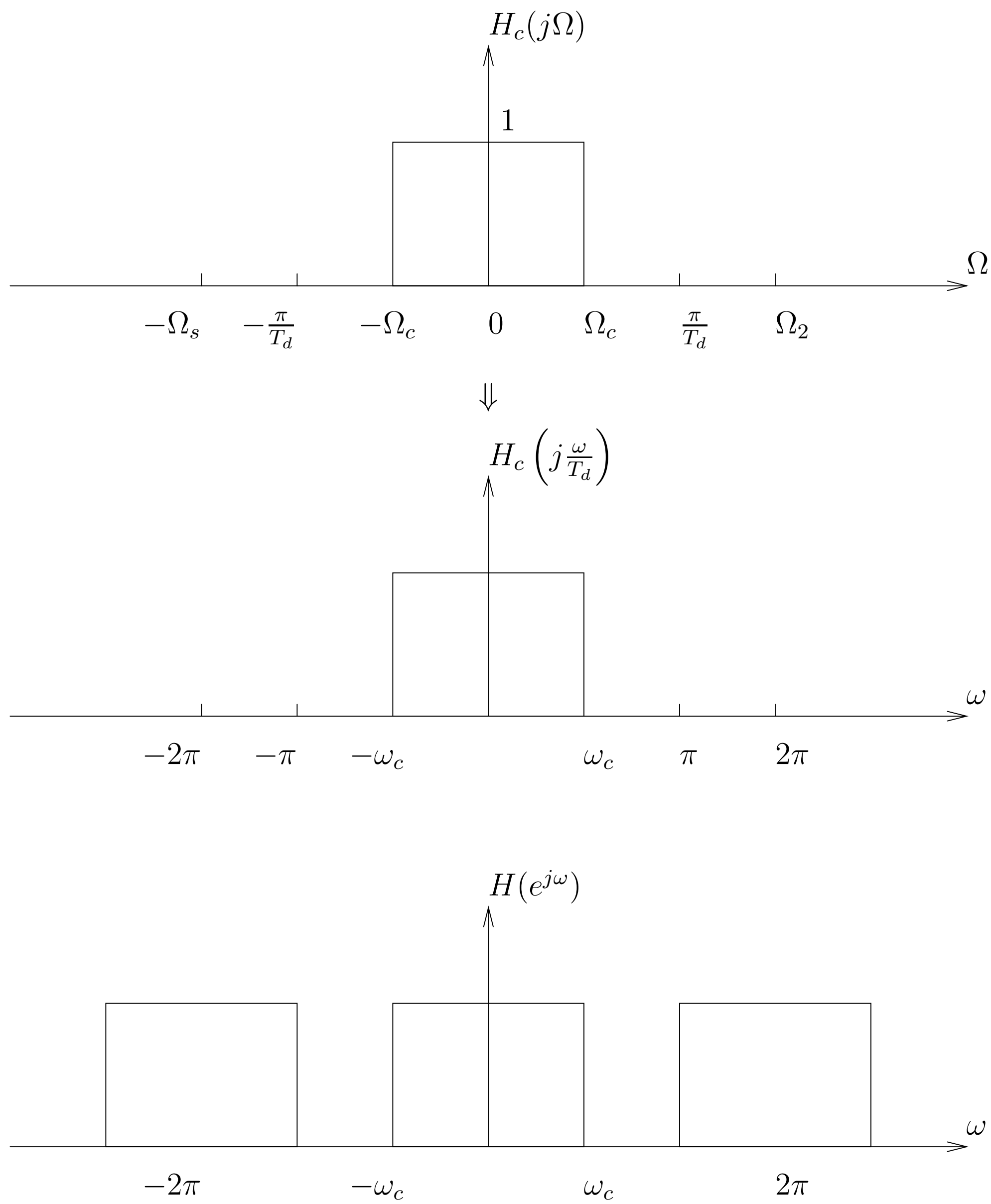


Figure 9.2: Aliasing in bandlimited filters

Unfortunately, any practical continuous-time filter cannot be exactly bandlimited and aliasing will occur in $H(e^{j\omega})$. And in this case, the "sampling" period T_d cannot be used (reduced) to control aliasing. This is due to the fact that the specifications are in terms of discrete-time frequency (ω). If the sampling rate is increased (i.e., T_d is made smaller) then the cut-off frequency of the continuous-time filter ($\Omega_c = \frac{\omega_c}{T_d}$) must increase in proportion.

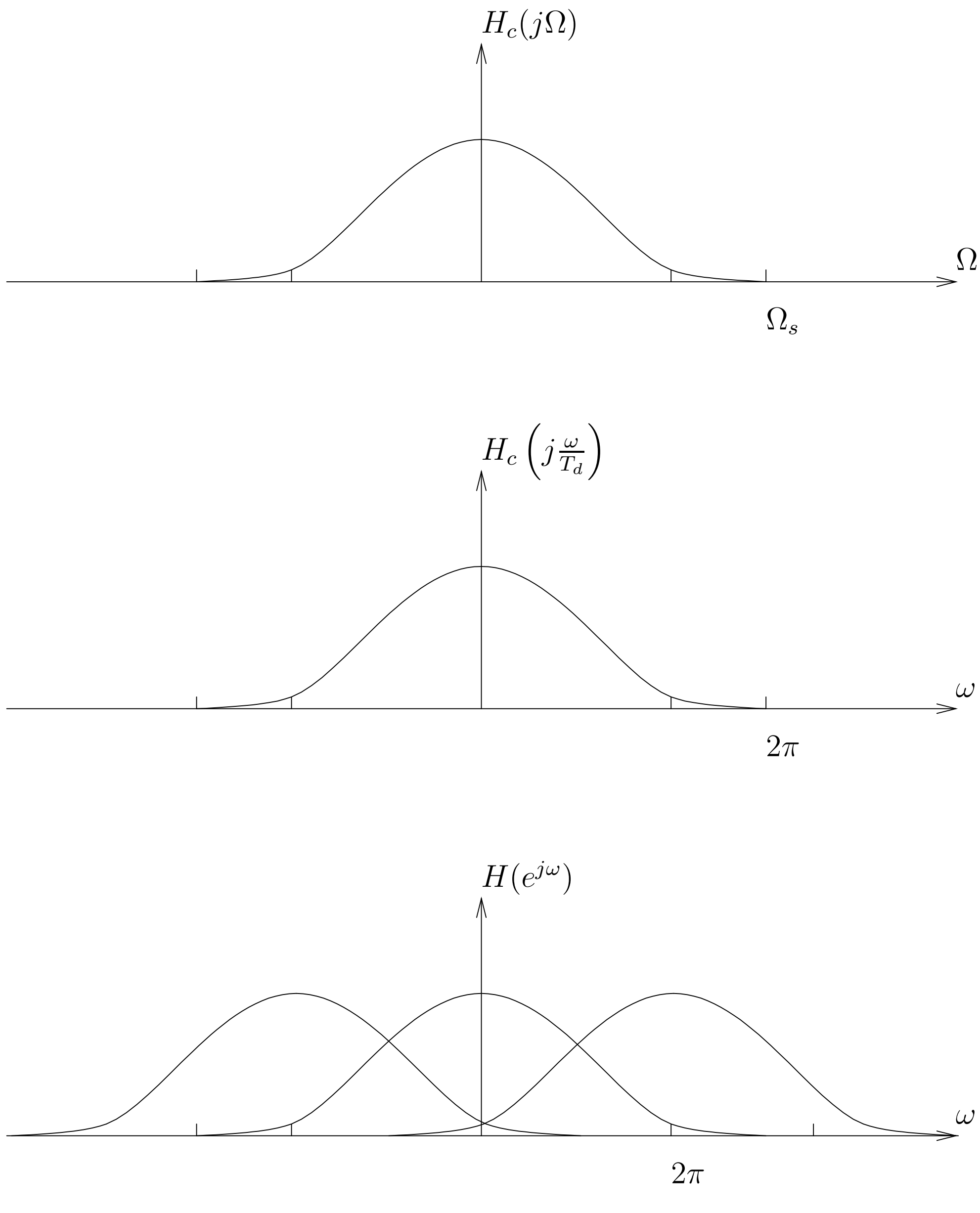


Figure 9.3: Aliasing in bandlimited filters

The best that we can do, in this case, is to make the continuous-time filter approach zero at high frequencies so that the aliasing may be negligibly small and a useful discrete-time filter can result from the sampling of the impulse response of a continuous-time filter. In practice, to compensate for aliasing that might occur in the transformation from $H_c(s)$ to $H(z)$, the continuous time filter may be somewhat overdesigned, i.e., designed to exceed the specifications, particularly in the stopband.

The direct relationship between the system function $H_c(s)$ of the continuous-time filter and the of $H(z)$, the desired discrete-time filter can be established as follows.

Assume that $H_c(s)$ is expressed in terms of a partial fraction expansion

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \tag{9.1-25}$$

where s_k are the single poles of the $H_c(s)$. The corresponding impulse response is

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t} & t \geqslant 0 \\ 0 & t < 0 \end{cases} \tag{9.1-26}$$

The impulse response of the discrete-time filter obtained by sampling $T_d h_c(t)$ is

$$h[n] = T_d h_c(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] \tag{9.1-27}$$

$$= \sum_{k=1}^{N} T_d A_k \left(e^{s_k T_d}\right)^n u[n] \tag{9.1-28}$$

where $u[n]$ is the unit step function. Therefore the system function of the corresponding discrete-time filter is

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}} \tag{9.1-29}$$

Comparing Equation 9.1-25 and Equation 9.1-29 it is seen that a pole at $s = s_k$ in the s-plane transforms to a pole at $z = e^{s_k T_d}$ in the z-plane. Therefore if $H_c(s)$ is stable ($\Re s_k < 0$ for all k), $H(z)$ is also stable ($|e^{s_k T_d}| < 1$ for all k).

EXAMPLE 9.1-1 *Given the specifications for a desired discrete-time low pass filter as shown in Figure 9.4 with $\delta_1 = 0.10875$, $\delta_2 = 0.17783$, $\omega_p = 0.2\pi$ and $\omega_s = 0.3\pi$, design a low-pass discrete-time filter by applying impulse invariance to an appropriate Butterworth continuous time filter.*

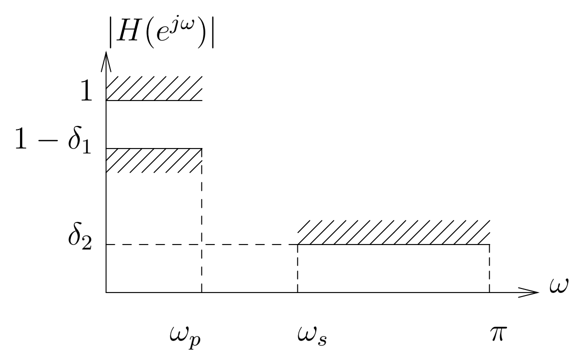


Figure 9.4: Discrete-time low-pass filter specification

Solution:

In general the specifications for the analog filter to be designed then transformed by impulse invariance are as shown in Figure 9.5. Since T_d cancels in the impulse invariance procedure, T_d is chosen to be 1 so that $\omega = \Omega$.

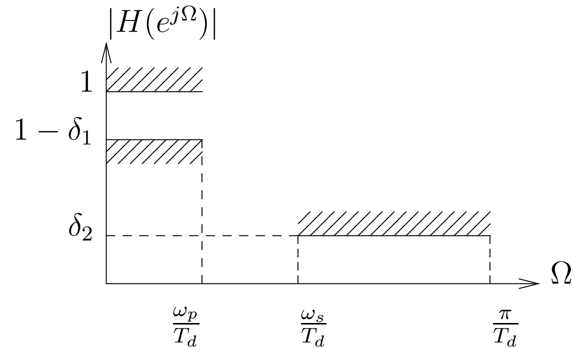


Figure 9.5: Low-pass analog filter specification

From the given specification,

$$0.89125 \leqslant H(e^{j\omega}) \leqslant 1 \qquad 0 \leqslant |\omega| \leqslant 0.2\pi \tag{9.1-30}$$

$$|H(e^{j\omega})| \leqslant 0.17783 \qquad 0.3\pi \leqslant |\omega| \leqslant \pi \tag{9.1-31}$$

we have specifications for the analog Butterworth filter to be designed as

$$0.89125 \leqslant |H_c(j\Omega)| \leqslant 1 \qquad 0 \leqslant |\Omega| \leqslant 0.2\pi \tag{9.1-32}$$

$$H_c(j\Omega) \leqslant 0.17783 \qquad 0.3\pi \leqslant |\Omega| \leqslant \pi \tag{9.1-33}$$

and

$$\alpha_p = -20\log_{10}(1-0.10875) \doteqdot 1\ dB \qquad 0 \leqslant |\Omega| \leqslant 0.2\pi \tag{9.1-34}$$

$$\alpha_s = -20\log_{10}(0.17783) \doteqdot 15\ dB \qquad 0.3 \leqslant |\Omega| \leqslant \pi \tag{9.1-35}$$

The standard analog Butterworth filter design formulas and procedure can be used to calculate the minimum order N can the cutoff frequency Ω_c as well as the system function $H_c(s)$. In this example, $N = 5.8858$ and corresponding $\Omega_c = 0.70474$. Since N must be an integer, we choose $N = 6$ and corresponding $\Omega_c = 0.7032$ and the system function is given by

$$H_c(s) = \frac{0.1209}{(s^2+0.3640s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)} \tag{9.1-36}$$

Express $H_c(s)$ as a partial fraction expansion and then transform $H_c(s)$ to obtain

$$H(z) = \frac{0.2871-0.4466z^{-1}}{1-1.2971z^{-1}+0.6949z^{-2}} + \frac{-2.1428+1.1455z^{-1}}{1-1.0691z^{-1}+0.3699z^{-2}} + \dots \tag{9.1-37}$$

$$\dots \frac{1.8557-0.6303z^{-1}}{1-0.9972z^{-1}+0.2570z^{-2}} \tag{9.1-38}$$

The frequency response of the discrete-time system can be calculated and plotted to see if the specifications are met.

Remark 9.1-2

- *If the resulting discrete-time filter fails to meet the specifications because of aliasing, there is NO alternative with impulse invariance but to try again with a higher-order filter or a different adjustment of the filter parameters, holding the order fixed.*
- *Impulse invariance can be extended to waveform invariance to preserve the output waveshape for a variety of inputs. (Step invariance, for instance).*
- *In the impulse invariance design procedure, the relationship between $H_c(j\Omega)$ and $H(e^{j\omega})$ is linear; as a result, the shape of the frequency response is preserved.*
- *The impulse invariance technique is only appropriate for bandlimited filters.*

9.1.2 IIR filter design by bilinear transformation

A discrete-time IIR filter $H(z)$ can be obtained from a corresponding analog IIR filter by using the bilinear transformation defined as

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{9.1-39}$$

that is

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] \tag{9.1-40}$$

where T_d is the "sampling period".

Since the specifications on $H(e^{j\omega})$ are given in the discrete-time domain, the effects of T_d will cancel when these specifications are mapped to continuous-time specifications and then the analog filter is mapped back to a digital filter.

(Note that the former corresponds to

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s} \tag{9.1-41}$$

and the later to Equation 9.1-39. Therefore, any convenient value of T_d can be chosen and used in the designing procedure.)

$$z = \frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2} \tag{9.1-42}$$

Substituting $s = \sigma + j\Omega$ into Equation 9.1-42, the following properties of the bilinear transformation can be seen.

1. if $\sigma < 0$, $|z| < 1$ for all Ω
2. if $\sigma > 0$, $|z| > 1$ for all Ω
3. if $\sigma = 0$, $|z| = 1$ for all Ω

Recall that $z = re^{j\omega}$, if $r = 1$ then $|z| = 1$.
Therefore

$$e^{j\omega} = \frac{1+j\Omega T_d/2}{1-j\Omega T_d/2} \tag{9.1-43}$$

or

$$
\begin{aligned}
j\Omega &= \frac{2}{T_d}\left\{\frac{1-e^{-j\omega}}{1+e^{j\omega}}\right\} && (9.1\text{-}44)\\
&= \frac{2}{T_d}\left\{\frac{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}+e^{-j\omega/2}\right)}\right\} && (9.1\text{-}45)\\
&= \frac{2}{T_d}\left\{\frac{2j\sin(\omega/2)}{2\cos(\omega/2)}\right\} && (9.1\text{-}46)\\
&= \frac{2}{T_d} j\tan(\omega/2) && (9.1\text{-}47)
\end{aligned}
$$

That is

$$\Omega = \frac{2}{T_d}\tan(\omega/2) \qquad (9.1\text{-}48)$$

or

$$\omega = 2\arctan\left(\frac{\Omega T_d}{2}\right) \qquad (9.1\text{-}49)$$

From Equation 9.1-48 and Equation 9.1-49 it can be seen that the entire $j\Omega$-axis of the s-plane maps onto the unit circle in the z-plane. When $z = re^{j\omega}$ is in its polar form $-\infty < \Omega < \infty$ maps to $-\pi < \omega < \pi$.

Consequently, the bilinear transformation avoids the aliasing problems which occurs when the impulse invariance is used to design IIR filters. However, this time the relationship between ω and Ω is nonlinear.

Since

$$|H(z)| = |\ H_c(s)|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}\ | \qquad (9.1\text{-}50)$$

if the corner frequencies of the analog filter $H_c(j\Omega)$ are prewarped using

$$\Omega = \frac{2}{T_d}\tan(\omega/2) \qquad (9.1\text{-}51)$$

then the digital filter $(H(e^{j\omega}))$ will meet the desired specifications when the analog filter is transformed to the digital filter using

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] \qquad (9.1\text{-}52)$$

REMARK 9.1-3 *A linear phase digital lowpass filter can not be obtained by applying the bilinear transformation to a linear phase lowpass analog filter due to the frequency warping represented by Equation 9.1-51.*

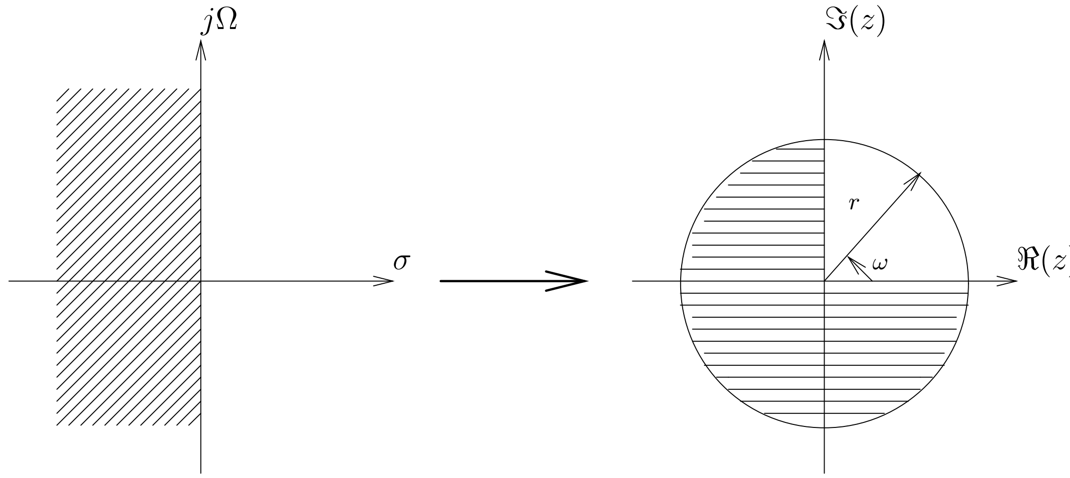
$$e^{-j\Omega\alpha} \to e^{-j\alpha\frac{2}{T_d}\tan(\omega/2)} \qquad (9.1\text{-}53)$$

EXAMPLE 9.1-2 *Given the specifications for the desired digital filter as*

$$
\begin{aligned}
&0.89125 \leqslant |H(e^{j\omega})| \leqslant 1 \qquad 0 \leqslant \omega \leqslant 0.2\pi && (9.1\text{-}54)\\
&|H(e^{j\omega}) \leqslant 0.17783 \qquad 0.3\pi \leqslant \omega \leqslant \pi && (9.1\text{-}55)
\end{aligned}
$$

design the filter and find its $H(z)$.

Solution:



*** X11-output-driver not found, switching to dumb terminal! *** If you want to use the X-output, please ins

Figure 9.6: Conversion from $j\Omega$ to z-plane

1. *Prewarp the corner frequencies for the corresponding analog filter.*

$$0.89125 \leqslant |H_c(j\Omega)| \leqslant 1 \qquad 0 \leqslant \Omega \leqslant \frac{2}{T_d} \tan\left(\frac{0.2\pi}{2}\right) \tag{9.1-56}$$

$$|H_c(j\Omega) \leqslant 0.17783 \qquad \frac{2}{T_d} \tan\left(\frac{0.3\pi}{2}\right) \leqslant \Omega \leqslant \infty \tag{9.1-57}$$

2. *For convenience choose $T_d = 1$, and use the Butterworth response.*

3. *Find the minimum order for the Butterworth filter.*

   Since the Butterworth filter has a monotonic magnitude response, it follows that it requires

$$|H_c(j2\tan(0.1\pi))| \geqslant 0.89125 \tag{9.1-58}$$

and

$$|H_c(j2\tan(0.15\pi))| \leqslant 0.17783 \tag{9.1-59}$$

The magnitude response of the Butterworth filter satisfies

$$|H_c(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \tag{9.1-60}$$

It follows that

$$1+\left(\frac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N} = (\ frac10.89125)^2 \tag{9.1-61}$$

and

$$1+\left(\frac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N} = (\ frac10.17783)^2 \tag{9.1-62}$$

Solve for N and Ω_c in the above equations.

$$N = \frac{\log\left[\left(\left(\frac{1}{0.17783}\right)^2 - 1\right) / \left(\left(\frac{1}{0.89125}\right)^2 - 1\right)\right]}{2\log\left[\frac{\tan(0.15\pi)}{\tan(0.1\pi)}\right]} \tag{9.1-63}$$

$$\doteqdot 5.30466 \tag{9.1-64}$$

Choose $N = 6$, $\Omega_c = 0.76622$.

4. *The system function of the analog filter is*

$$H_c(s) = \frac{0.20238}{(s^2+0.3966s+0.5871)(s^2+1.0836s+0.5871)(s^2+1.4802s+0.5871)} \tag{9.1-65}$$

5. *Convert $H_c(s)$ to $H(z)$ using the bilinear transformation*

$$H(z) = \frac{0.0007378(1+z^{-1})^6}{(1-0.9044z^{-1}+0.2155z^{-2})(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})} \tag{9.1-66}$$

9.2 Frequency Transformation of Low Pass IIR Filters

To design an IIR filter other than lowpass, a similar approach to the traditional analog frequency-selective filter design method can be taken.

If the bilinear transformation is used for the conversion from analog to digital filters, we may follow the following steps.

I.

$$\begin{array}{c}\text{Analog Lowpass prototype filter: } H_{lp}(s)\\ \Downarrow \leftarrow \text{s-plane transformation}\\ \text{Analog Highpass, bandpass or bandstop filter: } H'_{lp}(s),\ H_{hp}(s),\ H\\ \Downarrow \leftarrow \text{bilinear transformation}\\ \text{Digital highpass, bandpass or bandstop filter: } H'_{lp}(z),\ H_{hp},\ H_{bp}(\end{array}$$

II.

$$\begin{array}{c}\text{Design analog low pass prototype filter: } H_{lp}(s)\\ \Downarrow \leftarrow \text{bilinear transformation}\\ \text{Digital lowpass normalized filter: } H_{lp}(z)\\ \Downarrow \leftarrow z\text{-plane transformation}\\ \text{Digital highpass, bandpass or bandstop filter: } H'_{lp}(z), H_{hp}, H_{bp}(\end{array}$$

If the impulse invariance is used, then procedure I cannot be used because of the aliasing that results from sampling.

9.3 Design of FIR filters by windowing

9.3.1 Frequency response of FIR linear phase systems

• zero phase response, linear phase response and generalized linear phase response.

The ideal low pass filter has a magnitude response

$$H_{lp}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leqslant \pi \end{cases} \tag{9.3-67}$$

where ω_c is the cutoff frequency. Its phase response is assumed to be zero, i.e.,

$$\measuredangle H_{lp}(e^{j\omega}) = 0 \tag{9.3-68}$$

However, because its impulse response is

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n} \qquad -\infty < n < \infty \tag{9.3-69}$$

which is noncausal, it is not computationally realizable.

The impulse response of the ideal highpass filter is also noncausal. It can be shown as follows:

The ideal highpass filter is defined to have a frequency response

$$H_{hp}(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c < |\omega| \leqslant \pi \end{cases} \tag{9.3-70}$$

and

$$\measuredangle H_{hp}(e^{j\omega}) = 0 \tag{9.3-71}$$

To obtain its impulse response, apply the inverse Fourier transform

$$\begin{aligned}
h_{hp}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{hp}(e^{j\omega})e^{j\omega n}d\omega && (9.3\text{-}72)\\
&= \frac{1}{2\pi}\left\{\int_{-\pi}^{-\omega_c} e^{j\omega n}d\omega + \int_{\omega_c}^{\pi} e^{j\omega n}d\omega\right\} && (9.3\text{-}73)\\
&= \frac{1}{2\pi}\left\{\frac{1}{jn}e^{j\omega n}\Big|_{-\pi}^{-\omega_c} + \frac{1}{jn}e^{j\omega n}\Big|_{\omega_c}^{\pi}\right\} && (9.3\text{-}74)\\
&= \frac{1}{2\pi jn}\left\{\left[e^{-j\omega_c n} - e^{-jn\pi}\right] + \left[e^{j\pi n} - e^{j\omega_c n}\right]\right\} && (9.3\text{-}75)\\
&= \frac{1}{2\pi jn}\left[e^{jn\pi} - e^{-jn\pi}\right] + \frac{1}{2\pi jn}\left[e^{--j\omega_c n} - e^{j\omega_c n}\right] && (9.3\text{-}76)\\
&= \delta[n] + \frac{1}{2\pi jn}\{\cos(-\omega_c n) - j\sin(\omega_c n) - \cos(\omega_c n) - j\sin(\omega_c n)\} && (9.3\text{-}77)\\
&= \delta[n] - \frac{\sin(\omega_c n)}{n\pi} && (9.3\text{-}78)
\end{aligned}$$

Note:

$$\lim_{n\to 0}\frac{\left\{e^{jn\pi} - e^{-jn\pi}\right\}}{j2\pi n} = \lim_{n\to 0}\frac{j\pi\left\{e^{jn\pi} + e^{-jn\pi}\right\}}{j2\pi n} = \frac{2}{2} = 1 \tag{9.3-79}$$

and

$$h_{hp}[n] = \delta[n] - h_{lp}[n] \tag{9.3-80}$$

Causal approximations to ideal frequency-select filters must have nonzero phase response.

Linear phase response is our next best choice, since it corresponds to shifting the sequence in time.

EXAMPLE 9.3-1 *Given the impulse response of an ideal delay system*

$$h_{id}[n] = \delta[n - n_d] \tag{9.3-81}$$

its frequency response is

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d} \tag{9.3-82}$$

i.e.,

$$|H_{id}(e^{j\omega})| = 1 \tag{9.3-83}$$

and

$$\measuredangle H_{id}(e^{j\omega}) = -\omega n_d \tag{9.3-84}$$

where n_d is the delay in time and $|\omega| < \pi$. Therefore its group delay is a constant

$$\tau(\omega) = -\frac{d}{d\omega}\left\{\arg[H(e^{j\omega})]\right\} \tag{9.3-85}$$

$$= -\frac{d}{d\omega}\{-\omega n_d\} \tag{9.3-86}$$

$$= n_d \tag{9.3-87}$$

Hence the ideal lowpass filter with linear phase response is used in the design for the approximations. The frequency response is defined

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leqslant \pi \end{cases} \tag{9.3-88}$$

and its impulse response is

$$h_{lp}[n] = \frac{\sin \omega_c(n-n_d)}{\pi(n-n_d)} \qquad -\infty < n < \infty \tag{9.3-89}$$

Although $h_{lp}[n]$ is still noncausal, its magnitude and phase response can be approximated by a causal system.

Definition 9.3-1 *If the frequency response of a system can be expressed in the form*

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta} \tag{9.3-90}$$

where α and β are constants and $A(e^{j\omega})$ is a real function of ω, this system is a generalized linear phase system.

Example 9.3-2 *Given the impulse response of the moving average system*

$$h[n] = \begin{cases} \frac{1}{M+1} & 0 \leqslant n \leqslant M \\ 0 & \textit{otherwise} \end{cases} \tag{9.3-91}$$

its frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \tag{9.3-92}$$

$$= \frac{1}{M+1}\sum_{n=0}^{M} e^{-j\omega n} \tag{9.3-93}$$

$$= \frac{1}{M+1}\frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} \tag{9.3-94}$$

$$= \frac{1}{M+1}\frac{e^{-j\frac{\omega(M+1)}{2}}}{e^{-j\frac{\omega M}{2}}}\frac{e^{j\frac{\omega(M+1)}{2}} - e^{j\frac{\omega(M+1)}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \tag{9.3-95}$$

$$= \frac{1}{M+1}e^{-j\frac{\omega M}{2}}\frac{2j\sin\left[\frac{\omega(M+1)}{2}\right]}{2j\sin\left(\frac{\omega}{2}\right)} \tag{9.3-96}$$

$$= \frac{1}{M+1}\frac{\sin\left[\omega\frac{M+1}{2}\right]}{\sin\left(\frac{\omega}{2}\right)}e^{-j\omega M/2} \tag{9.3-97}$$

Therefore, the moving average system is a generalized linear phase system. (Due to the fact that at frequencies for which the factor

$$\frac{1}{M+1}\frac{\sin\left[\omega\frac{M+1}{2}\right]}{\sin\left(\frac{\omega}{2}\right)} \tag{9.3-98}$$

is negative, an additional phase of π radians is added to the total phase.)

Note that the moving average system is given by

$$y[n] = \frac{1}{M+1}\sum_{k=0}^{M} x[n-k] \tag{9.3-99}$$

REMARK 9.3-1 *Many of the advantages of linear phase systems apply to generalized linear phase systems.*

I. *If the impulse response of the system is defined as*

$$h[n] = \begin{cases} h[M-n] & 0 \leqslant n \leqslant M \\ 0 & \textit{otherwise} \end{cases} \tag{9.3-100}$$

(causal, symmetric about $M/2$), the frequency response is

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-jM/2\omega} \tag{9.3-101}$$

where $A_e(e^{j\omega})$ is a real, even, periodic function of ω. (With generalized linear phase response).

II. *If (causal, antisymmetric about $M/2$)*

$$h[n] = \begin{cases} -h[M-n] & 0 \leqslant n \leqslant M \\ 0 & \textit{otherwise} \end{cases} \tag{9.3-102}$$

then

$$H(e^{j\omega}) = jA_0(e^{j\omega})e^{-j\omega M/2} \tag{9.3-103}$$
$$= A_0(e^{j\omega})e^{-j\omega M/2+j\pi/2} \tag{9.3-104}$$

where $A_0(e^{j\omega})$ is a real, odd, periodic function of ω.

There are four types of FIR generalized linear phase systems:

9.3.1.1 Type I FIR linear phase systems

A type I system has a symmetric impulse response

$$h[n] = h[M-n] \qquad 0 \leqslant n \leqslant M \tag{9.3-105}$$

where M is an even integer. The frequency response is

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} && (9.3\text{-}106)\\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{n=M/2+1}^{M} h[n]e^{-j\omega n} && (9.3\text{-}107)\\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{k=0}^{M/2-1} h[m-k]e^{-j\omega(M-k)} && (9.3\text{-}108)\\
&= \left\{\sum_{k=0}^{M/2-1} h[k]e^{j\omega(M/2-k)} + h[M/2] + \sum_{k=0}^{M/2-1} h[k]e^{-j\omega(M/2-k)}\right\} e^{-j\omega M/2} && (9.3\text{-}109)\\
&= \left\{h[M/2] + \sum_{k=0}^{M/2-1} h[k]2\cdot\cos[(M/2-k)\omega]\right\} e^{-j\omega M/2} && (9.3\text{-}110)
\end{aligned}$$

Let $k' = M/2 - k$

$$\begin{aligned}
k = 0 &\Rightarrow k' = M/2 && (9.3\text{-}111)\\
k = M/2-1 &\Rightarrow k' = 1 && (9.3\text{-}112)\\
k = M/2 &\Rightarrow k' = 0 && (9.3\text{-}113)
\end{aligned}$$

thus

$$\begin{aligned}
H(e^{j\omega}) &= e^{-j\omega M/2}\left\{h[M/2]\cos(0) + \sum_{k'=1}^{M/2} 2h[M/2-k']\cos(k'\omega)\right\} && (9.3\text{-}114)\\
&= e^{-j\omega M/2}\sum_{k=0}^{M/2} a[k]\cos(k\omega) && (9.3\text{-}115)
\end{aligned}$$

where

$$a[k] = \begin{cases} h[M/2] & \text{for } k = 0\\ 2h[(M/2)-k] & \text{for } k = 1, 2, \ldots, M/2 \end{cases} \qquad (9.3\text{-}116)$$

That is that is is a generalized linear phase system with a delay $M/2$ which is an integer. ($\beta = 0$ or π).

9.3.1.2 Type II FIR linear phase systems

A type II system has a symmetric impulse response

$$h[n] = h[M-n] \qquad 0 \leqslant n \leqslant M \qquad (9.3\text{-}117)$$

where M is an odd integer. It can be shown that

$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k-\frac{1}{2}\right)\right]\right\} \qquad (9.3\text{-}118)$$

where

$$b[k] = 2h[(M+1)/2 - k] \qquad k = 1, 2, \ldots, (M+1)/2 \tag{9.3-119}$$

This is also a generalized linear phase system with a delay M/2 which is an integer plus one-half. ($\beta = 0$ or π).

9.3.1.3 Type III FIR linear phase systems

A type III system has an antisymmetric impulse response.

$$h[n] = -h[M-n] \qquad 0 \leqslant n \leqslant M \tag{9.3-120}$$

where M is an even integer, and it has a frequency response in the form of

$$H(e^{j\omega}) = je^{-j\omega M/2}\left\{\sum_{k=1}^{M/2} C[k]\sin\omega k\right\} \tag{9.3-121}$$

where

$$C[k] = 2h[(M/2) - k] \qquad k = 1, 2, \ldots, M/2 \tag{9.3-122}$$

It has a delay $M/2$ which is an integer. ($\beta = \pi/2$ or $3\pi/2$).

9.3.1.4 Type IV FIR linear phase systems

A type IV system has an antisymmetric impulse response.

$$h[n] = -h[M-n] \qquad 0 \leqslant n \leqslant M \tag{9.3-123}$$

where M is an odd integer, and its $H(e^{j\omega})$ is

$$H(e^{j\omega}) = je^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} d[k]\sin[\omega(k-\frac{1}{2})]\right\} \tag{9.3-124}$$

where

$$d[k] = 2h[(M+1)/2 - k] \qquad k = 1, 2, \ldots, (M+1)/2 \tag{9.3-125}$$

It has a delay $M/2$. ($\beta = \pi/2$ or $3\pi/2$).

9.3.2 Properties of symmetrical FIR linear phase systems

If $h[n]$ is symmetrical (Type I and II)

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} \tag{9.3-126}$$
$$= \sum_{n=0}^{M} h[M-n]z^{-n} \tag{9.3-127}$$
$$= \sum_{k=0}^{M} h[k]z^{k}z^{-M} \tag{9.3-128}$$
$$= z^{-M}\sum_{k=0}^{M} h[k](z^{-1})^{-k} \tag{9.3-129}$$
$$= z^{-M}H(z^{-1}) \tag{9.3-130}$$

Therefore, if z_0 is a zero of $H(z)$

$$H(z_0) = z_0^{-M}H(z_0^{-1}) = 0 \tag{9.3-131}$$

i.e.,

If $z_0 = re^{j\theta}$ is a zero of $H(z)$

$$z_0^{-1} = r^{-1}e^{-j\theta} \quad \text{is also a zero of } H(z) \tag{9.3-132}$$

If $h[n]$ is real

$$H^*(z) = \left(\sum_{n=0}^{M} h[n]z^{-n}\right)^* = \sum_{n=0}^{M} h^*\left(z^*\right)^{-n} \tag{9.3-133}$$
$$= \sum_{n=0}^{M} h[n]\left(z^*\right)^{-n} \tag{9.3-134}$$
$$= H(z^*) \tag{9.3-135}$$

Thus

$$H(z_0) = 0 \Rightarrow H^*(z_0) = 0 \tag{9.3-136}$$
$$\Rightarrow H(z_0^*) = 0 \tag{9.3-137}$$

i.e.,
if z_0 is a zero of $H(z)$, $z_0^* = re^{-j\theta}$ is also a zero of $H(z)$ and, $(z_0^*)^{-1} = r^{-1}e^{j\theta}$ is a zero of $H(z)$ as well.

In summary, if $h[n]$ is real, each complex zero that is not on the unit circle will be part of a set of four conjugate reciprocal zeros of the form

$$(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})(1-r^{-1}e^{j\theta}z^{-1})(1-r^{-1}e^{-j\theta}z^{-1}) \tag{9.3-138}$$

1. If a zero of $H(z)$ is on the unit circle Equation 9.3-138 reduces to

$$(1-e^{j\theta}z^{-1})(1-e^{-j\theta}z^{-1}) \tag{9.3-139}$$

as $r = r^{-1}$.

2. If a zero of $H(z)$ is real but not on the unit circle Equation 9.3-138 reduces to

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}) \tag{9.3-140}$$

as $e^{jk\pi} = e^{-jk\pi}$ (k is an integer).

3. If a zero of $H(z)$ is as $z = \pm 1$, Equation 9.3-138 reduces to

$$(1 \pm z^{-1}) \tag{9.3-141}$$

4. When $z = -1$, Equation 9.3-130 requires that

$$H(-1) = (-1)^M H(-1) \tag{9.3-142}$$

If M is odd, $H(-1)$ must be zero as Equation 9.3-142 requires

$$H(-1) = -H(-1) \tag{9.3-143}$$

Since $z = -1$ corresponds to $\omega = \pi$ and the frequency response of type II FIR is constrained to be zero, it cannot be used to approximate a highpass or bandstop filter.

9.3.3 Properties of antisymmetrical FIR linear phase systems

If $h[n]$ is antisymmetrical (Types III and IV)

$$H(z) = -z^{-M} H(z^{-1}) \tag{9.3-144}$$

1. The zeros of $H(z)$ are constrained in the same way as those for the symmetric case.

2. However, if $z = 1$, Equation 9.3-144 becomes

$$H(1) = -H(1) \tag{9.3-145}$$

i.e., $H(z)$ must have a zero at $z = 1$ for both M even and M odd.

3. If $z = -1$

$$H(-1) = (-1)^{M+1} H(-1) \tag{9.3-146}$$

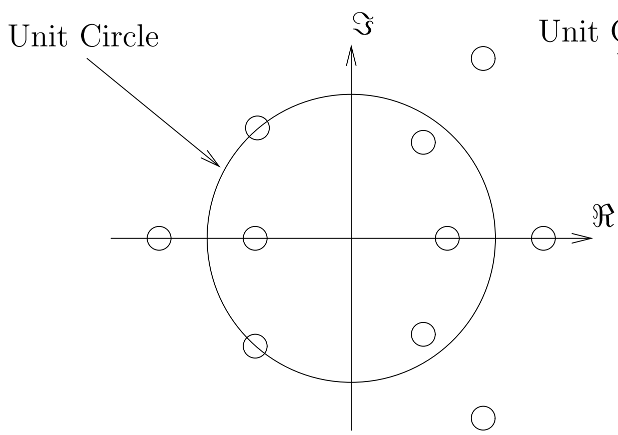
Equation 9.3-146 requires that if M is even (Type III) $H(z)$ must have a zero at $z = -1$.

Types III and IV are not used to approximate low filters and type IV is not used to approximate the high-pass or band-stop filters.

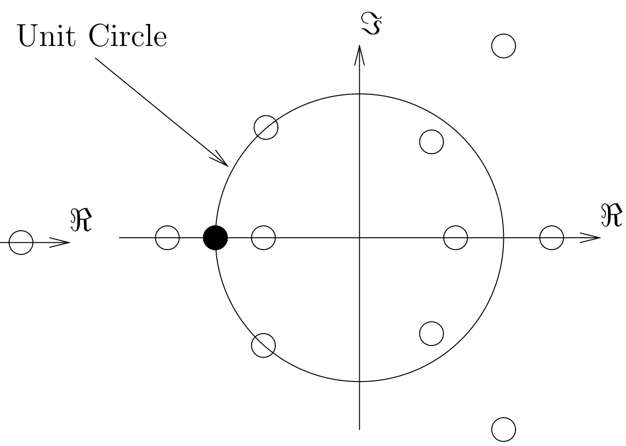
9.3.4 Design of FIR filters by windowing

Design techniques for FIR filters are based on directly approximating the desired frequency response of the discrete-time system.

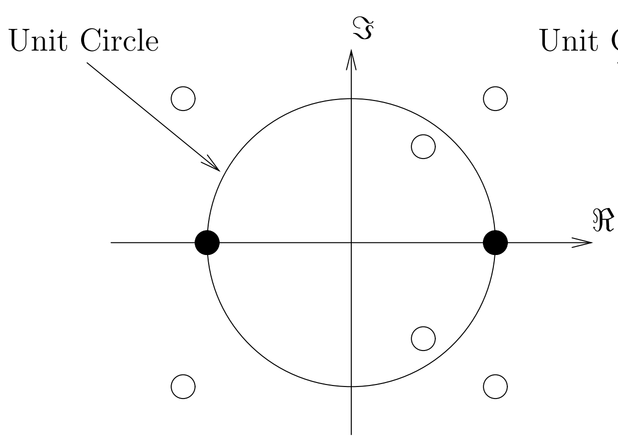
They assume a linear phase constraint.



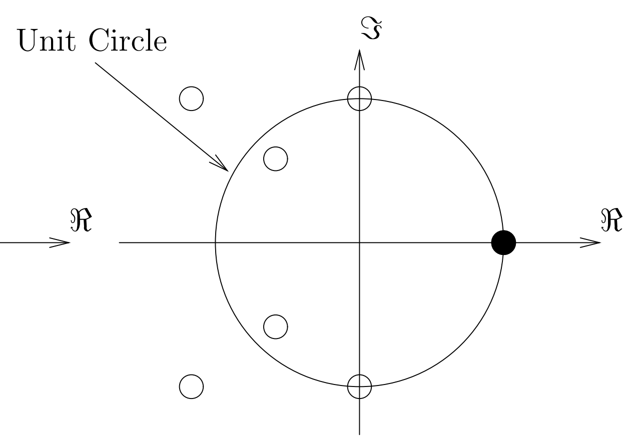
(a) Type I



(b) Type II



(c) Type III



(d) Type IV

Figure 9.7: Typical zero locations for linear phase systems

9.3.4.1 Window method

Given an ideal desired frequency range

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n} \tag{9.3-147}$$

where $h_d[n]$ is the corresponding impulse response sequence which can be expressed using the inverse Fourier transform

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \tag{9.3-148}$$

Many idealized systems have impulse responses that are non-causal and infinitely long due to discontinuities at the boundaries between bands.

Therefore, truncation of ideal response will be applied to obtain a causal FIR approximation.

Equation 9.3-147 can be thought of as a Fourier series representation of the periodic frequency response $H_d(e^{j\omega})$ with $h_d[n]$ being the Fourier coefficients. Thus, the approximation of an ideal filter by truncation of the ideal response is identical to the issue of convergence of Fourier series. (Gibb's phenomenon.)

Using the rectangular window $w[n]$ -

$$w[n] = \begin{cases} 1 & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-149}$$

a causal FIR filter with impulse response $h[n]$ can be obtained from $h_d[n]$

$$h[n] = h_d[n]w[n] \tag{9.3-150}$$

$$= \begin{cases} h_d[n] & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-151}$$

The frequency response $H(e^{j\omega})$ of the new FIR filter can be obtained from the windowing theorem.

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)}d\theta \tag{9.3-152}$$

where $W(e^{j\omega}$ is the Fourier transform of $w[n]$.

$$W(e^{j\omega}) = \sum_{n=0}^{\infty} e^{-j\omega n} = \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} \tag{9.3-153}$$

$$= e^{-j\omega M/2}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} \tag{9.3-154}$$

Note: $\Delta\omega_M = \frac{4\pi}{M+1}$

If M increases, $\Delta\omega_m$ decreases and $\left|\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}\right|$ increases.

In the extreme case, $w[n] = 1$ for all n

$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k) \tag{9.3-155}$$

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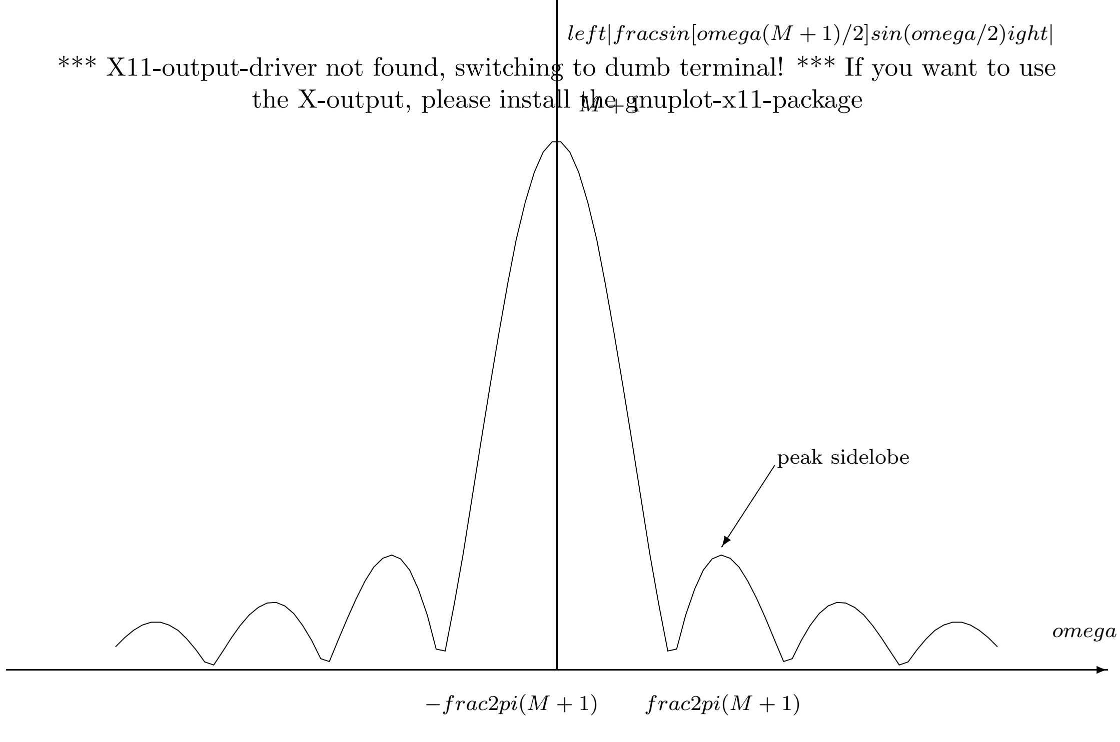


Figure 9.8: Frequency response of rectangular window.

which is a periodic impulse train with period 2π.

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})2\pi\delta(\omega-\theta)d\theta \tag{9.3-156}$$

$$= H_d(e^{j\omega}) \tag{9.3-157}$$

From the view point of the accuracy of the approximation, the narrower the $\Delta\omega_m$ of $W(e^{j\omega})$ the better, i.e., $W(e^{j\omega}) \to 2\pi\delta(\omega)$, $H(e^{j\omega}) \to H_d(e^{j\omega})$. This normally requires the increase of M, the length of the FIR filter.

From the implementation point of view, we would like to have $w[n]$ as short as possible in duration so as to minimize computation.

9.3.4.2 Other window in the FIR filter design

The Gibbs phenomenon can be moderated by tapering the window smoothly to zero at both ends. As a result, the mainlobe will become wider, and so will the transition at the discontinuity. The magnitude spectrums of various windows is shown in Figures 9.9 to 9.18.

Rectangular Window

$$w[n] = \begin{cases} 1 & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-158}$$

Bartlett (triangular) Window

$$w[n] = \begin{cases} 2n/M & 0 \leqslant n \leqslant M/2 \\ 2-2n/M & M/2 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-159}$$

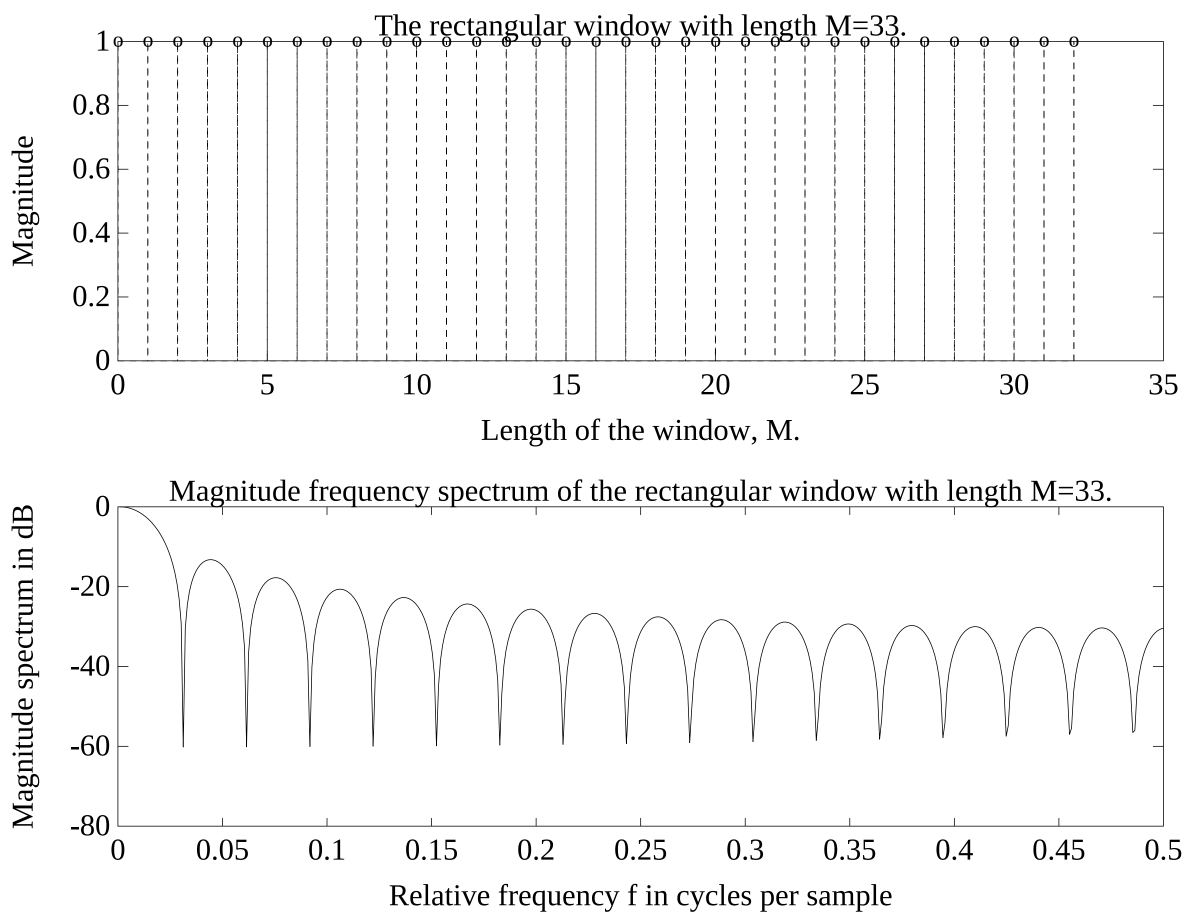


Figure 9.9: Rectangular Window length M=33

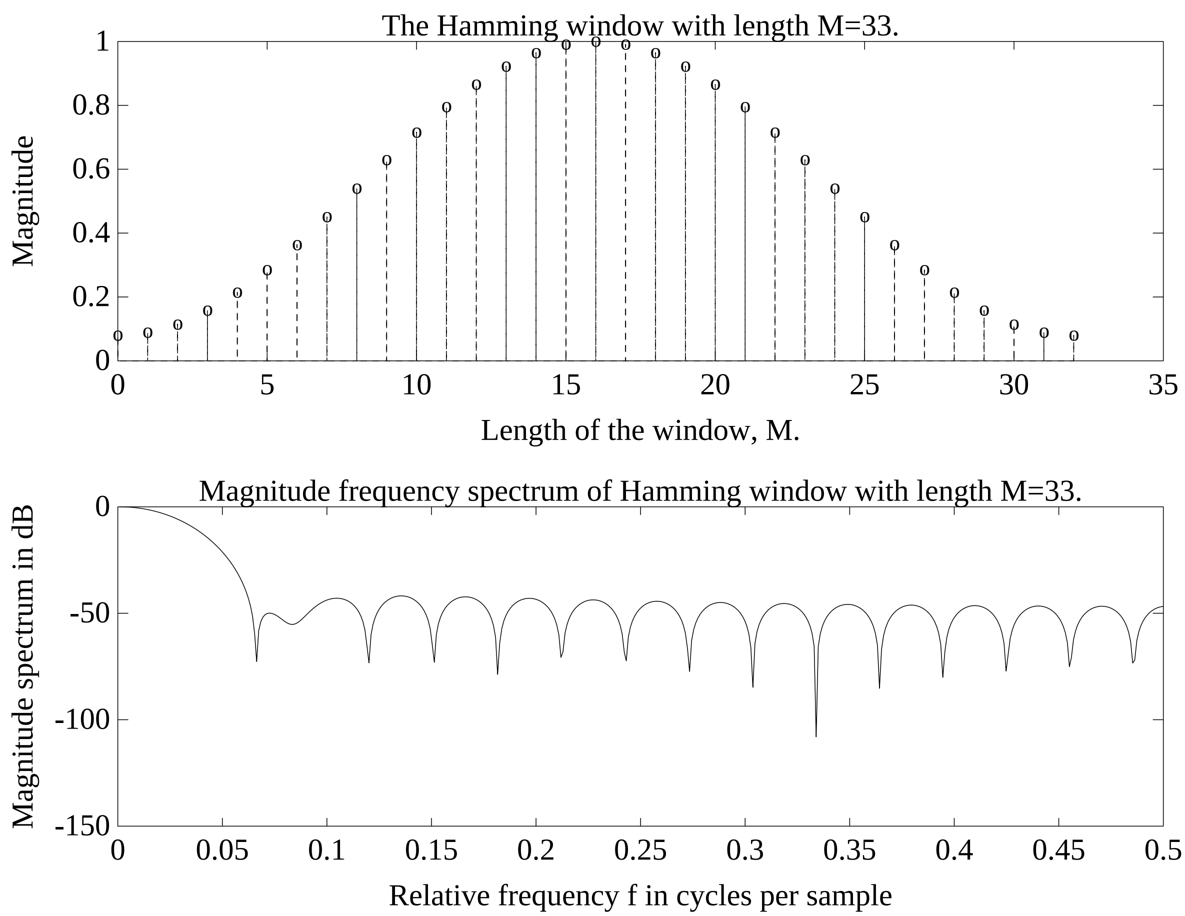


Figure 9.10: Hamming Window length M=33

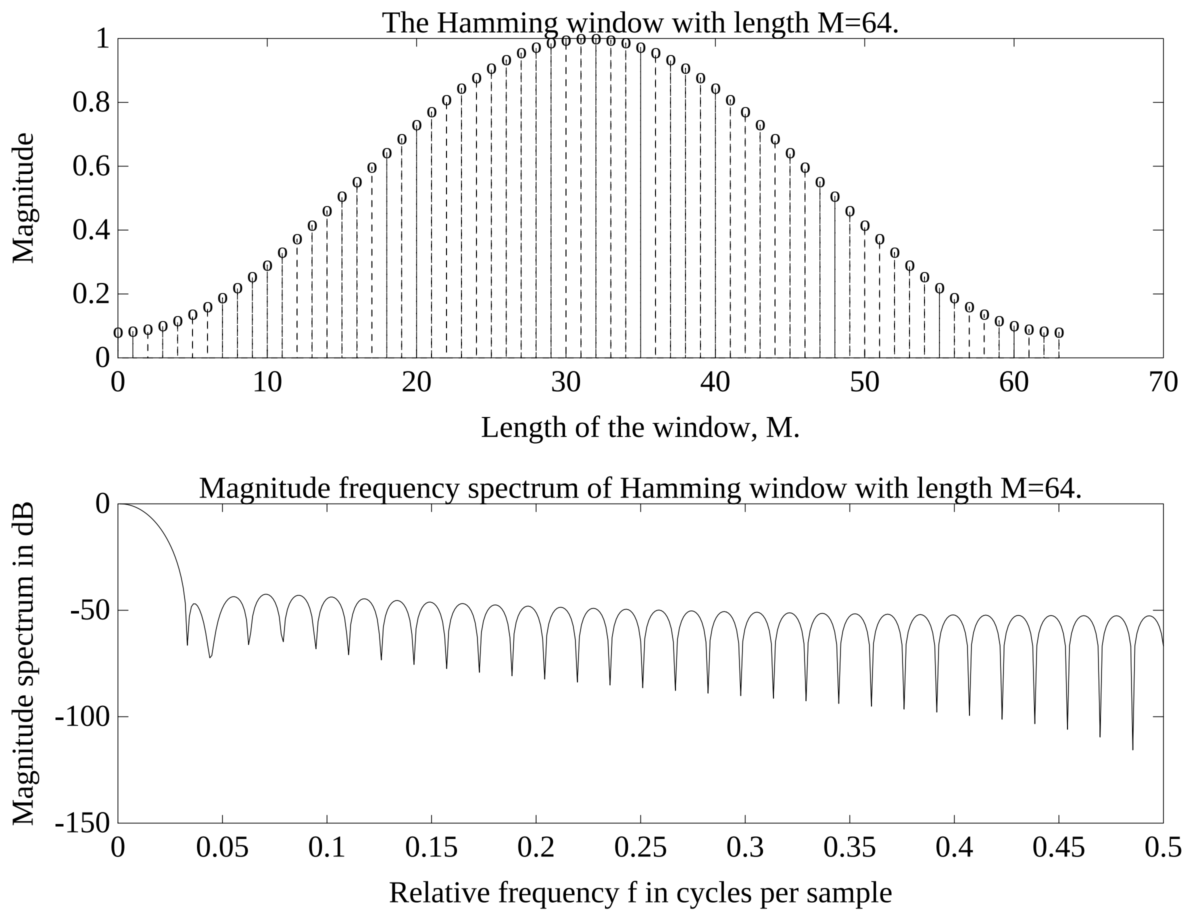


Figure 9.11: Hamming Window length M=64

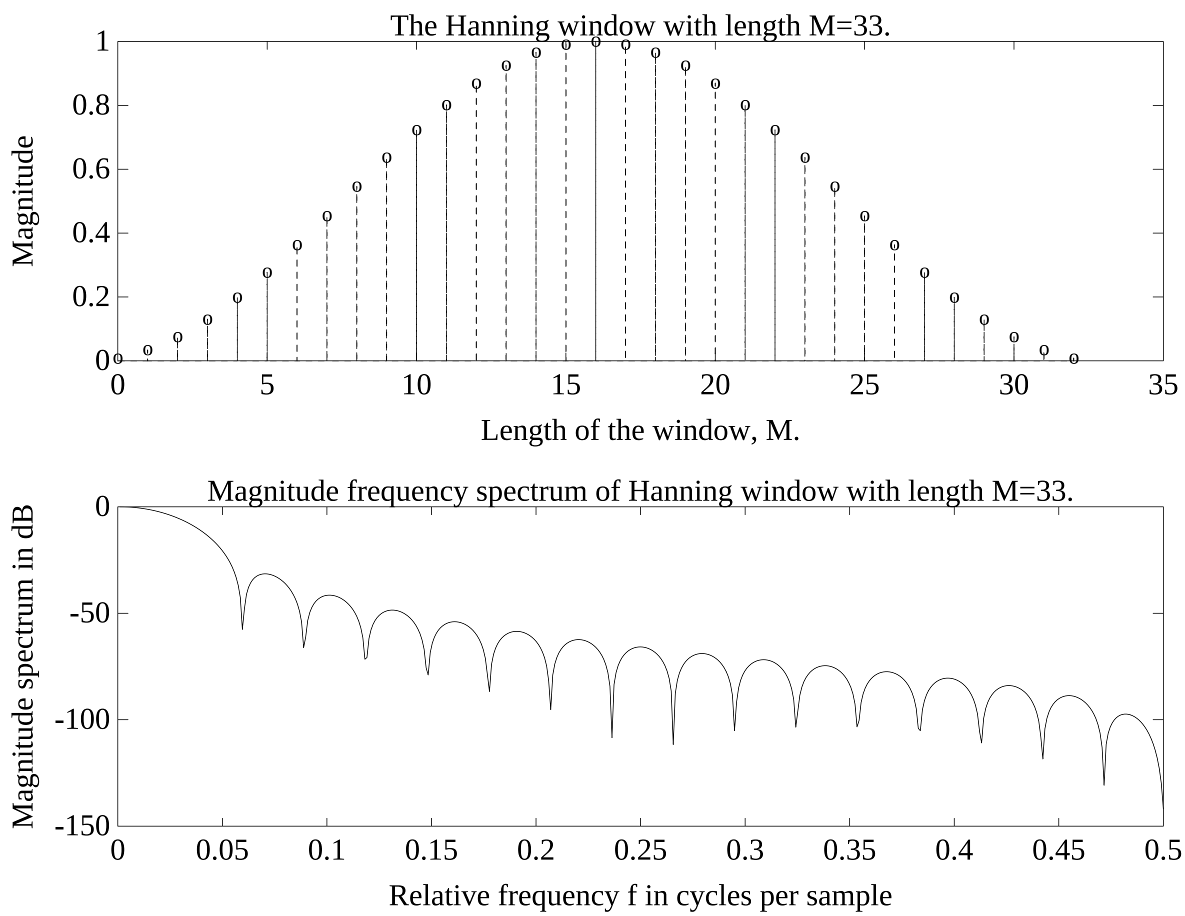


Figure 9.12: Hanning Window length M=33

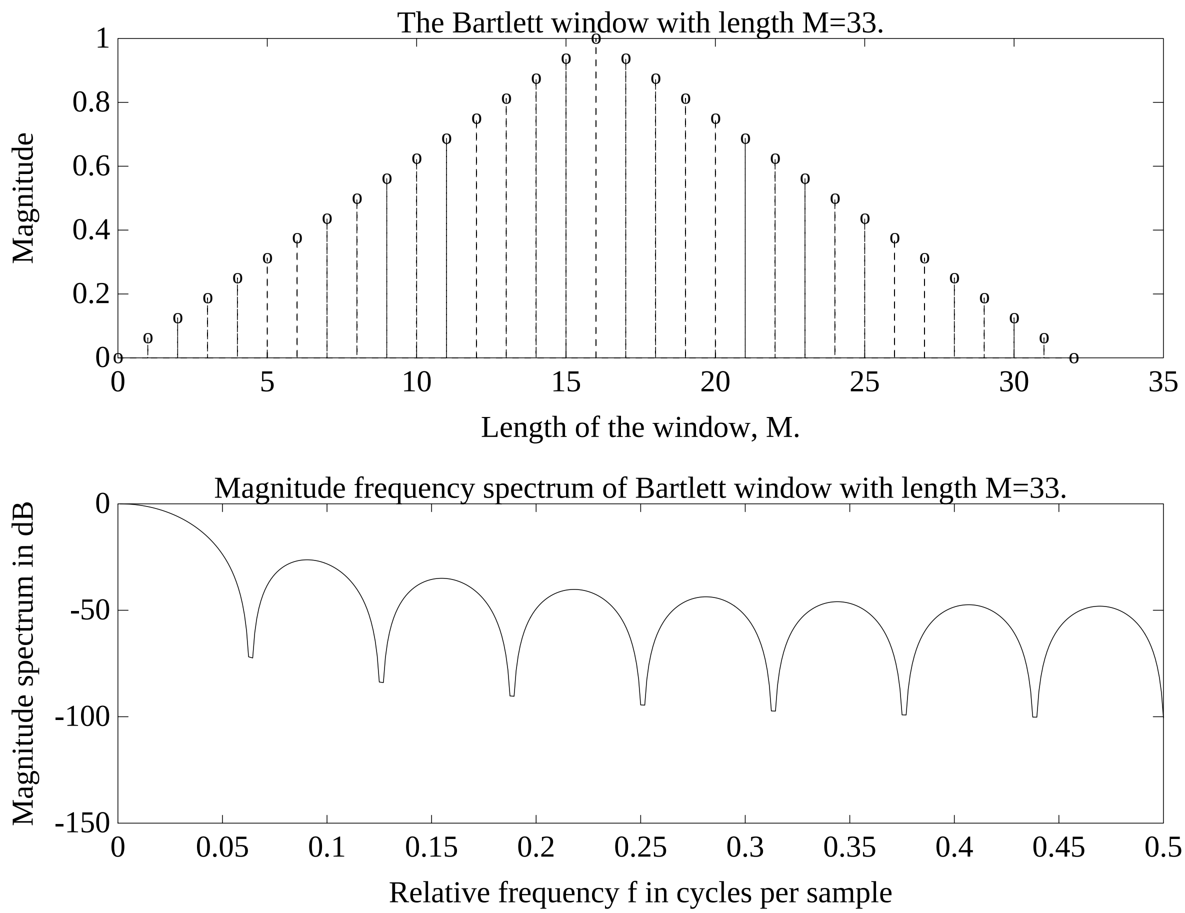


Figure 9.13: Bartlett Window length M=33

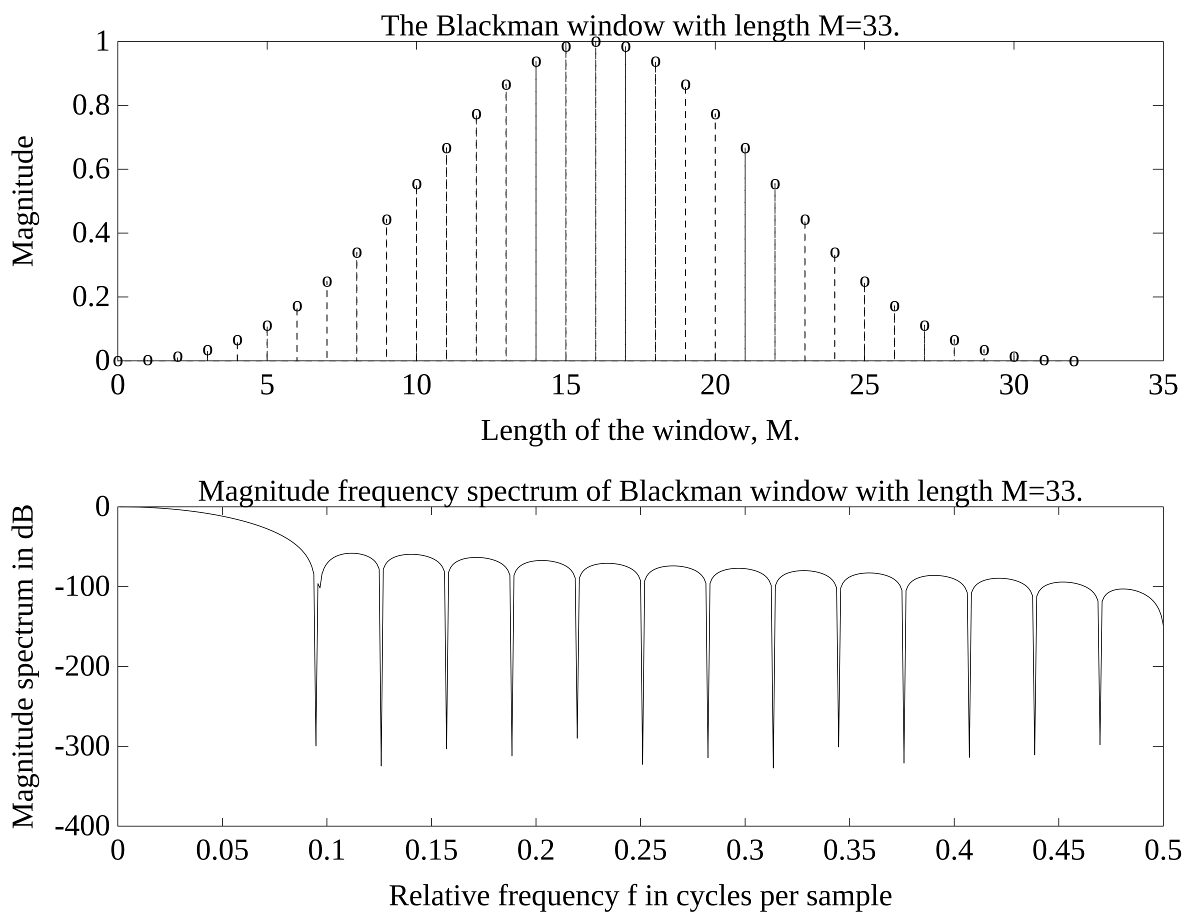


Figure 9.14: Blackman Window length M=33

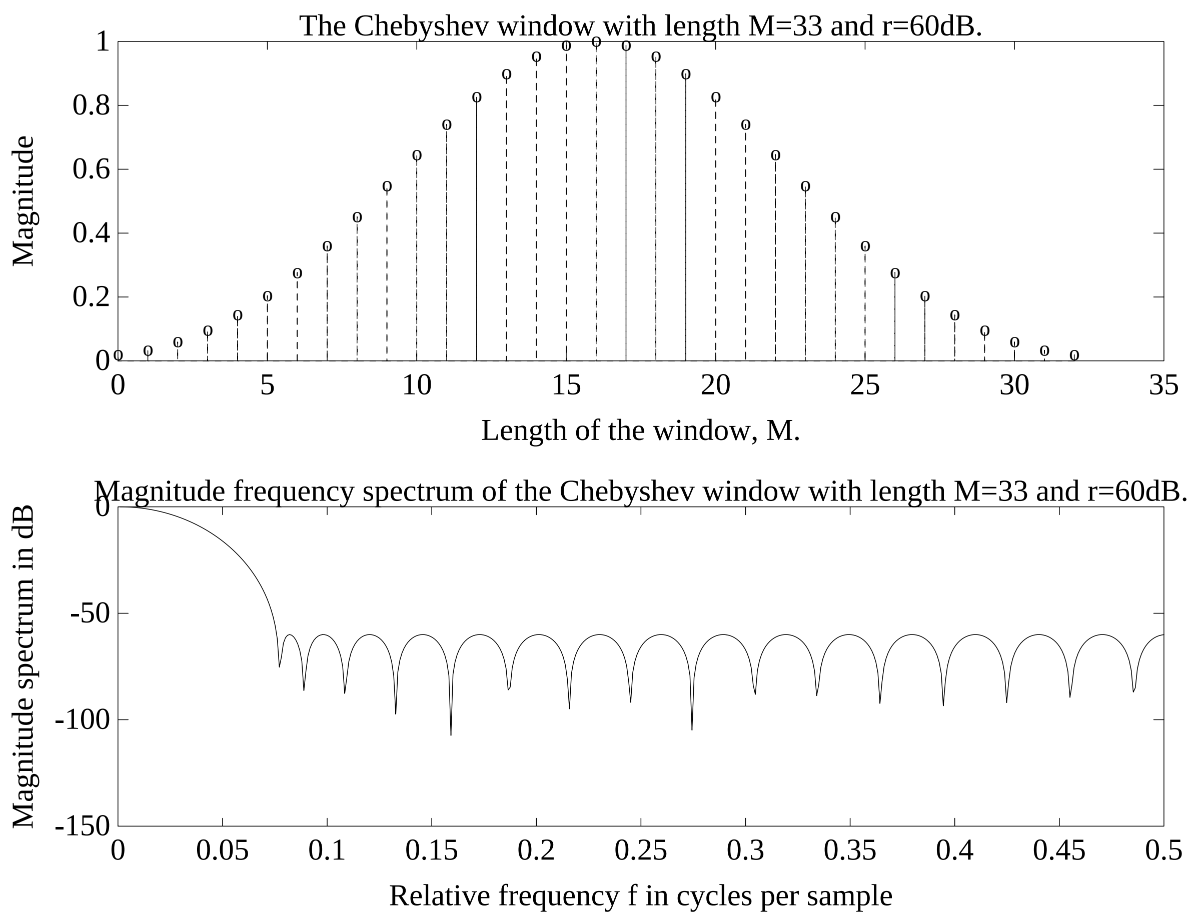


Figure 9.15: Chebychev Window length M=33

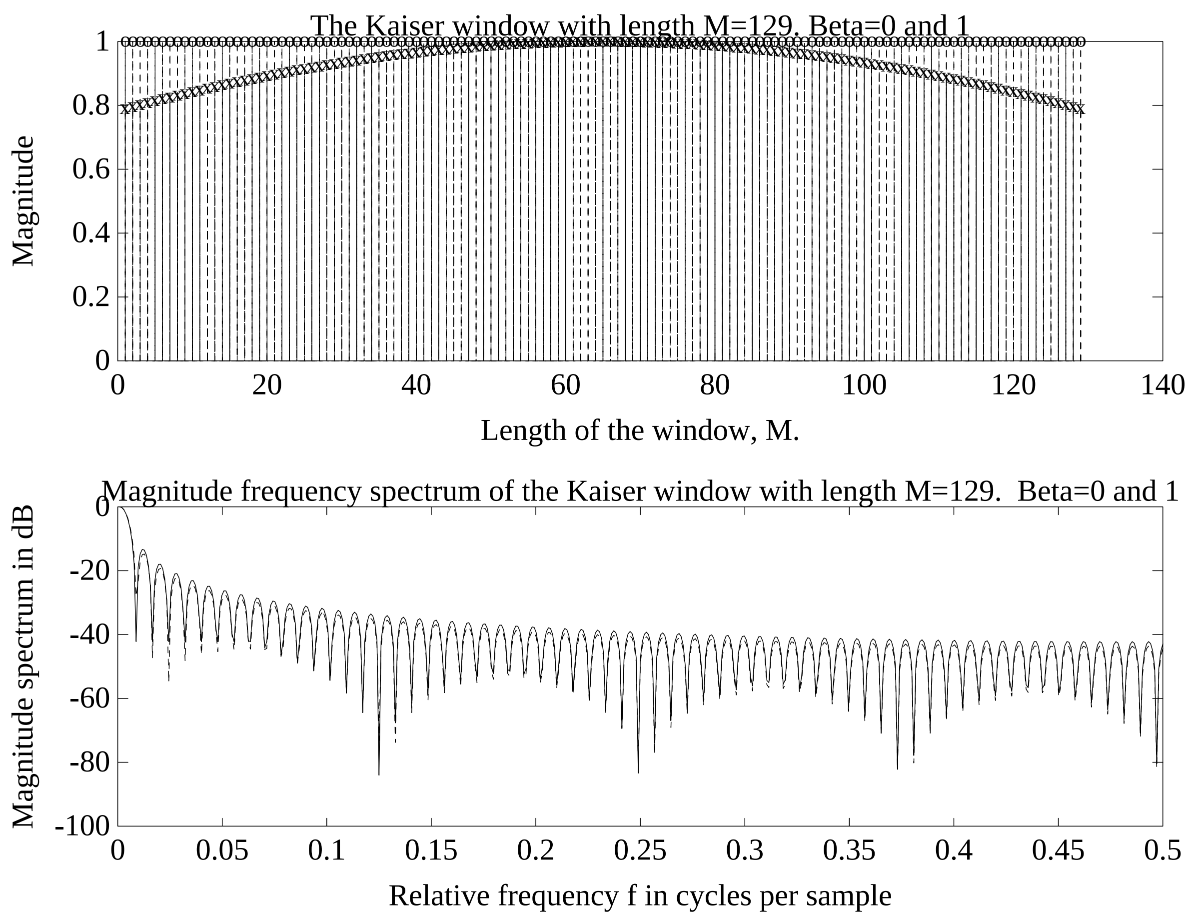


Figure 9.16: Kaiser Window length M=129. $\beta = 0, 1$.

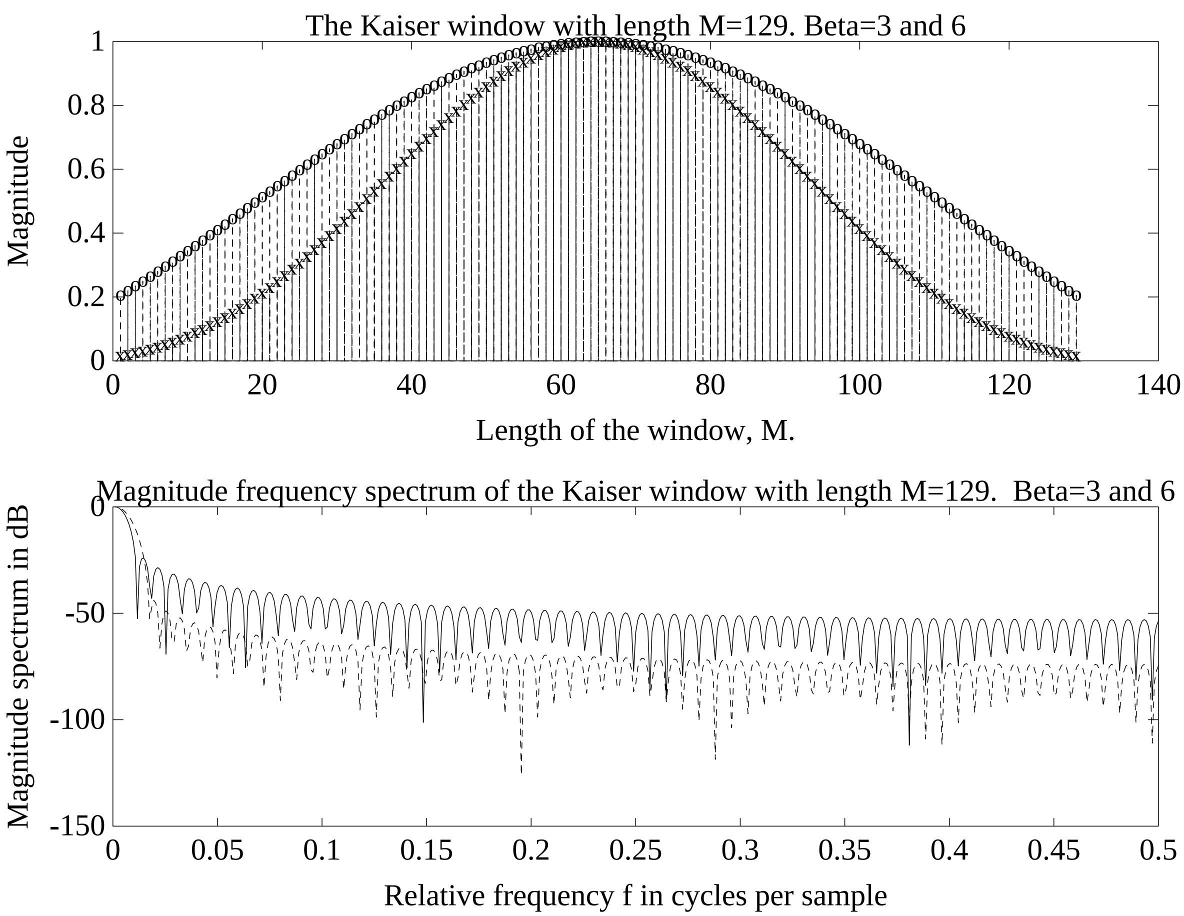


Figure 9.17: Kaiser Window length M=129. $\beta = 3, 6$.

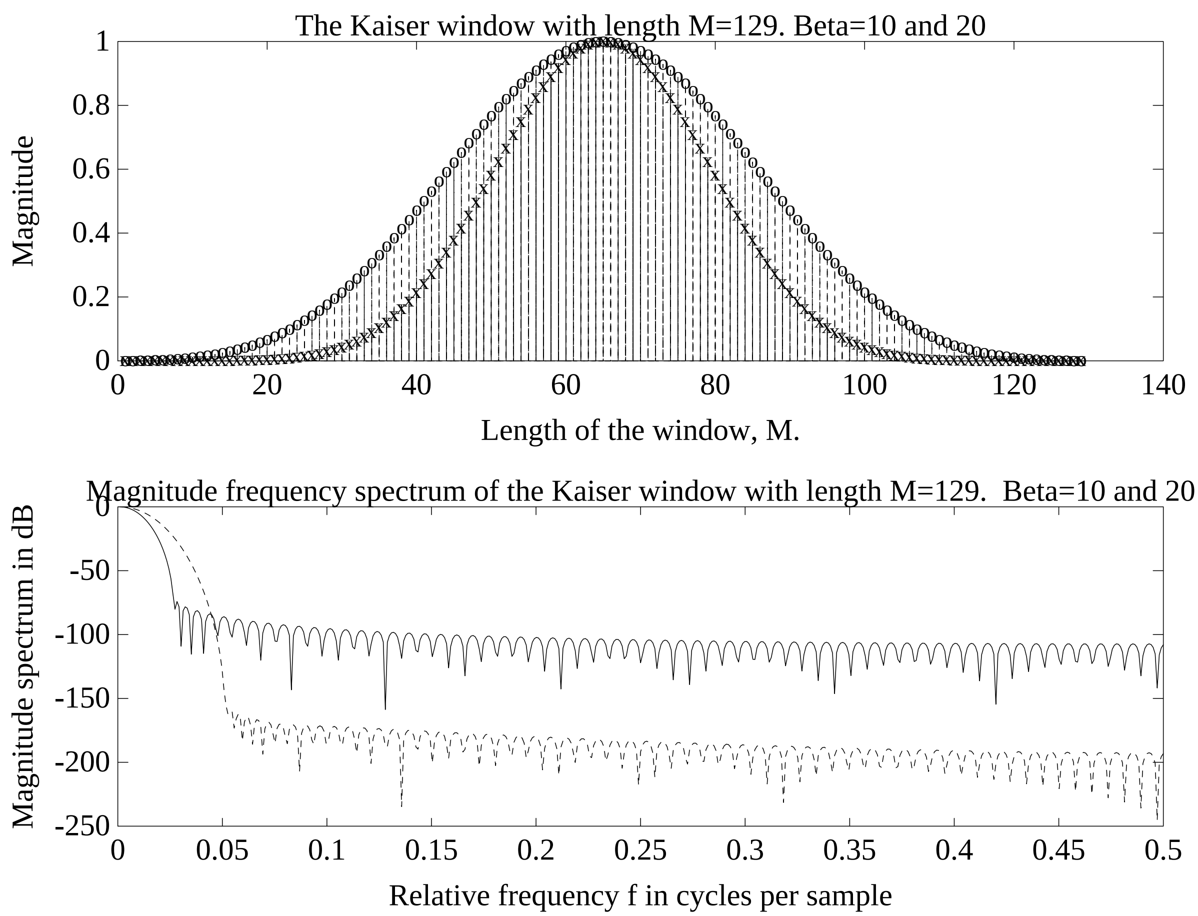


Figure 9.18: Kaiser Window length M=129. $\beta = 10, 20$.

Hanning Window

$$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M) & 0 \leqslant n \leqslant M \\ 0\text{otherwise} \end{cases} \tag{9.3-160}$$

Hamming Window

$$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M) & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-161}$$

Blackman Window

$$w[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) + 0.08\cos(4\pi n/M) & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-162}$$

Kaiser Window

$$w[n] = \begin{cases} \frac{I_0\left[\beta\left(1-[(n-\alpha)/\alpha]^2\right)^{\frac{1}{2}}\right]}{I_0(\beta)} & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-163}$$

where $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind, $\alpha = M/2$, when $\beta = 0$ the Kaiser window reduces to the rectangular window.

Window Type	Peak Sidelobe Amplitude (Relative)	Approximate Width of Mainlobe	Peak Approximation Error $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window β	Transition Width of Equivalent Kaiser Window
Rectangular	-13	$4\pi/(M+1)$	-21	0	$1.81\pi/M$
Bartlett	-25	$8\pi/M$	-25	1.33	$2.37\pi/M$
Hanning	-31	$8\pi/M$	-44	3.86	$5.01\pi/M$
Hamming	-41	$8\pi/M$	-53	4.86	$6.27\pi/M$
Blackman	-57	$12\pi/M$	-74	7.04	$9.19\pi/M$

Table 9.1: Comparison of commonly used windows

All the above mentioned windows are symmetric about the point $M/2$, i.e.,

$$w[n] = \begin{cases} W[M-n] & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-164}$$

and causal as well.

Therefore they possess generalized linear phase response.

$$W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2} \tag{9.3-165}$$

where $W_e(e^{j\omega})$ is a real and even function of ω.

If the desired impulse response $h_d[n]$ is also symmetric about $M/2$

$$h_d[n] = \begin{cases} h_d[M-n] & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-166}$$

then the windowed impulse response is also symmetric about $M/2$.

$$h[n] = h_d[n]w[n] \tag{9.3-167}$$
$$= \begin{cases} h_d[M-n]w[M-n] & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-168}$$
$$= \begin{cases} h[M-n] & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-169}$$

The resulting frequency response will also have generalized linear phase, i.e.,

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2} \tag{9.3-170}$$

where $A_e(e^{j\omega})$ is a real and even function of ω.

If the desired impulse response is antisymmetric about $M/2$,

$$h_d[n] = \begin{cases} -h_d[M-n] & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-171}$$

then the windowed impulse response is also antisymmetric provided that $w[n]$ is symmetric about $M/2$.

$$h[n] = h_d[n]w[n] \tag{9.3-172}$$
$$= \begin{cases} -h_d[M-n]w[M-n] \\ 0 \end{cases} \tag{9.3-173}$$
$$= \begin{cases} -h_d[M-n] & 0 \leqslant n \leqslant M \\ 0 & \text{otherwise} \end{cases} \tag{9.3-174}$$

Example 9.3-3 (FIR approximation of the ideal lowpass filter using windows) *Assume that*

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leqslant \pi \end{cases} \tag{9.3-175}$$

The impulse response is

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega M/2} d\omega \tag{9.3-176}$$
$$= \frac{\sin[\omega_c(n-M/2)]}{\pi(n-M/2)} \qquad \textit{for } -\infty < n < \infty \tag{9.3-177}$$
$$h_{lp}[M-n] = \frac{\sin[\omega_c(M-n-M/2)]}{\pi(M-n-M/2)} \tag{9.3-178}$$
$$= \frac{\sin[\omega_c(M/2-n)]}{\pi(M/2-n)} \tag{9.3-179}$$
$$= \frac{-\sin[\omega_c(M/2-n)]}{-\pi(n-M/2)} \tag{9.3-180}$$
$$= h_{lp}[n] \tag{9.3-181}$$

i.e., $h_{lp}[n]$ is symmetric about $M/2$. When a symmetric window is used the windowed impulse response

$$h[n] = \frac{\sin[\omega_c(n-M/2)]}{\pi(n-M/2)}w[n] \tag{9.3-182}$$

will have a linear phase.

Remark 9.3-2

1. *This method applies accurately when ω_c is not close to zero or to π and when the width of the mainlobe ($\Delta\omega_m$) is smaller than $2\omega_c$.*

2. *The windows with the smaller sidelobes yield better approximations at a discontinuity of the ideal response.*

3. *Increasing M will produce a narrower transition region.*

Example 9.3-4 (FIR approximation of the ideal highpass filter using windows)
The ideal highpass filter with generalized phase is given by

$$H_{hp}(e^{j\omega}) = \begin{cases} 0 & 0 \leqslant |\omega| \leqslant \omega_c \\ e^{-j\omega M/2} & \omega_c < |\omega| \leqslant \pi \end{cases} \tag{9.3-183}$$

$$= e^{-j\omega M/2} - H_{lp}(e^{j\omega}) \tag{9.3-184}$$

The corresponding impulse response is

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{hp}(e^{j\omega})d\omega \tag{9.3-185}$$

$$= \frac{1}{2\pi}\left\{\int_{-\pi}^{-\omega_c} e^{-j\omega M/2}e^{j\omega n}d\omega + \int_{\omega_c}^{\pi} e^{-j\omega M/2}e^{j\omega n}d\omega\right\} \tag{9.3-186}$$

$$= \frac{1}{2\pi j(n-M/2)}\left\{ e^{-j\omega(M/2-n)}\Big|_{-\pi}^{-\omega_c} + e^{-j\omega(M/2-n)}\Big|_{\omega_c}^{\pi}\right\} \tag{9.3-187}$$

$$= \frac{1}{2\pi j(n-M/2)}\left\{e^{-j\omega_c(M/2-n)} - e^{j\pi(M/2-n)} + e^{-j\pi(M/2-n)} - e^{-j\omega_c(M/2-n)}\right\} \tag{9.3-188}$$

$$= \frac{1}{2\pi j(n-M/2)}\left\{-2j\sin[\pi(M/2-n)] + 2j\sin[\omega_c(M/2-n)]\right\} \tag{9.3-189}$$

$$= \frac{1}{\pi(n-M/2)}\left\{\sin[\pi(n-M/2)] - \sin[\omega_c(n-M/2)]\right\} \tag{9.3-190}$$

the windowed impulse response is

$$h[n] = \frac{1}{\pi(n-M/2)}\left\{\sin[\pi(n-M/2)] - \sin[\omega_c(n-M/2)]\right\} w[n] \tag{9.3-191}$$

It is to prove that $h_{hp}[n]$ is symmetric about $M/2$, i.e.,

$$h[n] = \begin{cases} h[M-n] & 0 \leqslant n \leqslant M \\ 0 & \textit{otherwise} \end{cases} \tag{9.3-192}$$

if $w[n]$ is symmetric.

Note that M has to be even since if M is odd, its system function $H(z)$ has a zero at $z = -1$ ($\omega = \pi$).

9.3.5 The Kaiser window filter design method

When the Kaiser Window is used to design FIR filters, a set of empirical formulas can be used to determine the β and the required filter length M.

Given the passband and stopband edge frequencies ω_p and ω_s at which the peak approximation error is δ,

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \leqslant A \leqslant 50 \\ 0.0 & A < 21 \end{cases} \tag{9.3-193}$$

and

$$M = \frac{A-8}{2.285\Delta\omega} \tag{9.3-194}$$

where

$$\Delta\omega = \omega_s - \omega_p \tag{9.3-195}$$

(for lowpass filter) and

$$A = -20\log_{10}\delta \tag{9.3-196}$$

Example 9.3-5 (Design lowpass FIR filter using Kaiser window) *Given the passband edge frequency $\omega_p = 0.4\pi$ at which $\delta_1 = 0.01$ and the stopband edge frequency $\omega_s = 0.6\pi$ at which $\delta_2 = 0.001$, determine β, M and $h[n]$ for the Kaiser windowed FIR filter.*

Solution:

1. *Since FIR filters designed by the window method inherently have $\delta_1 = \delta_2$, we must set $\delta = \delta_2 = 0.001$.*

2. *Due to the symmetry of the approximation at the discontinuity of $H_d(e^{j\omega})$*

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.4\pi + 0.6\pi}{2} = 0.5\pi \tag{9.3-197}$$

and

$$\Delta\omega = \omega_s - \omega_p = 0.2\pi \tag{9.3-198}$$

$$A = -20\log_{10}\delta = 60 \tag{9.3-199}$$

3.

$$\beta = 0.1102(A-8.7) \tag{9.3-200}$$

$$= 5.653 \tag{9.3-201}$$

since $A > 50$

And,

$$M = \frac{60-8}{2.285 \times 0.2\pi} = 37 \tag{9.3-202}$$

4. The impulse response of the filter is obtained as

$$h[n] = h_d[n]w[n] \tag{9.3-203}$$

$$= \begin{cases} \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)} \cdot \frac{I_0\left[\beta\left(1-[(n-\alpha)/\alpha]^2\right)^{\frac{1}{2}}\right]}{I_0(\beta)} & 0 \leqslant n \leqslant M \\ 0 & \textit{otherwise} \end{cases} \tag{9.3-204}$$

where $\alpha = M/2 = 37/2 = 18.5$.

Note that Equation 9.3-204 is a linear phase type II system since $h[n]$ *is symmetric about* $M/2$ *and* M *is odd.*

Chapter 10

The Discrete Fourier Transform

Contents

The discrete Fourier transform is an alternative Fourier representation for finite duration sequences.

10.1 The Discrete Fourier Series Representation of Periodic Sequences

Given a periodic sequence $\tilde{x}[n]$ with period N. i.e.

$$\tilde{x}[n] = \tilde{x}[n + rN] \tag{10.1-1}$$

where r is an integer, the Discrete Fourier Series (DFS) representation $\tilde{X}[k]$ is defined as:

Synthesis equation:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \tag{10.1-2}$$

and

Analysis equation:

$$\tilde{X}[k] = \sum_{k=0}^{N-1} \tilde{x}[n] W_N^{kn} \tag{10.1-3}$$

where $W_n = e^{-j(\frac{2\pi}{N})}$ and the set of N periodic complex exponentials $e_0[n], e_1[n], \ldots, e_{N-1}[n]$ defines all the distinct periodic complex exponentials with frequencies that are integer multiples of $\frac{2\pi}{N}$.

Note $e_k[n] = e^{j(\frac{2\pi}{N})kn} = e_k[n+rN] = e_{k+lN}[n]$ with r, l, k and n being integer values.

It is natural to represent a periodic sequence using a series of harmonics which are also periodic.

$$\tilde{x}[n] = \frac{1}{N} \sum_k \tilde{X}[k] e^{j(2\pi N)kn} \tag{10.1-4}$$

since

$$e_k[n] = e_{k+lN}[n] \tag{10.1-5}$$

i.e. there are only N distinct harmonics or complex exponentials, e_k with $k = 0, 1, \ldots, N-1$, Equation (10.1-4 reduces to

$$)\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(\frac{2\pi}{N})kn} \tag{10.1-6}$$

To determine the Fourier series coefficients $\tilde{X}[k]$ multiply Equation (10.1-6 by $e^{-j(\frac{2\pi}{N})rn}$) and summing n from 0 to $N-1$ to obtain

$$\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(\frac{2\pi}{N})rn} = \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(\frac{2\pi}{N})(k-r)n} \right\} \tag{10.1-7}$$

Thus:

$$\sum_{k=0}^{N-1} \tilde{X}[k] \left\{ \frac{1}{N} \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})(k-r)n} \right\} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(\frac{2\pi}{N})rn} \tag{10.1-8}$$

From the orthogonality of the complex exponentials:

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})(k-r)n} = \begin{cases} 1 & k-r = mN \qquad m \text{ an integer} \\ 0 & \text{otherwise} \end{cases} \tag{10.1-9}$$

and note when $k = 0, \ldots, N-1$, $k - r = 0$ or $k = r$, Equation (10.1-9 becomes

$$)\tilde{X}[r] = \sum n = 0^{N-1} \tilde{x}[n] e^{-j(\frac{2\pi}{N})rn} \tag{10.1-10}$$

or

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(\frac{2\pi}{N})kn} \tag{10.1-11}$$

To prove the orthogonality of the complex exponentials, we first note that

A. if $k - r = mN$,

$$e^{j(\frac{2\pi}{N})(k-r)n} = e^{j(\frac{2\pi}{N})mNn} = ej2\pi mn \tag{10.1-12}$$
$$= (1)^{mn} = 1 \tag{10.1-13}$$

as m and n are integers, thus

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})(k-r)n} = \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1 \tag{10.1-14}$$

for $k - r = mN$.

B. In general

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})ln} = \frac{1}{N}\frac{1 - e^{j(\frac{2\pi}{N})lN}}{1 - e^{j(\frac{2\pi}{N})l}} \tag{10.1-15}$$

$$= \begin{cases} \frac{1}{N} \cdot \frac{-j2\pi e^{j2\pi l}}{-j(\frac{2\pi}{N})e^{j(\frac{2\pi}{N})l}} = \frac{1}{N} \cdot N = 1 & \text{for } l = mN \\ \frac{1}{N} \cdot \frac{1-e^{j2\pi l}}{1-e^{j(\frac{2\pi}{N})l}} = 0 & \text{for } l \neq mN \end{cases} \tag{10.1-16}$$

Replace l by $k - r$ in Equation (10.1-16) resulting in Equation (10.1-9).

Example 10.1-1 *Given a periodic impulse train*

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n + rN] \tag{10.1-17}$$

find its DFS.

Solution:

Since

$$\tilde{x}[n] = \delta[n] \qquad \text{for } 0 \leqslant n \leqslant N - 1 \tag{10.1-18}$$

$\tilde{x}[n]$*, therefore, can be also expressed as*

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n + rN] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-kn} \tag{10.1-19}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} e^{j(\frac{2\pi}{N})kn} \tag{10.1-20}$$

10.1.1 The relation between the discrete-time Fourier transform and the discrete Fourier series

Define

$$x[n] = \begin{cases} \tilde{x}[n] & 0 \leqslant n \leqslant N - 1 \\ 0 & \text{otherwise} \end{cases} \tag{10.1-21}$$

the discrete-time Fourier transform is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\omega n} \tag{10.1-22}$$

It can be seen

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega=2\pi\frac{k}{N}} \tag{10.1-23}$$

i.e. The DFS corresponds to sampling the Fourier transform at N equally spaced frequencies between $\omega = 0$ and $\omega = 2\pi$ with a frequency spaceing of $\frac{2\pi}{N}$.

EXAMPLE 10.1-2 *Given a periodic sequence $\tilde{x}[n]$ as shown in Figure 10.1 find $\tilde{X}[k]$ and $X(e^{j\omega})$ for $\tilde{x}[n]$ and $x[n]$ respectively.*

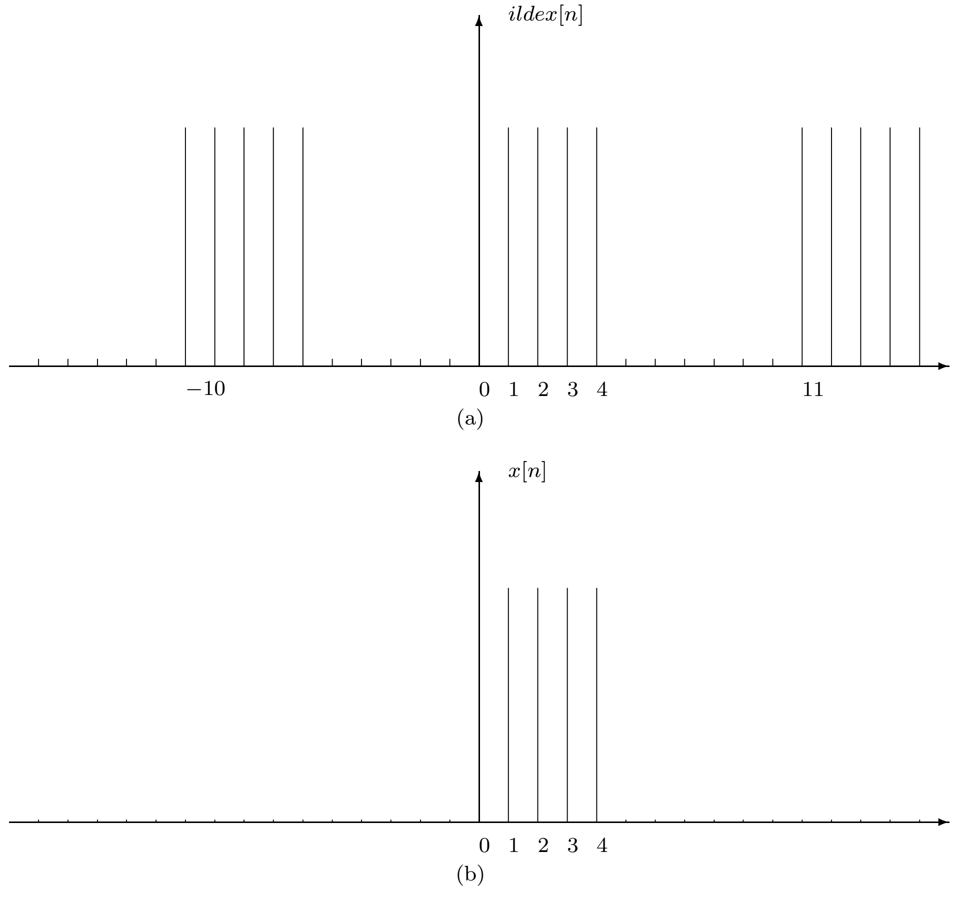


Figure 10.1: Periodic sequence $\tilde{x}[n]$

Solution:

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j(\frac{2\pi}{10})kn} \tag{10.1-24}$$

$$= \frac{1 - W_{10}^{5k}}{1 - W_{10}^{k}} = e^{-j(4\pi\frac{k}{10})}\frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{10})} \tag{10.1-25}$$

and

$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = e^{-j2\omega} \frac{\sin(\frac{5\omega}{2})}{\sin(\frac{\omega}{2})} \tag{10.1-26}$$

or:

$$\tilde{X}[k] = X(e^{j\omega})\big|_{\omega=2\pi\frac{k}{10}} \tag{10.1-27}$$

10.1.2 Periodic Convolution

Given two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, their DFSs are denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$, respectively. The periodic convolution of $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ is defined as

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \tag{10.1-28}$$

The DFS of $\tilde{x}_3[n]$ can be obtained by

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} W_N^{kn} \left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \right) \tag{10.1-29}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m] \sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{k(n'+m)} \tag{10.1-30}$$

$$\overset{n'=n-m}{=} \sum_{m=0}^{N-1} \tilde{x}_1[m] \sum_{n'=-m}^{N-1-m} \tilde{x}_2[n'] W_N^{k(n'+m)} \tag{10.1-31}$$

$$= \left(\sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \right) \left(\sum_{n'=-m}^{N-1-m} \tilde{x}_2[n'] W_N^{kn'} \right) \tag{10.1-32}$$

Note that $\tilde{x}_2[n]$ and W_N^{kn} are periodic with period N.

$$\sum_{n'=-m}^{N-1-m} \tilde{x}_2[n'] W_N^{kn'} = \sum_{n'=-m}^{-1} \tilde{x}_2[n'] W_N^{kn'} + \sum_{n'=0}^{N-1-m} \tilde{X}_2[n'] W_N^{kn'} \tag{10.1-33}$$

$$= \sum_{n'=-m}^{-1} \tilde{x}[n'+N] W_N^{k(n'+N)} + \sum_{n'=0}^{N-1-m} \tilde{x}_2[n'] W_N^{kn'} \tag{10.1-34}$$

$$\overset{n''=n'+N}{=} \sum_{n''=N-m}^{N-1} \tilde{x}_2[n''] W_N^{kn''} + \sum_{n'=0}^{N-1-m} \tilde{x}_2[n'] W_N^{kn'} \tag{10.1-35}$$

$$= \sum_{n=0}^{N-1} \tilde{x}_2[n] W_N^{kn} \tag{10.1-36}$$

Therefore Equation (10.1-32 becomes

$$)\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k] \tag{10.1-37}$$

i.e.

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \overset{DFS}{\leftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k] \tag{10.1-38}$$

REMARK 10.1-1 *1. The convolution sum is over the finite interval* $0 \leqslant m \leqslant N-1$

2. The values of $\tilde{x}_2[n-m]$ *in the interval* $0 \leqslant m \leqslant N-1$ *repeat periodically for* m *outside that interval.*

10.2 Sampling the Fourier Transform

The aim of this section is to establish the general relationship between an aperiodic sequence with Fourier transform $X(e^{j\omega})$ and the periodic sequence for which the DFS coefficients corresponds to samples of $X(e^{j\omega})$ equally spaced in frequency.

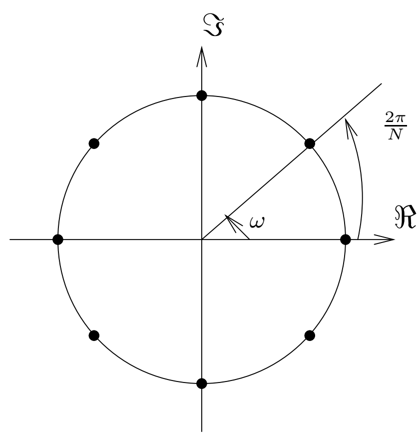
Given an aperiodic sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, we may define

$$\tilde{X}[k] = X(z)\big|_{z=e^{j(\frac{2\pi}{N})k}} \tag{10.2-39}$$

$$= X(e^{j\omega})\big|_{\omega=\frac{2\pi}{N}k} \tag{10.2-40}$$

$$= X(e^{j(\frac{2\pi}{N})k}) \tag{10.2-41}$$

Note Equation (10.2-41 is obtained by sampling $X(e^{j\omega}))$ at $\omega_k = \frac{2\pi}{N}$ and tje Fourier transform is equal to the z-transform evaluated on the unit circle.



The $\tilde{X}[k]$ so defined in Equation (10.2-41) could be seen as the DFS coefficients of a periodic sequence $\tilde{x}[n]$, and

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn} \tag{10.2-42}$$

Since

$$X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \tag{10.2-43}$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left\{X\left(e^{j(\frac{2\pi}{N})k}\right)\right\}W_N^{-kn} \tag{10.2-44}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left\{\sum_{m=-\infty}^{\infty} x[m]e^{-j(\frac{2\pi}{N})k}\right\}W_N^{-kn} \tag{10.2-45}$$

$$= \frac{1}{N}\sum_{m=-\infty}^{\infty} x[m]\left\{\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right\} \tag{10.2-46}$$

Note that

$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)} = \sum_{r=-\infty}^{\infty} \delta[n-m+rN] \tag{10.2-47}$$

Therefore

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m] \sum_{r=-\infty}^{\infty} \delta[n-m+rN] \tag{10.2-48}$$

$$= x[n] * \sum_{r=-\infty}^{\infty} \delta[n+rN] \tag{10.2-49}$$

$$= \sum_{r=-\infty}^{\infty} x[n+rN] \tag{10.2-50}$$

Assume that the length of $X[n]$ is M and the period of $\tilde{x}[n]$ is N. If $M \leqslant N$, $x[n]$ can be extracted from one period of $\tilde{x}[n]$. If $M > N$, then time-domain aliasing will occur.

10.3 The Discrete Fourier Transform of Finite-Duration Sequences

Given a finite-duration sequence $x[n]$, the Discrete Fourier Transform (DFT) analysis and synthesis equations are defined as:

$$\text{Analysis Equation :} \qquad X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \tag{10.3-51}$$

$$\text{Synthesis Equation :} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \tag{10.3-52}$$

where $W_N = e^{-j\frac{2\pi}{N}}$.

Note that $X[k] = 0$ for k outside the interval $0 \leqslant k \leqslant N-1$ and $x[n] = 0$ for n outside the interval $0 \leqslant n \leqslant N-1$.

$$x[n] \overset{DFT}{\leftrightarrow} X[k] \tag{10.3-53}$$

REMARK 10.3-1 *We are only interested in $x[n]$ for $0 \leqslant n \leqslant N-1$ as $x[n]=0$ for values of n outside the interval.*

We are only intertested in $X[k]$ for $0 \leqslant k \leqslant N-1$ as there are the values needed in Equation (10.3-52).

EXAMPLE 10.3-1 *Given a finite-duration sequence $x[n]$ with length N, where $N=5$,*

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN] \tag{10.3-54}$$

$$= \sum_{r=-\infty}^{\infty} x[n+r5] \tag{10.3-55}$$

$$= 1 \tag{10.3-56}$$

The DFS of $\tilde{x}[n]$ is

$$\tilde{X}[k] = \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi k}{N}\right)n} \tag{10.3-57}$$

$$= \frac{1-e^{-j2\pi k}}{1-e^{-j\left(\frac{2\pi k}{N}\right)}} \tag{10.3-58}$$

$$= \begin{cases} N & \text{for } k = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{10.3-59}$$

The DFT of $x[n]$ is

$$X[k] = \sum_{n=0}^{N-1} X[n] W_N^{kn} \tag{10.3-60}$$

$$= \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi k}{N}\right)n} \tag{10.3-61}$$

$$= \begin{cases} N & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{10.3-62}$$

The FT of $x[n]$ is

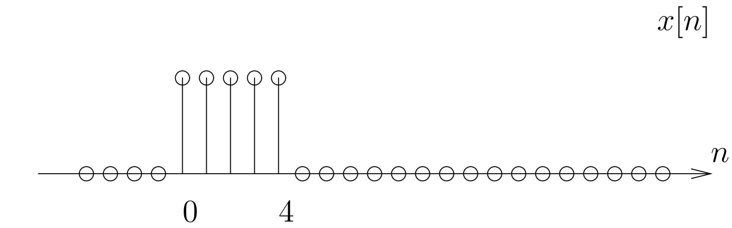
$$X\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \tag{10.3-63}$$

$$= \sum_{n=0}^{N-1} e^{-j\omega n} \tag{10.3-64}$$

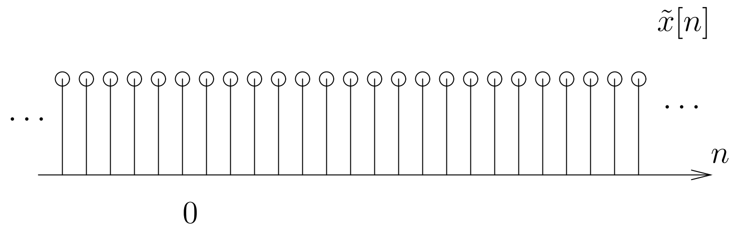
$$= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \tag{10.3-65}$$

$$= e^{-j\omega\left(\frac{N}{2}-\frac{1}{2}\right)} \frac{\sin\left(\omega\frac{N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \tag{10.3-66}$$

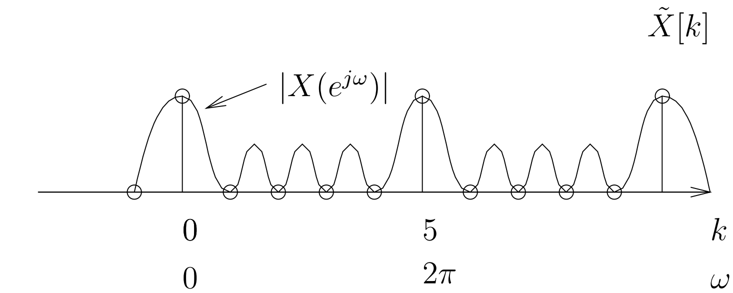
$$\underset{FT}{X\left(e^{j\omega}\right)} \quad \overset{\omega_k=\frac{2\pi k}{N}}{\Rightarrow} \quad \underset{DFS}{\tilde{X}[k]} \quad \overset{\text{one period}}{\Rightarrow} \quad \underset{DFT}{X[k]}$$



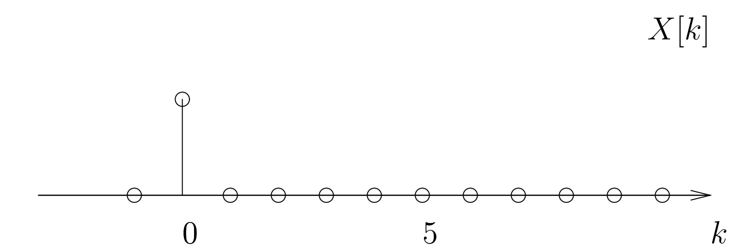
(a) Finite Length Sequence



(b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$



(c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasise that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown.



(d) DFT of $x[n]$

Figure 10.2: Illustration of the DFT

If we assume that $N = 10$ *for* $x[n]$,

$$X\left(e^{j\omega}\right) = \frac{1 - e^{-j\omega 10}}{1 - e^{-j\omega}} \tag{10.3-67}$$

$$= e^{-j\omega 4.5}\frac{\sin(5\omega)}{\sin\left(\frac{\omega}{2}\right)} \tag{10.3-68}$$

$$\tilde{X}[k] = e^{-j\left(\frac{4\pi k}{10}\right)}\frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{10}\right)} \tag{10.3-69}$$

$$X[k] = \begin{cases} \tilde{X}[k] & \text{for } k = 0, 1, \ldots, 9 \\ 0 & \text{otherwise} \end{cases} \tag{10.3-70}$$

10.4 Circular Convolution of Finite-Duration Sequences

Given a finite-duration sequence $x[n]$ with length N, a periodic sequence $\tilde{x}[n]$ may be defined such that

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN] \tag{10.4-71}$$

Since there is no overlap between the terms $X[n + rN]$ for different values of r, Equation (10.4-71) can be written as

$$\tilde{x}[n] = x[((n))_N] \tag{10.4-72}$$

where the notation $((n))_N$ denoted n modulo N.

By the same token

$$\tilde{X}[k] = X[((k))_N] \tag{10.4-73}$$

where $\tilde{X}[k]$ and $X[k]$ are the DFS of $\tilde{x}[n]$ and the DFT of $x[n]$ respectively.

In summary,

$$\tilde{x}[n] = x[((n))_N] \overset{DFS}{\leftrightarrow} \tilde{X}[k] = X[((k))_N] \tag{10.4-74}$$

DEFINITION 10.4-1 *The N-point circular convolution of two finite-duration sequences* $x_1[n]$ *and* $x_2[n]$ *is defined as*

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2]((n - m))_N] \tag{10.4-75}$$

or

$$X_3[n] = X_1[n] \, ⓝ \, X_2[n] \tag{10.4-76}$$

Note that the sequence $x_2[n]$ *is time-reversed and linear shifted in Equation (10.4-76).*

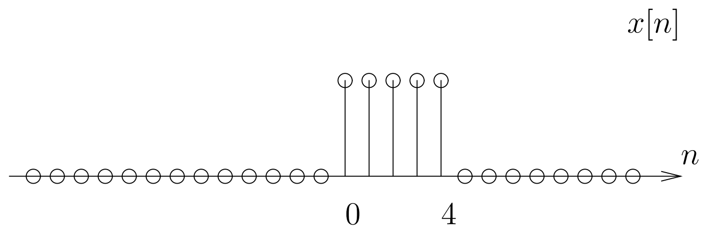
It can be shown that if $X_1[k]$ *and* $X_2[k]$ *are DFTs of* $x_1[n]$ *and* $x_2[n]$ *respectively, the DFT of* $x_3[n]$ *is obtained by*

$$X_3[k] = X_1[k]X_2[k] \tag{10.4-77}$$

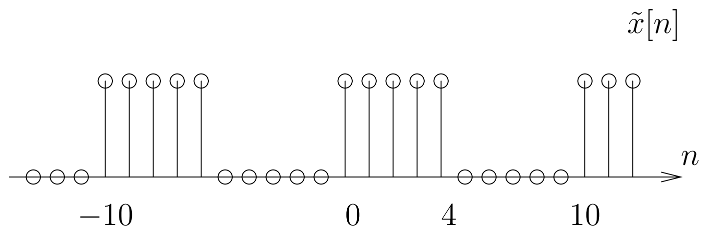
REMARK 10.4-1 *The linear convolution is done for all* m*, while the circular convolution is only done for* $0 \leqslant m \leqslant N - 1$.

EXAMPLE 10.4-1 *Given*

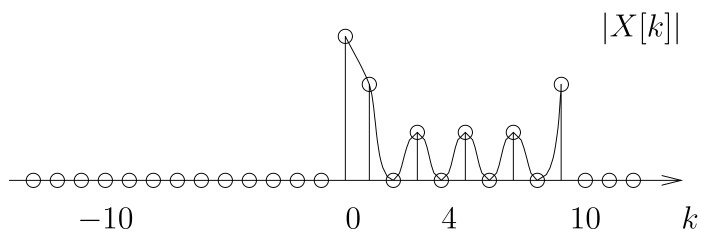
$$X_1[n] = X_2[n] = \begin{cases} 1 & 0 \leqslant n \leqslant L - 1 \\ 0 & \text{otherwise} \end{cases} \tag{10.4-78}$$



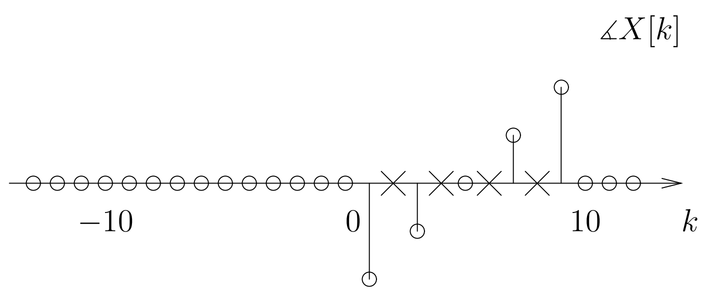
(a) Finite Length Sequence



(b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$



(c) DFT magnitude.



(d) DFT phase. ($\times$ indicate indeterminate value)

Figure 10.3: Illustration of the DFT

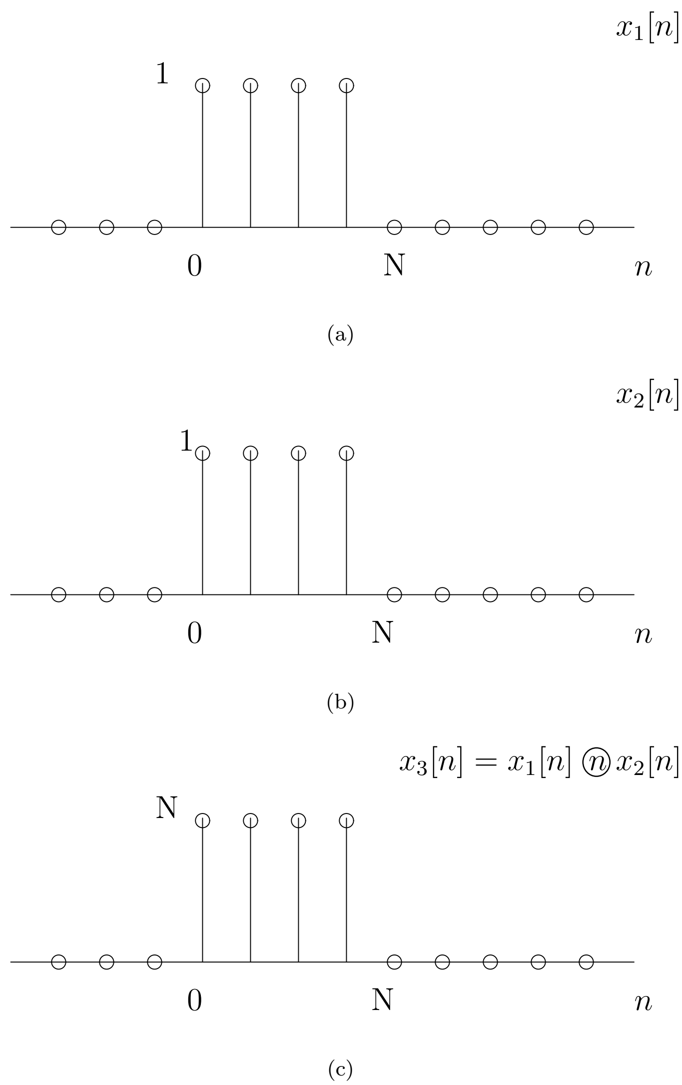


Figure 10.4: N-point circular convolution of two constant sequences of length N

1. *if $N = L$, find*

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

2. *if $N = 2L$, find*

$$x_3[n]$$

Solution:

1. *When $N = L$, the N-point DFTs are*

$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} W_N^{kn} \tag{10.4-79}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi k}{N}n} \tag{10.4-80}$$

$$= \begin{cases} N & k = 0 \\ 0 & otherwise \end{cases} \tag{10.4-81}$$

Therefore

$$X_3[k] = X_1[k]X_2[k] \tag{10.4-82}$$

$$= \begin{cases} N^2 & k = 0 \\ 0 & otherwise \end{cases} \tag{10.4-83}$$

where $X_3[k]$ is the DFT of $x_3[n]$.

$$x_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k]W_N^{-kn} \tag{10.4-84}$$

$$= \frac{1}{N}N^2 W_N^0 = N \qquad 0 \leqslant n \leqslant N-1 \tag{10.4-85}$$

2. *When $N = 2L$, $x_1[n]$ and $x_2[n]$ are augmented by L zeros. The result is shown in Figure 10.5.*

From the previous Example 10.4-1, it can also be seen that whether a circular convolution corresponding to the product of two N-point DFTs is the same as the linear convolution of the corresponding finite-length sequences depends on the length of the DFT in relation to the length of the two given finite-length sequences.

Remark 10.4-2 *Given*

$$x_{3l}[n] = x_1[n] * x_2[n] \overset{FT}{\leftrightarrow} X_3\left(e^{j\omega}\right) = X_1\left(e^{j\omega}\right) X_2\left(e^{j\omega}\right) \tag{10.4-86}$$

and

$$x_3[n] = x_1[n] \, ⓝ \, x_2[n] \overset{DFT}{\leftrightarrow} X_3[k] = X_1[k]X_2[k] \tag{10.4-87}$$

if the length of the DFTs N satisfies

$$N \geqslant L + P * 1 \qquad x_3[n] = x_{3l}[n]$$

Otherwise time aliasing in the circular onvolution may occur.

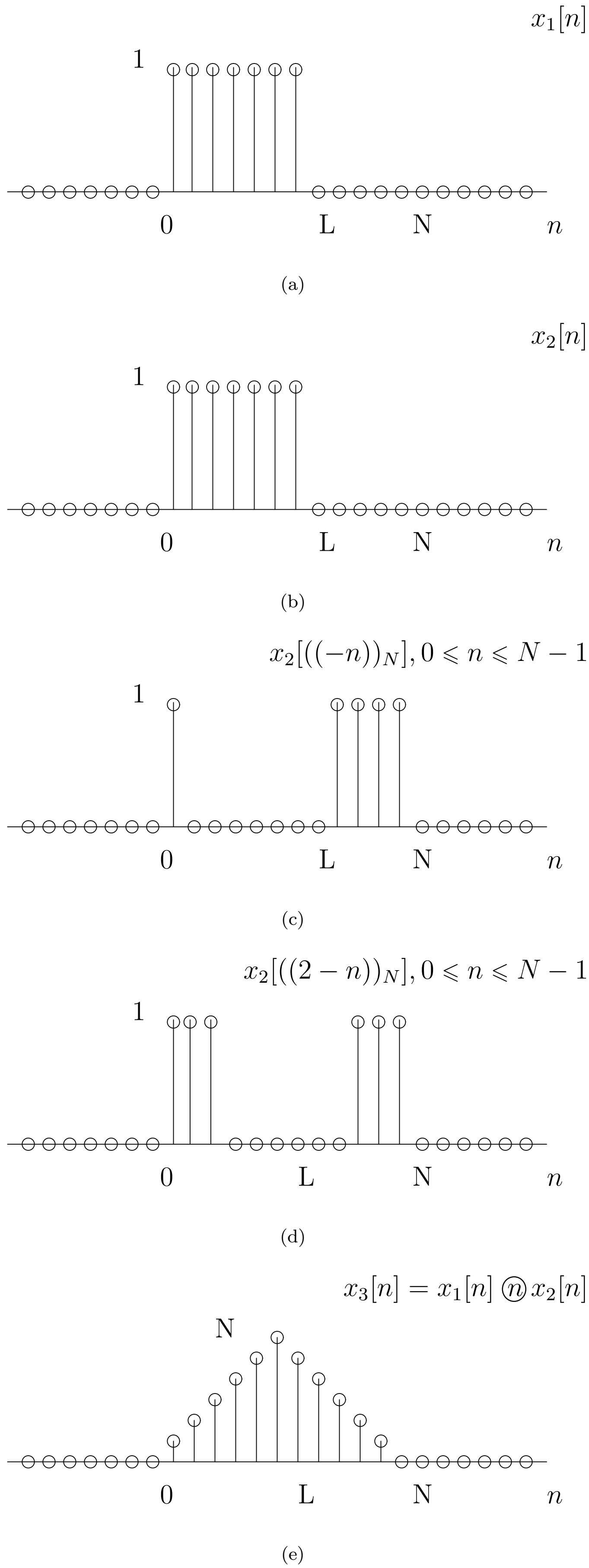
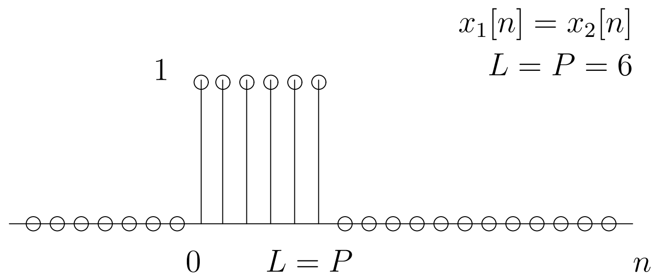
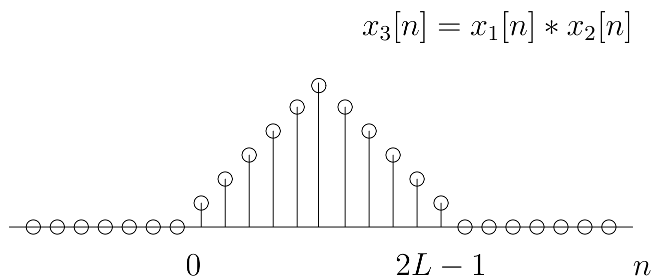


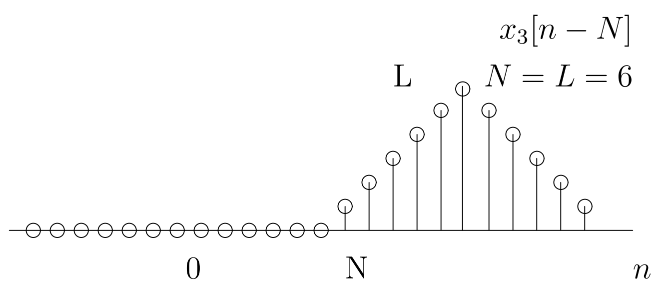
Figure 10.5: $2L$-point circular convolution of two constant sequences of length L



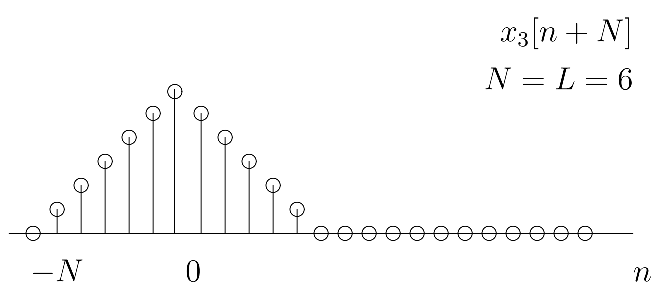
(a) The sequences $x_1[n]$ and $x_2[n]$ to be convolved



(b) The linear convolution of $x_1[n]$ and $x_2[n]$.

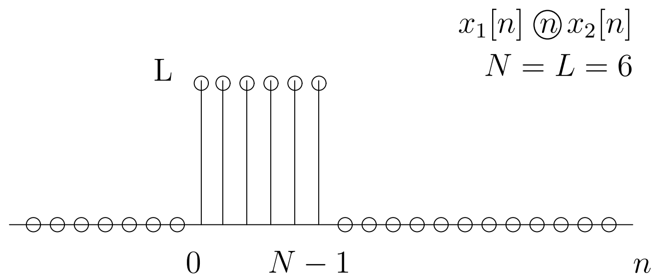


(c) $x_3[n - N]$ for $N = 6$

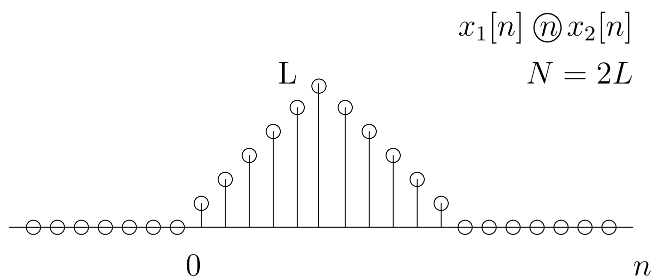


(d) $x_3[n + N]$ for $N = 6$

Figure 10.6: Illustration that circular convolution is equivalent to linear convolution followed by aliasing



(e) $x_1[n]$ ⑥ $x_2[n]$, which is equal to the sum of (b), (c) and (d) in the interval $0 \leqslant n \leqslant 5$.



(f) $x_1[n]$ ⑫ $x_2[n]$.

Figure 10.7: Illustration that circular convolution is equivalent to linear convolution followed by aliasing continued

10.5 Linear Convolution Using the DFT

To implement a linear time-invariant system, a linear convolution is normally required.

We have seen that

$$X_3[k] = X_1[k]X_2[k] \overset{DFT}{\Leftrightarrow} x_3[n] = x_1[n] \textcircled{n} x_2[n] \tag{10.5-88}$$

To obtain a linear convolution, we have to ensure that a circular convolution has the effect of linear convolution.

10.5.1 Linear convolution of two finite-length sequences

Given a sequence $x_1[n]$ with length L and another $x_2[n]$ with length P, the linear convolution of $x_1[n]$ and $x_2[n]$ is defined as

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] \tag{10.5-89}$$

It is obvious that the product

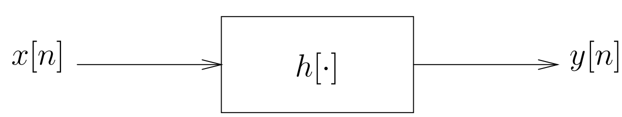
$$X_1[m]X_2[n-m]$$

is zero for all m whenever $n < 0$ and $n > L+P-2$. That is $(L+P-1)$ is the maximum length of $x_3[n]$.

10.6 Implementing Linear Time-Invariant Systems Using DFT

Since linear time-invariant systems can be implemented by convolution, circular convolution can be used to implement the systems.

Given an input sequence $x[n]$ with length L and a P-point impulse response $h[n]$, the system response $y[n]$ has a length $(L+P-1)$.



In this case for $\textcircled{n}$ and $*$ operations to be identical, the length of $\textcircled{n}$ must be at least $(L+P-1)$, i.e. the length of DFTs must be $(L+P-1)$ at least.

To compute $(L+P-1)$-point DFTs, bith $x[n]$ and $h[n]$ must be augmented using zero padding. ($x[n]$, $P-1$ zeros; $h[n]$, $L-1$ zeros).

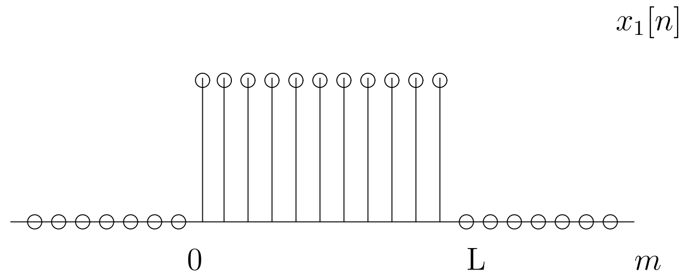
10.6.1 Block convolution

Given the impulse response $h[n]$ with length P, assume $x[n]$ is causal and has very long length,

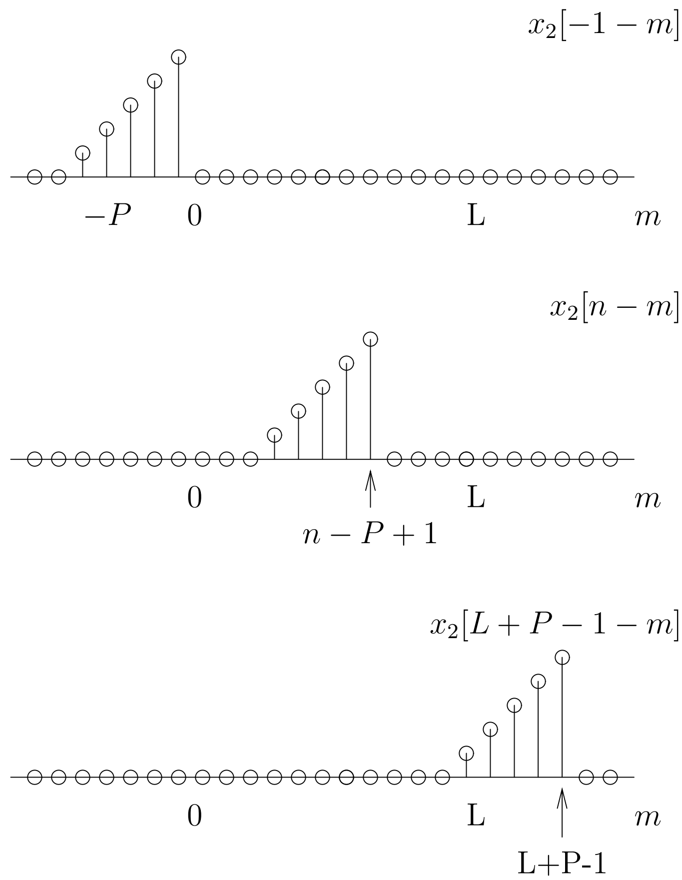
$$x[n] = \sum_{r=0}^{\infty} x_r[n-rL] \tag{10.6-90}$$

where

$$x_r[n] = \begin{cases} x[n+rL] & 0 \leqslant n \leqslant L-1 \\ 0 & \text{otherwise} \end{cases} \tag{10.6-91}$$



(a) Finite-length sequence $x_1[n]$



(b) $x_2[n-m]$ for several values of n

Figure 10.8: Example of linear convolution of two finite-length sequences showing that the result is such that $x_3[n] = 0$ for $n \leqslant 1$ and for $n \geqslant L+P-1$

Because convolution is a linear time-invariant operation it follows that

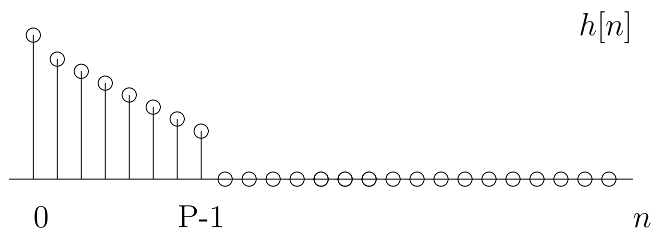
$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} y_r[n - rL] \tag{10.6-92}$$

where

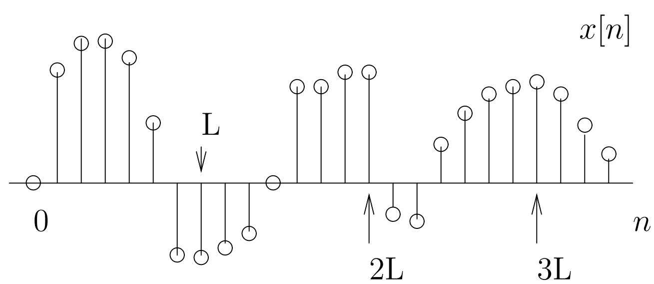
$$y_r[n] = x_r[n] * h[n] \tag{10.6-93}$$

with length $(L + P - 1)$.

The linear convolution can be computed using N-point DFTs where $N \geqslant L + P - 1$..



(a) Finite length impulse response $h[n]$



(b) Indefinite length signal $x[n]$ to be filtered

Figure 10.9: Example signal for convolution

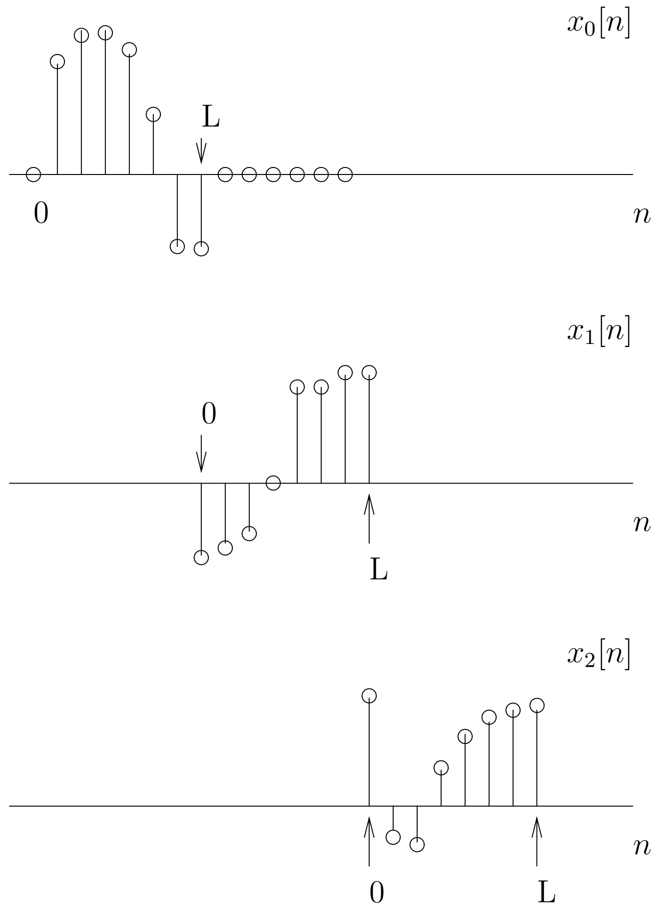
Since the nonzero points in the filtered sections overlaps by $(P - 1)$ points and these overlap samples must be added to obtain $y[n]$. Therefore this method of block linear convolution is called *the overlap-add method*.

10.6.2 The overlap-save method

From the discussion on the circular convolution, it can be shown that if an L-point sequence is circularly convolved with a P-points sequence $(P < L)$, then the first $(P - 1)$ points of the results are "incorrect" while the remaining points are identical to those that would be obtained had we implemented a linear convolution.

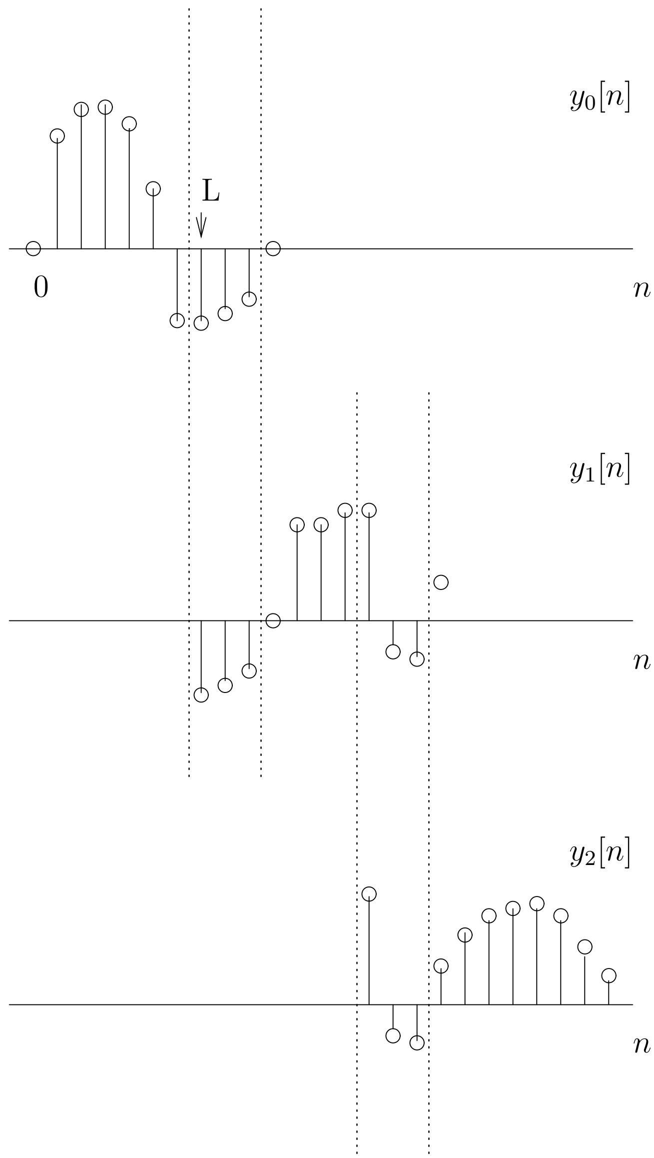
If $x[n]$ is divided into sections with length L so that the input sections overlaps the preceeding section by $(P - 1)$ points, i.e.,

$$x[n] = \sum_{r=0}^{\infty} x[n + r(L - P + 1) - P + 1] \qquad 0 \leqslant n \leqslant L - 1 \tag{10.6-94}$$



(c) Decomposition of $x[n]$ in Figure 10.9 into non-overlapping sections of length L

Figure 10.10: Example of the overlap-add method



(a) Result of convolving each section with $h[n]$

Figure 10.11: Example of the overlap-add method continued

with $(P-1)$ zeros added to the beginning of the original sequence, then

$$y[n] = \sum_{r=0}^{\infty} y_r[n - r(L-P+1) + P - 1] \tag{10.6-95}$$

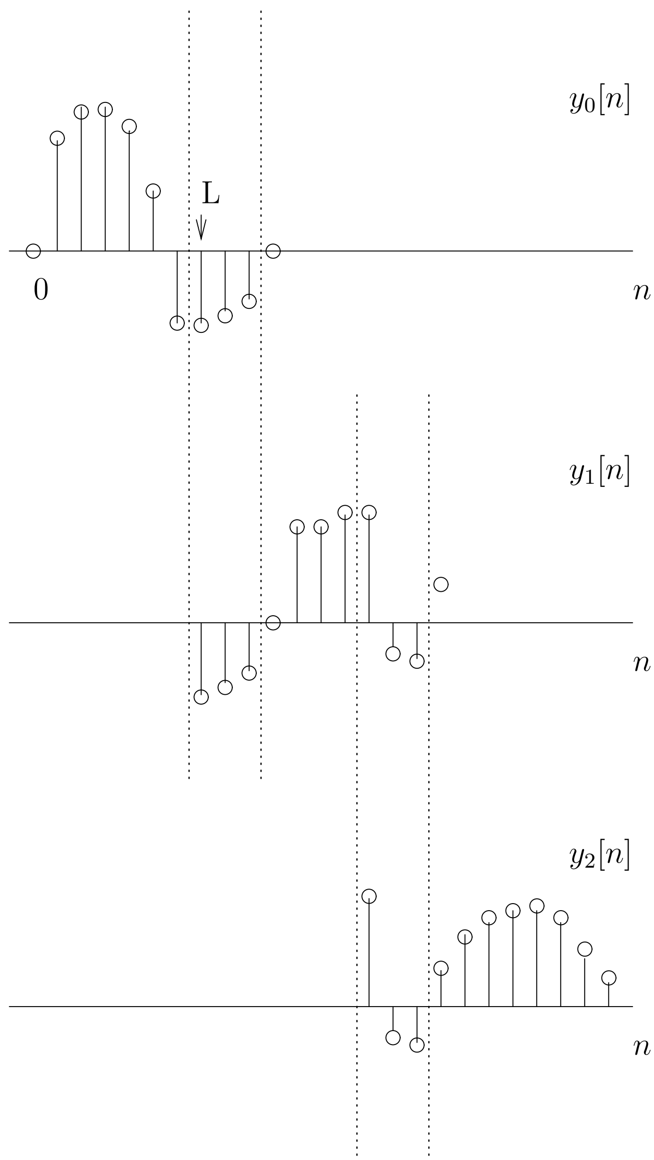
where

$$y_r[n] = \begin{cases} y_{rp}[n] & P-1 \leqslant n \leqslant L-1 \\ 0 & \text{otherwise} \end{cases} \tag{10.6-96}$$

and

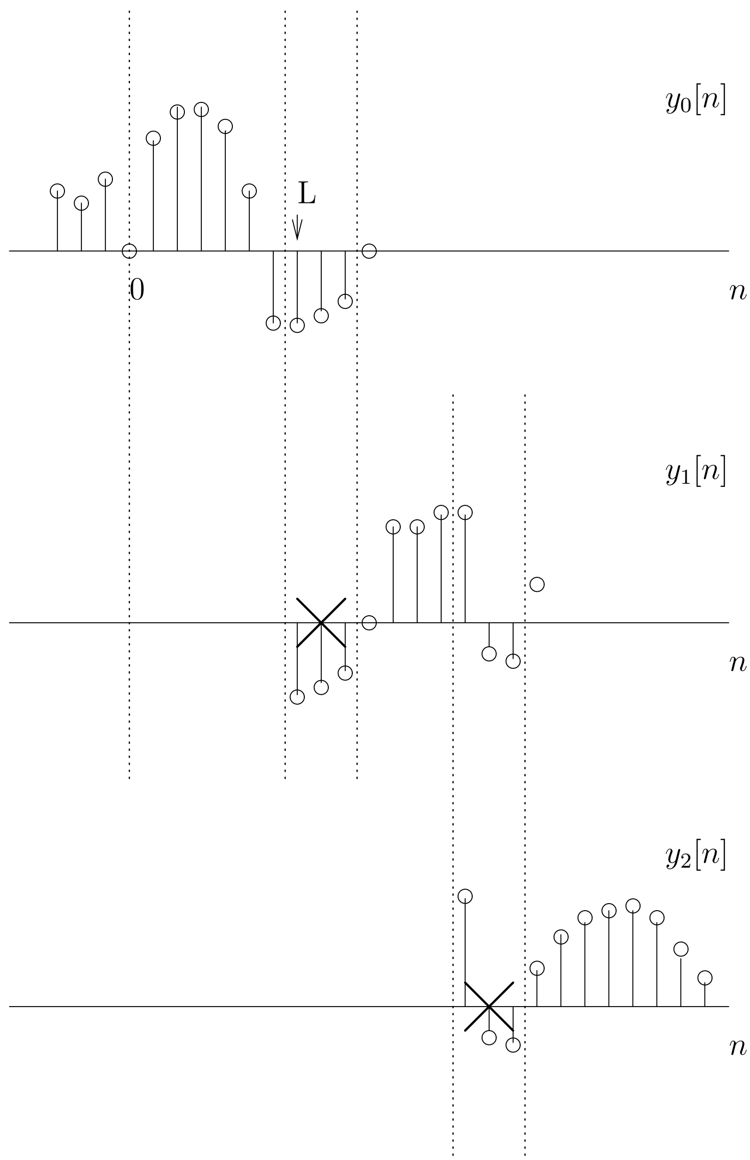
$$y_{rp}[n] = x_r[n] \; n \; h[n] \tag{10.6-97}$$

The output section $0 \leqslant n \leqslant P-2$ (first $(P-1)$ points) must be discarded (or "saved"). Therefire the name of the method is *the overlap-save method*.



(a) Decomposition of $x[n]$ of Figure 10.9 into overlapping sections of length L.

Figure 10.12: Example of the overlap-save method



(b) Result of convolving each section with $h[n]$. The portions of each filtered section to be discarded in forming the linear convolution are indicated.

Figure 10.13: Example of the overlap-save method continued

Chapter 11

Computation of Discrete Fourier Transforms

Contents

"It has been said that the rediscovery of the Fast Fourier Transform (FFT) algorithm was one of the saviours of the IEEE Signal Processing Society and marked the beginning of modern digital signal processing."

11.1 Introduction to Fast Algorithms

An algorithm, like most engineering devices, can be described either by an input/output relationship or a detailed explanation of its internal construction.

Algorithms for fast computations related to digital signal processing is our main concern.

Fast algorithms have been designed for computations of:

- Digital Filtering
- DFTs
- Correlation
- Spectral Analysis etc.

Why do we need fast algorithms?

EXAMPLE 11.1-1 *Relative performance of some two dimensional Fourier transform algorithms are shown in Table 11.1.*

Algorithm	Multiplications/Pixel	Additions/Pixel
Discrete Computation of Discrete Fourier Transform 1000×1000	8,000	4,000
Basic Cooley-Tukey FFT 1024×1024	40	60
Hybrid Cooley-Tukey/ Winograd FFT 1000×1000	40	72.8
Winograd FFT 1008×1008	6.2	91.6
Nussbaumer-Quandalle FFT 1008×1008	4.1	79

Table 11.1: Relative performance of some two-dimensional Fourier Transform Algorithms

DEFINITION 11.1-1 (FAST ALGORITHMS) *A fast algorithm is a detailed description of a computation procedure that is not the obvious way to compute the required output from the input.*

A fast algorithm usually gives up a conceptually clear computation in favour of one that is computationally efficient.

EXAMPLE 11.1-2 *Compute a number A given by*

$$A = ac + ad + bc + bd \tag{11.1-1}$$

A direct implementation of Equation 11.1-1 requires 4 multiplications (mult.) and 3 additions (adds.)

If we need to compute A many times with different sets of date, an equivalent form can be found to reduce the number of multiplications and additions.

$$A = (a + b)(c + d) \tag{11.1-2}$$

Equation 11.1-2 only requires 1 multiplication and 2 additions.

EXAMPLE 11.1-3 *Compute the complex product*

$$(e + jf) = (a + jb)(c + jd) \tag{11.1-3}$$

It can be written in terms of real multiplications and additions

$$e = (ac - bd) \tag{11.1-4}$$
$$f = (ad + bc) \tag{11.1-5}$$

According to Equation 11.1-4 and 11.1-5 we need 4 multiplications and 2 additions to obtain 1 complex multiplication

A more efficient "algorithm" is

$$e = (a - b)d + a(c - d) \tag{11.1-6}$$
$$f = (a - b)d + b(c + d) \tag{11.1-7}$$

whenever multiplication is harder than addition. It requires 3 multiplications and 5 additions.

• *Furthermore, if the complex number $(c + jd)$ is constant or c and d are constant for a series of complex multiplications such as FFT, the terms $(c - d)$ and $(c + d)$ are constants as well and they can be computed off-line. In this case using Equation 11.1-6 and 11.1-7, we only need 3 real multiplications and 3 real additions to compute one complex multiplication.*

We have traded 1 multiplication for 1 addition.

EXAMPLE 11.1-4 (FAST ALGORITHMS AND MATRIX DECOMPOSITION) *A complex multiplication represented by*

$$(e + jf) = (a + jb)(c + jd) \tag{11.1-8}$$

can be written as a matrix product

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{11.1-9}$$

where the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ *represents the complex number $(a + jb)$, the matrix* $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ *represents the complex number $(c + jd)$ and the vector* $\begin{bmatrix} e \\ f \end{bmatrix}$ *represents the complex number $(e + jf)$.*

The "fast algorithm" in the previous example can be expressed in matrix form as

$$\begin{bmatrix} e \\ f \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{post\ additions} \underbrace{\begin{bmatrix} (c-d) & 0 & 0 \\ 0 & (c+d) & 0 \\ 0 & 0 & d \end{bmatrix}}_{\substack{diagonal\ matrix \\ (multiplications)}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}}_{preadditions} \begin{bmatrix} a \\ b \end{bmatrix} \tag{11.1-10}$$

The algorithm can be thought of as nothing more than the usual matrix factorization:

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} (c-d) & 0 & 0 \\ 0 & (c+d) & 0 \\ 0 & 0 & d \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}}_{A} \tag{11.1-11}$$

and

$$\begin{bmatrix} e \\ f \end{bmatrix} = BDA \begin{bmatrix} a \\ b \end{bmatrix} \tag{11.1-12}$$

REMARK 11.1-1 *Many of the best computational procedures for convolution and for the discrete Fourier transform can be put into this form. These fast algorithms will have the structure of a batch of additions followed by a batch of multiplications followed by another batch of additions.*

11.2 Efficient Computation of the DFT

The DFT of a finite-length sequence of length N is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \tag{11.2-13}$$

where $W_n = e^{-j(2\pi/N)}$. The inverse DFT is defined as

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \qquad n = 0, 1, \ldots, N-1 \tag{11.2-14}$$

Both $x[n]$ and $X[k]$ may be complex. The only difference between Equation 11.2-13 and 11.2-14 are the signs of the exponent of W_N and the scale factor

$$x[n] = \left(\frac{1}{N}\sum_{k=0}^{N-1} X^*[k]W_N^{kn}\right)^* \tag{11.2-15}$$

The computational procedures for Equation 11.2-13 applies to Equation 11.2-14 with modifications and vice versa.

Note:

$$\begin{aligned} W_N &= e^{-j(2\pi/N)} & (11.2\text{-}16)\\ &= \cos(2\pi/N) - j\sin(2\pi/N) & (11.2\text{-}17)\\ (W_N)^* &= \left\{e^{-j(2\pi/N)}\right\}^* & (11.2\text{-}18)\\ &= \{\cos(2\pi/N) - j\sin(2\pi/N)\}^* & (11.2\text{-}19)\\ &= \cos(2\pi/N) + j\sin(2\pi/N) & (11.2\text{-}20)\\ &= W_N^{-1} & (11.2\text{-}21)\end{aligned}$$

11.2.1 Direct evaluation of the DFT

For each value $X[k]$ of the DFT, N complex multiplications and $(N-1)$ complex additions are required for the computation. Thus, to compute all N values of the DFT requires N^2 complex multiplications and $N(N-1)$ complex additions.

In terms of real number operations we need $4N^2$ real multiplications and $N(4N-2)$ real additions, Since

$$\begin{aligned} X[k] = \sum_{n=0}^{N-1} \big[&\left(\Re\{x[n]\}\Re\{W_N^{kn}\} - \Im\{x[n]\}\Im\{W_N^{kn}\}\right) \\ &+ j\left(\Re\{x[n]\}\Im\{W_N^{kn}\} + \Im\{x[n]\}\Re\{W_N^{kn}\}\right)\big] \qquad k = 0, 1, \ldots, N-1 \end{aligned} \tag{11.2-22}$$

(Note: 1 complex multiplication $\Rightarrow$ 4 real multiplications and 2 real additions and 1 complex addition $\Rightarrow$ 2 real additions.)

The amount of computation is approximately proportional to N^2 or on the order $O(N^2)$.

Most fast DFT algorithms exploit the symmetry and periodicity properties of W_N^{kn}.

Complex conjugate symmetry $W_N^{k[N-n]} = W_N^{-kn} = \left(W_N^{kn}\right)^*$

Periodicity in n and k $W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$

For example, in Equation 11.2-22

$$\begin{aligned} &\Re\{x[n]\}\Re\{W_N^{kn}\} + \Re\{x[N-n]\}\Re\left\{W_N^{k[N-n]}\right\} \\ &= \left(\Re\{x[n]\} + \Re\{x[N-n]\}\right)\Re\{W_N^{kn}\} \end{aligned} \tag{11.2-23}$$

Note:

$$\Re\{W_N^{kn}\} = \Re\left\{\left(W_N^{kn}\right)^*\right\} \tag{11.2-24}$$

and symmetry property is used.

$$\begin{aligned} & -\Im\{x[n]\}\Im\{W_N^{kn}\} - \Im\{x[N-n]\}\Im\left\{W_N^{k[N-n]}\right\} \\ =& -\left(\Im\{x[n]\} + \Im\{x[N-n]\}\right)\Im\{W_N^{kn}\} \end{aligned} \tag{11.2-25}$$

Also note that in W_N^{kn}, some values of the product kn will result in sin and cos functions taking on value 1 or 0, i.e., $W_N^{kn} = \pm 1$ or $\pm j$. In these cases, we do not need multiplications at all.

11.3 The Goertzel Algorithm

The Goertzel algorithm for the computation of the DFT is derived using the periodicity of the sequence W_N^{kn} to reduce computation.

Note that

$$W_N^{-kN} = e^{j(2\pi/N)Nk} = e^{j2\pi k} = 1 \tag{11.3-26}$$

$$X[k] = \underbrace{W_N^{-k_N N-1}}_{1} \sum_{r=0}^{N-1} x[r] W_N^{kr} = \sum_{r=0}^{N-1} x[r] W_N^{-k(N-r)} \tag{11.3-27}$$

Equation 11.3-27 is equivalent to the DFT definition. If we define the sequence:

$$y_k[n] = \sum_{r=-\infty}^{\infty} x[r] W_N^{-k(n-r)} u[n-r] \tag{11.3-28}$$

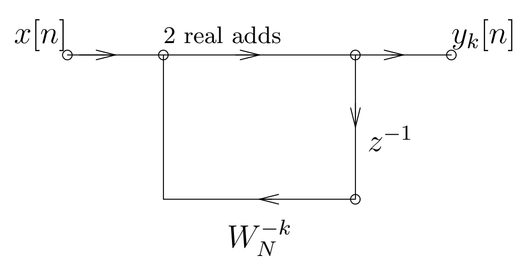
it follows that

$$X[k] = y_k[n]\big|_{n=N} \tag{11.3-29}$$

as $x[n] = 0$ for $n < 0$ and $n \geqslant N$.

Equation 11.3-28 can be interpreted as a discrete convolution of $x[n]$, for $0 \leqslant n \leqslant N-1$ with $W_N^{-kn} u[n]$.

Consequently, $y_k[n]$ is the response of a system with $h[n] = W_N^{-kn} u[n]$ to a finite length input sequence $x[n]$. And $X[k]$ is the value of the output when $n = N$.



4 mults+2 adds (real)

To compute $X[k] = y_k[N]$ we need $4N$ real multiplications and $4N$ real additions. To obtain

$$X[k] \qquad \text{for } k = 0, 1, \ldots, N-1 \tag{11.3-30}$$

we need $4N^2$ real multiplications and $4N^2$ real additions.

The system function is given by

$$H_k[z] = \frac{1}{1 - W_N^{-k} z^{-1}} \tag{11.3-31}$$

To achieve the fast algorithm, we note

$$H_z[z] = \frac{1 - W_N^k z^{-1}}{(1 - W_N^{-k} z^{-1})(1 - W_N^k z^{-1})} \tag{11.3-32}$$

$$H_k[z] = \frac{1 - W_N^k z^{-1}}{1 - 2\cos(2\pi k/N) z^{-1} + z^{-2}} \tag{11.3-33}$$

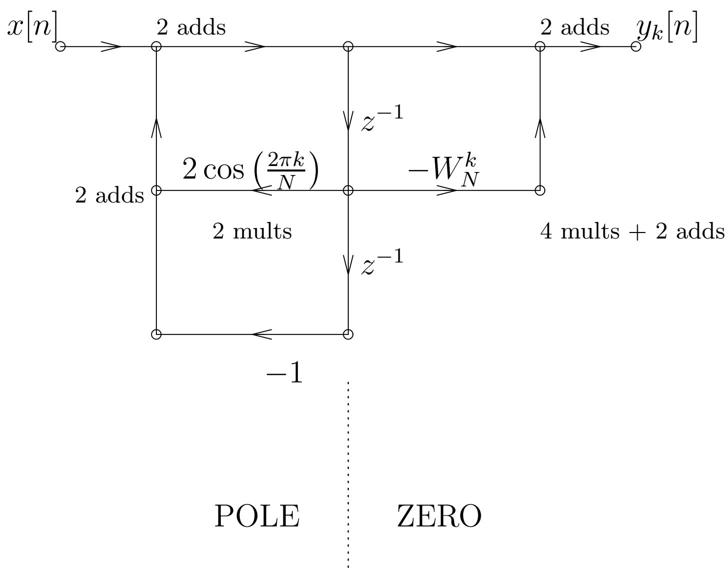


Figure 11.1: SFG of DFT

$2N$ real multiplications and $4N$ additions for poles, and 4 real multiplications and 4 additions for the zero as shown in Figure 11.1.

A total of $2(N+2)$ real multiplications and $4(N+1)$ real additions for the computation of $X[k]$.

Therefore, for the computation of all N values of the DFT using the Goertzel algorithm, we need $2N^2$ real additions and N^2 real multiplications.

11.4 Decimation-in-Time FFT Algorithms

Definition 11.4-1 *Algorithms in which the decomposition of the DFT computation is based on decomposing the sequence $x[n]$ into successively smaller subsequences, are called decimation-in-time FFT algorithms.*

Assume that the 1-D N-points DFT $X[k]$ is defined as

$$X[k] = \sum_{n=0}^{N-1} s[n] W_N^{nk} \tag{11.4-34}$$

where

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}, \quad N = 2^\nu, \quad k = 0, 1, \ldots, N-1 \tag{11.4-35}$$

To separate $x[n]$ into its even-odd numbered points so that

$$X[k] = \sum_{\text{n even}} x[n]W_N^{nk} + \sum_{\text{n odd}} x[n]W_N^{nk} \tag{11.4-36}$$

set

$$n = n_1 \times 2 + n_2 \qquad \text{(decimation with radix-2)} \tag{11.4-37}$$

where $n_1 = 0, 1, \ldots, N/2 - 1$ and $n_2 = 0, 1$.

Substitute Equation 11.4-37 into Equation 11.4-34

$$X[k] = \sum_{n_1=0}^{N/2-1} \sum_{n_2=0}^{1} x[n_1 \times 2 + n_2]W_N^{(n_1 \times 2 + n_2)k} \tag{11.4-38}$$

$$= \sum_{n_1=0}^{N/2-1} x[2n_1]W_N^{2n_1 k} + \sum_{n_1=0}^{N/2-1} x[2n_1+1]W_N^{(2n_1+1)k} \tag{11.4-39}$$

$$X[k] = \sum_{n_1=0}^{N/2-1} x[2n_1]\left(W_N^2\right)^{n_1 k} + W_N^k \sum_{n_1=0}^{N/2-1} x[2n_1+1]\left(W_N^2\right)^{n_1 k} \tag{11.4-40}$$

$$= \sum_{n_1=0}^{N/2-1} x[2n_1]W_{N/2}^{n_1 k} + W_N^k \sum_{n_1=0}^{N/2-1} x[2n_1+1]W_{N/2}^{n_1 k} \tag{11.4-41}$$

$$= \underbrace{G[k]}_{\text{even}} + \underbrace{W_N^k H[k]}_{\text{odd}} \tag{11.4-42}$$

note:

$$W_N^2 = e^{-2j(2\pi/N)} = e^{-j2\pi/(N/2)} = W_{N/2} \tag{11.4-43}$$

and $G[k]$ and $H[k]$ are $N/2$-point or half-length DFTs.

As well, $X[k]$ is periodic in k with period N and $G[k]$ and $H[k]$ are periodic in k with period $N/2$.

Remark 11.4-1 (Computational Complexity) *If the (N/2)-points DFTs are computed using direct matrix multiplication method, we require $N + 2(N/2)^2 = N + N^2/2$ complex multiplications and approximately $N + 2(N/2)^2 = N + N^2/2$ complex additions.*

(Note: an N-points DFT using direct methods need N^2 complex multiplications and $N(N-1)$ complex additions.)

For $N > 2$,

$$N + N^2/2 < N^2 \tag{11.4-44}$$

From Equation 11.4-36, $G[k]$ and $H[k]$ are $N/2$-point DFTs and can be expressed as

$$G[k] = \sum_{n_1=0}^{N/2-1} g[n_1]W_{N/2}^{n_1 k} \tag{11.4-45}$$

and

$$H[k] = \sum_{n_1=0}^{N/2-1} h[n_1] W_{N/2}^{n_1 k} \tag{11.4-46}$$

where $g[n_1] = x[2n_1]$, $h[n_1] = x[2n_1 + 1]$.

Since N is equal to a power of 2, further decimation can be conducted, i.e., $N/2$-pont DFTs can be computed by $N/$-point DFTs. Set $N_1 = n_{11} \times 2 + n_{12}$.

$$G[k] = \sum_{n_{11}=0}^{N/4-1} \sum_{n_{12}=0}^{1} g[2n_{11} + n_{12}] W_{N/2}^{(2n_{11}+n_{12})k} \tag{11.4-47}$$

$$= \sum_{n_{11}=0}^{N/4-1} g[2n_{11}] W_{N/2}^{2n_{11}k} + W_{N/2}^{k} \sum n_{11} = 0^{N/4-1} g[2n_{11}+1] W_{N/2}^{2n_{11}k} \tag{11.4-48}$$

$$= \sum_{n_{11}=0}^{N/4-1} g[2n_{11}] W_{N/4}^{n_{11}k} + W_{N/2}^{k} \sum_{n_{11}=0}^{N/4-1} g[2n_{11}+1] W_{N/4}^{n_{11}k} \tag{11.4-49}$$

and by the same token

$$H[k] = \sum_{n_{11}=0}^{N/4-1} h[2n_{11}] W_{N/4}^{n_{11}k} + W_{N/2}^{k} \sum_{n_{11}=0}^{N/4-1} h[2n_{11}+1] W_{N/4}^{n_{11}k} \tag{11.4-50}$$

This decimation is repeated until it comes down to a simple 2-point DFT computation.

It is known that the computational complexity of $(N/4)$-point DFTs are

$$(N/4)^2 \text{multiplications and} N(N/4)^2 \text{additions} \tag{11.4-51}$$

$$N + 2(N/2)^2 = N + 2(N/2 + 2(N/4)^2) \tag{11.4-52}$$

$$= N + n + 4(N/4)^2 \tag{11.4-53}$$

For $N = 2^\nu$, this process can be at most done $\nu = \log_2 N$ times. Namely, the eventual numbers of complex multiplications will be $N \log_2 N$ and $\approx N \log_2 N$ additions.

Note: Further reduction of multiplications can be achieved using

$$W_N^{N/2} = e^{-j(2\pi/N)N/2} = e^{-j\pi} = -1 \tag{11.4-54}$$

or

$$W_N^{r+N/2} = W_N^{N/2} W_N^r = -W_N^r \tag{11.4-55}$$

11.5 Decimation-in-Frequency FFT Algorithms

Definition 11.5-1 *Algorithms in which the decomposition of the DFT computation is based on decomposing the sequence $X[k]$ into successively smaller subsequences, are called decimation-in-frequency FFT algorithms.*

Given the DFT $X[k]$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \qquad \text{where } k = 0, 1, \ldots, N-1 \tag{11.5-56}$$

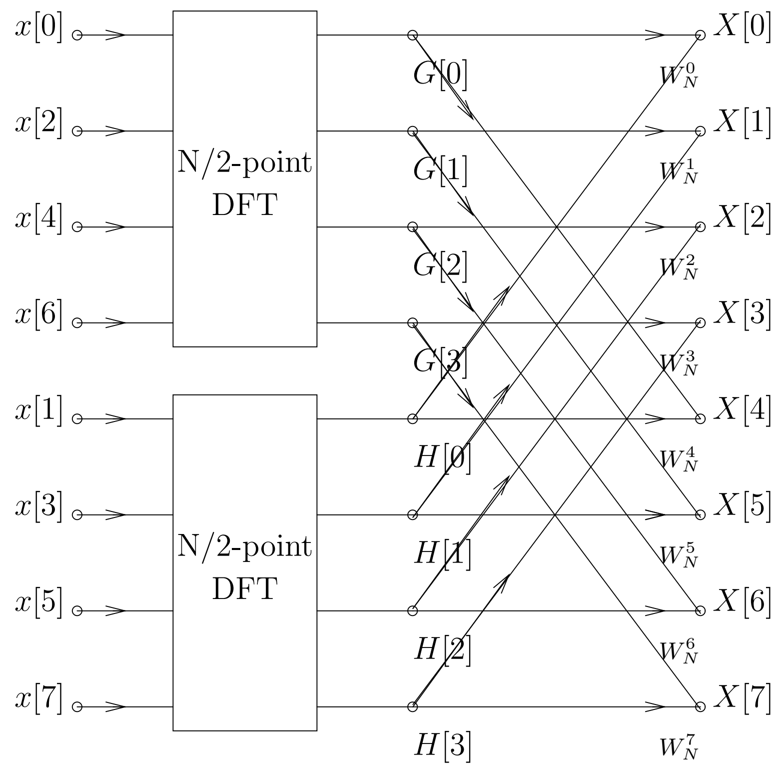


Figure 11.2: SFG of decimation-in-time of N-point DFT into 2 N/2-points DFTs

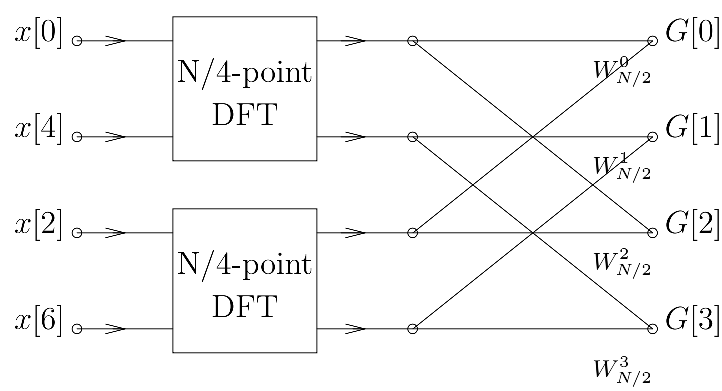


Figure 11.3: SFG of decimation-in-time of N/2-point DFT into 2 N/4-points DFTs

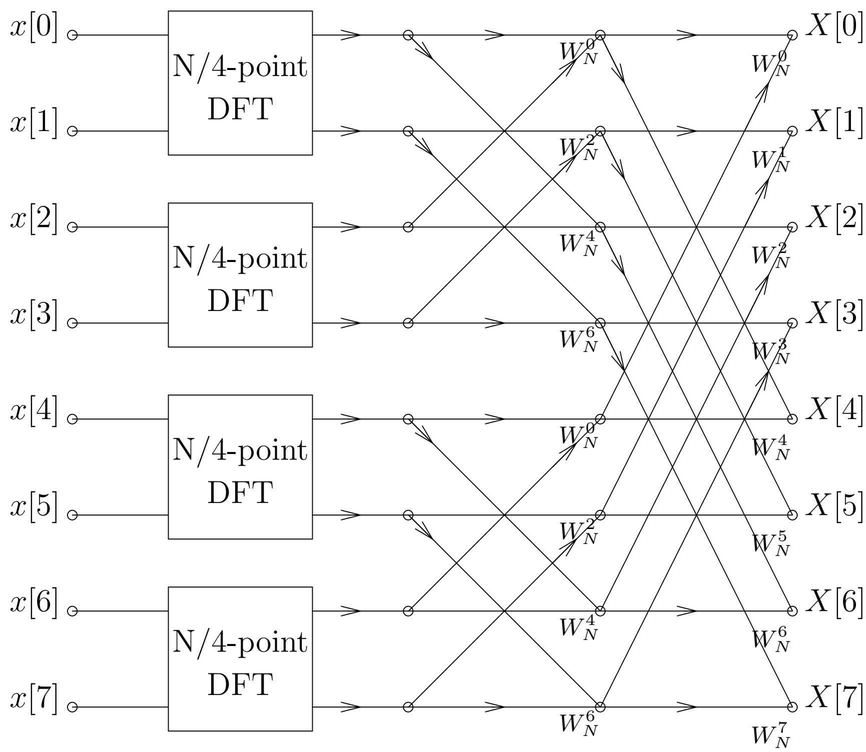


Figure 11.4: Result of Substituting Figure 11.2 into Figure 11.3

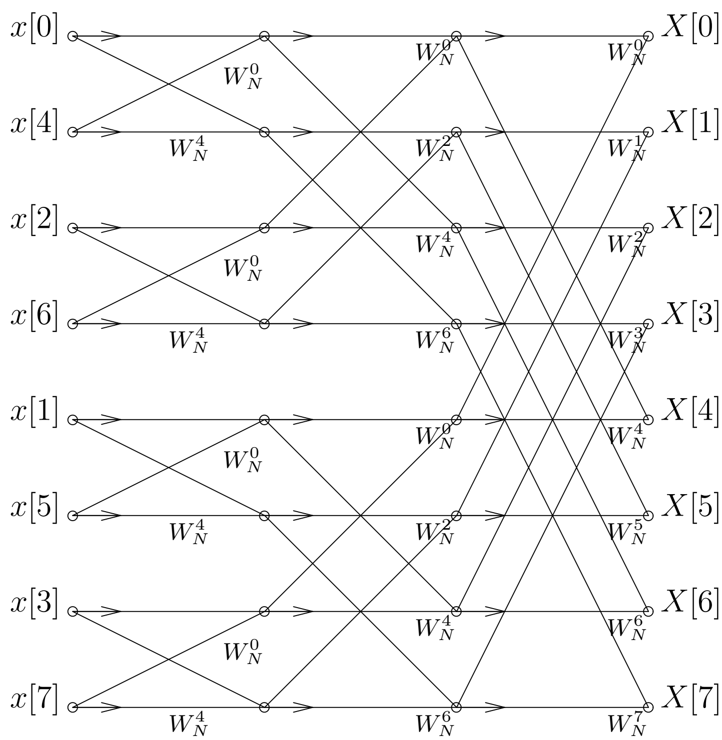


Figure 11.5: SFG of complete DIT decomposition of an 8-point DFT computation

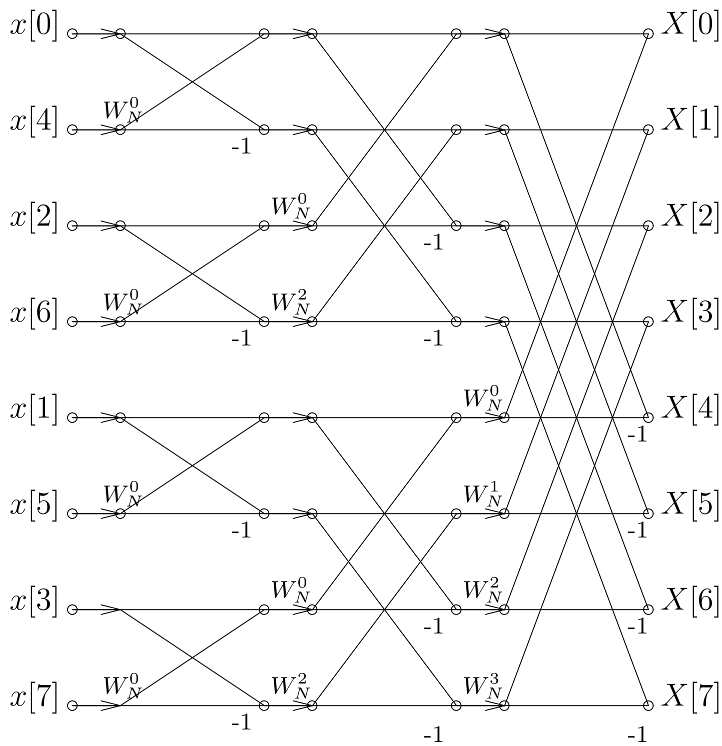


Figure 11.6: SFG of complete DIT using butterfly computation

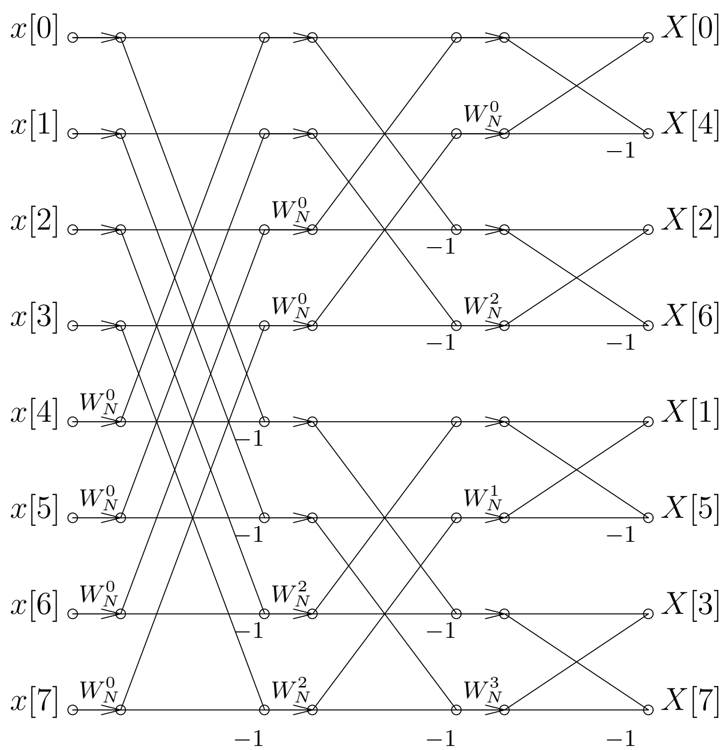


Figure 11.7: Rearrangement of Figure 11.6 with input in normal order and output in bit-reversed order

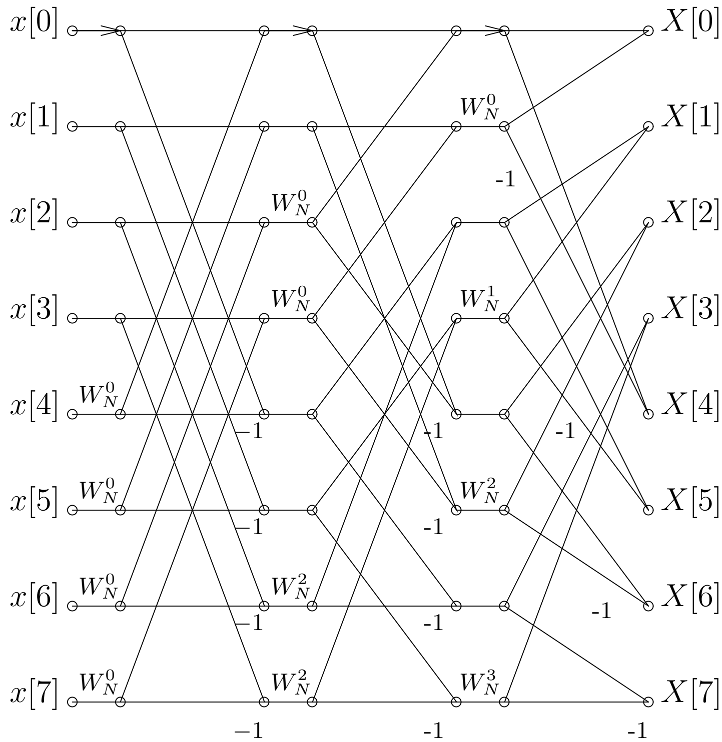
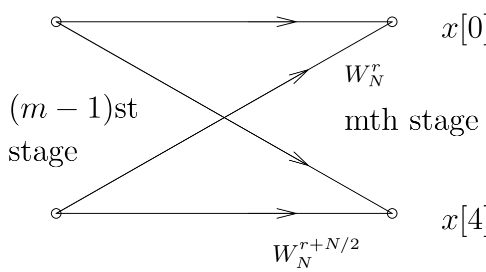
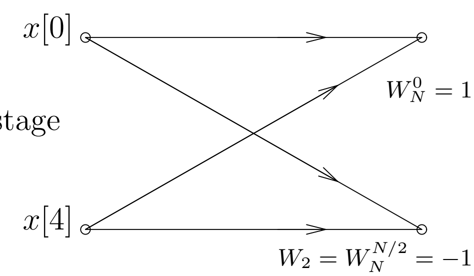


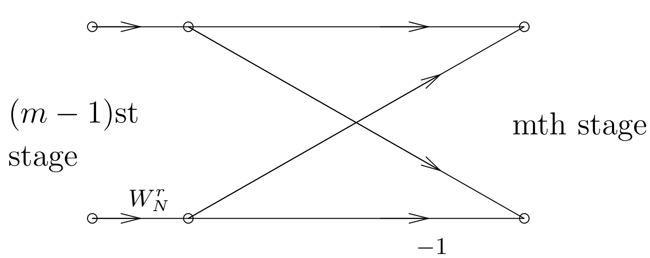
Figure 11.8: Rearrangement of Figure 11.6 with both input and output in normal order



(a) Flow graph of basic butterfly computation



(b) Flow graph of 2-point DFT



(c) Flow graph of simplified butterfly computation requiring only onecomplex multiplication

Figure 11.9: Butterfly computations

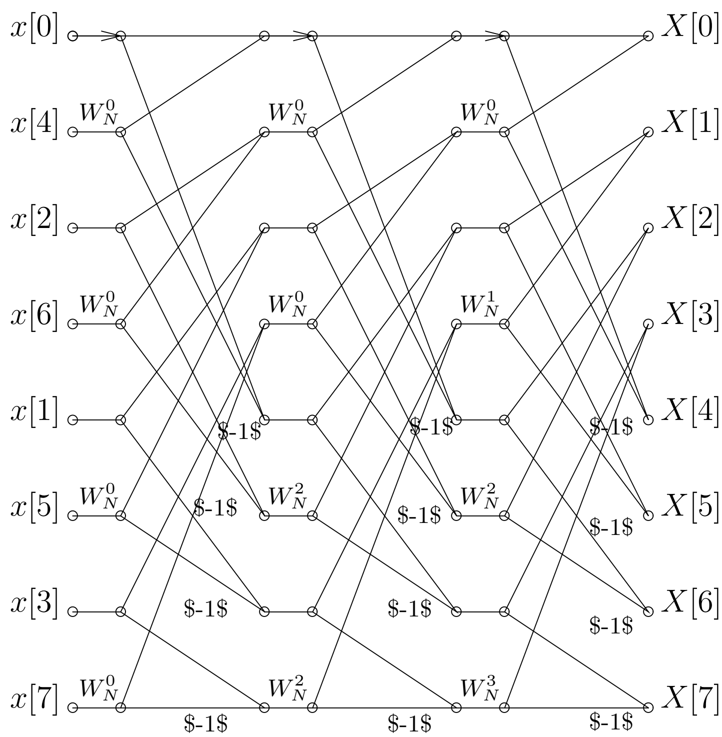


Figure 11.10: Rearrangement of Figure 11.6 having the same geometry for each stage thereby permitting sequential data accessing and storage

Set $k = 2 \times k_1 + k_2$ for $K_1 = 0, 1, \ldots, N/2 - 1$ and $K_2 = 0, 1$

For $K_2 = 0$:

$$X[2k_1] = \sum_{n=0}^{N-1} x[n] W_N^{n(2k_1)} \tag{11.5-57}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{2nk_1} + \sum_{n=N/2}^{N-1} x[n] W_N^{2nk_1} \tag{11.5-58}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{2nk_1} + \sum_{n'=0}^{N/2-1} x[n' + N/2] W_N^{2(n'+N/2)k_1} \tag{11.5-59}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{2nk_1} + \sum_{n=0}^{N/2-1} x[n + N/2] W_N^{2nnk_1} W_N^{Nk_1} \tag{11.5-60}$$

$$= \sum_{n=0}^{N/2-1} \{x[n] + x[n + N/2]\} W_{N/2}^{nk_1} \tag{11.5-61}$$

Note that

$$W_N^2 = W_{N/2} \tag{11.5-62}$$

And for $k_2 = 1$

$$X[2k_1 + 1] = \sum_{n=0}^{N-1} x[n] W_N^{n(2k_1+1)} \tag{11.5-63}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{n(2k_1+1)} + \sum_{n=N/2}^{N-1} x[n] W_N^{n(2k_1+1)} \tag{11.5-64}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{n(2k_1+1)} + \sum_{n'=0}^{N/2-1} x[n' + N/2] W_N^{(n'+N/2)(2k_1+1)} \tag{11.5-65}$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{n(2k_1+1)} + W_N^{N/2(2k_1+1)} \sum_{n=0}^{N/2-1} x[n + N/2] W_N^{n(2k_1+1)} \tag{11.5-66}$$

$$= \sum_{n=0}^{N/2-1} \{x[n] - x[n + N/2]\} W_N^{n(2k_1+1)} \tag{11.5-67}$$

$$= \sum_{n=0}^{N/2-1} \{(x[n] - x[n + N/2]) W_N^{n}\} W_{N/2}^{nk_1} \tag{11.5-68}$$

Note that $W_N^{2nk_1} = W_{N/2}^{nk_1}$.
The above decimation can be repeated until it reduces to 2-point DFTs.

Remark 11.5-1 *The DIF FFT algorithms are transposes of the DIT FFTs and vice versa.*

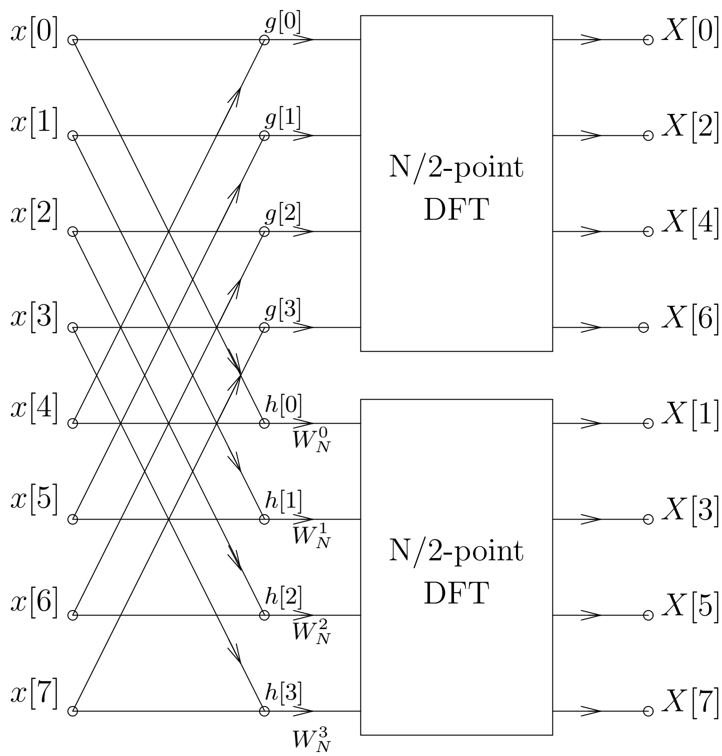


Figure 11.11: SFG of decimation-in-frequency decomposition of an N-point DFT computation into two N/2-point DFT computations (N=8)

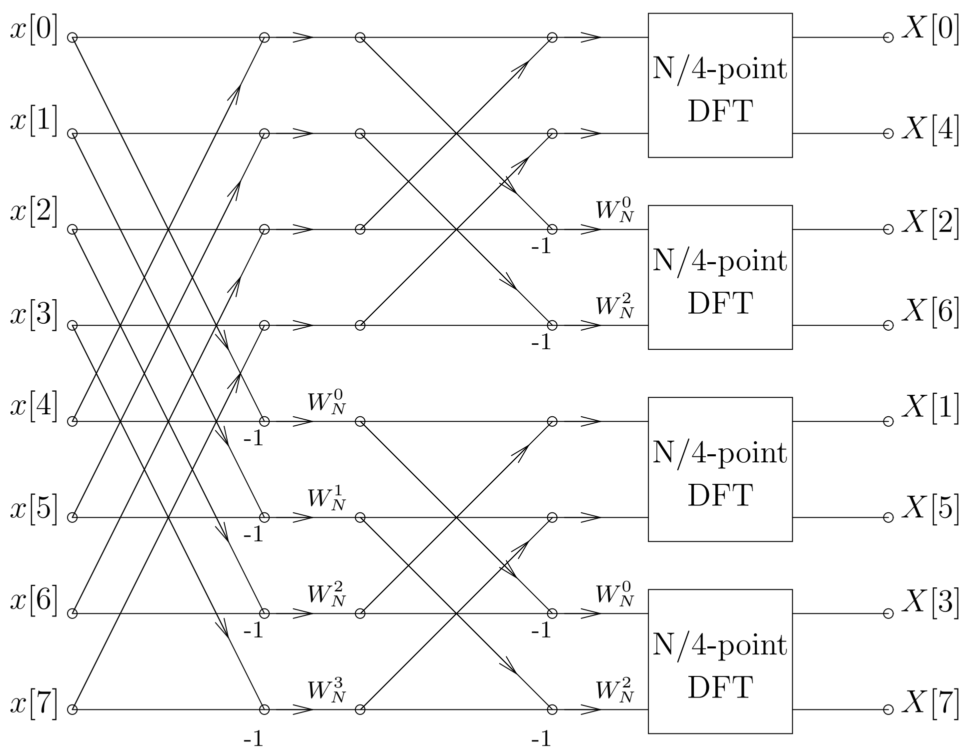


Figure 11.12: SFG of decimation-in-frequency decomposition of an 8-point DFT into two 4-point DFT computations

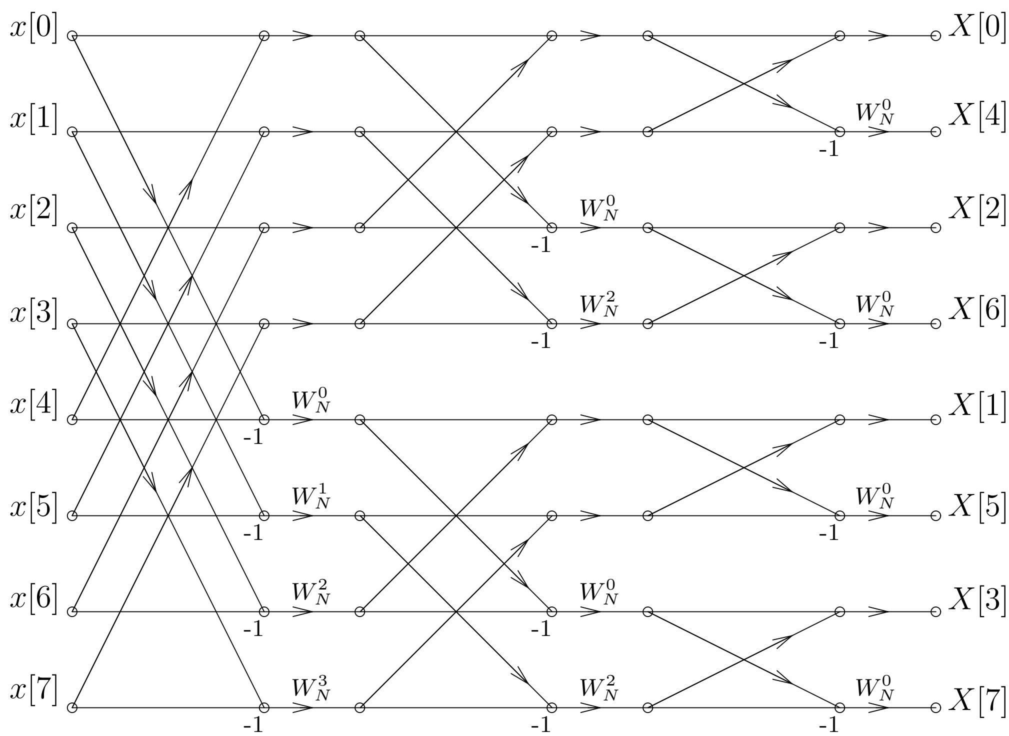


Figure 11.13: SFG of complete decimation-in-frequency decomposition of an 8-point DFT

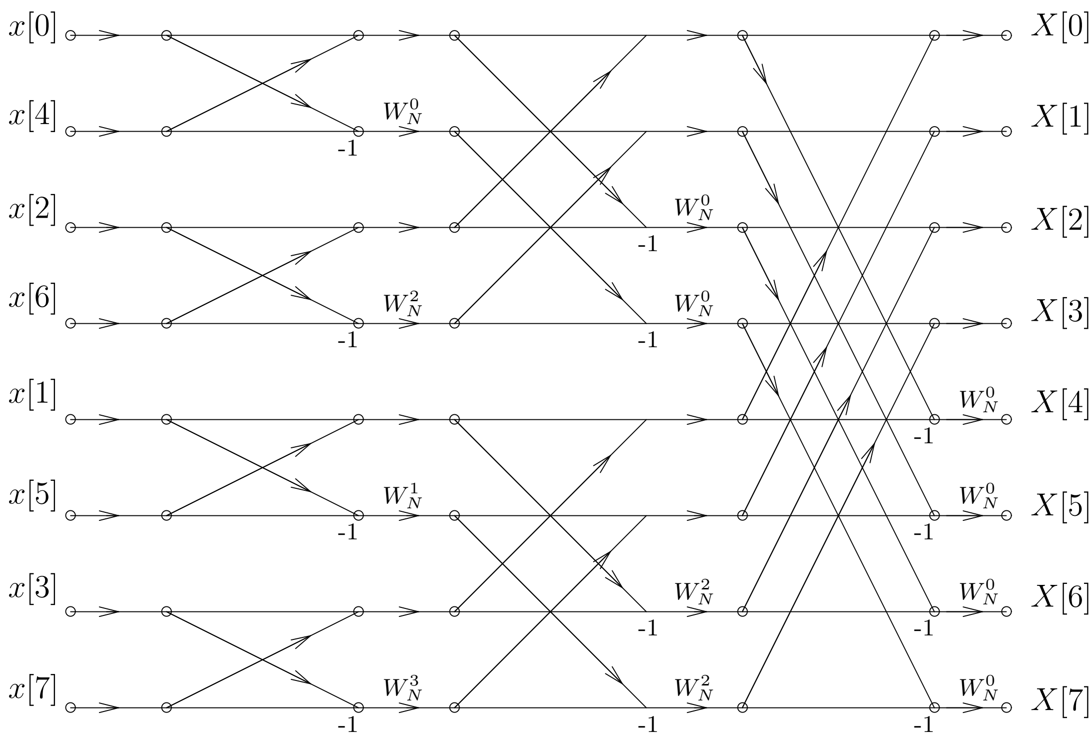


Figure 11.14: SFG of 8-point DFT using the butterfly computation

11.6 Implementation of FFT Algorithms

To implement FFT algorithms. the following issues have to be taken care of

- bit-reversed order or indexing
- in-place computation
- generation of coefficients (LUT or run-time)
- fixed-point arithmetic and real additions and real multiplications

When FFT algorithms are used in the FIR filter implementation, the convolution can be implemented using the forward and inverse transform pair as shown in Figure 11.17.

11.6.1 Implementation of the FFT BF with scaling

For an N-point DFT according to Parseval's Theorem

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2 \tag{11.6-69}$$

where $x[n]$ is the input sequence and $X[k]$ is the DFT of $x[n]$. When $x[n]$ is real we have

$$\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2 \tag{11.6-70}$$

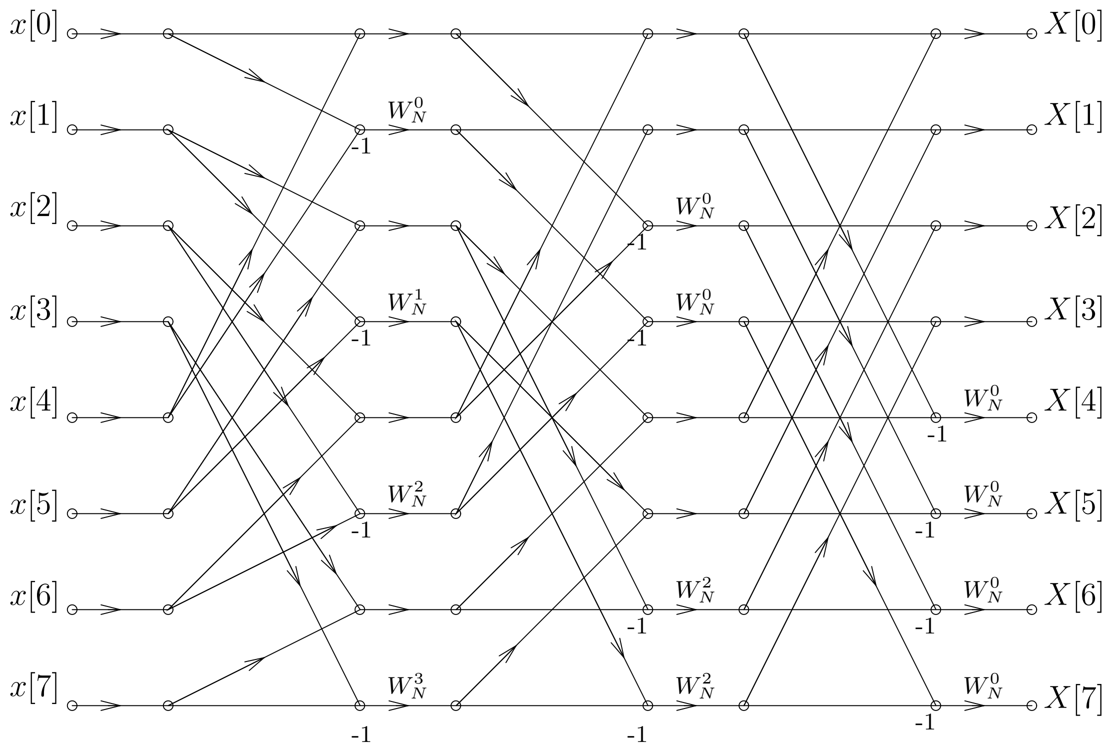


Figure 11.15: Rearrangement so both input and output in normal order

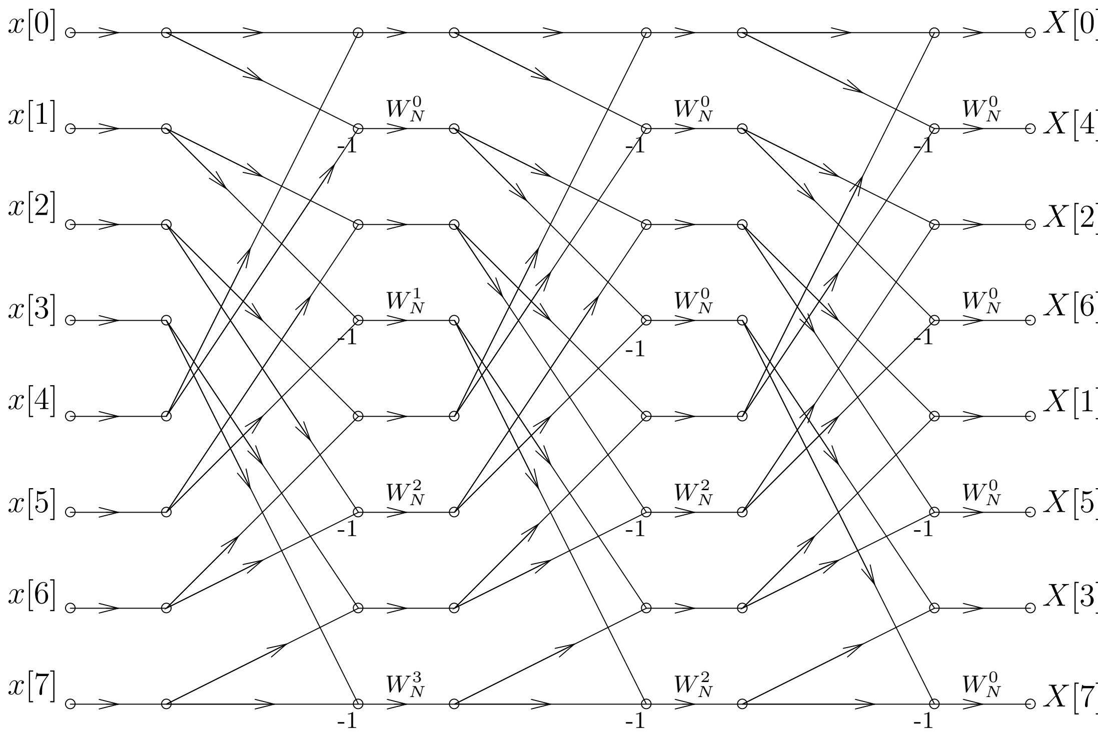


Figure 11.16: Rearrangement so that each stage has the same geometry thereby permitting sequential data accessing and storage

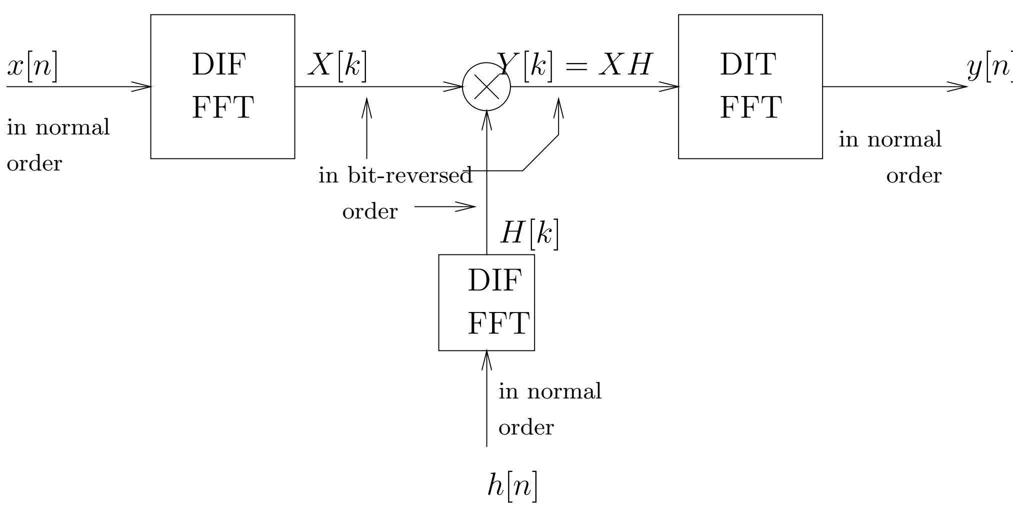


Figure 11.17: FFT implementation of convolution

```
C   FORTRAN subroutine for decimation-in-time FFT algorithm
C       X is an N=2**NU point complex array that initially
C       contains the input and finally contains the DFT.

    SUBROUTINE    DITFFT(X,NU)
    COMPLEX X(1024),U,W,T
    N=2**NU
    NV2=N/2
    NM1=N-1
    PI=3.1415926538979
C***********************************************************
    J=1
    DO 7 I=1,NV2
        IF(I.GE.J) GO TO 5
            T=X(J)
            X(J)=X(I)
            X(I)=T
5       K=NV2
6       IF(K.GE.J) GO TO 7
        J=J-K
        K=K/2
        GO TO 6
7   J=J+K
C***********************************************************
    DO 20 L=1,NU
        LE=2**L
        LE1=LE/2
        U=(1.0,0.0)
        W=CMPLX(COS(PI?FLOAT(LE1)),-SIN(PI/FLOAT(LE1)))
            DO 20 J=1,LE1
                DO 10 I=J,N,LE
                    IP=I+LE1
                    T=X(IP)*U
                    X(IP)=X(I)-T
10              X(I)=X(I)+T
20      U=U*W
C***********************************************************
    RETURN
    END
```

Figure 11.18: FORTRAN program for decimation-in-time algorithm

```
C   FORTRAN subroutine for decimation-in-frequency FFT algorithm
C       X is an N=2**NU point complex array that initially
C       contains the input and finally contains the DFT.

    SUBROUTINE    DIFFFT(X,NU)
    COMPLEX X(1024),U,W,T
    N=2**NU
    PI=3.1415926538979
C***********************************************************
    DO 20 L=1,NU
    LE=2**(NU+1-L)
    LE1=LE/2
    U=(1.0,0.0)
    W=CMPLX(COS(PI/FLOAT(LE1)),-SIN(PI/FLOAT(LE1)))
        DO 20 J=1, LE1
            DO 10 I=J,N,LE
                IP=I+LE1
                T=X(I)+X(IP)
                X(IP)=(X(I)-X(IP))*U
10          X(I)=T
20      U=U*W
C***********************************************************
    NV2=N/2
    NM1=N-1
    J=1
    DO 30 I=1,NM1
        IF(I.GE.J) GO TO 25
            T=X(J)
            X(J)=X(I)
            X(I)=T
25      K=NV2
26      IF(K.GE.J) GO TO 30
            J=J-K
            K=K/2
        GO TO 26
30  J=J+K
C***********************************************************
    RETURN
    END
```

Figure 11.19: FORTRAN program for decimation-in-frequency algorithm

or

$$N\left\{\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]\right\} = \left\{\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2\right\} \tag{11.6-71}$$

Equation 11.6-71 states that the mean-squared value of $X[k]$ is N times that of the input $x[n]$. Consequently, in computing the DFT of the input sequence $x[n]$, overflow may occur when fixed-point arithmetic is used without appropriate scaling.

Consider the general radix-2 butterfly in the mth stage of an N-point DIT FFT as shown in Figure 11.20.

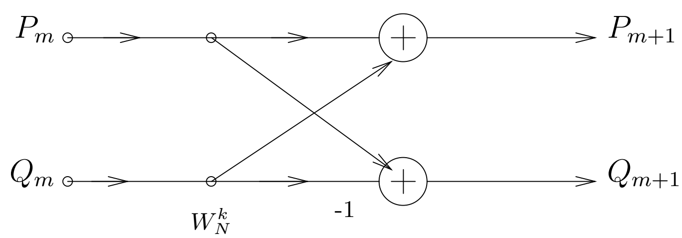


Figure 11.20: General radix-2 butterfly

i.e.,

$$\begin{cases} P_{m+1} &= P_m + W_N^k Q_m \\ Q_{m+1} &= P_m - W_N^k Q_m \end{cases} \tag{11.6-72}$$

where P_m, Q_m, P_{m+1}, Q_{m+1} and W_N^k are in general complex.

Assume that

$$\begin{cases} P_m &= PR + jPI \\ Q_m &= QR + jQI \\ W_N^k &= e^{-j(2\pi/N)k} = \cos(X) - j\sin(X) \end{cases} \tag{11.6-73}$$

Where $X = (2\pi/N)k$ and $j = \sqrt{-1}$.

Substitute Equation 11.6-73 into Equation 11.6-72

$$\begin{cases} P_{m+1} &= \{PR + QR\cos(X) + QI\sin(X)\} + j\{PI + QI\cos(X) - QR\sin(X)\} \\ Q_{m+1} &= \{PR - QR\cos(X) - QI\sin(X)\} + j\{PI - QI\cos(X) + QR\sin(X)\} \end{cases} \tag{11.6-74}$$

Although $|PR|$, $|PI|$, $|QR|$ or $|QI| \leqslant 1$,

$$max\{|PR_{m+1}|, |PI_{m+1}|, |QR_{m+1}|, |QI_{m+1}|\} \tag{11.6-75}$$
$$= 1 + 1\sin(45^\circ) + 1\cos(45^\circ) \tag{11.6-76}$$
$$= 2.414213562 \tag{11.6-77}$$

Each stage of the FFT is scaled down by a factor of 2 to avoid the possibility of overflow.

For M stage FFTs, the output is scaled down by $2^M = N$, where N is the length of the FFT.

Even with scaling by 2, overflow is still possible as

$$max\{\cdot\} = \frac{1}{2} + \frac{1}{2}\sin(45^\circ) + \frac{1}{2}\cos(45^\circ) \tag{11.6-78}$$
$$= 1.207106781 = a \tag{11.6-79}$$

If the input signal is further scaled down by a, the output of the last FFT stage by the same factor.

If the input signal is real, the above additional scaling is not necessary as the above maximum value cannot be obtained.

11.7 Other fast algorithms for computation of DFT

Other fast algorithms to compute DFTs include

- Radix-4, Radix-8, ... algorithms
- Split-radix FFT algorithms
- Prime factor algorithms
- The Winograd Fourier transform algorithm (WFTA) - a benchmark
- The chirp transform algorithm

In the last two algorithms, convolution is used to compute the DFT.

11.7.1 The Chirp-z Transform Algorithm

The DFT can be viewed as the z-transform of $x[n]$ evaluated at N equally spaced points on the unit circle in the z-plane. It can also be viewed as N equally spaced samples of the Fourier transform of $x[n]$.

Now consider the evaluation of $X(z)$ on other contours in the z-plane, including the unit circle.

Compute the values of the z-transform of $x[n]$ at a set of points $\{z_k\}$,

$$X(z_k) = \sum_{n=0}^{N-1} x[n] z_k^{-n} \qquad \text{where } k = 0, 1, \ldots, L-1 \tag{11.7-80}$$

If the contour is a circle of radius r and the z_k are N equally spaced points them

$$z_k = re^{j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \tag{11.7-81}$$
$$X(z_k) = \sum_{n=0}^{N-1} \left\{x[n]r^{-n}\right\} e^{-j2\pi kn/N} \tag{11.7-82}$$

In this case the FFT algorithm can be applied on the modified sequence $x[n]r^{-n}$.

More generally, assume that $\{z_k\}$ in the z-plane fall on an arc which begins at

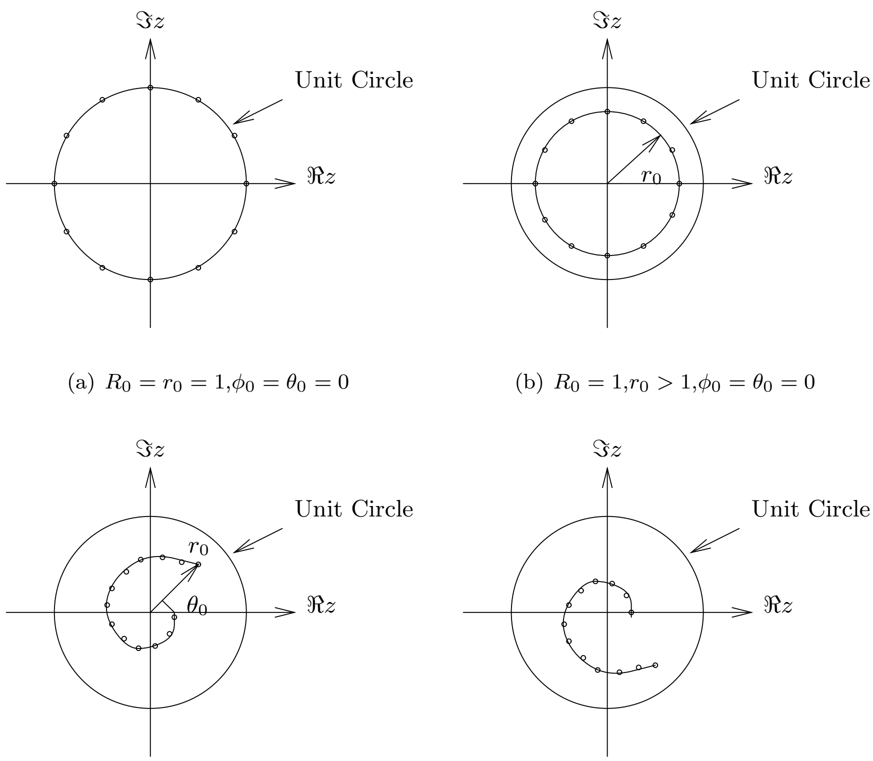
$$z_0 = r_0 e^{j\theta_0} \tag{11.7-83}$$

and spirals wither in toward the origin or out away from the origin such that the points $\{z_k\}$ are defined as

$$z_k = r_0 e^{j\theta_0} \left(R_0 e^{j\Phi_0}\right)^k \qquad k = 0, 1, \ldots, L-1 \tag{11.7-84}$$

```
********************************************************
* TMS32020 code for a general radix-2 DIT FFT butterfly *
********************************************************
*
* Equates for the real and imaginary parts if Xm(P) and Xm(Q).
* The locations PR, PI, QR, and QI are used both for the input
* and the output data.
*
PR EQU 0 Re(Pm) stored in location 0 in data memory
PI EQU 1 Im(Pm) stored in location 1 in data memory
QR EQU 2 Re(Qm) stored in location 2 in data memory
QI EQU 3 Im(Qm) stored in location 3 in data memory
*
* Equates for the real and imaginary parts of the twiddle factor
*
COSX EQU 4 COS(X) stored in location 4 in data memory
SINX EQU 5 SIN(X) stored in location 5 in data memory
*
* Initialize system
*
AORG >20
INIT SPM 0 No shifts at output of P-register
SSXM Select sign extension mode
ROVM Reset overflow mode
LDPK 4 Choose data page 4
*
* Calculate (QR COS(X)+ QI SIN(X)); store result in QR.
*
BTRFLY LT QR Load T-Register with QR
MPY COSX P-Register = (1/2) QR COS(X)
LTP QI ACC=(1/2) QR COS X ; Load T-Register with QI
MPY SINX P-Register = (1/2) QI SINX
APAC ACC=(1/2)(QR COSX+QI SINX)
MPY COSX P-Register = (1/2) QI COSX
LT QR Load T-Register with AR
SACH QR AR=(1/2)(QR COSX+ QI SINX)
*
* Calculate (QI COS(X) - QR SIN(X)); Store result in QI
*
PAC ACC=(1/2) QI COSX
MPY SINX P-Register - (1/2) QR SINX
SPAC ACC=(1/2)(QI COSX - QR SINX)
SACH QI QI=(1/2)(QI COSX - QR SINX)
*
* Calculate Re(Pm+1) and Re(Qm+1); Store result in PR and QR.
*
LAC PR,14 ACC=(1/4) PR
ADD QR,15 ACC=(1/4)(PR+QR COSX + QI SINX)
SACH PR,1 PR=(1/2)(PR+QR COSX + QI SINX)
SUBH QR ACC=(1/4)(PR-QR COSX - QI SINX)
SACH QR,1 QI=(1/2)(QI COSX - QR SINX)
*
* Calculate Im[Pm+1] and Im[Qm+1]; Store result in PI and QI.
*
LAC PI,14 ACC=(1/4) PI
ADD QI,15 ACC=(1/4)(PI + QI COSX - QR SINX)
SACH PI,1 PI =(1/2)(PI + QI COSX - QR SINX)
SUBH QI ACC=(1/4)(PI - QI COSX + QR SINX)
SACH QI,1 ACC=(1/2)(PI - QI COSX + QR SINX)
*
```

Figure 11.21: TMS32020 code for a general radix-2 DIT FFT butterfly



(a) $R_0 = r_0 = 1, \phi_0 = \theta_0 = 0$

(b) $R_0 = 1, r_0 > 1, \phi_0 = \theta_0 = 0$

(c) $R_0 < 1$

(d) $R_0 > 1$

Figure 11.22: Some examples of contours on which we may evaluate the z-transform

Define

$$V = R_0 e^{j\phi_0} \tag{11.7-85}$$

the z-transform becomes

$$X(z_k) = \sum_{n=0}^{N-1} x[n] z_k^{-n} \tag{11.7-86}$$

$$= \sum_{n=0}^{N-1} x[n] \left(r_0 e^{j\theta_0}\right)^{-n} v^{-nk} \tag{11.7-87}$$

Equation 11.7-87 can be expressed in the form of a convolution by substituting

$$nk = \frac{1}{2}\left[n^2 + k^2 - (k-n)^2\right] \tag{11.7-88}$$

into Equation 11.7-87

$$X(z_k) = V^{-k^2/2} \sum_{n=0}^{N-1} \left\{ x[n] \left(r_0 e^{j\theta_0}\right)^{-n} V^{-n^2/2} \right\} V^{(k-n)^2/2} \tag{11.7-89}$$

Define a new sequence $g[n]$

$$g[n] = x[n] \left(r_0 e^{j\theta_0}\right)^{-n} V^{-n^2/2} \qquad n = 0, 1, \ldots, N-1 \tag{11.7-90}$$

Equation 11.7-89 becomes

$$X(z_k) = V^{-k^2/2} \sum_{n=0}^{N-1} g[n] V^{(k-n)^2/2} \tag{11.7-91}$$

Equation 11.7-91 may be interpreted as the convolution of $g[n]$ with the impulse response $h[n]$ of a filter where

$$h[n] = V^{n^2/2} \tag{11.7-92}$$

and a post-multiplication. OR,

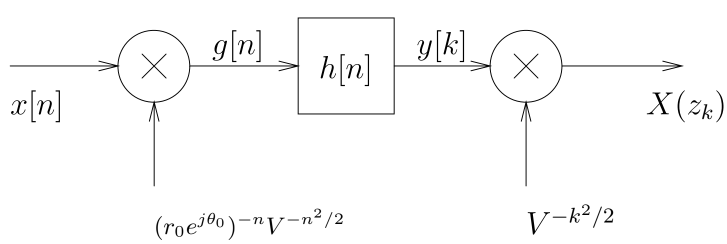
$$X(z_k) = V^{-k^2/2} y[k] \tag{11.7-93}$$

$$= \frac{y[k]}{h[k]} \qquad k = 0, 1, \ldots, L-1 \tag{11.7-94}$$

where $y[k]$ is the output of the filter

$$y[k] = \sum_{n=0}^{N-1} g[n] h[k-n] \qquad k = 0, 1, \ldots, L-1 \tag{11.7-95}$$

Note that both $g[n]$ and $h[n]$ are complex valued sequences.



When $R_0 = 1$, the sequence

$$h[n] = V^{n^2/2} = \left(R_0 e^{j\phi_0}\right)^{n^2/2} \tag{11.7-96}$$
$$= e^{j\phi_0 n^2/2} \tag{11.7-97}$$
$$= e^{j(\phi_0 n/2)n} e^{j\omega n} \tag{11.7-98}$$

has a form of a complex exponential. The quantity $\omega = \phi_0 n/2$ represents the frequency of the complex exponential signal which is increasing linearly with time (n). Such signals are used in radar systems and are called chirp signals. Hence the z-transform evaluated as in Equation 11.7-91 is called the *chirp-z transform*.

Equation 11.7-95 represents a linear convolution and it can most efficiently be computed using the FFT algorithm. In Equation 11.7-95, the new sequence $g[n]$ is of length N. Although $h[n]$ has infinite duration, only a portion of $h[n]$ is required to compute the L values of $X(z_k)$. (i.e., $h[n]$ for $n = -(N-1), \ldots, (L-1)$.)

The circular convolution of the N-point sequence $g[n]$ with an M-point section of $h[n]$, where $M > N$, will be used to compute the linear convolution in Equation 11.7-95 if the FFT is to apply.

As it is known, the first $N-1$ points contain time aliasing when the circular convolution is used to compute a linear convolution, and the remaining $M-n+1$ points are identical to the result that would be obtained from linear convolution of $h[n]$ with $g[n]$ (Note the order.) Therefore, we should select a DFT of size

$$M = L + N - 1 \tag{11.7-99}$$

which would yield L valid points and $N-1$ points corrupted by aliasing. $h[n]$, as it is is a noncausal sequence thus we define a new causal sequence of length M as

$$h_1[n] = h[n - N + 1] \qquad n = 0, 1, \ldots, M-1 \tag{11.7-100}$$

Computation Procedure:

1. Compute the M-point DFT of $h_1[n]$ using the FFT algorithm to obtain $H_1(k)$.
2. Form $g[n]$ from $x[n]$ as
$$g[n] = x[n]\left(r_0 e^{j\theta_0}\right)^{-n} V^{-n^2/2} \qquad n = 0, 1, \ldots, N-1 \tag{11.7-101}$$
and pad $g[n]$ with $L-1$ zeros.
3. Compute the M-point DFT of $g[n]$ to yield $G(k)$.
4. Find the product:
$$Y_1(k) = G(k)H_1(k) \tag{11.7-102}$$
5. Calculate the IDFT of $Y_1(k)$ using the FFT algorithm to obtain the M-point sequence.
$$y_1[n] \qquad n = 0, 1, \ldots, M-1 \tag{11.7-103}$$
(The first $(N-1)$ points are corrupted by aliasing and should be discarded.)
6. The linear convolution
$$y[n] = y_1[n + N - 1] \qquad n = 0, 1, \ldots, L-1 \tag{11.7-104}$$
7. Compute the complex values
$$X(z_k) = \frac{y[k]}{h[k]} \qquad k = 0, 1, \ldots, L-1 \tag{11.7-105}$$

An alternative is to define a new sequence $h_2[n]$, as

$$h_2[n] = \begin{cases} h[n] & 0 \leqslant n \leqslant L-1 \\ h[n-N-L+1] & L \leqslant n \leqslant M-1 \end{cases} \tag{11.7-106}$$

The M-point DFT of $h_2[n]$ is $H_2(k)$ and $Y_2(k) = G(k)H_2(k)$.

$$y_s[n] = DFT^{-1}\{Y_2(k)\} \qquad 0 \leqslant n \leqslant M-1 \tag{11.7-107}$$

and

$$y[n] = y_2[n] \qquad 0 \leqslant n \leqslant L-1 \tag{11.7-108}$$

The rest is the same as the previous method.

REMARK 11.7-1 *Computational complexity of the chirp-z transform algorithm described above is of the order of $M \log_2 M$ complex multiplications where $M = N + L - 1$. Compared with NL when the computations are performed by direct evaluation of the z-transform, the chirp-z transform algorithm is more efficient if L is large.*

Computation of DFT using the chirp-z transform (CZT) method.
Select $r_0 = R_0 = 1$, $\theta_0 = 0$, $\phi_0 = 2\pi/N$ and $L = N$.
Thus

$$V^{-n^2/2} = e^{-j\pi n^2/N} \tag{11.7-109}$$

$$= \cos\frac{\pi n^2}{N} - j\sin\frac{\pi n^2}{N} \tag{11.7-110}$$

The chirp filter with impulse response

$$h[n] = V^{n^2/2} \tag{11.7-111}$$

$$= \cos\frac{\pi n^2}{N} + j\sin\frac{\pi n^2}{N} \tag{11.7-112}$$

$$= h_r[n] + jh_i[n] \tag{11.7-113}$$

can be implemented as a pair of FIR filters with coefficients $h_r[n]$ and $h_i[n]$ respectively.

Note that the premultiplications $V^{-n^2/2}$ and the postmultiplications $V^{-k^2/2}$ can be implemented by storing the cosine and sine sequences in ROM.

REMARK 11.7-2 *If only the magnitude of the DFT is requires the postmultiplications are unnecessary as*

$$|X(z_k)| = \left|V^{-k^2/2}y[k]\right| \tag{11.7-114}$$

$$= \left|V^{-k^2/2}\right| |y[k]| \tag{11.7-115}$$

$$= |y[k]| \tag{11.7-116}$$

where

$$V^{-k^2/2} = \cos\frac{\pi k^2}{N} - j\sin\frac{\pi k^2}{N} \tag{11.7-117}$$

Chapter 12

Advanced Digital Signal Processing

Contents

Numerical effects caused by using finite word length digital computation can be classified into the following two groups.

Quantization Errors

- Signal quantization
- Coefficient quantization

Roundoff Errors

- Truncation
- Rounding
- Others

Theoretically equivalent system structures may behave differently when implemented with finite numerical precision.

12.1 Quantization in Data Aquisition/Representation

Analog-to-Digital conversion in an integral part of many discrete-time systems.

The A/D converter in a physical device that converts a voltage or current amplitude at its input into a binary code representing a qunatized amplitude value closes to the amplitude of the input at its output.

The A/D consists of a quantizer and a coder

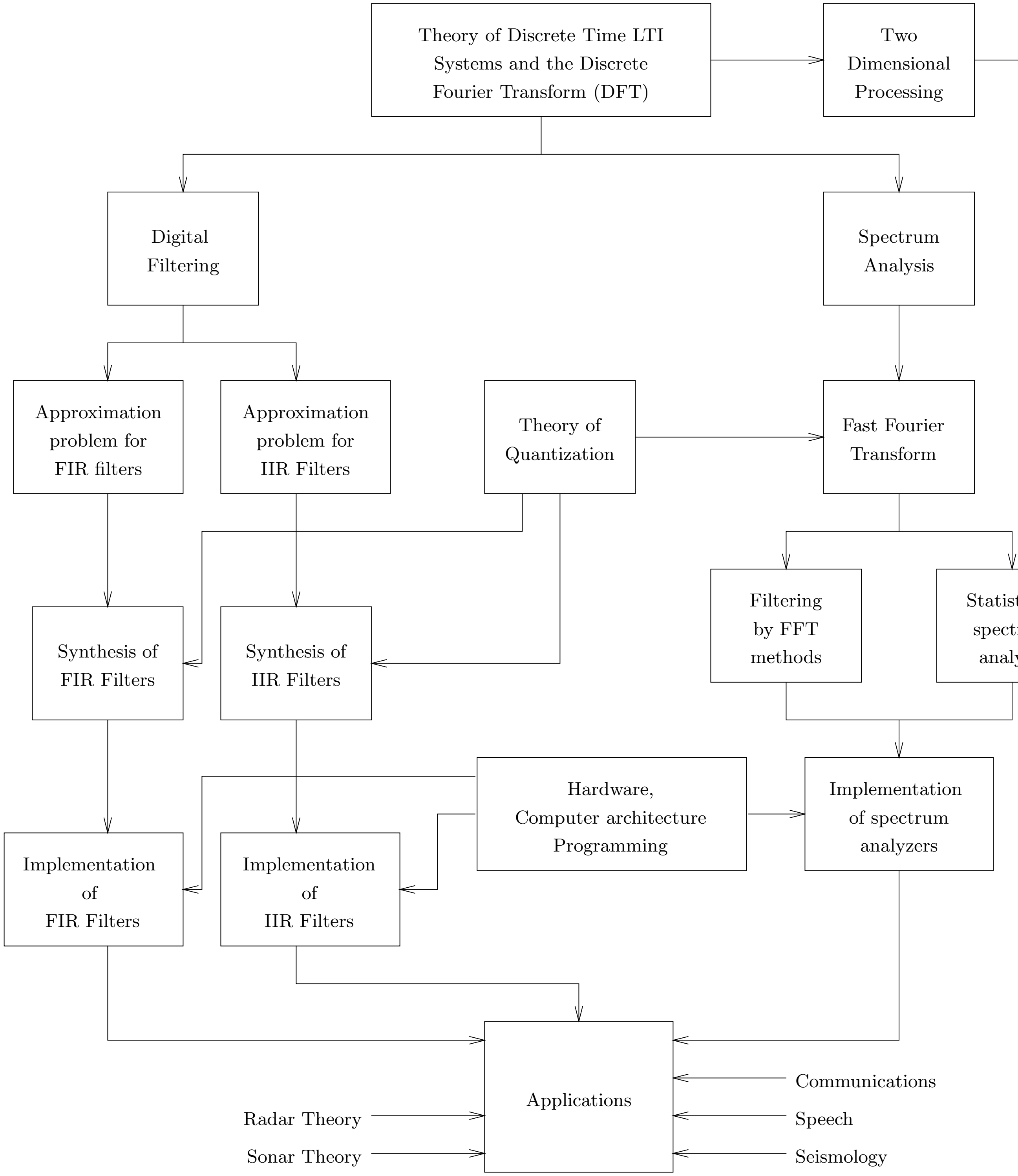
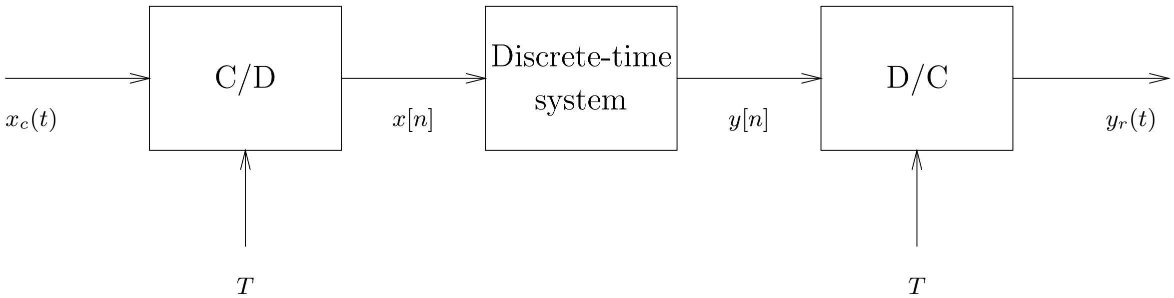
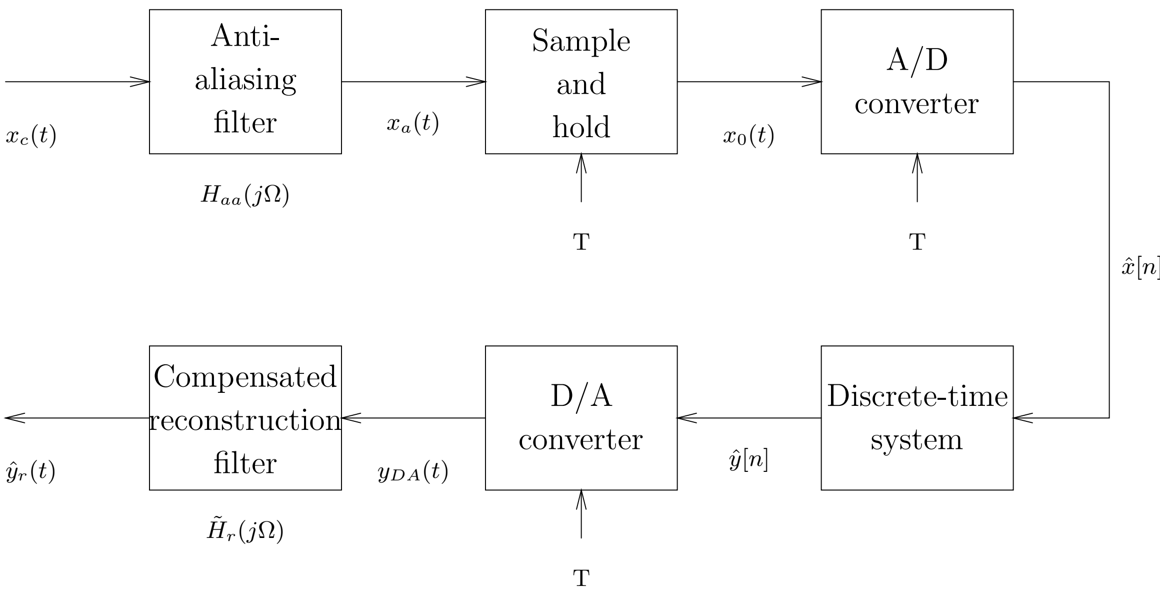


Figure 12.1: Overview of digital signal processing

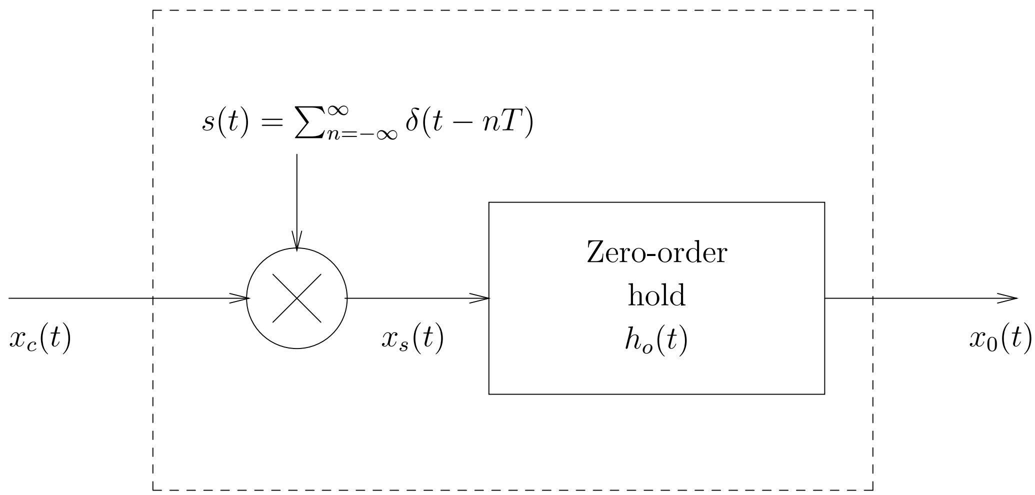


(a) Discrete-time filtering of continuous-time signals

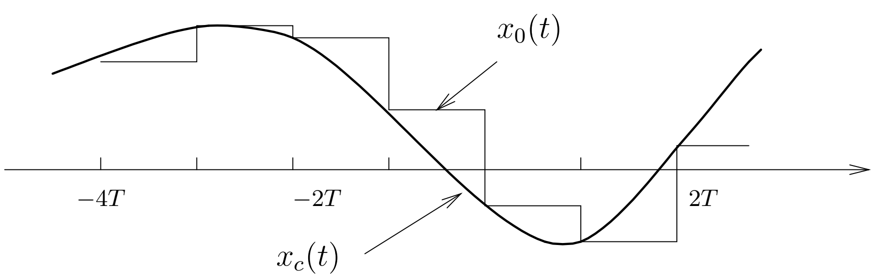


(b) A more realistic model

Figure 12.2: Discrete-time filtering of continuous-time signals

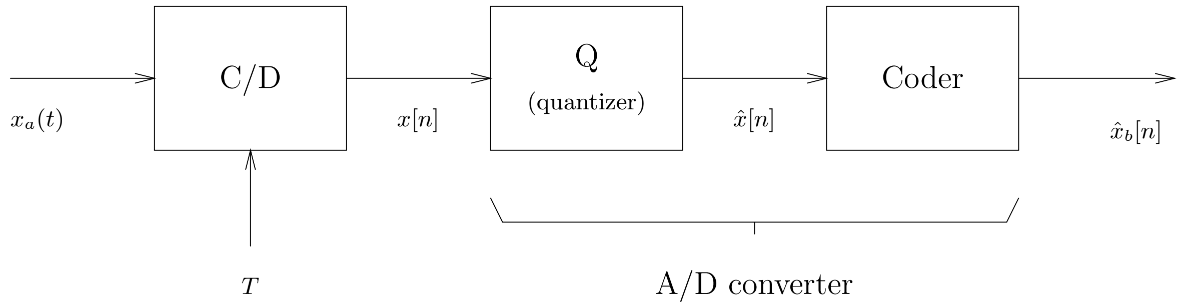


(a) Representation of an ideal sample and hold



(b) Typical input and output signals for the sample and hold

Figure 12.3: Sample and Hold



where $x[n] = x_a(nT)$ are the ideal samples of $x_a(t)$. The sample and hold are normally uised for C/D conversion.

The quantizer is a non-linear system with an operation defined as

$$\hat{x}[n] = Q(x[n]) \tag{12.1-1}$$

where $\hat{x}[n]$ is the quantized sample.

The quantizer can be further classified as

- Uniform quantizer where quantization levels are uniformly spaced;
- Non-uniform quantizer where quantiztion levels are non-uniformly spaced.

Proper fractions are normally assumed in the fixed-point arithmetic. For a $(B+1)$ bit binary two's complement number

$$a_0 a_1 a_2 \dots a_B \tag{12.1-2}$$

its value is

$$-a_0 2^0 + a_1 2^{-1} + a_2 2^{-2} + \cdots + a_B 2^{-B} \tag{12.1-3}$$

For $B = 2$

Binary	Numeric Value ($\hat{x}_B$)
0 1 1	$\frac{3}{4}$
0 1 0	$\frac{1}{2}$
0 0 1	$\frac{1}{4}$
0 0 0	0
1 1 1	$-\frac{1}{4}$
1 1 0	$-\frac{1}{2}$
1 0 1	$-\frac{3}{4}$
1 0 0	-1

The relationship between $\hat{x}_B$ (code words) and the quantized levels depends on X_m.

The quantization step size is defined as

$$\Delta = \frac{2X_m}{2^{B+1}} = \frac{X_m}{2^B} = 2^{-B} X_m \tag{12.1-4}$$

From Equation (12.1-4), it can be seen that the smallest quantization levels $(\pm\Delta)$ correspond to the least significant bit (a_B) of the binary code word.

The numeric relationship between the quantization samples and the code words is

$$\hat{x}[n] = X_m \hat{x}_B[n] \tag{12.1-5}$$

since $\hat{x}_B[N]$ is in two's complement and a proper fraction, i.e. $-1 \leqslant \hat{x}_B[n] < 1$.

If the input signal is normalized to X_m, the number values of $\hat{x}[n]$ and $\hat{x}_B[n]$ will be identical.

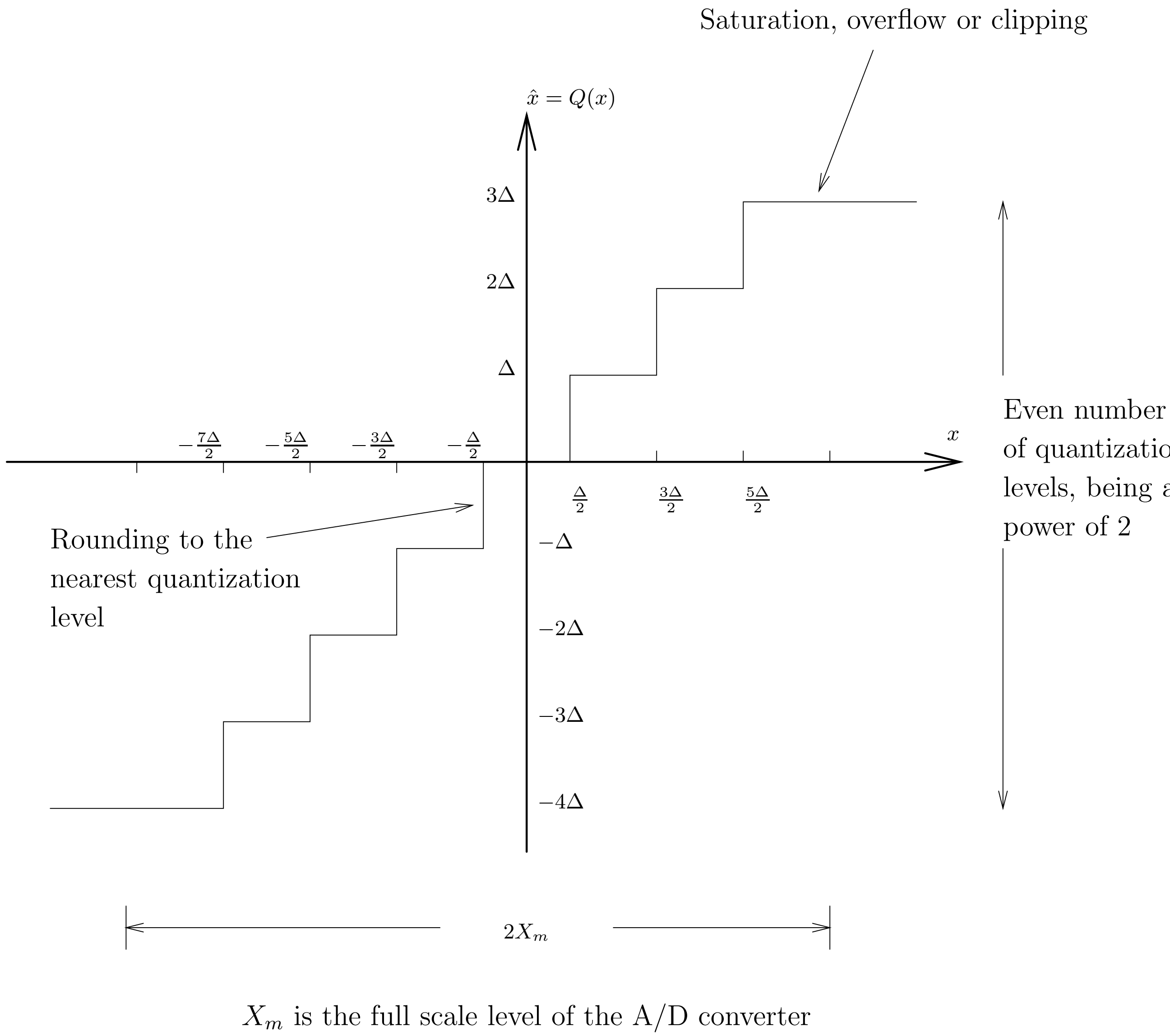


Figure 12.4: Typical quantizer for A/D conversion

12.2 Analysis of Quantization Errors

DEFINITION 12.2-1 *The quantization error $e[n]$ is defined as*

$$e[n] = \hat{x}[n] - x[n] \tag{12.2-6}$$

where $x[n]$ is the sample value, and $\hat{x}[n]$ is the quantized sample.

- When rounding is applied in quantization, i.e.,

$$\hat{x}[n] = i\Delta \qquad \text{if } (i - \tfrac{1}{2})\Delta < x[n] \leqslant (i + \tfrac{1}{2})\Delta \quad i \in (-4, -3, -2, \ldots, 0, 1, 2, 3) \tag{12.2-7}$$

it follows that

$$-\frac{\Delta}{2} < e[n] \leqslant \frac{\Delta}{2} \tag{12.2-8}$$

provided that

$$(-X_m - \frac{\Delta}{2}) < x[n] \leqslant (x_m - \frac{\Delta}{2}) \tag{12.2-9}$$

If the sample $x[n]$ is outside this range, the quantization error will be larger than $\frac{\Delta}{2}$ is magnitude.

- When truncation is used, i.e.,

$$\hat{x}[n] = i\Delta \qquad \text{if } (i - 1)\Delta < x[n] \leqslant (i)\Delta \quad i \in (-(2^B - 1), \ldots, 0, \ldots 2^B) \tag{12.2-10}$$

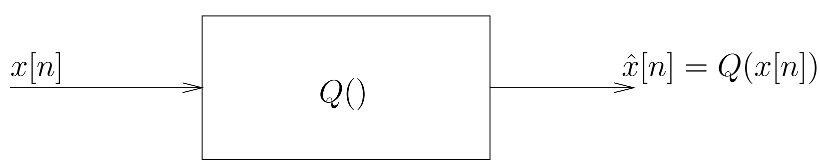
then

$$-\Delta < e[n] \leqslant 0 \tag{12.2-11}$$

provided that

$$-X_m < x[n] \leqslant X_m \tag{12.2-12}$$

A statistical model of the quantization error $e[n]$ is useful in representing the effects of quantization.



When rounding is used, $e[n]$ in the model is usually assumed to be a uniformly distributed white-noise sequence with the following probability density function:

Note that $\Delta = 2^{-B} X_m$.

The mean value of $e[n]$ is

$$M_e = \mathcal{E}\{\rceil\} = \int_{-\frac{\cdot}{\in}}^{\frac{\cdot}{\in}} \rceil \mathcal{P}_{\rceil\backslash}(\rceil)\lceil\rceil \tag{12.2-13}$$

$$= \frac{1}{\Delta}\left(\frac{1}{2}e^2\right)\Bigg|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \tag{12.2-14}$$

And its variance is

$$var[e] = \mathcal{E}\{|(\rceil - \Updownarrow_{\rceil})|^{\in}\} = \sigma_{\rceil}^{\in} \tag{12.2-15}$$

$$= \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 P_{en}(e)de = \frac{1}{\Delta}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 de \tag{12.2-16}$$

$$= \frac{1}{3\Delta} e^3 \Big|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{1}{3\Delta}\left\{\left(\frac{\Delta}{2}\right)^3 - \left(-\frac{\Delta}{2}\right)^3\right\} \tag{12.2-17}$$

$$= \frac{\Delta^3}{12\Delta} = \frac{\Delta^2}{12} \tag{12.2-18}$$

For a $(B+1)$ bit generator with X_m, tje noise variance or noise power is

$$\sigma_e^2 = \frac{\Delta^2}{12} = \frac{\left(2^{-B}X_m\right)^2}{12} \tag{12.2-19}$$

Consequently, the signal-to-noise ratio (SNR) defined as the ratio of the signal variance (or power) to noise variance is

$$SNR = 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right) = 10\log_{10}\left(\frac{122^{2B}\sigma_x^2}{X_m^2}\right) \tag{12.2-20}$$

$$= 10\log_{10} 2^{2B} + 10\log 10(12) - 10\log_{10}\left(\frac{X_m^2}{\sigma_x^2}\right) \tag{12.2-21}$$

$$= 6.02B + 10.8 - 20\log_{10}\left(\frac{x_m}{\sigma_x}\right) \tag{12.2-22}$$

Remark 12.2-1 (Equation (12.2-22))

1. *SNR increases approximately 6 dB for each bit added to the wordlength of the quantized samples or for each doubling of the number of quantization levels.*

2. σ_x *is the root mean square (RMS) value defined as*

$$\sigma_x = \sqrt{\frac{1}{T}\int_0^T x^2(t)dt} \tag{12.2-23}$$

for periodic signals, representing the average power.

For a sinusoid, with peak amplitude X_p

$$\sigma_x = X_p/\sqrt{2} = 0.707X_p \tag{12.2-24}$$

(Note:

$$x(t) = X_p\cos(\omega t + \theta) \tag{12.2-25}$$

$$\sigma_x = \sqrt{\frac{1}{T}\int_0^T X_p^2\cos^2(\omega t + \theta)dt} \tag{12.2-26}$$

and use

$$\cos^2 X = \frac{1}{2}(1 + \cos 2X) \tag{12.2-27}$$

)

If σ_x *is too large,* $X_p >> X_m$ *and Equation (12.2-22) is not valid and severe distortion results. If* σ_x *is too small,* $\sigma_x << X_M$*, SNR decreases. SNR decreases by 6 dB if* σ_x *is halved.*

3. *The signal amplitude must be matched to the X_m of the A/D converter.*

   For speech or music, the probability that $X_p > 4\sigma_x$ is very low (0.064% if Gaussian distribution is assumed). The gain of filters/amplifiers preceding the A/D converter may be set so that $\sigma_x = X_m/4$.

   Then
$$SNR \doteqdot 6B - 1.25dB \tag{12.2-28}$$

   16-bit quantization is required to obtain a SNR of 90 – 96 dB.

Quantization and Overflow in A/D process

Both quantization and overflow introduce errors in the A/D process. Given the number of bits for an A/D converter, to minimize overflow (increase the dynamic range) X_m must be increased resulting in larger quantization errors.

In order to achieve wider dynamic range (larger X_m) and lower quantization erros, the number of bits for the A/D converter must be increased.

In many signal processing implementations, the fixed-point arithmetic is applied so that both signals and system coefficients are proper fractions.

The full scale amplitude X_m of the A/D conver is used as a scale factor so that analog signal amplitudes will be mapped into the range $-1 \leqslant \hat{X_B} < 1$.

(On the C-25 board input is set to 2 V p-p, $X_m = 1$)

Generally, $X_m = 2^c$.

12.3 Quantization in Implementing Systems

Given an LTI system
$$y[n] - ay[n-1] = x[n] \tag{12.3-29}$$

its system function is
$$H(z) = \frac{1}{1 - az^{-1}} \tag{12.3-30}$$

and its impulse response is
$$h[n] = a^n u[n] \tag{12.3-31}$$

The system function of the IIR system is given for both direct forms
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \tag{12.3-32}$$

where a_k and b_k are the ideal infinite-precision coefficients. In the finite-word-length implementation, the coefficients have to be quantized and the system function becomes
$$\hat{H}(z) = \frac{\sum_{k=0}^{M} \hat{b_k} z^{-k}}{1 - \sum_{k=1}^{N} \hat{a_k} z^{-k}} \tag{12.3-33}$$

where $\hat{a_k} = a_k + \Delta a_k$ and $\hat{b_k} = b_k + \Delta b_k$ are the quantized coefficients.

The quantization error in a given coefficient affects all the poles of $H(z)$.

Assume that all the poles are first order and they are at $z = z_i$, $i = 1, 2, \ldots N$. From Equation (12.3-33),
$$A(z) = 1 - \sum_{k=1}^{N} a_k z^{-k} = \prod_{j=1}^{N} (1 - z_j z^{-1}) \tag{12.3-34}$$

12.4 Effects of Coefficient Quantization

The selection of an implementation structure for LTI discrete-time systems depends on the following criteria:

- hardware and software complexity;
- the quantization noise generated internally in the system.

Some structures are more sensitive than others to perturbation of the system coefficients.

12.4.1 Effects of coefficient quantization in IIR systems

System responses (or the poles and zeros of the system function) may be changed due to the quantization of coefficients.

If the system implementation structure is highly sensitive to perturbations of the coeffecients, the resulting system may no longer meet the original design specifications, or worse, even an IIR system might become unstable.

The error in the location of the ith polse due to the quantization errors in $\{a_k\}$ can be expressed as

$$\Delta Z_i = \sum_{k=1}^{N} \frac{jz_i}{ja_k} \Delta a_k \tag{12.4-35}$$

The partial derivatives $\frac{jz_i}{ja_k}$, for $k = 1, 2, \ldots, N$ can be obtained by differentiating $A(z)$ with repsect to each of the $\{a_k\}$.

$$\left(\frac{jA(z)}{ja_k}\right)_{z=z_i} = \left(\frac{jA(z)}{jz}\right)_{z=z_i} \left(\frac{jz_i}{ja_k}\right) \tag{12.4-36}$$

Therefore

$$\frac{jz_i}{ja_k} = \frac{\left(\frac{jA(z)}{ja_k}\right)}{\left(\frac{jA(z)}{jz}\right)_{z=z_i}} \tag{12.4-37}$$

From Equation (12.3-34)

$$\left(\frac{jA(z)}{ja_k}\right)_{z=z_i} = \left(\frac{j(1 - \sum_{k=1}^{N} a_k z^{-k})}{ja_k}\right)_{z=z_i} \tag{12.4-38}$$

$$= (-z^{-k})_{z=z_i} = -z_i^{-k} \tag{12.4-39}$$

And

$$\left(\frac{jA(z)}{jz}\right)_{z=z_i} = \left\{\frac{j}{jz}\left[\prod_{j=1}^{N}(1 - z_j z^{-1})\right]\right\}_{z=z_i} \tag{12.4-40}$$

$$= \left\{\sum_{k=1}^{N} \frac{z_k}{z^2} \prod_{j=1, j\neq k}^{N} (1 - z_j z^{-1})\right\}_{z=z_i} \tag{12.4-41}$$

$$= \frac{1}{z_i^N} \prod_{j=1, j\neq i}^{N} (z_i - z_j) \tag{12.4-42}$$

Thus,

$$\frac{jz_i}{ja_k} = \frac{-z_i^{N-k}}{\prod_{j=1,j\neq i}^{N}(z_i - z_j)} \tag{12.4-43}$$

The total perturbation error is

$$\Delta z_i = -\sum_{k=1}^{N} \frac{z_i^{N-k}}{\prod_{j=1,j\neq i}^{N}(z_i - z_j)} \Delta a_k \tag{12.4-44}$$

From Equation (12.4-44), if $|z_i - z_j|$ is very small, Δz_i will be large. Namely, if the poles are tightly clustered, it is possible that small errors in the denominator coeffecients can cause large shifts of poles for the direct form structures.

REMARK 12.4-1

- *The larger the number of clustered poles; the greater the sensitivity;*
- *Second-order subsystems are not extremely sensitive to quantization;*
- *In both cascade and parallel form system functions, each pair of complex conjugate poles is realized independently of all other poles. Hence, they are generally much less sensitive to coefficient quantization than the equivalent direct form realization.*

EXAMPLE 12.4-1 *Design a bandpass IIR elliptic filter meeting the following specifications:*

$$\begin{aligned} 0.99 \leqslant |H(e^{j\omega})| &\leqslant 1.01 \qquad 0.3\pi \leqslant \omega \leqslant 0.4\pi \\ |H(e^{j\omega})| &\leqslant 0.01 \qquad \omega \leqslant 0.29\pi \end{aligned}$$

and

$$|H(e^{j\omega})| \leqslant 0.01 \qquad 0.41\pi \leqslant \omega \leqslant \pi$$

A filter of order 12 is required.

12.4.2 Effects of coefficient quantization in FIR systems

For an FIR system,

$$H(z) = \sum_{m=0}^{M} h[n]z^{-n} \tag{12.4-45}$$

the direct form structure is commonly used in the implementation.

The quantized coefficients can be expressed by

$$\hat{h}[n] = h[n] + \Delta h[n] \tag{12.4-46}$$

The system function for the quantized system is

$$\hat{H}(z) = \sum_{n=0}^{M} \hat{h}[n]z^{-n} = H(z) + \Delta H(z) \tag{12.4-47}$$

where

$$\Delta H(z) = \sum_{n=0}^{M} \Delta h[n] z^{-n} \tag{12.4-48}$$

The system function and its frequency response of the quantized system is linearly related to the quantization errors in $h[m]$.

• The reason that the direct form FIR system is widely used is that for most linear phase FIR filters, the zeros are more or less uniformly spread in the z-plane.

EXAMPLE 12.4-2 *Design an FIR filter of low pass meeting the following specifications*

$$0.99 \leqslant |H(e^{j\omega})| \leqslant 1.01 \qquad 0 \leqslant \omega \leqslant 0.4\pi \tag{12.4-49}$$

$$|H(e^{j}\omega)| \leqslant 0.001 \qquad 0.6\pi \leqslant \omega \leqslant \pi \tag{12.4-50}$$

The frequency response of an FIR system with quantized coefficients is given by

$$\hat{H}(e^{j\omega}) = H(e^{j\omega}) + \Delta H(e^{j\omega}) \tag{12.4-51}$$

where

$$\Delta H(e^{j\omega}) = \sum_{k=0}^{M} \Delta h[n] e^{-j\omega n} \tag{12.4-52}$$

If the impulse response coefficients $h[n]$ are scaled and represented by proper fractions, when $\{h[n]\}$ are rounded to $(B+1)$ bits, the quantization error is

$$-2^{-(B+1)} < \Delta h[n] \leqslant 2^{-(B+1)} \tag{12.4-53}$$

It follows that

$$|\Delta H(e^{j\omega})| = |\sum_{n=0}^{M} \Delta h[n] e^{-j\omega n}| \leqslant \sum_{n=0}^{M} |\Delta h[n]||e^{-j\omega n}| \tag{12.4-54}$$

$$\leqslant (M+1)2^{-(B+1)} \tag{12.4-55}$$

That is that $(M+1)2^{-(B+1)}$ is the bound on the size of the frequency response error. In example 12.4-2, M+1=28

$$|\Delta H_{16-bit}(e^{j\omega})| \leqslant 0.000427 \tag{12.4-56}$$

$$|\Delta H_{14-bit}(e^{j\omega})| \leqslant 0.001709 \tag{12.4-57}$$

$$|\Delta H_{13-bit}(e^{j\omega})| \leqslant 0.003418 \tag{12.4-58}$$

and

$$|\Delta H_{8-bit}(e^{j\omega})| \leqslant 0.109375 \tag{12.4-59}$$

REMARK 12.4-2

- *The deviations of the system response introduced by the quantization errors are less thatn the corresponding bounds in all the above cases;*
- *The $|\Delta H(e^{j\omega})|$ increases linearly with the length of the impulse response $(M+1)$.*

- *Since a linear phase FIR system has*

$$h[M-n] = h[n] \qquad \textit{symmetric} \tag{12.4-60}$$

or

$$h[M-n] = -h[n] \qquad \textit{antisymmetric} \tag{12.4-61}$$
$$\tag{12.4-62}$$

The linear phase confitions are preserved regardless of the coarseness of the quantization.

12.5 Effects of Roundoff Noise in Digital Filters

12.5.1 Analysis of direct form IIR structures

12.5.2 Scaling in fixed-point implementations of IIR systems

12.6 Zero-Input Limit Cycles in Fixed-Point Realizations of IIR Digital Filters

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