01.

Forward commitments: A forward is an agreement between two parties to buy or sell an asset at a pre-determined future time for a certain price.

- Forward price for a forward contract is defined as the delivery price, which make the value of the contract at initiation be zero.
- The buyer of a forward contract has a "long position" in the asset/commodity.

2. PRINCIPLES OF ARBITRAGE-FREE PRICING AND VALUATION OF FORWARD COMMITMENTS

Forward commitment pricing: Forward commitment pricing involves determining the appropriate forward commitment price or rate at which the forward commitment contract is initiated.

Forward commitment valuation: Forward commitment valuation involves determining the appropriate value of the forward commitment once it has been initiated. Forward value refers to the monetary value of an existing forward or futures contract.

Key assumptions made in pricing and valuation of contracts:

i. Replicating instruments are identifiable and investable;
ii. There are no market frictions;
iii. Short selling is allowed with full use of proceeds;
iv. Borrowing and lending are available at a known risk-free rate.

Note: Cash inflows to the arbitrageur have a positive sign and outflows are negative.

Carry arbitrage models used for forward commitment pricing and valuation are based on the no-arbitrage approach.

- Arbitrage occurs when equivalent assets or combination of assets sell for two different prices.
- The law of one price states that two identical goods must sell for the same current price in the absence of transaction costs. According to law of one price, arbitrage will drive prices of equivalent assets to a single price so that no riskless profits can be earned. The law of one price is based on the value ***additivity principle***, according to which the value of a portfolio is simply the sum of the values of each instrument held in the portfolio.
- Arbitrage opportunities should disappear quickly in an efficient and frictionless market.

3. PRICING AND VALUING FORWARD AND FUTURES CONTRACTS

3.1 Our Notation

Notation:

- 0 = today, T = expiration, underlying asset = $S_{0 (\text{or } t \text{ or } T)}$, forward = F(0,T)
- S_0 denotes the underlying price at the time of forward contract initiation
- S_T denotes the underlying price when the forward contract expires.
- $F_0(T)$ denote the forward price established at the initiation date, 0, and expiring at date T, where T represents a period of time later.
- Uppercase "F" denotes the forward price, whereas lowercase "f" denotes the futures price. Similarly, uppercase "V" denotes the forward value, whereas lowercase "v" denotes the futures value.

- Forward contracts are traded over-the-counter, no money changes hand initially and during the life time of the contract. Hence, the contract value at the initiation of the contract is ZERO. The forward contract value when initiated is expressed as $V_0(T) = v_0(T) = 0$.
- The contract price is set such that the value of the contract is Zero, that is,

  Present value of contract price = Prevailing spot price of the underlying
- Subsequent to the initiation date, the value can be significantly positive or negative.

At Market Contract: The forward contracts having value of zero at contract initiation are referred to as at market.

Property of Convergence: According to property of convergence, at Time T (expiration), both the forward price and the futures price are equivalent to the spot price, that is,

$$F_T(T) = f_T(T) = S_T$$

Important to Remember:

- The market value of a **long position** in a forward contract value <u>at maturity</u> is $V_T(T) = S_T – F_0(T)$.
- The market value of a **short position** in a forward contract value <u>at maturity</u> is $V_T(T) = F_0(T) – S_T$.
- The market value of a **long position** in a futures contract value *before marking to market* is $vt(T) = f_t(T) – f_{t-}(T)$.
- The market value of a short position in a futures contract value *before marking to market* is $v_t(T) = f_{t-}(T) – f_t(T)$.
- The futures contract value after daily settlement is $v_t(T) = 0$.
- If value of underlying > initial forward price, a long position in a forward contract will have a positive value.
- If value of underlying < initial forward price, a short position in a forward contract will have a positive value at expiration.

<u>Note:</u> The forward value and the futures value will be different because futures contracts are marked to market while forward contracts are not being marked to market.

3.2 No-Arbitrage Forward Contracts

3.2.1.) Carry Arbitrage Model When There Are *No* Underlying Cash

Carry arbitrage model is based on following two rules:

1) Do not use your own money, i.e. borrow money to buy the underlying.
2) Do not take any price risk (here refers to market risk); i.e. invest the proceeds from short selling transactions at risk-free rate or in other words, lend the money by selling the underlying).

Cash Flows related to Carrying the Underlying through Calendar Time:

If an arbitrageur enters a forward contract to sell an underlying instrument for delivery at Time T, then this exposure can be hedged by buying the underlying instrument at Time 0 with borrowed funds and carry it to the forward expiration date so it can be sold under the terms of the forward contract.

The table below shows Cash Flows Related to Carrying the Underlying through Calendar Time.

	Underlying Purchased	Underlying Sold
	0	T
Underlying:	$-S_0$	$+S_T$
Borrow:	$+S_0$	$-FV(S_0)$
Forward:	0	$F_0(T) - S_T$
Net:	0	$F_0(T) - FV(S_0)$

- The above figure shows that arbitrageur borrows the money to buy the asset, so at Time T, he will pay back $FV(S_0)$, based on the risk-free rate.
- When $S_T < FV(S_0)$, the arbitrageur will suffer a loss.
- When $S_T = FV(S_0)$, there will be breakeven.
- If we assume continuous compounding (r_c), then $FV(S_0) = S_0e^{r_cT}$
- If we assume annual compounding (r), then $FV(S_0) = S_0(1 + r)^T$

Carry Arbitrage Model Steps[1]: Assuming $S_0 = 100$, r = 5%, T = 1, and $S_T = 90$

1) Purchase one unit of the underlying at Time 0 and sell at T:
   At Time 0: cash outflow of $-S_0 = -100$.
   At Time T: cash inflow of $+S_T = +90$.
2) Borrow the purchase price at Time 0 and repay with interest at Time T.
   At Time 0: cash inflow of $+S_0 = +100$.
   At Time T: cash outflow of $-FV(S_0) = -100(1 + 0.05)^1 = -105$
   Net Cash Flows for Financed Position in the Underlying Instrument
   - Net Cash flow at Time 0: zero
   - Net Cash flow at Time T: $+S_T - FV(S_0) = 90 - 105 = -15$
3) Sell a forward contract on the underlying. Assuming, the forward price is trading at 105.
   At Time 0: Cash inflow of $+V_0(T)$
   At Time T: $V_0(T) = F_0(T) - S_T = 105 - 90 = 15$
4) Pre-capture your arbitrage profit (or in other words borrow it) by bringing it to the present so as to receive it at Time 0. The amount borrowed is forward price minus the future value of the spot price when compounded at the risk-free rate[2].
   At Time 0: Cash inflow of $+PV[F_0(T) - FV(S_0)]$
   At Time T: $V_0(T) = -[F_0(T) - FV(S_0)] = -[105 - 100(1 + 0.05)] = 0$

1Note that all four transactions are done simultaneously not sequentially.
[2]Note that if the forward contract is priced correctly, there will be no arbitrage profit and, hence, no Step 4.
The lending case is not discussed here because it would occur only if a strategy is executed to capture a certain loss.

Net Cash flow:

At Time 0: $+V_0(T) + PV[F_0(T) - FV(S_0)]$

At Time T: $V_0(T) = -[F_0(T) - FV(S_0)] = 0$ (for every underlying value)

The no-arbitrage forward price is simply the future value of the underlying as stated below:

$$F_0(T) = FV(S_0)$$

- If $F_0(1) = 106$, which is higher than that determined by the carry arbitrage model ($F_0(T) = FV(S_0) = 105$). This shows that market forward price is too high and should be sold.
- If the forward price were 106, the value of the forward contract at time 0 would be $V_0(T) = PV[F_0(T) - FV(S_0)] = (106 - 105)/(1 + 0.05) = 0.9524$.
- If the counterparty enters a long position in the forward contract at a forward price of 106, then the forward contract seller has the opportunity to receive the 0.9524 with no liability in the future.

Cash Flows with Forward Contract Market Price Too High Relative to Carry Arbitrage Model

1) Sell forward contract on underlying at $F_0(T) = 106$
   At Time 0: $V_0(T) = 0$
   At Time T: $V_0(T) = V_T(T) = F_0(T) - S_T = 106 - 90 = 16$
2) Purchase underlying at 0 and sell at T:
   At Time 0: $-S_0 = -100$
   At Time T: $-FV(S_0) = +S_T = 90$
3) Borrow funds for underlying purchase
   At Time 0: $+S_0 = 100$
   At Time T: $-FV(S_0) = -100(1 + 0.05) = -105$
4) Borrow arbitrage profit
   At Time 0: $+PV[F_0(T) - FV(S_0)] = (106 - 105)/(1+0.05) = 0.9524$
   At Time T: $-[F_0(T) - FV(S_0)] = -[106 - 100(1+0.05)] = -1$

Net cash flow
At Time 0: 0.9524
At Time T: 16 + 90 – 105 – 1 or –4 + 110 – 105 – 1 = 0

Reverse Carry Arbitrage:
Suppose forward price of $F_0(T) = 104$, which is less than the forward price determined by the carry arbitrage model (105). In this case, the opposite strategy – named "***Reverse Carry Arbitrage"*** is followed. It involves the following steps:

1) Buy a forward contract, and the value at T is $S_T - F_0(T)$.
2) Sell short the underlying instrument.
3) Lend the short sale proceeds.
4) Borrow the arbitrage profit.

Important to Remember:
- If $F_0(T) \neq FV(S_0)$, there is an arbitrage opportunity.
- If $F_0(T) > FV(S_0)$, then the forward contract is sold and the underlying is purchased.
- If $F_0(T) < FV(S_0)$, then the forward contract is purchased and the underlying is sold short.
- If the forward contract price is equal to its equilibrium price, there will be no arbitrage profit and thus no Step 4.
- The quoted forward price does not directly reflect expectations of future underlying prices.

Relationship between Forward price and interest rate:
Forward price is directly related to interest rates – i.e., when interest rate falls (rises), forward price decreases (increases). This relationship between forward prices and interest rates will generally hold except for interest rates forward contracts.

Practice: **Example 1, Reading 39, Curriculum.**

Cash Flows for the Valuation of a Long Forward Position:

Steps	Cash Flow at Time 0	Value at Time t	Cash Flow at Time T
1. Buy forward contract at 0 at $F_0(T)$	0	$V_t(T)$	$V_T(0, T) = S_T - F_0(T)$
2. Sell forward contract at t at $F_t(T)$	NA	0	$V_T(t, T) = F_t(T) - S_T$
Net cash flows/Value	0	$V_t(T)$	$+F_t(T) - F_0(T)$

- "Value at Time t" represents the value of the forward contracts.
  $$F_t(T) = FV_{t,T}(S_t)$$
- The value observed at Time t of the original forward contract initiated at Time 0 and expiring at Time T is simply the present value[3] of the difference in the forward prices, as stated below.

$$V_t(T) = \text{Present value of difference in forward prices}$$
$$= PV_{t,T}[F_t(T) - F_0(T)]$$

Alternatively,

$$\mathbf{V_t(T) = S_t - PV_{t,T}[F_0(T)]}$$

Practice: **Example 2, Reading 39, Curriculum.**

3.2.2.) Carry Arbitrage Model When Underlying Has Cash Flows

[3] Present value is calculated over the remaining life of the contract.

In Carry arbitrage, we are required to pay the interest cost, whereas in reverse carry arbitrage, we receive the interest benefit.

- Let γ denote the carry benefits (for example, dividends, foreign interest, and bond coupon payments that would arise from certain underlyings).

Future value of underlying carry benefits = $\gamma_T = FV_{0,T}(\gamma_0)$

Present value of underlying carry benefits = $\gamma_0 = PV_{0,T}(\gamma_T)$

- Let θ denote the carry costs. These refer to additional costs to hold the commodities, like storage, insurance, deterioration, etc. These can be considered as negative dividends. Carry costs are zero for financial instruments but holding these assets does involve opportunity cost of interest.

Future value of underlying costs = $\theta_T = FV_{0,T}(\theta_0)$

Present value of underlying costs = $\theta_0 = PV_{0,T}(\theta_T)$

Forward price is the future value of the underlying adjusted for carry cash flows. **Forward pricing equation** is stated as below:

$$F_0(T) = \text{Future value of underlying adjusted for carry cash flow} = FV_{0,T}(S_0 + \theta_0 - \gamma_0)$$

- ➢ Carry costs (e.g. interest rate) are added to forward price because they increase the cost of carrying the underlying instrument through time.
- ➢ Carry benefits are subtracted from forward price because they decrease the cost of carrying the underlying instrument through time.

Example: Suppose, $S_0 = 100$, $r = 5\%$, $T = 1$, and $S_T = 90$.Assuming the underlying will distribute 2.9277 at Time t = 0.5: $\gamma t = 2.9277$. The time until the distribution of 2.9277 is t, and hence, the present value is

$$\gamma_0 = 2.9277/(1 + 0.05)^{0.5} = 2.8571$$

The time between the distribution and the forward expiration is $T - t = 0.5$, and thus, the

$$\text{Future value} = \gamma_T = 2.9277(1 + 0.05)^{0.5} = 3$$

Cash Flows for Financed Position in the Underlying with Forward: The steps involved in this strategy are as below:

1) Purchase the underlying at Time 0, receive the dividend at Time t = 0.5 and sell the underlying at Time T.
2) Reinvest the dividend received at Time t = 0.5 at the risk-free interest rate until Time T.
3) Borrow the initial cost of the underlying.
4) Sell a forward contract at Time 0 and the underlying will be delivered at Time T.
5) Borrow the arbitrage profit.

Cash flows are reflected in the table:

Steps	Cash Flow at Time 0	Cash Flow at Time t	Cash Flow at Time T
1. Purchase underlying at 0, sell at T	$-S_0 = -100$	$+\gamma_t = 2.9277$	$+S_T = 90$ or $+S_T = 110$
2. Reinvest distribution		$-\gamma_t = -2.9277$	$+\gamma_T = 2.9277(1 + 0.05)^{0.5} = 3$
3. Borrow funds	$+S_0 = 100$		$-FV(S_0) = -100(1 + 0.05)^1 = -105$
4. Sell forward contract	$V_0(T)$		$V_T(T) = F_0(T) - S_T = 102 - 90 = 12$ or $= 102 - 110 = -8$
5. Borrow arbitrage profit	$+PV[F_0(T) + \gamma_T - FV(S_0)]$		$-[F_0(T) + \gamma_T - FV(S_0)]$
Net cash flows	$V_0(T) + PV[F_0(T) + \gamma_T - FV(S_0)]$	0	$+S_T + \gamma_T - FV(S_0) + F_0(T) - S_T - [F_0(T) + \gamma_T - FV(S_0)] = 0$

The value of the cash flow at Time 0 is zero, or

$$V_0(T) + PV[F_0(T) + \gamma_T - FV(S_0)] = 0$$

and

$$V_0(T) = -PV[F_0(T) + \gamma T - FV(S_0)].$$

If theForward contract has zero value, then

Forward Price = $F_0(T)$ = Future value of underlying – Future value of carry benefits= $FV(S_0) - \gamma_T$

Initial forward price = Future value of the underlying - Value of any ownership benefits at expiration

or

$$F_0(T) = FV_{0,T}(S_0 - \gamma_0)$$

Forward value for a long position is estimated using the following:

$$V_t(T) = \text{Present value of difference in forward prices} = PV_{t,T}[F_t(T) - F_0(T)]$$

Where, $F_t(T) = FV_{t,T}(S_t + \theta_t - \gamma_t)$

Annual compounding and continuous compounding: The equivalence between annual compounding and continuouscompounding can be expressed as follows:

$$(1 + r)^T = e^{rct}$$

or

$$r_c = \ln[(1 + r)^T]/T = \ln(1 + r);$$

If the quoted interest rate is 5% based on annual compounding, then the implied interest rate based on continuous compounding is

$$r_c = \ln(1 + r) = \ln(1 + 0.05) = 0.0488\text{, or } 4.88\%$$

- ➢ This implies that a cash flow compounded at 5% annually is equivalent to being compounded at

4.88% continuously.
- Continuous compounding results in a lower quoted rate.

Carry arbitrage model with continuous compounding:
The carry arbitrage model with continuous compounding is expressed as

$$F_0(T) = S_0 e^{(r_c+\theta-\gamma)T} \text{ (Future value of the underlying adjusted for carry)}$$

The future value of the underlying adjusted for carry, i.e., the dividend payments, is $F_0(T) = S_0 e^{(r_c-\gamma)T}$

- If a dividend payment is announced between the forward's valuation and expiration dates, assuming the news announcement does not change the current underlying price, the forward value will most likely decrease.
- If a new dividend is imposed, the new forward price will decrease and consequently, the value of the old forward contract will be lower.

3.3 Equity Forward and Futures Contracts

Since, futures contracts are marked to market daily, the equity futures value is zero each day after settlement has occurred.

Practice: Example 3, 4 & 5 Reading 39, Curriculum.

3.4 Interest Rate Forward and Futures Contracts

Libor, which stands for London Interbank Offered Rate, is a widely used interest rate that serves as the underlying for many derivative instruments. It represents the rate at which London banks can borrow from other London banks.

- When these loans are in dollars, they are known as ***Eurodollar*** time deposits, with the rate referred to as dollar Libor.
- Average Libor rates are derived and posted each day at 11:30 a.m. London time.
- Libor is stated on an actual over 360-day count basis (often denoted ACT/360) with interest paid on an add-on basis.

Let,
$L_i(m)$ = Libor on an m-day deposit observed on day i
NA = notional amount, quantity of funds initially deposited
NTD = number of total days in a year, used for interest calculations (always 360 in the Libor market)
t_m = accrual period, fraction of year for m-day deposit—tm = m/NTD
TA = terminal amount, quantity of funds repaid when the Libor deposit is withdrawn

Example: Suppose day i is designated as Time 0, and we are considering a 90-day Eurodollar deposit (m = 90). Dollar Libor is quoted at 2%; thus, $L_i(m) = L_0(90) = 0.02$. $50,000 is initially deposited, i.e. NA = $50,000. Hence,

$$t_m = 90/360 = 0.25$$
$$TA = NA\,[1 + L_0(m)t_m] = \$50{,}000[1 + 0.02(90/360)] = \$50{,}250$$
$$\text{Interest paid} = TA - NA = \$50{,}250 - \$50{,}000 = \$250$$

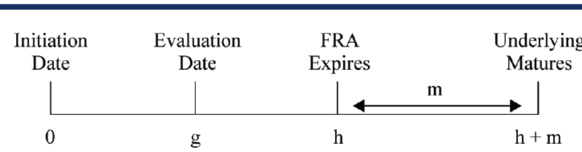
Forward market for Libor: A forward rate agreement (FRA) is an over-the-counter (OTC) forward contract in which the underlying is an interest rate on a deposit. An FRA involves two counterparties: the fixed receiver (short) and the floating receiver (long).

- Being long the FRA means that we gain when Libor rises.
- The fixed receiver counterparty receives an interest payment based on a fixed rate and makes an interest payment based on a floating rate.
- The floating receiver counterparty receives an interest payment based on a floating rate and makes an interest payment based on a fixed rate.
- FRA price is the fixed interest rate such that the FRA value is zero on the initiation date.
- The underlying of an FRA is an interest payment.
- It is also important to understand that the parties to an FRA do not necessarily engage in a Libor deposit in the spot market. Rather, a Libor spot market is simply the benchmark from which the payoff of the FRA is determined.

A 3 × 9 FRA is pronounced as "3 by 9." It implies that FRA expires in ***three*** months and the payoff of the FRA is 6months Libor (i.e. 9 -3) when the FRA expires in 3 months.

- A short (long) FRA will effectively replicate going short (long) a nine-month Libor deposit and long (short) a three-month FRA deposit.
- FRA value is the market value on the evaluation date and reflects the fair value of the original position.

Important FRA Dates, Expressed in Days from Initiation



Example: A 30-day FRA on 90-day Libor would have h = 30, m = 90, and h + m = 120. If we want to value the FRA prior to expiration, g could be any day between 0 and 30.

- FRA (0, h, m) denotes the fixed forward rate set at Time 0 that expires at Time h wherein the underlying Libor deposit has m days to maturity at expiration of the FRA.

- Thus, the rate set at initiation of a contract expiring in 30 days in which the underlying is 90-day Libor is denoted FRA (0, 30, 90).
- Like all standard forward contracts, at initiation, no money changes hands, implying value is zero.
- We can estimate price of FRA by determining the fixed rate [FRA(0,30,90)] such that the value is zero on the initiation date.

How to settle interest rate derivative at expiration: There are two ways to settle an interest rate derivative when it expires:

1) **Advanced set, settled in arrears:** Advanced set implies that the reference interest rate is set at the time the money is deposited. The term settled in arrears means that the interest payment is made at Time h + m, (i.e. at the maturity of the underlying instrument). Swaps and interest rate options are normally based on advanced set, settled in arrears.
2) **Advanced set, advanced settled:** FRAs are typically settled based on advanced set, advanced settled. In an FRA, the term "advanced" refers to the fact that the interest rate is set at Time h, the FRA expiration date, which is the time when the underlying deposit starts. Here, advanced settled means the settlement is made at Time h. Libor spot deposits are settled in arrears, whereas FRA payoffs are settled in advance.

The **settlement amounts for advanced set, advanced settled** are determined in the following manner:

- Settlement amount at h for receive-floating: NA{[(m) L_h – FRA(0,h,m)]t_m}/[1 + $D_h(m)t_m$]
- Settlement amount at h for receive-fixed: NA{[FRA(0,h,m) – $L_h(m)$]t_m}/[1 + $D_h(m)t_m$]

Where, 1 + $D_h(m)t_m$ is a discount factor applied to the FRA payoff. It reflects that the rate on which the payoff is determined, $L_h(m)$, is obtained on day h from the Libor spot market, which uses settled in arrears, that is, interest to be paid on day h + m.

Example: In 30 days, a UK company expects to make a bank deposit of £10,000,000 for a period of 90 days at 90-day Libor set 30 days from today. The company is concerned about a possible decrease in interest rates. The company enters into a £10,000,000 notional amount 1 × 4 receive-fixed FRA that is advanced set, advanced settled. This implies that an instrument that expires in 30 days and is based on 90-day (4 – 1) Libor. The discount rate for the FRA settlement cash flows is 0.40%. After 30 days, 90-day Libor in British pounds is 0.55%.

TA = 10,000,000[1 + 0.0055(0.25)] = £10,013,750.
Interest paid at maturity = TA – NA = £10,013,750 - £10,000,000 = £13,750.

- If the FRA was initially priced at 0.60%, the payment received to settle it will be closest to:

m = 90 (number of days in the deposit)
tm = 90/360
h = 30 (number of days initially in the FRA)

The settlement amount of the 1 × 4 FRA at h for receive-fixed = [10,000,000(0.0060 – 0.0055)(0.25)]/[1 + 0.0040(0.25)] = £1,248.75

- If the FRA was initially priced at 0.50%, the payment received to settle it will be closest to as follows:

Settlement amount of the 1 × 4 FRA at h for receive-fixed = [10,000,000(0.0050 – 0.0055)(0.25)]/[1 + 0.0040(0.25)] = –£1,248.75

- In pay floating FRA, the long benefits when interest rate declines.

Practice: Example 6, Reading 39, Curriculum.

FRA pricing: Steps are as follows:

Step 1: Deposit funds for h + m days:

- **At Time 0:** deposit an amount = 1/[1 + $L_0(h)t_h$], the present value of 1 maturing in h days, in a bank for h+ m days at an agreed upon rate of $L_0(h + m)$.
- **After h + m days**, withdraw an amount = [1 + $L_0(h + m)t_{h+m}$]/[1 + $L_0(h)t_h$]

Step 2: Borrow funds for h days:

- **At Time 0:** Borrow {1/[1 + $L_0(h)t_h$]}, for h days so that the net cash flow at Time 0 is zero.
- In h days, this borrowing will be worth 1.

Step 3: At Time h, roll over the maturing loan in Step 2 by borrowing funds for m days at the rate $L_h(m)$. At the end of m days, we will owe [1 + $L_h(m)t_m$].
In order to mitigate the risk of increase in interest rate, we would enter into a receive-floating FRA on m-day Libor that expires at Time h and has the rate set at FRA(0,h,m) as defined in step 4.

Step 4: Enter a receive-floating FRA and roll the payoff at h to h + m at the rate $L_h(m)$. The payoff at Time h will be ([$L_h(m)$ – FRA(0,h,m)]t_m)/(1 + $L_h(m)t_m$). There will be no cash flow from this FRA at Time h because this amount will be rolled forward at the rate $L_h(m)t_m$. Therefore, the value realized at Time h + m will be [$L_h(m)$ – FRA(0,h,m)]t_m.

Cash Flow Table for Deposit and Lending Strategy with FRA

Steps	Cash Flow at Time 0	Cash Flow at Time h	Cash Flow at Time h + m
1. Make deposit for h + m days	$-1/[1 + L_0(h)t_h]$ = −0.996264	0	$+[1 + L_0(h + m)t_{h+m}]/[1 + L_0(h)t_h]$ = 1.006227
2. Borrow funds for h days	$+1/[1 + L_0(h)t_h]$ = +0.996264	−1	
3. Borrow funds for m days initiated at h		+1	$-[1 + L_h(m)t_m]$ = −1.0075
4. Receive-floating FRA and roll payoff at $L_h(m)$ rate from h to h + m	0	0	$+[L_h(m) - FRA(0, h, m)]t_m$ = [0.03 − FRA(0,h,m)] (90/360)
Net cash flows	0	0	$+[1 + L_0(h + m)t_{h+m}]/[1 + L_0(h)t_h] - [1 + L_h(m)t_m] + [L_h(m) - FRA(0,h,m)]t_m$

The terminal cash flows as expressed in the table can be used to solve for the FRA fixed rate. Because the transaction starts off with no initial investment or receipt of cash, the net cash flows at Time h + m should equal zero; thus,

$$+[1 + L_0(h+m)t_{h+m}]/[1 + L_0(h)t_h] - [1 + L_h(m)t_m] + [L_h(m) - FRA(0,h,m)]t_m = 0$$

FRA fixed rate:

$$FRA(0,h,m) = \{[1 + L_0(h+m)t_{h+m}]/[1 + L_0(h)t_h] - 1\}/t_m$$

E.g. $FRA(0,90,90) = \{[1 + L_0(180)t_{180}]/[1 + L_0(90)t_{90}] - 1\}/t_{90}$

Practice:Example 7, Reading 39, Curriculum.

Valuing an existing FRA:If we are long the old FRA, we will receive the rate $L_h(m)$ at h. We will go short a new FRA that will force us to pay $L_h(m)$ at h. Suppose that we initiate an FRA that expires in 90 days and is based on 90-day Libor. The fixed rate at initiation is 2.49%. Thus, tm = 90/360, and FRA (0,h,m) = FRA(0,90,90) = 2.49%.

- When the FRA expires and makes its payoff, assume that we roll it forward by lending it (if a gain) or borrowing it (if a loss) from period h to period h + m at the rate $L_h(m)$. We then collect or pay the rolled forward value at h + m. Thus, there is no cash realized at Time h.
- Assume 30 days later, the rate on an FRA based on 90-day Libor that expires in 60 days is 2.59%. Thus, FRA (g, h – g, m) = FRA(30,60,90) = 2.59%. We go short this FRA, and as with the long FRA, we roll forward its payoff from Time h to h + m. Therefore, there is no cash realized from this FRA at Time h.

Value of the offset position = (2.59% – 2.49%) = 10 bps times 90/360 paid at Time h + m

- To determine the fair value of the original FRA at Time g, we need the present value of this Time h + m cash flow at Time g.

Value of the old FRA = Present value of the difference in the new FRA rate and the old FRA rate

Hence, the value is

$$V_g(0,h,m) = \{[FRA(g,h-g,m) - FRA(0,h,m)]t_m\} / [1 + D_g(h+m-g)t_{h+m-g}]$$

$$FRA(g,h-g,m) = \{[1 + L_g(h+m-g)t_{h+m-g}]/[1 + L_g(h-g)t_{h-g}] - 1\}/t_m$$

Where, $V_g(0,h,m)$ is the value of the FRA at Time g that was initiated at Time 0, expires at Time h, and is based on m-day Libor. Dg(h + m – g) is the discount rate. Traditionally, it is assumed that the discount rate, $D_g(h + m - g)$, is equal to the underlying floating rate, $L_g(h + m - g)$, but that is not necessary.

Example: Suppose a 60- day rate of 3% on day g. Thus, $L_g(h - g) = L_{30}(60) = 3\%$. Then the value of the FRA would be

$$V_g(0,h,m) = V_{60}(0,90,90) = 0.00025/[1 + 0.03(60/360)] = 0.000249$$

Cash Flows for FRA Valuation are as following:

Steps	Flow at Time g	Time h	Cash Flow at Time h + m
1. Receive-floating FRA (settled in arrears) at Time 0; roll forward at Rate $L_h(m)$ from h to h + m		0	$+\{[L_h(m) - FRA(0,h,m)]t_m\}$ = $+(L_h(m) - 0.0249)(90/360)$
2. Receive-fixed FRA (settled in arrears) at Time g; roll forward at Rate $L_h(m)$ from h to h + m	0	0	$+[FRA(g,h - g,m) - L_h(m)]t_m$ = $+[0.0259 - L_h(m)](90/360)$
Net cash flows	0	0	$+[FRA(g,h - g,m) - FRA(0,h,m)]t_m$ = +(0.0259 – 0.0249)(90/360) = 0.00025

Practice:Example 8, Reading 39, Curriculum.

3.5 Fixed-Income Forward and Futures Contracts

Accrued interest = Accrual period × Periodic coupon amount

or

AI = (NAD/NTD) × (C/n)

Where NAD denotes the number of accrued days since the last coupon payment, NTD denotes the number of total days during the coupon payment period, n denotes the number of coupon payments per year, and C is the stated annual coupon amount.

Example: After two months (60 days), a 3% semi-annual coupon bond with par of 1,000 would have accrued interest of AI = (60/180) × (30/2) = 5.

Important to remember:

- The accrued interest is expressed in currency (not percent) and the number of total days (NTD) depends on the coupon payment frequency (semi-annual on 30/360 day count convention would be 180).

We know that Forward price is equal to Future value of underlying adjusted for carry cash flows, as stated below:

$$= FV_{0,T}(S_0 + \theta_0 - \gamma_0)$$

- For the fixed-income bond, let T + Y denote the underlying instrument's current time to maturity. Therefore, Y is the time to maturity of the underlying bond at Time T, when the contract expires.
- Let $B_0(T + Y)$ denote the quoted price observed at Time 0 of a fixed-rate bond that matures at Time T + Y and pays a fixed coupon rate.
- For bonds quoted without accrued interest, let AI_0 denote the accrued interest at Time 0.
- The carry benefits are the bond's fixed coupon payments, γ_0 = present value of all coupon interest paid over the forward contract horizon from Time 0 to Time T = $PVCI_{0,T}$.
- Future value of these coupons is $\gamma_T = FVCI_{0,T}$.
- Assuming no carry costs, $\theta_0 = 0$.

$$S_0 = \text{Quoted bond price + Accrued interest} = B_0(T + Y) + AI_0 \quad (1)$$

Fixed-income futures contracts: Fixed-income futures contracts often have more than one bond that can be delivered by the seller. These bonds are usually traded at different prices based on maturity and stated coupon, therefore, an adjustment known as the ***conversion factor*** is used to make prices of all deliverable bonds equal (roughly, not exactly).

In Fixed-incomefutures contracts markets, the futures price, $F_0(T)$, is defined as

Quoted futures price × conversion factor= $QF_0(T) \times CF(T)$

In general, the futures contract are settled against the quoted bond price without accrued interest. Thus, the total profit or loss on a long futures position = $B_T(T + Y) - F_0(T)$. Based on above equation (1), this profit or loss can be expressed as follows:

$$(S_T - AI_T) - F_0(T)$$

Adjusted Price of fixed-income forward or futures price including the conversion factor can be expressed as

$F_0(T) = QF_0(T)\, CF(T)$ =Future value of underlying adjusted for carry cash flows = $FV_{0,T}[S_0 - PVCI_{0,T}]$ = Future value (Quoted bond price + accrued interest - coupon payments made during the life of the contract) = $FV_{0,T}[B_0(T+Y) + AI_0 - PVCI_{0,T}]$

<u>Steps of Carry arbitrage in the bond market:</u>

Step 1: Buy the underlying bond, requiring S_0 cash flow.
Step 2: Borrow an amount equivalent to the cost of the underlying bond, S_0.
Step 3: Sell the futures contract at $F_0(T)$.
Step 4: Borrow the arbitrage profit.

Exhibit 11. Cash Flows for Fixed Rate Coupon Bond Futures Pricing

Steps	Cash Flow at Time 0	Cash Flow at Time T
1. Buy bond	$-S_0 = -[B_0(T + Y) + AI_0]$ $= -[107 + 0.07]$ $= -107.07$	$S_T + FVCI_{0,T}$ $= 110.20 + 0.0$ $= 110.20$
2. Borrow	$+S_0 = 107.07$	$-FV_{0,T}(S_0)$ $= -(1+0.002)^{0.25}(107.07)$ $= -107.12$
3. Sell futures	0	$F_0(T) - B_T(T + Y)$ $= 108 - 110$ $= -2$
4. Borrow arbitrage profit	$+PV_{0,T}[F_0(T) - FV_{0,T}(S_0) + AI_T + FVCI_{0,T}]$ $= (1 + 0.002)^{-0.25}[108 - 107.12 + 0.20 + 0.0]$ $= 1.0795$	$-[F_0(T) - FV_{0,T}(S_0) + AI_T + FVCI_{0,T}]$ $= -[108 - 107.12 + 0.20 + 0.0]$ $= -1.08$
Net cash flows	$+PV_{0,T}[F_0(T) - FV_{0,T}(S_0) + AI_T + FVCI_{0,T}]$ $=1.0795$	0

- ➢ The value of the Time 0 cash flows ***should be zero*** or else there is an arbitrage opportunity.
- ➢ If the value in the Time 0 column for net cash flows is positive, then we buy bond, borrow, and sell futures.
- ➢ If the Time 0 column is negative, then we conduct the reverse carry arbitrage strategy, i.e. short sell bond, lend, and buy futures.

In equilibrium, to eliminate an arbitrage opportunity, $PV_{0,T}[F_0(T) - FV_{0,T}(S_0) + AI_T + FVCI_{0,T}] = 0$ or $F_0(T) = FV_{0,T}(S_0) - AI_T - FVCI_{0,T}$

$QF_0(T)$ = Conversion factor adjusted future value of underlying adjusted for carry = $[1/CF(T)]\{FV_{0,T}[B_0(T + Y) + AI_0] - AI_T - FVCI_{0,T}\}$

<u>Practice:</u>Example 9, Reading 39, Curriculum.

Cash Flows for Offsetting a Long Forward Position:

Steps	Cash Flow at Time 0	Cash Flow at Time t	Cash Flow at Time T
1. Buy bond forward contract at 0	0	$V_t(T)$	$V_T(0,T) = B_T(T + Y) - F_0(T)$ $= 108 - 107.12 = 0.88$
2. Sell bond forward	NA	0	
contract at t			$V_T(t,T) = F_t(T) - B_T(T + Y)$ $= 107.92 - 108 = -0.08$
Net cash flows	0	$V_t(T)$	$F_t(T) - F_0(T)$ $= 107.92 - 107.12 = 0.8$

Practice: Example 10, Reading 39, Curriculum.

3.6 Currency Forward and Futures Contracts

The carry arbitrage model with foreign exchange presented here is also known as covered interest rate parity and sometimes just ***interest rate parity***. We will discuss two strategies here.

Strategy #1: Invest one currency unit in a domestic risk-free bond. Thus, at Time T, we have the original investment grossed up at the domestic interest rate or the future value of 1DC, denoted FV(1DC). Future value at Time T of this strategy is expressed as $FV_{£,T}(1)$, given British pounds as the domestic currency.

Strategy #2:

1) Firstly, the domestic currency is converted at the current spot exchange rate, $S_0(FC/DC)$, into the foreign currency (FC), that is, $S_0(DC/FC) = 1/S_0(FC/DC)$.
2) Then, FC is invested at the foreign risk-free rate until Time T. For example, the future value at Time T of this strategy can be expressed as $FV_{€,T}(1)$ - denoting the future value of one euro, given that the euro is the foreign currency.
3) And then, we enter into a forward foreign exchange contract to sell the foreign currency at Time T in exchange for domestic currency with the forward rate denoted $F_0(DC/FC,T)$. So, for example, $F_0(£/€,T)$ is the rate on a forward commitment at Time 0 to sell one euro for British pounds at Time T. This transaction is equivalent to taking short position in the euro in pound terms or being long the pound in euro terms for delivery at Time T.

Based on the two strategies, the value at Time T follows:

Strategy 1: Future value at Time T of investing £1: $FV_{£,T}(1)$

Strategy 2: Future value at Time T of investing £1: $F_0(£/€,T)FV_{€,T}(1)S_0(€/£)$

Solving for the forward foreign exchange rate, the forward rate can be expressed as

$F_0(£/€,T)$ = Future value of spot exchange rate adjusted for foreign rate

$$= FV_{£,T}(1)/\left[FV_{€,T}(1)S_0(€/£)\right] = S_0(£/€)FV_{£,T}(1)/FV_{€,T}(1)$$

- The higher the foreign interest rate, the greater the benefit, and hence, the lower the forward or futures price.

Practice: Example 11, Reading 39, Curriculum.

Assumingannual compounding and denoting the risk-free rates r£ and r€, respectively, we have

$$F_0(£/€,T) = S_0(£/€)(1 + r£)^T/(1 + r€)^T$$

Assuming continuous compounding and denoting these risk-free rates in domestic (UK) and eurozone as $r_{£,c}$ and $r_{€,c}$, respectively, we have

$$F_0(£/€,T) = S_0(£/€)\, e^{(r_{£,c}-r_{€,c})T} \text{ (Continuously compounded version)}$$

Carry arbitrage model: The carry arbitrage model based on $S_0(FC/DC) = 1/S_0(DC/FC)$ and $F_0(FC/DC) = 1/F_0(DC/FC)$ can be expressed as follows:

$$F_0(DC/FC,T) = S_0(DC/FC)\frac{(1+r_{DC})^T}{(1+r_{FC})^T} \text{ or } F_0(FC/DC,T) = S_0(FC/DC)\frac{(1+r_{FC})^T}{(1+r_{DC})^T}$$

For, continuous compounding:

$$F_0(DC/FC,T) = S_0(DC/FC)e^{(r_{DC,c}-r_{FC,c})T}$$

$$F_0(FC/DC,T) = S_0(FC/DC)e^{(r_{FC,c}-r_{DC,c})T}$$

- The interest rate in the numerator should be the rate for the country whose currency is specified in the spot rate quote. The interest rate in the denominator is the rate in the other country.
- Similarly, in continuous compounding formula, the first interest rate in the exponential will be the rate for the country whose currency is specified in the spot rate quote.

In equilibrium,

$$F_0(£/€,T) = S_0(£/€)FV_£(1)/FV_€(1)$$

Please refer to following table for cash flows for offsetting a long forward position:

Cash Flows for Offsetting a Long Forward Position

Steps	Cash Flow at Time 0	Cash Flow at Time t	Cash Flow at Time T
1. Buy forward contract at 0	0	$V_t(T)$	$V_T(0,T) = S_T(£/€) - F_0(£/€,T)$ $= 1.2 - 0.804 = 0.396$
2. Sell forward contract at t	NA	0	$V_T(t,T) = F_t(£/€,T) - S_T(£/€)$ $= 0.901 - 1.2 = -0.299$
Net cash flows	0	$V_t(T)$	$+F_t(£/€,T) - F_0(£/€,T)$ $= 0.901 - 0.804 = 0.097$

The forward value observed at t of a T maturity forward contract = Present value of the difference in foreign exchange forward prices. That is,

$$V_t(T) = \text{Present value of the difference in forward prices}$$
$$= PV_{£,t,T}[F_t(£/€,T) - F_0(£/€,T)]$$

Practice: Example 12, Reading 39, Curriculum.

3.7 Comparing Forward and Futures Contracts

Forward pricing: $F_0(T) = FV_{0,T}(S_0 + \theta_0 - \gamma_0)$

Note that the price of a forward commitment is a function of the price of the underlying instrument, financing costs, and other carry costs and benefits.

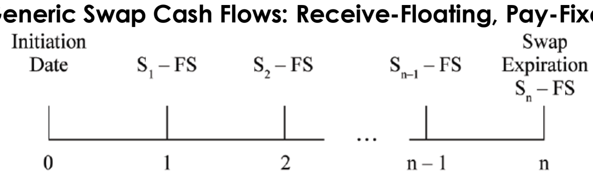
Forward valuation: $V_t(T) = PV_{t,T}[F_t(T) - F_0(T)]$

Futures prices are generally found using the same model, but unlike forwards, futures values are zero at the end of each day because daily market to market settlement.

4. PRICING AND VALUING SWAP CONTRACTS

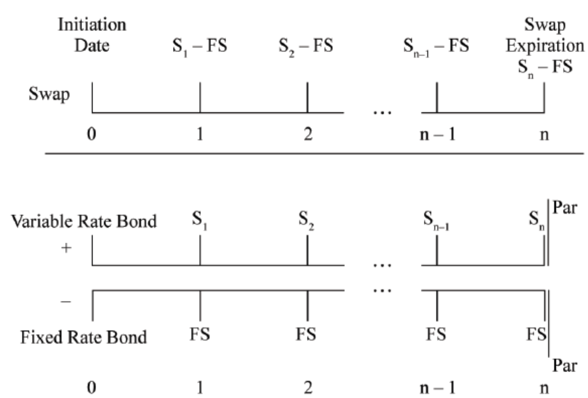
Swap contracts can be synthetically created by either a portfolio of underlying instruments or a portfolio of forward contracts. Thus, swaps can be viewed as a portfolio of futures contracts. A swap can also be viewed as a portfolio of option because a single forward contract can be viewed as a portfolio of a call and a put option.

Generic Swap Cash Flows: Receive-Floating, Pay-Fixed



- A ***receive-floating, pay-fixed swap*** is equivalent to being long a floating-rate bond and short a fixed-rate bond. If both bonds are purchased at par, the initial cash flows are zero and the par payments at the end offset each other. Also, note that the coupon dates on the bonds match the settlement dates on the swap and the maturity date matches the expiration date of the swap.

Receive-Floating, Pay-Fixed as a Portfolio of Bonds



Uses of Swaps: Swaps can be used to manage interest rate risk. E.g. we can create a synthetic floating-rate bond by entering a receive-fixed, pay-floating interest rate swap. This swap can be used to hedge exposure to fixed rate loan. The two fixed rate payments (i.e. on loan and swap) cancel each other, leaving on net the floating-rate payments.

There are also currency swaps and equity swaps. Currency swaps can be used to manage both interest rate and currency exposures. Equity swaps can be used to manage equity exposure.

Like OTC products, swaps can be designed with an infinite number of variations. A swap can have both semi-annual payments and quarterly payments, as well as actual day counts and day counts based on 30 days per month. Also, the notional amount can vary across the maturities. Due to differences in payment frequency and day count methods as well as identifying the appropriate discount rate to apply to the future cash flows, the pricing and valuation of swaps is a bit tricky.

4.1 Interest Rate Swap Contracts

Interest rate swaps have two legs, typically a floating leg (FLT) and a fixed leg (FIX). The floating leg cash flow (denoted Si) can be expressed as follows:

$$S_i = CF_{FLT,i} = AP_{FLT,i}\, r_{FLT,i} = \left(\frac{NAD_{FLT,i}}{NTD_{FLT,i}}\right) r_{FLT,i}$$

The fixed leg cash flow (denoted FS) can be expressed as follows:

$$FS = CF_{FIX,i} = AP_{FIX,i}\, r_{FIX} = \left(\frac{NAD_{FIX,i}}{NTD_{FIX,i}}\right) r_{FIX}$$

Where,

- CF_i represents cash flows
- APi denotes the accrual period
- r denotes the observed floating rate appropriate for Time i
- NADi denotes the number of accrued days during the payment period
- NTDi denotes the total number of days during the year applicable to cash flow i
- rFIX denotes the fixed swap rate.

Types of day count methods: The two most popular day count methods are known as 30/360 and ACT/ACT.

- As the name suggests, 30/360 treats each month as having 30 days, and thus a year has 360 days.
- ACT/ACT treats the accrual period as having the actual number of days divided by the actual number of days in the year (365 or 366).

In swap market, the floating interest rate is assumed to be advanced set and settled in arrears; thus, $r_{FLT,i}$ is set at the beginning of period and paid at the end. If we assume constant accrual periods, the receive-fixed, pay-floating net cash flow can be expressed as follows:

$$FS - S_i = AP(r_{FIX} - r_{FLT,i})$$

And the receive-floating, pay-fixed net cash flow can be expressed as follows:

$$S_i - FS = AP(r_{FLT,i} - r_{FIX})$$

Example: Suppose, a fixed rate is 5%, the floating rate is 5.2%, and the accrual period is 30 days based on a 360 day year, the payment of a receive-fixed, pay-floating swap is calculated as (30/360)(0.05 – 0.052) = –0.000167 per notional of 1. Because the floating rate > fixed rate, the party that pays floating (and receives fixed)would pay this amount to the party that receives floating (and pays fixed).

Swap pricing: Swap pricing involves determining the equilibrium fixed swap rate. The fixed swap rate is simply one minus the final present value term divided by the sum of present values (as discussed in detail below). Suppose the arbitrageur enters a receive-fixed, pay-floating interest rate swap with some initial value V. Please see the cash flows for receive-fixed swap hedge with bonds as stated below:

Cash Flows for Receive-Fixed Swap Hedge with Bonds

Steps	Time 0	Time 1	Time 2	...	Time n
1. Receive fixed swap	–V	+FS – S_1	+FS – S_2	...	+FS – S_n
2. Buy floating-rate bond	–VB	+S_1	+S_2	...	+S_n + Par
3. Short sell fixed-rate bond	+FB	–FS	–FS	...	–(FS + Par)
Net cash flows	–V – VB + FB	0	0	0	0

In equilibrium, we must have –V – VB + FB = 0 or else there is an arbitrage opportunity.
For a receive fixed and pay floating swap, the value of the swap is

V = Value of fixed bond – Value of floating bond = FB – VB

- The value of a receive-fixed, pay-floating interest rate swap is simply the value of buying a fixed-rate bond and issuing a floating-rate bond.
- The value of a floating-rate bond, assuming we are on a reset date and the interest payment matches the discount rate, is par, assumed to be 1 here.
- The value of a fixed bond is as follows:

$$\text{Fixed bond rate: } FB = C\sum_{i=1}^{n} PV_{0,t_i}(1) + PV_{0,t_n}(1)$$

Where, C denotes the coupon amount for the fixed-rate bond and $PV_{0,ti}$ (1) is the appropriate present value factor for the i^{th} fixed cash flow.

$$\text{Swap pricing equation: } r_{FIX} = \frac{1 - PV_{0,t_n}(1)}{\sum_{i=1}^{n} PV_{0,t_i}(1)}$$

- The fixed swap leg cash flow for a unit of notional amount is simply the fixed swap rate adjusted for the accrual period, i.e. $FS_i = AP_{FIX,i}r_{FIX}$.
- The annualized fixed swap rate = fixed swap leg cash flow / fixed rate accrual period, or
$r_{FIX,i} = FS/AP_{FIX}$
- The fixed swap payment will vary if the accrual period varies across the swap payments.

Practice: Example 13, Reading 39, Curriculum.

Interest rate swap valuation:
The value of a fixed rate swap at some future point in Time t is simply the sum of the present value of the difference in fixed swap rates times the stated notional amount (denoted NA), or

$$V = NA(FS_0 - FS_t)\sum_{i=1}^{n'} PV_{t,t_i}$$

➢ Positive (negative) value of FS_0 represents value to the party receiving (paying) fixed.

Please refer below to the Cash Flows for Receive-fixed Swap Valued at Time t:

Steps	Time t	Time 1	Time 2	...	Time n′
1. Receive fixed swap (Time 0)	$-V$	$+FS_0 - S_1$	$+FS_0 - S_2$	...	$+FS_0 - S_{n'}$
2. Receive floating swap (Time t)	0	$S_1 - FS_t$	$S_2 - FS_t$	...	$S_{n'} - FS_t$
Net cash flows	$-V$	$FS_0 - FS_t$	$FS_0 - FS_t$	...	$FS_0 - FS_t$

Practice: Example 14, Reading 39, Curriculum.

4.2 Currency Swap Contracts

A currency swap is a contract in which two counterparties agree to exchange future interest payments in different currencies. There are four major types of currency swaps:

1. Fixed-for-fixed
2. Floating for- fixed
3. Fixed-for-floating
4. Floating-for-floating

Important to Remember:

- Currency swaps often (but do not always) involve an exchange of notional amounts at both the initiation of the swap and at the expiration of the swap.
- The payment on each leg of the swap is in a different currency unit, such as euros and dollars, and the payments are not netted.
- Each leg of the swap can be either fixed or floating.

Currency swap pricing has three key variables: These include two fixed interest rates and one notional amount.

The value of a fixed-rate bond in Currency k can be expressed as

$$FB_k = C_k \sum_{i=1}^{n} PV_{0,t_i,k}(1) + PV_{0,t_n,k}(Par_k)$$

Where k = a or b, C_k denotes the periodic fixed coupon amount in Currency k,

$$\sum_{i=1}^{n} PV_{0,t_i,k}(1)$$

denotes the present value from Time 0 to Time t_i discounting at the Currency k risk-free rate, and Par_k denotes the k currency unit par value.

Here, par is not assumed to be equal to 1 because the notional amounts are typically different in each currency within the currency swap. Please refer to table below for cash flows for currency swaps hedged with Bonds.

Cash Flows for Currency Swap Hedged with Bonds

Steps	Time 0	Time 1	Time 2	...	Time n
1. Enter currency swap	$-V_a$	$+FS_a - S_1FS_b$	$+FS_a - S_2FS_b$	...	$+FS_a + NA_a - S_n(FS_b + NA_b)$
2. Short sell bond in Currency a	$+FB_a(C_a = FS_a)$	$-FS_a$	$-FS_a$	...	$-(FS_a + Par_a)$
3. Buy bond in Currency b	$-S_0FB_b(C_b = FS_b)$	$+S_1FS_b$	$+S_2FS_b$	...	$+S_n(FS_b + Par_b)$
Net cash flows	$-V_a + FB_a - S_0FB_b$	0	0	0	0

Based on this table, in equilibrium we must have $-V_a + FB_a - S_0FB_b = 0$

Fixed-for-fixed currency swap value is $V_a = FB_a - S_0FB_b$ or else there is an arbitrage opportunity.
Note that the exchange rate S_0 is the number of Currency a units for one unit of Currency b at Time 0; thus, S_0FB_b is expressed in Currency a units.

Swap value after initiation = $V_a = FB_a - S_0FB_b$

In equilibrium, the notional amounts of the two legs of the currency swap are $NA_b = Par_b$ and $NA_a = Par_a = S_0Par_b$.

In order to determine the fixed rates of the swap such that the current swap value is zero, we have

$$FB_a(C_{0,a}, Par_a) = S_0FB_b(C_{0,b}, Par_b)$$

The equilibrium fixed swap rate equations for each currency:

$$r_{FIX,a} = \frac{1 - PV_{0,t_n,a}(1)}{\sum_{i=1}^{n} PV_{0,t_i,a}(1)}$$

and

$$r_{FIX,b} = \frac{1 - PV_{0,t_n,b}(1)}{\sum_{i=1}^{n} PV_{0,t_i,b}(1)}$$

The fixed swap rate in each currency is simply one minus the final present value term divided by the sum of present values.

Numerical Example of Currency Swap Hedged with Bonds

Steps	Time 0	Time 1	Time 2	...	Time 10
1. Enter currency swap	0	+\$10 – (\$1.5/€)€9 = –\$3.5	+\$10 – (\$1.1/€)€9 = \$0.1	...	+\$10 + \$1,300 –(\$1.2/€) (€9 + €1,000) = \$99.2
2. Short sell US dollar bond	+\$1,300	–\$10	–\$10	...	–(\$10 + \$1,300)
3. Buy euro bond	– (\$1.3/€) €1,000	+(\$1.5/€)€9	+(\$1.1/€)€9	...	+(\$1.2/€) (€9 + €1,000)
Net cash flows	0	0	0	0	0

- ➢ If the initial swap value is positive, then we would follow the set of transactions stated in the table above.
- ➢ If the initial swap value is negative, then the opposite set of transactions would be implemented, that is, we would enter into a pay-US dollar, receive-euro swap, buy Currency a bonds, and short sell Currency b bonds.

Practice: Example 15, Reading 39, Curriculum.

Fixed-for-floating currency swap: A fixed-for-floating currency swap is simply a fixed-for-fixed currency swap paired with a floating-for-fixed interest rate swap.

Cash Flows for Currency Swap Hedged with Bonds

Steps	Time t	Time 1	Time 2	...	Time n′
1. Currency swap	$-V_a$	$+FS_{a,0} - S_1FS_{b,0}$	$+FS_{a,0} - S_2FS_{b,0}$	...	$+FS_{a,0} + NA_{a,0} - S_{n'}(FS_{b,0} + NA_{b,0})$
2. Short sell bond (a)	$+FB_a$	$-FS_{a,0}$	$-FS_{a,0}$	...	$-(FS_{a,0} + NA_{a,0})$
3. Buy bond (b)	$-S_tFB_b$	$+S_1FS_{b,0}$	$+S_2FS_{b,0}$	...	$-S_{n'}(FS_{b,0} + NA_{b,0})$
Net cash flows	0	0	0		0

Value of a fixed-for-fixed currency swap = $V_a = FB_a - S_0FB_b$

$$= FS_{a,0}\sum_{i=1}^{n'} PV_{t,t_i,a} + NA_{a,0}PV_{t,t_{n'},a} - S_t\left(FS_{b,0}\sum_{i=1}^{n'} PV_{t,t_i,b} + NA_{b,0}PV_{t,t_{n'},b}\right)$$

The currency swap valuation equation can be expressed as

$$V_a = NA_{a,0}\left(r_{FIX,a,0}\sum_{i=1}^{n'} PV_{t,t_i,a} + PV_{t,t_{n'},a}\right) - S_tNA_{b,0}\left(r_{FIX,b,0}\sum_{i=1}^{n'} PV_{t,t_i,b} + PV_{t,t_{n'},b}\right)$$

Practice: Example 16, Reading 39, Curriculum.

4.3 Equity Swap Contracts

An equity swap is an OTC derivative contract in which two parties agree to exchange a series of cash flows whereby one party pays variable cash flows based on an equity and the other party pays either (1) a variable cash flows based on a different equity or rate or (2) a fixed cash flow. Equity swaps are widely used in equity portfolio investment management to modify returns and risks.

Three common types of equity swaps are

i. Receive-equity return
ii. Pay-fixed; receive equity return
ii. Pay-floating; and receive-equity return, pay-another equity return. It can be viewed simply as a receive-equity a, pay-fixed swap combined with a pay-equity b, receive-fixed swap. The fixed payments cancel out each other – remaining with equity portion.

Important to Remember:

- The underlying reference instrument for the equity leg of an equity swap can be an individual stock, a

published stock index, or a custom portfolio.
- The equity leg cash flow can include or exclude dividends.
- Like interest rate swaps, equity swaps have a fixed or floating interest rate leg.

The equity swap cash flows can be expressed as follows:

For receive-equity, pay-fixed = NA (Equity return – Fixed rate)

For receive-equity, pay-floating = NA(Equity return – Floating rate)

For receive-equity, pay-equity = NA(Equity return$_a$ – Equity return$_b$)

Where, a and b denote different equities.

- When the equity leg of the swap is negative, then the receive-equity counterparty must pay both the equity return as well as the fixed rate.
- Equity swaps may cause liquidity problems because if the equity return is negative, then the receive-equity return, pay-floating or pay-fixed swap may result in a large negative cash flow.

Practice: Example 17, Reading 39, Curriculum.

The ***cash flows for the equity leg of an equity swap*** can be expressed as

$$S_i = NA_E R_{Ei}$$

Where, NA_E denotes the notional amount and R_{Ei} denotes the periodic return of the equity either with or without dividends as specified in the swap contract.

The ***cash flows for the fixed interest rate leg of the equity swap*** can be expressed as

$$FS = NA_E AP_{FIX} r_{FIX}$$

Where AP_{FIX} denotes the accrual period for the fixed leg for which we assume the accrual period is constant and r_{FIX} denotes the fixed rate on the equity swap.

Please refer to the table below for Cash Flows for Receive-Fixed Equity Swap Hedged with Equity and Bond.

Steps	Time 0	Time 1	Time 2	...	Time n
1. Enter equity swap	$-V$	$+FS - S_1$	$+FS - S_2$	...	$+FS - S_n$
2. Buy NA_E equity	$-NA_E$	$+S_1$	$+S_2$	...	$+S_n + NA_E$
3. Short sell fixed-rate bond	$+FB(C = FS)$	$-FS$	$-FS$	...	$-(FS + Par)$
4. Borrow arbitrage profit	$-PV(Par - NA_E)$				$Par - NA_E$
Net cash flows	$-V - NA_E + FB - PV(Par - NA_E)$	0	0	0	0

Equity swap value is $V = -NA_E + FB - PV(Par - NA_E)$

The fixed swap rate can be expressed as

$$r_{FIX} = \frac{1 - PV_{0,t_n}(1)}{\sum_{i=1}^{n} PV_{0,t_i}(1)}$$

- In a pay-floating swap, there is no need to calculate price of the swap because the floating side effectively prices itself at par automatically at the start.
- If the swap involves paying one equity return against another, there would be no need to price the swap because this arrangement can be viewed as paying equity a and receiving a fixed rate as specified above and receiving equity b and paying the same fixed rate. The fixed rates would cancel each other.
- Valuing an equity swap after the swap is initiated (Vt) is similar to valuing an interest rate swap except that rather than adjust the floating-rate bond for the last floating rate observed (remember, advanced set), the value of the notional amount of equity is adjusted as below.

$$V_t = FB_t(C_0) - (S_t/S_{t-})NA_E - PV(Par - NA_E)$$

Where,
- $FB_t(C_0)$ denotes the Time t value of a fixed-rate bond initiated with coupon C_0 at Time 0,
- S_t denotes the current equity price,
- S_{t-} denotes the equity price observed at the last reset date, and
- PV() denotes the present value function from Time t to the swap maturity time.

Practice: Example 18, Reading 39, Curriculum.

1. INTRODUCTION

A contingent claim is a derivative instrument whose payoff depends on occurrence of a future event. In a contingent claim (unlike forward and futures contracts), one party to the contract receives the right – not the obligation – to buy or sell an underlying asset from another party. The purchase price is fixed over a specific period of time and will eventually expire. Contingent claims include options.

- Options derive their value from an underlying asset, which has value. E.g. the payoff on a call (put) option occurs only if the value of the underlying asset is greater (lesser) than an exercise price that is specified at the time the option is created. If this contingency does not occur, the option is worthless.
- Like forward, futures, and swaps contracts, option valuation models are based on the principle of no arbitrage. Option valuation models typically use two approaches.

1) **Binomial model** – based on discrete time. The binomial model is used to value path-dependent options, which are options whose values depend both on the value of the underlying at expiration and how it got there. Such as American options, which can be exercised prior to expiration. (*Discussed in detail in Section 3*).
2) **Black–Scholes–Merton (BSM) model**, which is based on continuous time. The BSM model is based on the key assumption that the value of the underlying instrument follows a statistical process called ***geometric Brownian motion***. Geometric Brownian motion implies a lognormal distribution of the return, which implies that the continuously compounded return on the underlying is normally distributed. The BSM model values only ***path-independent*** options (i.e. European options), which depend on only the values of their respective underlyings at expiration.

2. PRINCIPLES OF A NO-ARBITRAGE APPROACH TO VALUATION

As discussed in Reading 40, Arbitrage is based on following two fundamental rules as well as law of one price.
Rule #1: Do not use your own money.
Rule #2: Do not take any price risk.

Key assumptions in Option Valuation[1]: In this reading, we will make following key assumptions in estimating values of Options:

1) Replicating instruments are identifiable and investable.
2) There are no market frictions, i.e. transaction costs and taxes.
3) Short selling is allowed with full use of proceeds.
4) The underlying instrument follows a known statistical distribution.
5) Borrowing and lending at a risk-free interest rate is available.

The option payoffs can be replicated with a dynamic portfolio of the underlying instrument and financing.

- **Dynamic Portfolio:** A dynamic portfolio is one whose composition changes over time.

[1] Throughout this reading, cash outflows are treated as negative and inflows as positive.

3. BINOMIAL OPTION VALUATION MODEL

The binomial option valuation model is based on the no-arbitrage approach to valuation.

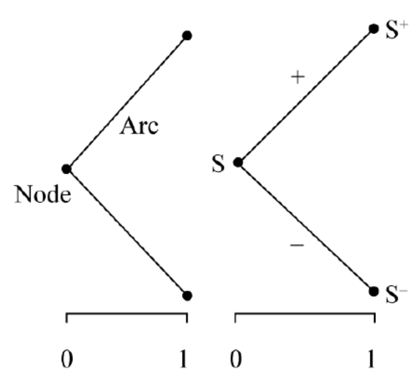
Value of Call Option at expiration: $c_T = Max(0, S_T - X)$
Value of Put Option at expiration: $p_T = Max(0, X - S_T)$
Where,

- S_t denote the underlying instrument price observed at Time t, where t is expressed as a fraction of a year. E.g. a call option had 60 days to expiration when purchased (T = 60/365), but now only has 35 days to expiration (t = 25/365).
- S_T denotes the underlying instrument price observed at the option expiration date, T.
- c_t denote a European-style call price at Time t and with expiration on Date t = T, where both t and T are expressed in years.
- C_t denote an American-style call price.
- X denote the exercise price.

If the option values deviate from these expressions, then there will be arbitrage profits available.
Since, European options cannot be exercised until expiration, they do not technically have exercise values prior to expiration.
Time Value of Options: The time value is always nonnegative for options because of the asymmetry of option payoffs at expiration. For example, for a call option, the upside is unlimited, whereas the downside is limited to zero. At expiration, time value is zero.

3.1 One-Period Binomial Model

The following figure represents a tree for possible outcomes in one-period Binomial Model.



- Each dot represents a particular outcome at a particular point in time in the binomial lattice. These dots are termed ***nodes.***
- At the Time 0 node, there are only two possible future paths in the binomial process, an ***up move*** and a ***down move***, termed as ***arcs***.
- At Time 1, there are only two possible outcomes: S^+ denotes the outcome when the underlying goes up, and S^- denotes the outcome when the underlying goes down.

$$u = \frac{S^+}{S} \text{ (up factor)}$$

$$d = \frac{S^-}{S} \text{ (down factor)}$$

- The up factors and down factors are the total returns.
- The magnitudes of the up and down factors are based on the volatility of the underlying. In general, the higher the volatility, the higher will be the up values and the lower will be the down values.

At expiration, Option value is either

$$c^{++} = Max(0, S^{++} - X) = Max(0, u^2S - X)$$

or

$$c^{+-} = Max(0, S^{+-} - X) = Max(0, udS - X)$$

- The value of a call option is positively related to the value of the underlying. That is, if the underlying goes up (down), value of call option increases (decreases). This implies that in order to ***hedge position***, a trader to be a ***long position*** in the underlying. Specifically, the trader buys a certain number of units, '**h**', of the underlying. The symbol h represents a **hedge ratio** – as estimated below.

$$h = \frac{c^+ - c^-}{S^+ - S^-} \geq 0$$

  - The above formula states that Hedge ratio is the value of the call if the underlying goes up minus the value of the call if the underlying goes down divided by the value of the underlying if it goes up minus the value of the underlying if it goes down.
  - Hedge ratio is non-negative because call prices are positively related to changes in the underlying price.

Writing One Call Hedge with h Units of the Underlying and Finance: The following table shows payoffs of writing one call hedge with h units of the underlying and finance.

Strategy	Time Step 0	Time Step 1 Down Occurs	Time Step 1 Up Occurs
1) Write one call option	+c	−c−	−c+
2) Buy h underlying units	−hS	+hS−	+hS+
3) Borrow or lend	−PV(−hS− + c−) = −PV(−hS+ + c+)	−hS− + c−	−hS+ + c+
Net Cash Flow	+c − hS −PV(−hS− + c−)	0	0

- At Time 0, the value of the net portfolio should always be zero, else there will be an arbitrage opportunity.
- If the net portfolio has positive value, then arbitrageurs will write call option, long the "h" underlying units, and then finance his transaction through borrowing.
- If the net portfolio has negative value, then arbitrageurs will buy call option, short sell the "h" underlying units, and then lend (or invest the proceeds) – pushing the call price up and the underlying price down until the net cash flow at Time 0 is no longer positive.

Long a call option = Owning 'h' shares of stock partially financed

Where,

Financed amount = PV $(-hS^- + c^-)$

or using the per period rate,

$(-hS^- + c^-) / (1 + r)$

<u>Single-period call option valuation using no-arbitrage approach:</u>

$$c = hS + PV(-hS^- + c^-)$$

or, equivalently,

$$c = hS + PV(-hS^+ + c^+)$$

Replicating a Call option: A call option can be replicated with the underlying and financing. Specifically, the call option is equivalent to a leveraged position in the underlying. The trading strategy that will generate the payoffs of taking a ***long position*** in a call option within a single-period binomial framework is as follows:

Buy h = (c+ – c–)/(S+ – S–) units of the underlying and financing of – PV(–hS⁻ + c⁻)

Please refer to table below:

Strategy	Time Step 0	Time Step 1 Down Occurs	Time Step 1 Up Occurs
Buy 1 call option	−c	+c−	+c+
OR A REPLICATING PORTFOLIO			
Buy h underlying units	−hS	+hS−	+hS+
Borrow or lend	−PV(−hS− + c−) = −PV(−hS+ + c+)	−hS− + c−	−hS+ + c+
Net	−hS − PV(−hS− + c−)	+c−	+c+

<u>Practice:</u> Example 1, Reading 40, Curriculum.

<u>No-arbitrage single-period put option valuation equation is as follows:</u>

$$p = hS + PV(-hS^- + p^-)$$

or, equivalently,

$$p = hS + PV(-hS^+ + p^+)$$

Where,

$$h = \frac{p^+ - p^-}{S^+ - S^-} \leq 0$$

- For put options, the hedge ratio is negative because p^+ is less than p^-.
- In order to hedge a ***long put*** position, the arbitrageur will ***short sell*** the underlying (–h = –(p+ – p–)/(S+ – S–) units of the underlying) and lend a portion of the proceeds.
- Note that since –h is positive, the value –hS results in a positive cash flow at Time Step 0.

Please refer to table below.

Strategy	Time Step 0	Time Step 1 Down Occurs	Time Step 1 Up Occurs
Buy 1 Put Option	−p	+p−	+p+
OR A REPLICATING PORTFOLIO			
Short sell −h Underlying Units	−hS	+hS−	+hS+
Borrow or Lend	−PV(−hS− + p−) = −PV(−hS+ + p+)	−hS− + p−	−hS+ + p+
Net	−hS − PV(−hS− + p−)	+p−	+p+

<u>Practice:</u> Example 2, Reading 40, Curriculum.

Expectations Approach: The expectations approach results in an identical value as the no-arbitrage approach, but it is usually easier to compute. The formulas are given as follows:

$$\mathbf{c = PV\,[\pi c^{+} + (1 - \pi)\, c^{-}]}$$

and

$$\mathbf{p = PV\,[\pi p^{+} + (1 - \pi)\, p^{-}]}$$

Where,

Probability of an up move = π = [FV(1) – d]/(u – d)

Expected terminal option payoffs: The option values are present value of the expected terminal option payoffs. The expected terminal option payoffs can be expressed as follows:

$$E(c_1) = \pi c^{+} + (1 - \pi)c^{-}$$

and

$$E(p_1) = \pi p^{+} + (1 - \pi)p^{-}$$

Where c1 and p1 are the values of the options at Time 1.

The option values based on the expectations approach can be expressed as follows:

$$c = PV_r[E(c_1)]$$

and

$$p = PV_r[E(p_1)]$$

Difference between Expectations approach and discounted cash flow approach to securities valuation: The expectations approach is often regarded as superior method to the discounted cash flow approach because it is based on objective measures as follows.

i. The expectation is not based on the investor's beliefs regarding the future course of the underlying – implying that the probability, **π**, is ***objectively*** determined and not based on the investor's personal view. This probability is referred to as **risk-neutral (RN) probability** – reason being the expectations approach is not based on assumption regarding risk preferences.

ii. In expectations approach, the discount rate is ***not risk adjusted***, rather it is based on the estimated risk-free interest rate.

Note: The expectations approach can be applied to European-style options. The no-arbitrage approach can be applied to either European-style or American style options because it provides the intuition for the fair value of options.

Practice: Example 3, Reading 40, Curriculum.

Put–call parity:

$$\mathbf{S + p = PV(X) + c}$$

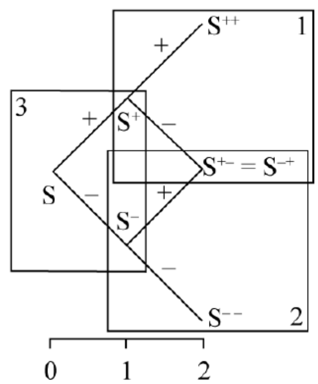
- The value of a put or call option can be found based on put–call parity.
- E.g. Call option can be expressed as a position in a stock, financing, and a put, i.e.

$$\mathbf{c = S - PV(X) + p}$$

Practice: Example 4, Reading 40, Curriculum.

3.2 Two-Period Binomial Model

Following figure reflects Two-Period Binomial Lattice as Three One-Period Binomial Lattices.



- For simplicity, it is assumed that the ***up and down factors are constant throughout the lattice***, that is $S^{+-} = S^{-+}$. For example, assume u = 1.25, d = 0.8, and S_0 = 100. Note that S^{+-} = 1.25(0.8)100 = 100 and S^{-+} = 0.8(1.25)100 = 100. So the middle node at Time 2 is 100 and can be reached from either of two paths.
- It is important to remember that Option valuation relies on self-financing, dynamic replication. Dynamic replication is obtained by using a portfolio of stock and the financing. The strategy is self-financing because the funds borrowed at Time 1 grew to 'x' amount.

Call Option Payoffs at Time 2:

$$c^{++} = \text{Max}\,(0, S^{++} - X) = \text{Max}(0, u^2S - X),$$

$$c^{+-} = \text{Max}\,(0, S^{+-} - X) = \text{Max}\,(0, udS - X), \text{ and}$$

$$c^{--} = \text{Max}\,(0, S^{--} - X) = \text{Max}\,(0, d^2S - X)$$

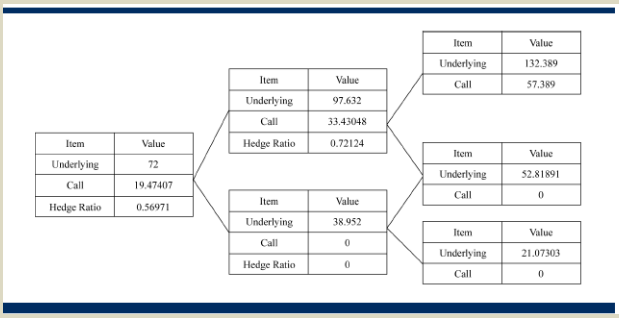
<u>Put Option Payoffs at Time 2:</u>

$$p^{++} = \text{Max}\ (0,\ X - S^{++}) = \text{Max}\ (0, X - u^2S),$$

$$p^{+-} = \text{Max}\ (0,\ X - S^{+-}) = \text{Max}\ (0, X - udS), \text{ and}$$

$$p^{--} = \text{Max}\ (0, X - S^{--}) = \text{Max}\ (0,\ X - d^2S)$$

<u>Example:</u> Following lattice shows the no-arbitrage approach for solving the two-period binomial call value. Suppose the annual interest rate is 3%, the underlying stock is S = 72, u = 1.356, d = 0.541, and the exercise price is X = 75. The stock does not pay dividends.



$$\mathbf{c = PV\ [\pi^2c^{++} + 2\pi\ (1 - \pi)\ c^{+-} + (1 - \pi)^2c^{--}]}$$
$$\mathbf{p = PV\ [\pi^2p^{++} + 2\pi\ (1 - \pi)\ p^{+-} + (1 - \pi)^2p^{--}]}$$

The expected terminal option payoffs are expressed as below:

$$\mathbf{E(c_2) = \pi^2c^{++} + 2\pi\ (1 - \pi)\ c^{+-} + (1 - \pi)^2c^{--}}$$
and
$$\mathbf{E(p_2) = \pi^2p^{++} + 2\pi\ (1 - \pi)p^{+-} + (1 - \pi)^2p^{--}}$$

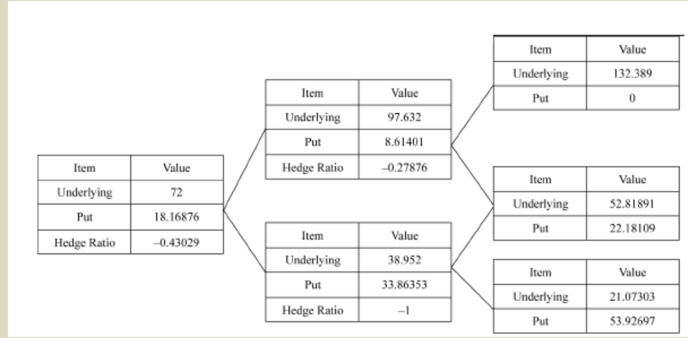
The two-period binomial option values based on the expectations approach are expressed as:

$$\mathbf{= PV_r[E\pi(c_2)]}$$
and
$$\mathbf{p = PV_r[E\pi(p_2)]}$$

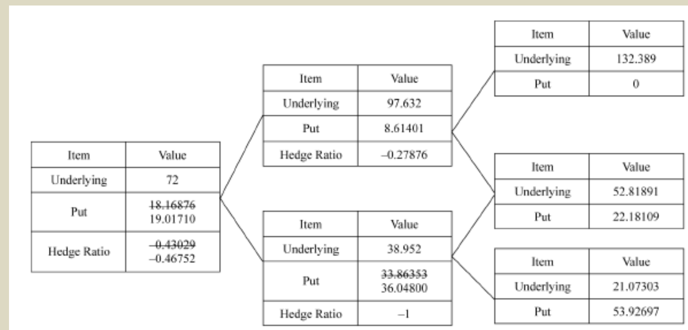
<u>Practice:</u> Example 5, Reading 40, Curriculum.

American-style options: American options are options which can be exercised prior to expiration. A non-dividend paying call options on stock will not be exercised early because the minimum price of the option exceeds its exercise value. However, this is not true for put options (particularly a deep in the- money put) because the sale proceeds can be invested at the risk-free rate and earn interest worth more than the time value of the put.

<u>Example:</u> Suppose the periodically compounded interest rate is 3%, the non-dividend-paying underlying stock is currently trading at 72, the exercise price is 75, u = 1.356, d = 0.541, and the put option expires in two years. Following lattice reflects **"Two-Period Binomial Model for a European-Style Put Option":**



Following lattice reflects **"Two-Period Binomial Model for an American-Style Put Option":**



Put value = $p = PV[\pi p^+ + (1 - \pi)p^-]$

<u>Practice:</u> Example 6, Reading 40, Curriculum.

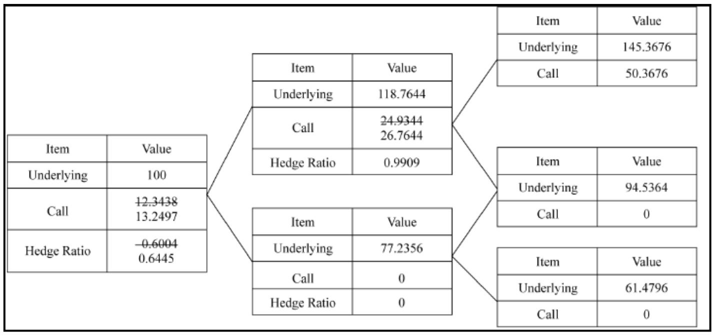
Escrow method: Dividends negatively affect the value of a call option because dividends lower the value of the stock. Most option contracts do not provide protection against dividends. Assuming dividends are perfectly predictable, we can split the underlying instrument into two components: the underlying instrument without the known dividends and the known dividends. For example, the current value of the underlying instrument ***without dividends*** can be expressed as follows:

$$\hat{S} = S - \gamma$$

Where,

γ denotes the present value of dividend payments. ^ symbol is used to denote the underlying instrument without dividends. At expiration, the underlying instrument value is the same, $\hat{S}_T = S_T$. because it is assumed that any dividends have already been paid. The value of an investment in the stock, however, would be $S_T + \gamma T$, which assumes the dividend payments are reinvested at the risk-free rate.

Following lattice reflects **"Two-Period Binomial Model for an American-Style Call Option with Dividends"**



- At Time 0, the present value of the US$3 dividend payment is US$2.970297 (= 3/1.01). Therefore, 118.7644 = (100 – 2.970297)1.224 is the stock value without dividends at Time 1, assuming an up move occurs.
- The exercise value for this call option, including dividends, is 26.7644 [= Max (0,118.7644 + 3 – 95)], whereas the value of the call option per the binomial model is 24.9344.
- The stock price just before it goes ex-dividend is 118.7644 + 3 = 121.7644, so the option can be exercised for 121.7644 – 95 = 26.7644.
- If not exercised, the stock drops as it goes ex-dividend and the option becomes worth 24.9344 at the ex-dividend price.

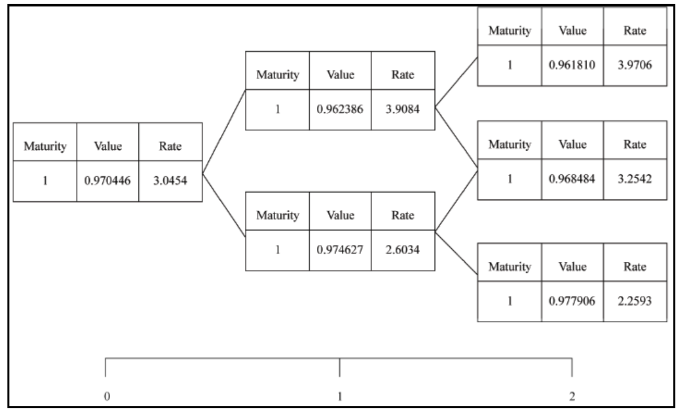
Important to Remember: This example tell us that the American-style call option is worth more than the European-style call option because at Time Step 1 when an up move occurs, the call is exercised early, capturing additional value. For non-dividend paying stocks, the American-style feature has no effect on either the hedge ratio or the option value. American-style put options on non-dividend-paying stock ***may be*** (not necessarily always) worth more than the analogous European style put options.

Practice: Example 7, Reading 40, Curriculum.

3.3 Interest Rate Options

- A call option on interest rates will be in the money when the current spot rate > exercise rate.
- A put option on interest rates will be in the money when the current spot rate < exercise rate.

Example: Following is the Two-Year Binomial Interest Rate Lattice by Year. Assume the notional amount of the options is US$1,000,000 and the call and put exercise rate is 3.25% of par and RN probability is 50%.



- The rates are expressed in annual compounding. Therefore, at Time 0, the spot rate is (1.0/0.970446) – 1 or 3.04540%.
- Note that at Time 1, the value in the column labeled "Maturity" reflects time to maturity not calendar time.

$$c^{++} = \text{Max}\ (0, S^{++} - X) = \text{Max}\ [0, 0.039706 - 0.0325] = 0.007206$$

$$c^{+-} = \text{Max}\ (0, S^{+-} - X) = \text{Max}\ [0, 0.032542 - 0.0325] = 0.000042$$

$$c^{--} = \text{Max}\ (0, S^{--} - X) = \text{Max}\ [0, 0.022593 - 0.0325] = 0.0$$

$$p^{++} = \text{Max}\ (0, X - S^{++}) = \text{Max}\ [0, 0.0325 - 0.039706] = 0.0$$

$$p^{+-} = \text{Max}\ (0, X - S^{+-}) = \text{Max}\ [0, 0.0325 - 0.032542] = 0.0$$

$$p^{--} = \text{Max}\ (0, X - S^{--}) = \text{Max}\ [0, 0.0325 - 0.022593] = 0.009907$$

At Time Step 1, we have

$c+ = PV_{1,2}[\pi c^{++} + (1 - \pi)c^{+-}] = 0.962386[0.5(0.007206) + (1 - 0.5)0.000042] = 0.003488$

$c- = PV_{1,2}[\pi c^{+-} + (1 - \pi)c^{--}] = 0.974627[0.5(0.000042) + (1 - 0.5)0.0] = 0.00002$

$p^{+} = PV_{1,2}[\pi p^{++} + (1 - \pi)p^{+-}] = 0.962386[0.5(0.0) + (1 - 0.5)0.0] = 0.0$

$p^{-} = PV_{1,2}[\pi p^{+-} + (1 - \pi)p^{--}] = 0.974627[0.5(0.0) + (1 - 0.5)0.009907] = 0.004828$

At Time Step 0, we have

$c = PV_{rf,0,1}[\pi c^{+} + (1 - \pi)c^{-}] = 0.970446[0.5(0.003488) + (1 - 0.5)0.00002] = 0.00170216$

$p = PV_{rf,0,1}[\pi p^{+} + (1 - \pi)p^{-}] = 0.970446[0.5(0.0) + (1 - 0.5)0.004828] = 0.00234266$

Because the notional amount is US$1,000,000, the call value is = US$1,000,000(0.00170216) = US$1,702.16 and the put value is = US$1,000,000(0.00234266) = US$2,342.66.

3.4 Multiperiod Model

The two-period model divides the expiration into two periods. The three-period model divides expiration into three periods and so forth. Similarly, the multi-period model divides expiration into multiple periods. Each time step is of equal length, i.e., with a maturity of T, if there are n time steps, then each time step is T/n in length.

4. BLACK–SCHOLES–MERTON OPTION VALUATION MODEL

4.2 Assumptions of the BSM model

The stochastic process (wherein value of instrument evolves over time) chosen by Black, Scholes, and Merton is called geometric Brownian motion (GBM).

Assumptions of the BSM model: The standard BSM model assumes a ***constant growth rate*** and ***constant volatility***. The specific assumptions of the BSM model are as follows:

a) The underlying follows a statistical process called ***geometric Brownian motion***, which implies a lognormal distribution of the return – meaning that the continuously compounded return is normally distributed.
b) Geometric Brownian motion implies ***continuous prices***, meaning that the price of underlying instrument does not jump from one value to another; rather, it moves smoothly from value to value.
c) The underlying instrument is ***liquid***, i.e. can be easily bought and sold.
d) ***Continuous trading is available***, i.e. we can trade at every instant.
e) ***Short selling*** of the underlying instrument with full use of the proceeds is allowed.
f) There are ***no market frictions***, i.e. transaction costs, regulatory constraints, or taxes.
g) ***No-arbitrage opportunities are available*** in the marketplace.
h) The ***options are European-style***, meaning that early exercise is not allowed.
i) The continuously compounded ***risk-free interest rate is known and constant***.
j) ***Borrowing and lending*** is allowed at the risk-free rate.
k) The ***volatility of the return*** on the underlying is ***known and constant***.
l) If the underlying instrument pays a yield, it is expressed as ***a continuous known and constant yield at an annualized rate***.

Practice: Example 9, Reading 40, Curriculum.

4.3 BSM model

The BSM model is a ***continuous time version*** of the discrete time binomial model and therefore, continuously compounded interest rate is used in this model. The volatility (σ) is also expressed in annualized percentage terms. The BSM model for stocks can be expressed as follows:

$$c = SN(d_1) - e^{-rT}XN(d_2)$$

and

$$p = e^{-rT}XN(-d_2) - SN(-d_1)$$

Where,

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- N(x) reflects the likelihood of observing values less than x from a random sample of observations taken from the ***standard normal distribution***. The ***standard normal distribution*** is a normal distribution with a mean of 0 and a standard deviation of 1.
- The ***normal distribution*** is a symmetric distribution with two parameters, the mean and standard deviation.

BSM model for call option is

$$c = PV_r[E(c_T)]$$

BSM model for put option is

$$p = PV_r[E(p_T)]$$

Where, $E(c_T) = Se^{rT}N(d_1) - XN(d_2)$ and $E(p_T) = XN(-d_2) - Se^{rT}N(-d_1)$. The present value term in this context is simply e^{-rT}.

BSM model can be described as having two components: a stock component and a bond component.

- For call options, the stock component is $SN(d_1)$ and the bond component is $e^{-rT}XN(d_2)$.

  BSM model call value = stock component - bond component
- For put options, the stock component is $SN(-d_1)$ and the bond component is $e^{-rT}XN(-d_2)$.

  BSM model put value = Bond component - Stock component

- The BSM model can be interpreted as a dynamically managed portfolio of the stock and zero-coupon bonds.
- For both call and put options, we can represent the initial cost of this replicating strategy as follows:

  Replicating strategy cost = $n_S S + n_B B$

Where,

- For calls, the equivalent number of underlying shares is $n_S = N(d_1) > 0$ and the equivalent number of bonds is $n_B = -N(d_2) < 0$.
- For puts, the equivalent number of underlying shares is $n_S = -N(-d_1) < 0$ and the equivalent number of bonds $n_B = N(-d_2) > 0$.
- The price of the zero-coupon bond is $B = e^{-rT}X$.

__**Important to remember:**__ If n is positive, we are buying the underlying and if n is negative we are selling (short selling) the underlying. The cost of the portfolio will exactly equal either the BSM model call value or the BSM model put value.

- A call option can be viewed as a leveraged position in the stock or calls because we are simply buying stock with borrowed money because $n_S > 0$ and $n_B < 0$.
- For call options, $-N(d_2)$ implies borrowing money or short selling $N(d_2)$ shares of a zero-coupon bond trading at $e^{-rT}X$.

- For put options, we are simply buying bonds with the proceeds from short selling the underlying because $n_S < 0$ and $n_B > 0$. A short put can be viewed as an over-leveraged or over-geared position in the stock because the borrowing exceeds 100% of the cost of the underlying. This is because a short position in a put will result in receiving money today and $n_S > 0$ and $n_B < 0$.
- For put options, $N(-d_2)$ implies lending money or buying $N(-d_2)$ shares of a zero-coupon bond trading at $e^{-rT}X$.

Comparison between BSM and Binomial Option Valuation Model: The following table summarized difference between BSM and Binomial Valuation model.

Option Valuation Model Terms	Call Option		Put Option	
	Underlying	Financing	Underlying	Financing
Binomial Model	hS	PV(–hS– + c–)	hS	PV(–hS– + p–)
BSM Model	$N(d_1)S$	$-N(d_2)e^{-rT}X$	$-N(-d_1)S$	$N(-d_2)e^{-rT}X$

- If the value of the underlying, S, increases, then the value of $N(d_1)$ also increases because S has a positive effect on d_1. Thus, the replicating strategy for calls requires continually buying shares in a ***rising market*** and selling shares in a ***falling market***.
- In practical, hedges are imperfect because (i) frequent rebalancing by buying and selling the underlying adds significant costs for the hedger because trading involves transaction costs; (ii) market may move discontinuously (contrary to the BSM model's assumption mentioned above) which requires continuous hedging adjustments, and (iii) volatility cannot be known in advance.

__Practice:__ Example 10, Reading 40, Curriculum.

Probability that the call option expires in the money: Probability that the call option expires in the money is denoted as $N(d_2)$, and correspondingly, $1 - N(d2) = N(-d_2)$ is the probability that the put option expires in the money.

Carry benefits: Carry benefits include dividends for stock options, foreign interest rates for currency options, and coupon payments for bond options. Carry benefits tend to lower the expected future value of the underlying.

Carry costs can be treated as negative carry benefits, i.e. storage and insurance costs for agricultural products. Because the BSM model assumes continuous time, these carry benefits can be modelled as a continuous yield, denoted as γ^c or simply γ.

Carry benefit-adjusted BSM model: The carry benefit-adjusted BSM model is expressed as follows:

$$c = Se^{-\gamma T}N(d_1) - e^{-rT}XN(d_2) \text{ and}$$

$$p = e^{-rT}XN(-d_2) - Se^{-\gamma T}N(-d_1)$$

Where,

$$d_1 = \frac{\ln(S/X) + \left(r - \gamma + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

d_2 can be expressed as $d_2 = d_1 - \sigma\sqrt{T}$

Value of a put option = $p + Se^{-\gamma T} = c + e^{-rT}X$

- $E(c_T) = Se^{(r-\gamma)T}N(d_1) - XN(d_2)$
- $E(p_T) = XN(-d_2) - Se^{(r-\gamma)T}N(-d_1)$
- The present value term is denoted as e^{-rT}.

The carry benefit adjusted BSM model can be described as having two components, a stock component and a bond component.

- For call options, the stock component is $Se^{-\gamma T}N(d_1)$ and the bond component is again $e^{-rT}XN(d_2)$.
- For put options, the stock component is $Se^{-\gamma T}N(-d_1)$ and the bond component is again $e^{-rT}XN(-d_2)$

Important to remember:

- If carry benefits increase, they lower the value of the call option and raise the value of the put option.
- The carry benefits tend to reduce d_1 and d_2, and consequently, the probability of being in the money with call options declines as the carry benefit rises.
- Dividends influence the dynamically managed portfolio by lowering the number of shares to buy for calls and lowering the number of shares to short sell for puts. Higher dividends will lower the value of d_1, thus lowering $N(d_1)$. In addition, higher dividends will lower the number of bonds to short sell for calls and lower the number of bonds to buy for puts.

BSM call model for a dividend-paying stock: The BSM call model for a dividend-paying stock can be expressed as follows:

$$Se^{-\delta T}N(d_1) - Xe^{-rT}N(d_2)$$

- The equivalent number of units of stock is $n_S = e^{-\delta T}N(d_1) > 0$ and the equivalent number of units of bonds remains $nB = -N(d_2) < 0$.

BSM put model for a dividend-paying stock: The BSM put model for a dividend-paying stock can be expressed as follows:

$$Xe^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

The equivalent number of units of stock is $n_S = -e^{-\delta T}N(-d_1) < 0$ and the equivalent number of units of bonds again remains $n_B = N(-d_2) > 0$.

Practice: Example 11 & 12, Reading 40, Curriculum.

Foreign exchange options: For foreign exchange options, $\gamma = r^f$, which is the continuously compounded foreign risk-free interest rate.

Currency options: In currency options, the underlying instrument is the foreign exchange spot rate. Here, the carry benefit is the interest rate in the foreign country because the foreign currency could be invested in the foreign country's risk-free instrument. With currency options, the underlying and the exercise price must be quoted in the same currency unit. The volatility in the model is the volatility of the log return of the spot exchange rate.

BSM model applied to currencies: The BSM model applied to currencies can be described as having two components, a foreign exchange component and a bond component.

- For call options, the foreign exchange component is $Se^{-r^f T}N(d_1)$ and the bond component is $e^{-rT}XN(d_2)$, where r is the domestic risk-free rate.
  BSM call model applied to currencies = Foreign exchange component - Bond component
- For put options, the foreign exchange component is $Se^{-r^f T}N(-d_1)$ and the bond component is $e^{-rT}XN(-d_2)$.
  BSM put model applied to currencies = Bond component - Foreign exchange component

Practice: Example 13, Reading 40, Curriculum.

5. BLACK OPTION VALUATION MODEL

5.1 European Options on Futures

Model for European-style futures options is as below:

$$c = e^{-rT}[F_0(T)N(d_1) - XN(d_2)]$$

$$p = e^{-rT}[XN(-d_2) - F_0(T)N(-d_1)]$$

Where,

$$d_1 = \frac{\ln[F_0(T)/X]+(\sigma^2/2)T}{\sigma\sqrt{T}} \text{ and}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- $F_0(T)$ denotes the futures price at Time 0 that expires at Time T, and σ denotes the volatility related to the futures price.

Futures option put–call parity can be expressed as

$$c = e^{-rT}[F_0(T) - X] + p$$

The Black model has two components, a futures component and a bond component.

- For call options, the futures component is $F_0(T)e^{-rT}N(d_1)$ and the bond component is again $e^{-rT}XN(d_2)$.
  Black call model = Futures component - Bond component
- For put options, the futures component is $F_0(T)e^{-rT}N(-d_1)$ and the bond component is again $e^{-rT}XN(-d_2)$.
  Black put model = Bond component - Futures component

Futures option valuation based on the Black model involves computing the present value of the ***difference between the futures price and the exercise price***.

- For call options, the futures price is adjusted by $N(d_1)$ and the exercise price is adjusted by $-N(d2)$.
- For put options, the futures price is adjusted by $-N(-d_1)$ and the exercise price is adjusted by $+N(-d_2)$.

Practice: Example 14, Reading 40, Curriculum.

5.2 Interest Rate Options

In interest rate options, the underlying instrument is a reference ***interest rate***, i.e. three-month Libor.

- An interest rate call option gains when the reference interest rate rises.
- An interest rate put option gains when the reference interest rate falls.

For an interest rate call option on three-month Libor with one year to expiration, the underlying interest rate is a forward rate agreement (FRA) rate that expires in one year. The underlying rate of the FRA is a 3-month Libor deposit that is investable in 12 months and matures in 15 months.

Interest rates are set in advance, but interest payments are made in arrears, which is referred to as *advanced set, settled in arrears*.

- The accrual period in FRAs is based on 30/360 whereas the accrual period based on the option is actual number of days in the contract divided by the actual number of days in the year (identified as ACT/ACT or ACT/365).

Example: In a bank deposit, the interest rate is usually set when the deposit is made, say t_{j-1}, but the interest payment is made when the deposit is withdrawn, say t_j. The deposit, therefore, has time until maturity = $t_m = t_j - t_{j-1}$.

Standard market model: In a standard market model, the prices of interest rate call and put options can be expressed as follows:

$$c = (AP)e^{-r(t_{j-1}+t_m)}\left[FRA(0,t_{j-1},t_m)N(d_1) - R_X N(d_2)\right]$$

And

$$p = (AP)e^{-r(t_{j-1}+t_m)}\left[R_X N(-d_2) - FRA(0,t_{j-1},t_m)N(-d_1)\right]$$

Where,

AP denotes the accrual period in years

$$d_1 = \frac{\ln\left[FRA(0,t_{j-1},t_m)/R_X\right] + (\sigma^2/2)t_{j-1}}{\sigma\sqrt{t_{j-1}}}$$

$$d_2 = d_1 - \sigma\sqrt{t_{j-1}}$$

- $FRA(0,t_{j-1},t_m)$ denote the fixed rate on a FRA at Time 0 that expires at Time t_{j-1}, where the underlying matures at Time t_j (= $t_{j-1} + t_m$), with all times expressed on an annual basis.
- RX denotes the exercise rate expressed on an annual basis.

- σ denotes the interest rate volatility. σ is the annualized standard deviation of the continuously compounded percentage change in the underlying FRA rate.
- Standard market model requires an adjustment when compared with the Black model for the accrual period, that is, $FRA(0,t_{j-1},t_m)$ or the strike rate, RX, are stated on an annual basis, as are interest rates in general.
- The actual option premium is adjusted for the accrual period.

Differences between Black Model and Standard Model:

1) The discount factor is applied to the maturity date of the FRA or t_j (= $t_{j-1} + t_m$), rather than to the option expiration, t_{j-1}.
2) The underlying is an interest rate, specifically a forward rate based on a forward rate agreement or $FRA(0,tj–1,tm)$. It is not a futures price.
3) The exercise price is a rate and reflects an interest rate, not a price.
4) The time to the option expiration, t_{j-1}, is used in the calculation of d_1 and d_2.
5) Both the forward rate and the exercise rate should be expressed in decimal form rather than as percent (for example, 0.01 and not 1.0).

Important to remember: In Black model, a forward or futures price is used as the underlying. In contrast, in BSM model, a spot price is used as the underlying.

Standard market model for calls:

$c = PV[E(c_{tj})]$

Standard market model for puts:

$p = PV[E(p_{tj})]$

Where,

- $E(c_{tj}) = (AP)[FRA(0,t_{j-1},t_m)N(d_1) – RXN(d_2)]$
- $E(p_{tj}) = (AP)\ [RXN(–d_2) – FRA(0,t_{j-1},t_m)N(–d_1)]$

Combinations created with interest rate options:

- If the exercise rate selected in interest rate option is equal to the current FRA rate, then long an interest rate call option and short an interest rate put option is equivalent to a receive-floating, pay-fixed FRA.
- If the exercise rate selected in interest rate option is equal to the current FRA rate, then long an interest rate put option and short an interest rate call option is equivalent to a receive-fixed, pay-floating FRA.
- An interest rate cap is a portfolio or strip of interest rate call options in which the expiration of the first underlying corresponds to the expiration of the second option and so forth. The underlying interest rate call options are called ***caplets***. Thus, a set of floating-rate loan payments can be hedged with a long position in an interest rate cap encompassing a series of interest rate call options.
- An interest rate floor is a portfolio or strip of interest rate put options in which the expiration of the first underlying corresponds with the expiration of the second option and so forth. The underlying interest rate put options are called ***floorlets***. Thus, a floating-rate bond investment or any other floating-rate lending situation can be hedged with an interest rate floor encompassing a series of interest rate put options.
- Long an interest rate cap and short an interest rate floor with the same exercise rate is equal to a receive-floating, pay-fixed interest rate swap. When the cap is in the money, the receive-floating counterparty will also receive an identical net payment. When the floor is in the money, the receive-floating counterparty will also pay an identical net payment.
- Long an interest rate floor and short an interest rate cap with the same exercise rate is equal to a receive-fixed, pay-floating interest rate swap. When the floor is in the money, the receive-fixed counterparty will also receive an identical net payment. When the cap is in the money, the receive-floating counterparty will also pay an identical net payment.
- If the exercise rate selected in interest rate option is set equal to the swap rate, then the value of the cap must be equal to the value of the floor. When an interest rate swap is initiated, its current value is zero and is known as an ***at-market swap***. When an exercise rate is selected such that the cap equals the floor, then the initial cost of being long a cap and short the floor is also zero.

Practice: Example 15, Reading 40, Curriculum.

5.3 Swaptions

A swap option or swaption is an option on a swap. It gives the holder the right, but not the obligation, to enter a swap at the ***pre-agreed*** swap rate (referred to as the exercise rate). Interest rate swaps can be either receive fixed, pay floating or receive floating, pay fixed.

Payer Swaption: A payer swaption is an option on a swap to pay fixed, receive floating.

Receiver Swaption: A receiver swaption is an option on a swap to receive fixed, pay floating.

Swap payments are ***advanced set, settled in arrears***.

Following equation represents the present value of an annuity matching the forward swap payment:

$$\text{PVA} = \sum_{j=1}^{n} PV_{0,t_j}(1)$$

Payer swaption valuation model is expressed as follows:

$$\text{PAY}_{\text{SWN}} = (\text{AP})\text{PVA}[\text{R}_{\text{FIX}}\text{N}(\text{d}_1) - \text{R}_{\text{X}}\text{N}(\text{d}_2)]$$

Receiver swaption valuation model is expressed as follows:

$$\text{REC}_{\text{SWN}} = (\text{AP})\text{PVA}[\text{R}_{\text{X}}\text{N}(-\text{d}_2) - \text{R}_{\text{FIX}}\text{N}(-\text{d}_1)]$$

where

$$d_1 = \frac{\ln(R_{FIX}/R_X)+(\sigma^2/2)T}{\sigma\sqrt{T}} \text{, and as always,}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The swaption model requires two adjustments, one for the accrual period and one for the present value of an annuity.

<u>Differences between Swaption Model and Black Model:</u>

i. The discount factor is absent in swaption model. The payoff is a series of payments. Thus, the present value of an annuity used here takes into account the option-related discount factor.
ii. The underlying is the fixed rate on a forward interest rate swap rather than a futures price,.
iii. The exercise price is expressed as an interest rate.
iv. Both the forward swap rate and the exercise rate are expressed in decimal form and not as percent (for example, 0.02 and not 2.0).

<u>Payer swaption model value is estimated as follows:</u>

$$\text{PAY}_{\text{SWN}} = \text{PV}[\text{E}(\text{PAY}_{\text{SWN,T}})]$$

<u>Receiver swaption model value is estimated as follows:</u>

$$\text{REC}_{\text{SWN}} = \text{PV}[\text{E}(\text{REC}_{\text{SWN,T}})]$$

Where,

$\text{E}(\text{PAY}_{\text{SWN,T}}) = e^{rT}\text{PAY}_{\text{SWN}}$ and

$\text{E}(\text{REC}_{\text{SWN,T}}) = e^{rT}\text{REC}_{\text{SWN}}$.

The swaption model can also be described as having two components, a swap component and a bond component.

- For payer swaptions, the swap component is $(\text{AP})\text{PVA}(\text{R}_{\text{FIX}})\text{N}(\text{d}_1)$ and the bond component is $(\text{AP})\text{PVA}(\text{R}_{\text{X}})\text{N}(\text{d}_2)$.
  Payer swaption model value = Swap component - Bond component
- For receiver swaptions, the swap component is $(\text{AP})\text{PVA}(\text{RFIX})\text{N}(-\text{d}1)$ and the bond component is $(\text{AP})\text{PVA}(\text{R}_{\text{X}})\text{N}(-\text{d}_2)$.
  Receiver swaption model value = Bond component - Swap component

<u>Combinations created with Swaptions:</u>

- Long a receiver swaption and short a payer swaption with the same exercise rate is equivalent to entering a receive-fixed, pay-floating forward swap.
- Long a payer swaption and short a receiver swaption with the same exercise rate is equivalent to entering a receive-floating, pay-fixed forward swap.
- If the exercise rate is selected such that the receiver and payer swaptions have the same value, then the exercise rate is equal to the ***at-market forward swap rate***.
- A long position in a callable fixed-rate bond can be viewed as being long a straight fixed-rate bond and short a receiver swaption. The receiver swaption buyer will benefit when rates fall and the swaption is exercised. Thus, the embedded call feature is similar to a receiver swaption.

<u>Practice:</u> Example 16, Reading 40, Curriculum.

6. OPTION GREEKS AND IMPLIED VOLATILITY

6.1 Delta

Option delta is the change in an option value for a ***given small change*** in the value of the underlying stock, holding everything else constant. The option deltas for calls and puts are as follows, respectively

$$\text{Delta}_c = e^{-\delta T}N(d_1)$$

$$\text{Delta}_p = -e^{-\delta T}N(-d_1)$$

- The delta of long one share of stock is +1.0, and the delta of short one share of stock is –1.0.
- Delta is a static risk measure because it does not tell us how likely this particular change would be.
- The range of call delta is 0 and $e^{-\delta T}$ and the range of put delta is $-e^{-\delta T}$ and 0.
- As the stock price increases, the call option goes deeper in the money and the value of $N(d_1)$ moves toward 1.
- As the stock price decreases, the call option goes deeper out of the money and the value of $N(d_1)$ moves toward zero.
- When the option gets closer to maturity, the delta will drift either toward 0 if it is out of the money or drift toward 1 if it is in the money.
- As the stock price changes and as time to maturity changes, the deltas also changes.

Delta neutral portfolio: A delta neutral portfolio refers to setting the portfolio delta to zero. Theoretically, the value of delta neutral portfolio does not change for small changes in the stock instrument.

- Delta neutral implies that the portfolio delta plus $N_H\text{Delta}_H$ is equal to zero. The optimal number of hedging units, N_H, is

$$N_H = -\frac{\text{Portfolio delta}}{\text{Delta}_H}$$

Where,
N_H denote the number of units of the hedging instrument;
Delta_H denote the delta of the hedging instrument, which could be the underlying stock, call options, or put options.

- If N_H is negative, then we should short the hedging instrument.
- If N_H is positive, then we should go long the hedging instrument.

Example: Suppose a portfolio consists of 100,000 shares of stock at US$10 per share. In this case, the portfolio delta is 100,000. The delta of the hedging instrument, stock, is +1. Thus, the optimal number of hedging units, N_H, is –100,000 (= –100,000/1) or short 100,000 shares.

If the portfolio delta is 5,000 and a particular call option with delta of 0.5 is used as the hedging instrument, then to arrive at a delta neutral portfolio, we need to sell 10,000 call options (= –5,000/0.5). Alternatively, if a portfolio of options has a delta of –1,500, then we need to buy 1,500 shares of stock to be delta neutral [= –(–1,500)/1]. If the hedging instrument is stock, then the delta is +1 per share.

Practice: Example 17, Reading 40, Curriculum.

Delta approximation Equation:

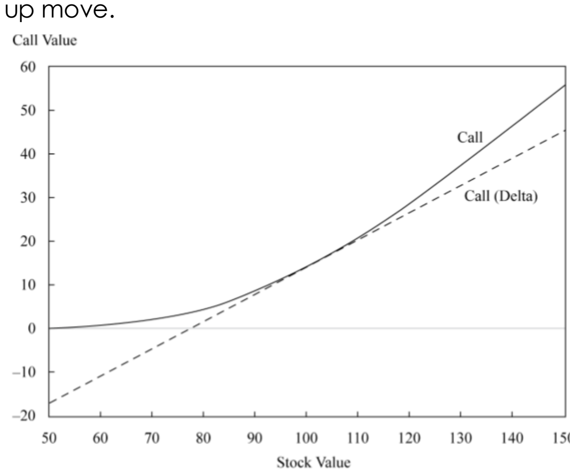
$$\hat{c} - c \cong \text{Delta}_c\left(\hat{S} - S\right) \text{ for calls}$$

or

$$\hat{c} = c + \text{Delta}_c\left(\hat{S} - S\right)$$

$$\hat{p} - p \cong \text{Delta}_p\left(\hat{S} - S\right) \text{ for puts}$$

The delta approximation is fairly accurate for ***very small*** changes in the stock. But as the change in the stock increases, the estimation error also increases. The delta approximation is biased low for both a down move and an up move.



The above chart shows that delta hedging is imperfect and gets worse as the underlying moves further away from its original value of 100.

Practice: Example 18, Reading 40, Curriculum.

6.2 Gamma

Option gamma refers to the change in a given option delta for a given small change in the stock's value, holding everything else constant. Option gamma is a ***measure of the curvature*** in the option price in relationship to the stock price. Gamma approximates the estimation error in delta for options because the option price with respect to the stock is non-linear and delta is a linear approximation. This implies that gamma measures the ***non-linearity*** risk. A gamma neutral portfolio implies the gamma is zero.

- The gamma of a long or short position in one share of stock is 0 because the delta of a share of stock never changes. The delta of stock is always +1 and –1 for a short position in the stock.
- The gamma for a call and put option are the same and can be expressed as below:

$$\text{Gamma}_c = \text{Gamma}_p = \frac{e^{-\delta T}}{S\sigma\sqrt{T}} n(d_1)$$

Where, $n(d_1)$ is the standard normal probability density function.

- The gamma of a call equals the gamma of a similar put based on put–call parity or $c - p = S_0 - e^{-rT}X$. Note that neither S0 nor e–rTX is a direct function of delta. Hence, the right-hand side of put–call parity has a delta of 1.
- Gamma is always non-negative.
- Gamma is largest near at the money.
- Options deltas do not change substantially for small changes in the stock price if the option is either deep in or deep out of the money.
- As the stock price changes and as time to expiration changes, the gamma also changes.
- Buying options (calls or puts) will always increase net gamma.
- Gamma Risk: It is the risk associated with non-continuous and un smooth change in stock prices.

__Important to remember:__ In delta neutral portfolio strategy, first we need to manage gamma to an acceptable level and then we neutralize the delta is neutralized. This hedging approach is more feasible because options, unlike stocks, have gamma. To alter the portfolio delta, we need to buy or sell stock. Because stock has a positive delta, but zero gamma, the portfolio delta can be brought to its desired level with no impact on the portfolio gamma.

Delta-plus-gamma approximation Equation:

$$\hat{c} - c \approx \text{Delta}_c\left(\hat{S} - S\right) + \frac{\text{Gamma}_c}{2}\left(\hat{S} - S\right)^2 \text{ for calls and}$$

$$\hat{p} - p \approx \text{Delta}_p\left(\hat{S} - S\right) + \frac{\text{Gamma}_p}{2}\left(\hat{S} - S\right)^2 \text{ for puts.}$$

The call value based on the delta approximation is estimated as below:

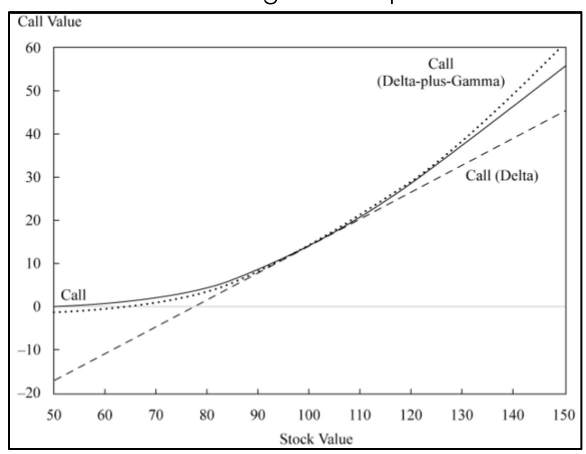
$$\hat{c} = c + \text{Delta}_c\left(\hat{S} - S\right)$$

Call value based on the delta-plus-gamma approximation is expressed as below:

$$\hat{c} = c + \text{Delta}_c\left(\hat{S} - S\right) + \frac{\text{Gamma}_c}{2}\left(\hat{S} - S\right)^2$$

- The delta approximation and the delta-plus-gamma approximations are fairly accurate for very small changes in the stock.

The chart below reflects that the call delta-plus-gamma estimated line is significantly closer to the BSM model call values. We can see that even for fairly large changes in the stock, the delta-plus-gamma approximation is accurate. As the change in the stock increases, the estimation error also increases. The chart also shows that the delta-plus-gamma approximation is biased low for a down move but biased high for an up move.



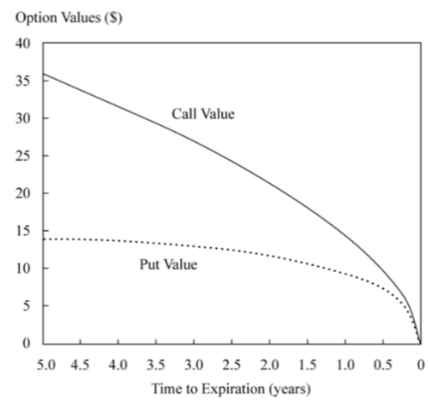
__Practice:__ Example 19, Reading 40, Curriculum.

6.3	Theta

Option theta is the change in an option value for a given small change in ***calendar time***, holding everything else constant. In other words, Option theta is the rate at which the option time value declines as the option approaches expiration. Stock theta is zero because stocks do not have an expiration date. Like gamma, theta cannot be adjusted with stock trades. Typically, theta is negative for options. That is, as calendar time passes, expiration time declines and the option value also declines.

Time decay: It refers to the gain or loss of an option portfolio in response to the mere passage of calendar time. Particularly with long options positions, often the mere passage of time without any change in other variables, such as the stock, will result is significant losses in value.

Please refer to the chart below to assess how the speed of the option value decline increases as time to expiration decreases.



6.4 Vega

Vega is the change in a given portfolio for a given ***small change in volatility***, holding everything else constant. Thus, vega measures the sensitivity of a portfolio to changes in the volatility used in the option valuation model. The vega of an option is positive, i.e., an increase in volatility results in an increase in the option value for both calls and puts.

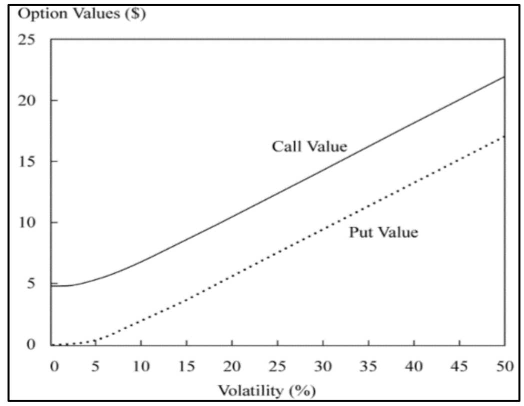
- Based on put–call parity, the vega of a call is equal to the vega of a similar put.
- Vega is high when options are at or near the money and are short dated.
- Volatility is usually only hedged with other options.
- Volatility is sometimes considered a separate asset class or a separate risk factor.

Unlike the delta, gamma, and theta, vega is based on an ***unobservable*** parameter, i.e. future volatility. Future volatility is a subjective measure similar to future value. Option's value is most sensitive to volatility changes.

When volatility is low, the option values tend toward their lower bounds.

- The lower bound of a European-style call option: Zero or the stock less the present value of the exercise price, whichever is greater.
- The lower bound of a European-style put option: Zero or the present value of the exercise price less the stock, whichever is greater.

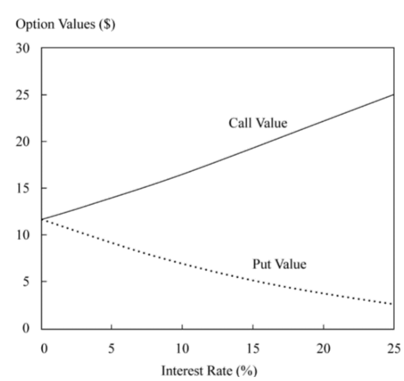
The chart given below shows that the call lower bound is 4.88 and the put lower bound is 0. The difference between the call and put can be explained by put–call parity.



6.5 Rho

Rho is the change in a given portfolio for a given small change in the ***risk-free interest rate***, holding everything else constant. Thus, rho measures the sensitivity of the portfolio to the risk-free interest rate.

- The rho of a call is positive because purchasing a call option allows an investor to earn interest on the money that otherwise would have gone to purchasing the stock. The higher the interest rate, the higher the call value.
- The rho of a put is negative because purchasing a put option rather than selling the stock deprives an investor of the potential interest that would have been earned from the proceeds of selling the stock. The higher the interest rate, the lower the put value.
- When interest rates are zero, the call and put option values are the same for at-the money options.
- As interest rates rise, the difference between call and put options increases.



The option prices not highly sensitive to changes in interest rates change when compared with changes in volatility and changes in the stock.

6.6 Implied Volatility

Implies volatility refers to the volatility estimated from option prices. Implied volatility is a measure of future volatility, whereas historical volatility is a measure of past volatility. The implied volatility can be estimated by using

BSM model. The that implied volatility provides us information regarding the perceived uncertainty going forward and thereby allows us to gauge collective opinions of investors on the volatility of the underlying and the demand for options.

- If the demand for options increases and the no-arbitrage approach is not perfectly reflected in market prices (e.g. due to transaction costs) then the option prices increase, and hence, the observed implied volatility also increases.
- If the implied volatility of a put increases, it indicates that it is more expensive to buy downside protection with a put. Hence, the market price of hedging rises.
- The original BSM model assumes constant volatility of underlying instrument. However, practically, the implied volatilities vary depending on exercise prices and observe different implied volatilities for calls and puts with the same terms. Implied volatility also varies across time to expiration as well as across exercise prices. Implied volatility is also not constant through calendar time.

There are two types of implied volatility:

1) **Term structure of volatility**: The implied volatility with respect to time to expiration is known as the term structure of volatility. The volatility surface is a ***three dimensional plot*** of the implied volatility with respect to both expiration time and exercise prices.
2) **Volatility smile**: The implied volatility with respect to the exercise price is known as the volatility smile or sometimes skew depending on the particular shape. The volatility smile is a ***two dimensional plot*** of the implied volatility with respect to the exercise price.

We can trade futures and options on various volatility indexes available in the market in order to manage our vega exposure in other options.

In the option markets, volatility can be used by investors as the medium in which to quote options. For example, rather than quote a particular call option as trading for €14.23, we may quote it as 30.00, where 30.00 denotes in percentage points the implied volatility based on a €14.23 option price. Quoting the option price in terms of implied volatility allows us to trade volatility.

Important to remember: Ignoring rounding errors, there is a one-to-one relationship between the implied volatility and the option price.

Uses of Implied Volatility:

- Implied volatility can be used to assess the relative value of different options, neutralizing the moneyness and time to expiration effects.
- Implied volatility can be used to revalue existing positions over time.
- Regulators, banks, compliance officers, and most option traders use implied volatilities to communicate information related to options portfolios because implied volatilities provide the "market consensus" valuation.

Example: The Chicago Board Options Exchange S&P 500 Volatility Index, known as the VIX, is a volatility index. The VIX is quoted as a percent and reflects the implied volatility of the S&P 500 over the next 30 days. VIX is often termed the ***fear index*** because it is viewed as a measure of market uncertainty. Thus, an increase in the VIX index is regarded as greater investor uncertainty.
Example: If a trader thinks that based on the current outlook, the implied volatility of S&P 500 (say 20%) should be 25%, it indicates that volatility is understated by the dealer. In this case, since the S&P 500 call is expected to increase in value. Hence, trader would buy the call.

Practice: Example 20 & 21, Reading 40, Curriculum.

1. INTRODUCTION

There are various types of derivative strategies; some of them are purely speculative which are designed to profit if a particular market change occurs, while other strategies are defensive, providing protection against an adverse event or removing the uncertainty around future events.

2. CHANGING RISK EXPOSURES WITH SWAPS, FUTURES, AND FORWARDS

Derivatives markets can be used to quickly and efficiently alter the underlying risk exposure of asset portfolios or forthcoming business transactions.

2.1 Interest Rate Swap/Futures Examples

Interest rate swaps and futures can be used to modify the risk and return of a fixed-income portfolio and can also be used in conjunction with an equity portfolio. Both interest rate swaps and futures are interest-sensitive instruments, so if they are added to a portfolio, they can increase or decrease the exposure of the portfolio to interest rates.

2.1.1.) Interest Rate Swap

It is an agreement to swap in which one party pays fixed interest rate payments (fixed rate is called ***swap rate***) and other party pays floating interest rate payments in exchange or when both parties pay floating-rate payments. When both parties pay floating rates then floating rates are different.

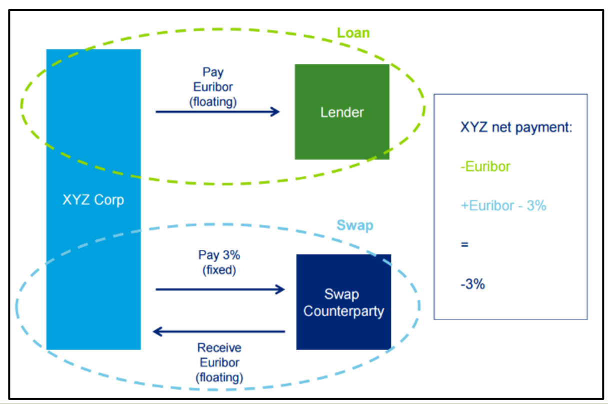
- Interest rate swaps have less credit risk relative to ordinary loans because interest payments are netted and there is no exchange of notional principal.
- However, it is important to note that netting reduces the credit risk but it does not prevent the LIBOR component of the net swap payment from offsetting the floating loan interest payment.

The period of time over which the payments are exchanged is called the ***swap tenor***. The swap expires at the end of this period.

Limitation: Swaps involve credit risk i.e. risk that counterparty may default on the exchange of the interest payments.

Example: XYZ Corp. has €100M of floating-rate debt at Euribor. XYZ would prefer to have fixed-rate debt. XYZ could enter a swap, in which they receive a floating rate and pay the fixed rate, which in the following example, is 3%.

- If a firm thought that rates would rise it would enter into a swap agreement to pay fixed and receive floating in order to protect it from rising debt-service payments.
- If a firm thought that rates would fall it would enter into a swap agreement to pay floating and receive fixed in order to take advantage of lower debt-service payments.
- The swap itself is not a source of capital but an alteration of the cash flows associated with payment.



Example: A portfolio manager has an investment portfolio containing $500 million of fixed-rate US Treasury bonds with an average duration of five years. He wants to reduce this duration to three over the next year but does not want to sell any of the securities.

- One way to do this would be with a pay-fixed interest rate swap in exchange for a floating-rate stream in order to lower the overall duration.
- Suppose the duration of the swap used by the manager is 1.5. This duration is less than the existing portfolio duration, so adding the swap to the portfolio will reduce the overall average duration.

2.1.2.) Interest Rate Futures

A forward contract is an agreement where one party promises to buy an asset from another party at a specified price at a specified time in the future. No money changes hands until the delivery date or maturity of the contract. The terms of the contract make it an obligation to buy the asset at the delivery date. The asset could be a stock, a commodity or a currency.

A futures contract is very similar to a forward contract. Futures contracts are usually traded through an exchange, which standardizes the terms of the contracts. The profit or loss from the futures position is calculated every day and the change in this value is paid from one party to the other.
Forwards, like swaps, have counterparty risk and can be customized. Futures are standardized and come with greater regulatory oversight and with a clearinghouse that makes counterparty risk virtually zero. These contracts are also sometimes referred to as ***bond futures*** because the underlying asset is often a bond. Hence, the futures price fairly consistently and proportionately moves with the yield that drives the underlying bond.

- We can reduce duration of our portfolio by selling bond futures.

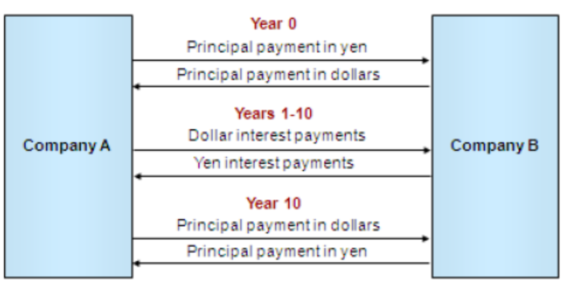
2.2 Currency Swap/Futures Examples

2.2.1.) Currency Swap

Currency swaps can be used by investors to manage exchange rate risk. In a currency swap, the interest rates are associated with different currencies and principal must be specified in each currency and the principal amounts are exchanged at the beginning and end of the life of the swap.

Currency Swaps can be used to transform a loan denominated in one currency into a loan denominated in another currency.

Example: Company B is a U.S. based firm and it borrows yen and engages in a swap with the company A that borrows dollars with parallel interest and principal repayment schedules:



Example:
A firm ABC needs ₤30 million expand into Europe. To implement this expansion plan, a firm needs to borrow Euros. Suppose current exchange rate is €1.62/₤. Thus, a firm needs to borrow €48.60 million. Instead of directly borrowing Euros, a firm can use currency swap e.g. if a firm issues fixed rate pound denominated bond for 30 million pounds with interest rate of 5% (annual interest payments). A firm enters into a currency swap contract in which it will pay 30 million pounds to dealer and receives 48.60 Euros. The terms of a swap are i.e.

- Firm will pay 3.25% in Euros to a dealer.
- Firm will receive 4.50% in pounds from a dealer.

Exchange of principals at contract initiation:

- Firm ABC will receive €48.60 million from currency swap dealer.
- Currency swap dealer will receive ₤30 million from Firm ABC.

Cash flows at each settlement:

- Interest payments on pound-denominated bond = ₤30,000,000 × 0.05 = ₤1,500,000.
- Interest payments due to Firm ABC from swap dealer = ₤30,000,000 × 0.045 = ₤1,350,000.
- Interest payments that Firm ABC owes swap dealer = €48,600,000 × 0.0325 = €1,579,500.

At swap and bond maturity:

- Firm ABC will receive ₤30 million from currency swap dealer and uses that amount to discharge its liabilities.
- Firm ABC will pay €48.60 million to currency swap dealer.

Difference between currency swaps and interest rate swaps:

- Currency swaps involve the payment of notional principal. However, it is important to note that not all currency swaps involve the payment of notional principal.
- Unlike interest rate swaps, interest payments in currency swaps are not netted as they are in different currencies.

2.2.2.) Currency Futures

Foreign currency futures can also be used in managing risk. Following table summarizes the appropriate strategy to pursue in managing foreign currency risk.

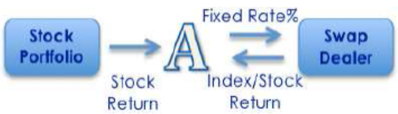
Currency Exposure	Position in Foreign Currency	Action taken to hedge currency risk
Receiving Foreign Currency	Long	Sell Futures Contract
Paying Foreign Currency	Short	Buy Futures Contract

Example: A firm expects to receive a payment in British pounds worth ₤10 million. Payment will be received in 60 days. Current spot exchange rate = \$1.45/ ₤ 60-days Futures exchange rate = \$1.47/ ₤
A firm is long foreign currency because it expects to receive foreign currency. Therefore, a firm should take short position in a futures contract i.e. using futures contract a firm will receive (after 60 days): ₤10,000,000 × \$1.47/₤ = \$14,700,000. This amount will be received by the firm irrespective of exchange rate at that time.

2.3 Equity Swap/Futures Examples

2.3.1.) Equity Swap

In an equity swap One party is obligated to make payments based on the total return of some equity index e.g. S&P 500 or an individual stock. The other party pays a fixed rate, a floating rate, or the return on another index.



- Equity swaps are created in the over-the-counter market, so they can be customized.

Strategies:

A. When investor has bearish outlook towards stock market and interest rates are falling → Swap equity return for fixed rate.

B. When investor has bearish outlook towards stock market and interest rates are increasing→ Swap equity return for floating rate.

Example: Consider the following table:

Pay the return on a $100 million equity portfolio

Receive six-month Libor, assumed to be 0.50%

Scenario 1: Equity portfolio rises 1%

Pay: $100 million × 1% =	$1,000,000
Receive: $100 million × 0.50% × 0.50 =	250,000
Net payment =	$750,000

Scenario 2: Equity portfolio declines 1%

Pay: $100 million × −1% =	($1,000,000)
Receive: $100 million × 0.50% × 0.50 =	250,000
Net receipt =	$1,250,000

- In the first scenario, the institutional investor would have an obligation to pay 1% × $100 million, or $1 million. On the Libor portion of the swap the investor would receive 0.50% × 0.50 × $100 million, or $250,000. The institutional investor would pay the netted amount of $750,000.
- In the second scenario, the return the institutional investor must pay is negative, which means it will receive money both from "paying" a negative return and from the Libor rate. It would receive $1 million from the "negative payment" and $250,000 from Libor, for a total of $1.25 million.

2.3.2.) Stock Index Futures

Stock index futures (unlike most other futures contracts) are cash settled at expiration. The market risk can be ***temporarily*** removed by selling stock index futures. One S&P 500 stock index futures contract is standardized as $250 times the index level.

Example: Assume that a one-month futures contract trades at 2,000 and that the portfolio carries average market risk, having a beta of 1.0. To fully hedge the $100,000,000 portfolio, the portfolio manager would want to sell $100,000 / ($250 × 2000) = 200 contracts.

- Suppose the S&P 500 stock index rises by 0.5% and thus, the index value is 2,012 at delivery time.

  Loss = −10 points per contract × $250 per point × 200 contracts = $500,000.

- If the stock index rises by 0.5%, the portfolio would also be expected to rise to = $100,000,000 × 0.5% = $500,000.

Practice: Example 1, Reading 41, Curriculum.

3. POSITION EQUIVALENCIES

3.1 Synthetic Long Asset

Synthetic long position = Buys a call + Writes a put = Long Call + Short Put

Where, both options have the same expiration date and the same exercise price.

- The long call creates the upside and the short put creates the downside of the underlying.
- The call exercises when the underlying is higher than the strike and turns into a synthetic position in the upside of the underlying.
- A short put obligates the writer to purchasing the stock at a higher price than its value from put buyer.

3.1 Synthetic Short Asset

Synthetic Short Position = Buy Put + Write Call = Long Put + Short Call

Where, both options have the same expiration date and the same exercise price.

3.3 Synthetic Assets with Futures/Forwards

Synthetic risk-free rate or Synthetic Cash = Long stock + Short futures

Or

Stock – Futures = Risk-free rate

Similarly, we can create a ***synthetic long position*** by investing in the risk-free asset and using the remaining funds to margin a long futures position, that is,

Stock = Risk-free rate + Futures

3.4 Synthetic Put

Synthetic Put = Short stock position + Long call

Important to Note: Any mispricing in a replicated put may make it cheaper or more expensive than a direct put.

3.5 Synthetic Call

Synthetic Call = Long stock position + Long put

➢ The long put eliminates the downside risk whereas the long stock leaves the profit potential unlimited.

3.6 Foreign Currency Options

Unlike forwards and future, options have asymmetrical payoffs. This implies that if someone wants to benefit from an appreciating currency "X" but do not want to lock in to a fixed rate, as with a futures or forward, he might buy a one-month call option on "X". Because the spot rate is quoted in "X", the strike will typically be quoted in "X". A foreign currency call option always has a put option that is an identical twin.

Practice: Example 2, Reading 41, Curriculum.

4. COVERED CALLS AND PROTECTIVE PUTS

Covered Call = Long stock position + Short call position

Covered Call[1] is appropriate to use when an investor:

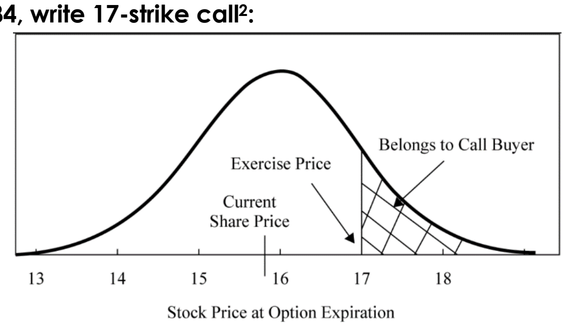
- Owns the stock and
- Expects that stock price will neither increase nor decrease in near future.

4.1 Investment Objectives of Covered Calls

Following are some of the investment objectives of Covered Call:

1) **Income Generation:** The most common motivation for writing covered calls is income generation as writing an option gives option writer option premium. There is a clear trade-off between the size of the option premium and the likelihood of option exercise. The option premium is higher for a longer-term option, but there is a greater chance that the option would move in the money, resulting in the option being exercised by the buyer.

Please refer to the return distribution below for a stock at 15.84, write 17-strike call[2]:



Note that if underlying goes up, the write of covered call bears opportunity loss.

2) **Improving on the Market:** If an investor has higher exposure in (say power sector) and wants to reduce it then he can write call option on those companies. By writing call option, he receives option premium. This income remains in his account regardless of what happens to the future stock price of those companies or whether or not the option is exercised by its holder. Hence, entering into covered call strategy provides her opportunity to reduce his exposure in power sector to desired level as well as generating additional income via option premium.

Option Premium:
The option premium is composed of two parts:

i. **Exercise value (also called intrinsic value):** The difference between the spot price of the underlying asset and the exercise price of the option is termed the intrinsic value of the option. E.g. the right to buy at 15 when the stock price is 15.50 is clearly worth 0.50. Thus, $0.50 is exercise value.

ii. **Time value:** The time value of an option is the difference between the premium of an option and its intrinsic value. E.g. say the option premium is $1.50, which is $1.0 more than the exercise value. This difference of $1.0 is called time value. Someone who writes covered calls to improve on the market is capturing the time value.

- When option is out of the money, the premium is entirely time value.

3) **Target Price Realization:** This strategy involves writing calls with an exercise price near the target price for the stock. Suppose a portfolio manager holds stock of company "X" in many of its accounts and that its research team believes the stock would be properly priced at 25/share, which is just slightly higher than its current price. So, if options trading is allowed, the portfolio manager may write ***near-term*** calls with an ***exercise price near the target price***, 25 in this case. Suppose an account holds 500 shares of "X". Writing 5 SEP 25 call contracts at 0.95 brings in 475 in cash. If the stock is above 25 in a month, the stock will be sold at its target price, with the option premium adding an additional 4% positive return to the account. If "X" fails to rise to 25, the manager might write a new OCT expiration call with the same objective in mind.

In short, covered calls can be used to *generate income*, to acquire shares at a lower-than market price, or to exit a position when the shares hit a target price.

Risks associated with this strategy: Although the covered call writing program potentially adds to the return, there is also the chance that the stock could fall substantially, resulting in an opportunity loss relative to the outright sale of the stock. The investor also would have an opportunity loss if the stock rises sharply above the exercise price and it was called away at a lower-than market price.

[1] If someone creates a call without owning the underlying asset, it is a **naked call**.
[2] 17-strike call" meaning a call option with an exercise price of 17.

4.1.4.) Profit and Loss at Expiration

Payoffs summary:

a) **Value at expiration** = Value of the underlying + Value of the short call = $V_T = S_T - \max(0, S_T - X)$

b) **Profit** = Profit from buying the underlying + Profit from selling the call = $V_T - S_0 + c_0$

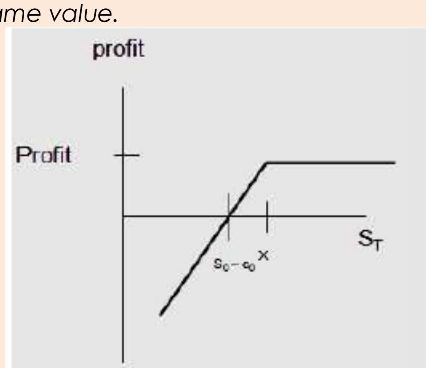
c) **Maximum Profit** = $X - S_0 + c_0$

d) Max loss would occur when $S_T = 0$. Thus, **Maximum Loss** = $S_0 - c_0$

- Even if the stock declines to nearly zero, the loss is less with the covered call because the option writer gets the option premium.

e) **Breakeven** $= S_T^* = S_0 - c_0$

Note that the breakeven price and the maximum loss are the same value.



The general shape of the profit and loss diagram for a covered call is the same as that of writing a put.

Practice: Example 3, Reading 41, Curriculum.

4.2 Investment Objective of Protective Puts

Protective Put = Long stock position + Long Put position

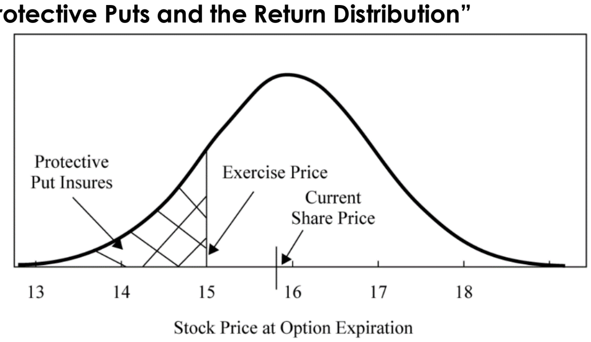
- This provides protection against a decline in value.
- It provides downside protection while retaining the upside potential.
- It requires the payment of cash up front in the form of option premium.
- The higher the exercise price of a put option, the more expensive the put will be and consequently the more expensive will be the downside protection.
- It is similar to "insurance" i.e. buying insurance in the form of the put, paying a premium to the seller of the insurance, the put writer.

Insurance Policy	Put Option
Premium	Time value
Value of asset	Price of stock
Face value	Exercise price
Term of policy	Time until option expiration
Likelihood of loss	Volatility of stock

- As with insurance policies, a put implies a deductible, which is the amount of the loss the insured is willing to bear. This implies that
  Deductible = Stock price - Exercise price
- The cost of insurance can be reduced by increasing the size of the deductible.
- Protective put strategy has a profit and loss diagram similar to that of a long call.

Protective put can be used when an investor expects a decline in the value of the stock in the near future but wants to preserve upside potential. The put value and its time until expiration does not have linear relationship. This implies that a two-month option does not sell for twice the price of a one-month option.

Please refer to the following diagram showing "**Protective Puts and the Return Distribution**"



The above diagram shows that the put provides protection from the left tail of the return distribution. It is important to note that the continuous purchase of protective puts is expensive.

4.2.2.) Profit and Loss at Expiration

Payoffs summary:

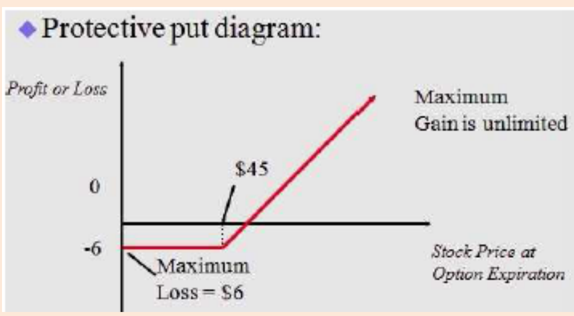
a) Value at expiration: $V_T = S_T + \max(0, X - S_T)$

b) Profit = $V_T - S_0 - p_0$

c) Maximum Profit = ∞ or unlimited because the stock can rise to any level

d) The maximum loss would occur when underlying asset is sold at exercise price. Thus, Maximum Loss = $S_0 + p_0 - X$ **or** "deductible" + cost of the insurance.

e) In order to breakeven, the underlying must be at least as high as the amount paid up front to establish the position. Thus, Breakeven = $S_T^* = S_0 + p_0$



Practice: Example 4, Reading 41, Curriculum.

4.3 Equivalence to Long Asset/Short Forward Position

Delta measures the change in option price due to the change in underlying asset price.

- A call option deltas range from 0 to 1 because call increases in value when value of underlying asset increases.
- A put option deltas range from 0 to -1 because put decreases in value when value of underlying asset increases.
- A long position in the underlying asset has a delta of 1.0, whereas a short position has a delta of –1.0.
- At-the-money option will have a delta that is ~0.5 (for a call) or ~–0.5 (for a put).
- Futures and forwards have delta of 1.0 for a long position and –1.0 for a short position.

4.4 Writing Cash-Secured Puts

Writing a cash secured put involves writing a put option and simultaneously depositing an amount of money equal to the exercise price into a designated account. This strategy is also called a fiduciary put. The escrow account provides assurance that the put writer will be able to purchase the stock if the option holder chooses to exercise. Cash in a cash-secured put is similar to the stock part of a covered call.

- This strategy is appropriate for someone who is bullish on a stock or who wants to buy shares at a particular price.
- When someone writes a put but does not escrow the exercise price, it is sometimes called a ***naked put***.

4.6 Collars

Collar refer to the strategy in which the cost of buying put option can be reduced by selling a call option. A collar is also called a ***fence or a hedge wrapper***. In a foreign exchange transaction, it might be called a risk reversal.

- When call option premium is equal to put option premium, no net premium is required up front. This strategy is known as a Zero-Cost Collar. For this reason, most collars are done in the over-the-counter market because the exercise price on the call must be a specific one.
- This strategy provides downside protection at the expense of giving up upside potential.

Typically,

- Put exercise price (e.g. X_1) < current value of the underlying.
- Call exercise price (e.g. X_2) must be > current value of the underlying.
- When price < X_1, put provides protection against loss.
- When price > X_2, short call reduces gains.
- When price lies between X_1 and X_2, both put and call are out-of-the-money.

Payoffs:

a) Initial value of the position = value of the underlying asset = $V_0 = S_0$

b) Value at expiration: V_T = Value of underlying S_T + Value of the put option + Value of the short call option = $S_T + \max(0, X_1 - S_T) - \max(0, S_T - X_2)$

c) Profit = $V_T - V_0 = V_T - S_0$

d) Maximum Profit = $X_2 - S_0$

e) Maximum Loss = $S_0 - X_1$

f) Breakeven = $S_T^* = S_0$

4.6.1.) Collars on an Existing Holding

A collar is typically established on an outstanding position. E.g. consider the risk–return trade-off for a shareholder who previously bought a stock at 12 and now buys the NOV 15 put for 1.46 and simultaneously writes the NOV 17 covered call for 1.44.

Stock price at expiration →	5	10	15	16	17	20
Profit/loss from long stock	−7.00	−2.00	3.00	4.00	5.00	8.00
Profit/loss from long 15 put	8.54	3.54	−1.46	−1.46	−1.46	−1.46
Profit/loss from short 17 call	1.44	1.44	1.44	1.44	1.44	−1.56
Total	2.98	2.98	2.98	3.98	4.98	4.98

- At or below the put exercise price of 15, the collar locks in a profit of 2.98.
- At or above the call exercise price of 17, the profit is constant at 4.98.

4.6.2.) Same-Strike Collar

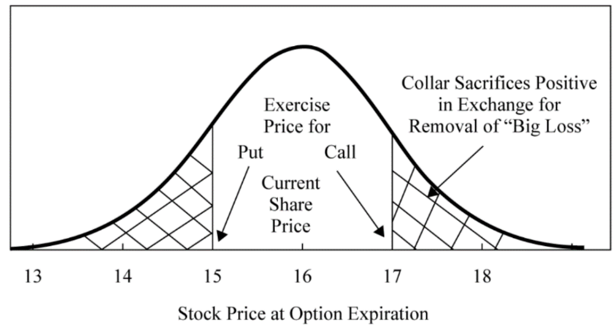
Long a put and short a call is a ***synthetic short position***. When a long position is combined with a synthetic short position, logically the risk is completely neutralized. Hence, if an investor combines a same-strike collar with a long position in the underlying asset, the value of combined position will be the option exercise price, regardless of the stock price at option expiration. *Please refer to the table below.*

Stock price at expiration →	0	20	40	50	60	80	100
Long 50-strike put payoff	50	30	10	0	0	0	0
Short 50-strike call payoff	0	0	0	0	–10	–30	–50
Long stock	0	20	40	50	60	80	100
Total payoff	50	50	50	50	50	50	50

4.6.3.) The Risk of a Collar

A collar forgoes the positive part of the return distribution in exchange for avoiding risk of adverse movement in stock price. See the diagram below (With stock at 15.84, write 17 call and buy 15 put):

- With the long put, the investor is protected against the left side of the distribution and the associated losses.
- With the short call option, the option writer sold the right side of the return distribution, which includes the most desirable outcomes.
- Hence, we can see that the collar tends to narrow the distribution of possible investment outcomes, which is risk reducing.



5. SPREADS AND COMBINATIONS

5.1 Bull Spreads and Bear Spreads

Spreads are classified in two ways, i) by market sentiment and ii) by the direction of the initial cash flows.

- **Bull spread:** A spread whose value increases when the price of the underlying asset rises is a bull spread.
- **Bear Spread:** A spread whose value increases when the price of the underlying asset declines.
- **Debit spread:** It is the spread which requires a cash payment. Debit spreads are effectively long because the long option value exceeds the short option value.
- **Credit spread:** If the spread initially results in a cash inflow, it is referred to as a credit spread. Credit spreads are effectively short because the short option value exceeds the long option value.

Any of these strategies can be created with puts or calls.

5.1.1.) Bull Spread

A spread strategy is appropriate to use with a volatile stock in a ***trending*** market.

Bull Call Spread: This strategy involves a combination of a long position in a call with a lower exercise price and a short position in a call with a higher exercise price i.e. Buy a call (X_1) with option cost c_1 and sell a call (X_2) with option cost c_2, where $X_1 < X_2$ and $c_1 > c_2$.

- Note that the lower the exercise price of a call option, the more expensive it is.

Rationale to use Bull Call Spread: Bull call spread is used when investor expects that the stock price or underlying asset price will increase in the near future.

- This strategy gains when stock price rises/ market goes up.
- Like covered call, it provides protection against downside risk but provides limited gain i.e. upside potential.
- It is similar to Covered call strategy i.e. in bull call spread, the short position in the call with a higher exercise price is covered by long position in the call with a lower exercise price.

<u>**Payoffs:**</u>

a) The initial value of the Bull call spread = $V_0 = c_1 - c_2$

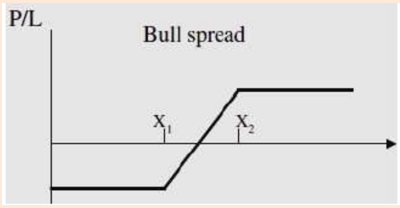
b) Value at expiration: VT = value of long call – Value of short call = $\max(0, S_T - X_1) - \max(0, S_T - X_2)$

c) Profit = Profit from long call + profit from short call. Thus, Profit = $V_T - c_1 + c_2$

d) Maximum Profit = $X_2 - X_1 - c_1 + c_2$

e) Maximum Loss = $c_1 - c_2$

f) Breakeven $=S_T^* = X_1 + c_1 - c_2$



Bull Put Spread: In bull put spread, investor buys a put with a lower exercise price and sells an otherwise identical put with a higher strike price.

5.1.2). Bear Spread

Bear Put Spread: This strategy involves a combination of a long position in a put with a higher exercise price and a short position in a put with a lower exercise price i.e. Buy a put (X_2) with option cost p_2 and sell a put (X_1) with option cost p_1, where $X_1 < X_2$ and $p_1 < p_2$.

- Note that the higher the exercise price of a put option, the more expensive it is.

Rationale to use Bear Put Spread: Bear Put spread is used when investor expects that the stock price or underlying asset price will decrease in the future.

Payoffs:

a) The initial value of the bear put spread $= V_0 = p_2 - p_1$

b) Value at expiration: V_T = value of long put – Value of short put $= \max(0, X_2 - S_T) - \max(0, X_1 - S_T)$

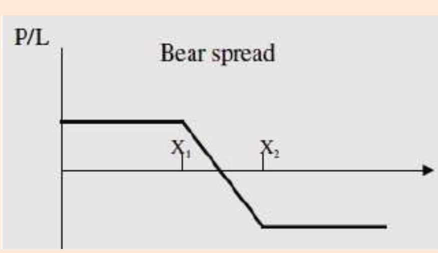
c) Profit = Profit from long put + profit from short put. Thus, Profit $= V_T - p_2 + p_1$

d) Maximum Profit occurs when both puts expire in-the-money i.e. when underlying price ≤ short put exercise price ($S_T \le X_1$),

- Short put is exercised and investor will buy an asset at X_1 and This asset is sold at X_2 when long put is exercised. Thus, Maximum Profit $= X_2 - X_1 - p_2 + p_1$

e) Maximum Loss occurs when both puts expire out-of-the-money and investor loses net premium i.e. when $S_T > X_2$. Thus, Maximum Loss $= p_2 - p_1$

f) Breakeven $=S_T^* = X_2 - p_2 + p_1$



Bear Call Spread: In bear call spread, investor sells a call with a lower exercise price and buys an otherwise identical call with a higher strike price.

Important to remember:

- With either a bull spread or a bear spread, both the maximum gain and the maximum loss are known and limited.
- Bull spreads with American puts have an additional risk, because the short put gets exercised early, whereas the long put is not yet in the money. In contrast, if the bull spread uses American calls and the short call is exercised, the long call is deeper in the money, which offsets that risk. A similar point can be applied to bear spreads using calls. Thus, with American options, bull spreads with calls and bear spreads with puts are generally preferred (but not necessarily required).
- If puts and calls are bought with different exercise prices, the position is called a **strangle**.

5.1.3.) Refining Spreads

5.1.3.1.) Adding a Short Leg to a Long Position

Suppose, a speculator in September paid a premium of 1.50 for a NOV 40 call when the underlying stock was selling for 37. A month later, in October, the stock has risen to 48. He observes the following premiums for one-month call options.

Strike	Premium
40	8.30
45	4.42
50	1.91

- The call he bought is now worth 8.30. So, his profit at this point is 8.30 – 1.50 = 6.80.
- He thinks the stock is likely to stabilize around its new level; so, he writes another call option with an exercise price of either 45 or 50, thereby converting his long call position into a bull spread.
- At stock prices of 50 or higher, the exercise value of the spread is 10.00 because both options would be in the money, and a call with an exercise price of 40 would always be worth 10 more than a call with an exercise price of 50. The initial cost of the call with an exercise price of 40 was 1.50, and there was a 1.91 cash inflow after writing the call with an exercise price of 50. Thus, the profit is 10.00 – 1.50 + 1.91 = 10.41.
- At stock prices of 40 or lower, the exercise value of the spread is zero; both options would be out of the money. Thus, the profit is 0 – 1.50 + 1.91 = 0.41.
- Between the two striking prices (40 and 50), the exercise value of the spread rises steadily as the stock price increases. For every unit

increase up to the higher striking price, the exercise value of this spread increases by 1.0. For instance, if the stock price remains unchanged at 48, the exercise value of the spread is 8.00. Thus, the profit is 8.00 – 1.50 + 1.91 = 8.41.

The above example tells us that the Bull spread "locks in a profit," but it does not completely hedge against a decline in the value of his new strategy.

Practice: Example given in section "5.1.3.2. Multiple strikes"

Practice: Example 5, Reading 41, Curriculum.

5.1.4.) The Risk of Spreads

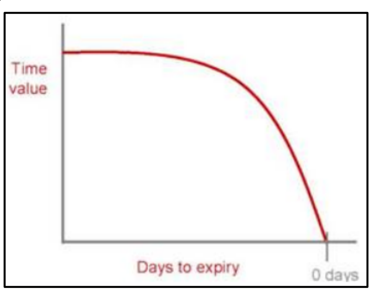
The shape of the profit and loss diagram for the bull spread is similar to that of the collar. Like collars, both the upside return potential and maximum loss is limited in bull spread.

5.2 Calendar Spread

Calendar spread involves ***selling (or writing) a near-dated*** call and ***buying a longer-dated*** call on the same underlying asset and with the same strike. Calendar spread can also be established using put options.

- When a more distant option is bought, it is a ***long calendar spread***.
- **Short calendar spread:** It involves buying a near-term option and selling a longer dated one.

Time value decays over time and approaches zero as the option expiration date approaches as reflected in chart below.



- Time decay is greater for a short-term option than that of a longer-term until expiration.
- A calendar spread trade seeks to exploit this characteristic by purchasing a longer-term option and writing a shorter-term option.

5.3 Straddle

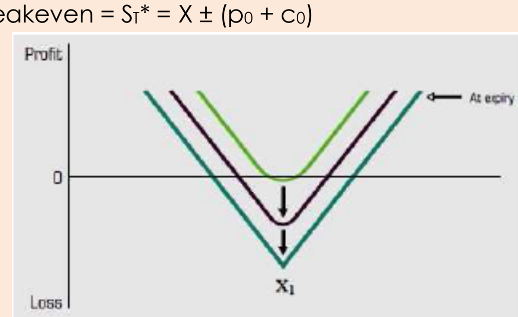
Long straddle: It involves buying a put and a call with same strike price on the same underlying with the same expiration; both options are at-the-money.

- Due to call option, the gain on upside is unlimited and due to put option, downside gain is quite large but limited.
- Straddle is a strategy that is based on the volatility of the underlying. It benefits from high volatility.
- A straddle is neither a bullish nor a bearish strategy; hence, the chosen options usually have an exercise price close to the current stock price.
- Straddle is a costly strategy because the straddle buyer pays the premium for two options. Hence, this implies that in order to make a profit, the underlying asset has to move either above or below the option exercise price by a significant amount (i.e. by the total amount spent on the straddle).
- In other words, in order to be profitable, the "true" underlying volatility of the underlying asset needs to be higher than the market consensus.

Rationale to use Straddle: Straddle is to be used only when the investor expects that volatility of the underlying will be relatively higher than what market expects but is not certain regarding the direction of the movement of the underlying price.

<u>Payoffs:</u>

a) Value at expiration: $V_T = \max(0, S_T - X) + \max(0, X - S_T)$
b) Profit = $V_T - p_0 - c_0$
c) Maximum Profit = ∞ or unlimited
d) Maximum Loss occurs when both call and put options expire at-the money and investor loses premiums on both options i.e. Maximum Loss = $p_0 + c_0$
e) Breakeven = $S_T^* = X \pm (p_0 + c_0)$



5.4 Consequences of Exercise

Options sellers (writers) have an obligation to perform if the option holder chooses to exercise the option. The option writer (seller) has no control over whether or not a contract is exercised, and he must recognize that exercise is possible at any time before expiration. The consequences of exercise can be significant. Hence, it is important to take into consideration those consequences before writing an option.

However, just as the buyer can sell an option back into the market rather than exercising it, a writer can purchase an offsetting contract to end their obligation to meet the terms of a contract provided they have not been assigned.

6. INVESTMENT OBJECTIVES AND STRATEGY SELECTION

6.1 The Necessity of Setting an Objective

For option holders, both the direction of the underlying and its volatility are important. A long call must have some upside volatility, and a long put must have some downside volatility.

Direction and Volatility with Options: The following table summarizes the direction and volatility of options.

		Direction		
		Bearish	**Neutral/No Bias**	**Bullish**
Volatility	High	Buy puts	Buy straddle	Buy calls
	Average	Write calls and buy puts	Spreads	Buy calls and write puts
	Low	Write calls	Write straddle	Write puts

6.2 Spectrum of Market Risk

Suppose a pension fund owns one million shares of ABC Holdings. The portfolio manager would like to temporarily reduce the position by 10%. There are a variety of ways to do this.

1) The manager can sell 100,000 shares, which is 10% of the holding. This strategy enables the manager to accomplish the goal of a 10% reduction. However, this action could create a tax problem for some investors or could result in inadvertently putting downward pressure on the stock price.
2) The manager can enter into a futures or forward contract to sell 100,000 shares. Forward contracts are simple and effective, but they involve counterparty risk and are difficult to cancel if later he needs to unwind this trade.
3) The portfolio manager can write call contracts sufficient to generate minus 100,000 delta points. By writing calls, he can earn cash premium, but this transaction exposes the manager to exercise risk.
4) The manager can buy put contracts sufficient to generate minus 100,000 delta points. Buying puts requires a cash outlay but protects the manager against exercise risk.
5) The manager can enter into a collar sufficient to generate minus 100,000 delta points.

6.3 Analytics of the Breakeven Price

The value of an option depends on variety of factors, including underlying market price, the exercise price of the option contract, the time left until option expiration, the current risk-free rate, and any dividends paid before expiration.

- The exercise price is known, and the underlying market price and risk-free rate are easily accessible.
- The time remaining until expiration is consistently changing but we know how much time is left until expiration.
- Dividends paid by companies are fairly stable.

However, the expected price movement or volatility of the underlying stock is not known with certainty. The higher the expected volatility, the higher the option premium would be.

Example: Suppose, a trader buying a ABC Mar 50 call and ABC Mar 50 put would be purchasing a ABC Mar 50 straddle for 1.95.

Buying a XYZ Mar 50 straddle would involve purchasing the XYZ Mar 50 call at 2.50 and XYZ Mar 50 put at 2.45 for a net cost of 4.95.

Breakeven at expiration for the ABC straddle occurs if the stock moves up or down 1.95 whereas breakeven for the XYZ straddle would require a move of 4.95.

In percentage terms, this means that breakeven for the ABC straddle is 3.90% (1.95/50.00) whereas breakeven for the XYZ straddle is 9.90% (4.95/50.00).

Suppose the underlying stock has an annual volatility of 30%. We know that in straddle, Breakeven occurs at X ± ($p_0 + c_0$). In order for the straddle to be profitable at expiration, the stock must move up or down by 2.29 (call option cost) plus 2.28 (put option cost) ➔ 4.57 units to allow the straddle buyer to breakeven (i.e. recover the cost of purchase of options) from the current price of 50, which is a 9.14% movement. With options, volatility is measured by the annual standard deviation[3]. Expiration is in 30 days, but this includes four weekends and possibly a holiday. Suppose there are only 21 trading days until expiration. We convert a 9.14% movement in 21 days to an annual volatility as follows:

[3] An annual variance (σ2) can be converted into a daily variance by dividing by 252, and an annual standard deviation (σ) can be converted into a daily standard deviation by dividing by √252.

$$\sigma_{annual} = 0.0914x\sqrt{\frac{252}{21}} = 32.6\%$$

Practice: Example 6, Reading 41, Curriculum.

6.4	6.4. Applications (Read examples from Curriculum)