E1 251 Project: Reconstruction from Non-Uniform Samples Using a DCT-l_p Prior

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1 Derivation of MM-CG Algorithm

$$J(x) = \|Wx - m\|_2^2 + \lambda \sum_{i=1}^{N} (\varepsilon + (\text{DCT } x)_i^2)^p$$

Focusing our attention the second term, we can see that $\lambda \sum_{i=1}^{N} (\varepsilon + (\text{DCT } x)_i^2)^p$ is concave for the given range $0.2 < p < 0.5$. So the cost function can be re-written as

$$J(\mathbf{x}) = f_0(\mathbf{x}) + f_{ccv}(\mathbf{x})$$

where $f_0(x)$ is the convex term $\|W\mathbf{x} - m\|_2^2$ and $f_{ccv}(\mathbf{x}^{(k)})$ is $\lambda \sum_{i=1}^{N} (\varepsilon + (\text{DCT}\mathbf{x})_i^2)^p$. For the concave term, the linear approximation at $\mathbf{x}^{(k)}$ maximizes the function around $\mathbf{x}^{(k)}$. We can thus write

$$f_{ccv}(\mathbf{x}) \leq f_{ccv}(\mathbf{x}^{(k)}) + \nabla f_{ccv}(\mathbf{x}^{(k)})^T (\mathbf{x} - \mathbf{x}^{(k)})$$

So, we can write

$$J(\mathbf{x}) \leq f_0(\mathbf{x}) + \nabla f_{ccv}(\mathbf{x}^{(k)})^T \mathbf{x} + \text{constant}$$

Let $z_i = (DCT\mathbf{x})_i^2$ and $g(\mathbf{z}) = \sum_{i=1}^{N} (\varepsilon + z_i)^p$, where $\mathbf{z} = [z_1 z_2 ... z_N]^T$. Then $f_{ccv}(\mathbf{x}) = \lambda \mathbf{z}$. Using the above equations we can write,

$$g(\mathbf{z}) \leq \nabla g(\mathbf{z}^{(k)})^T \mathbf{z} + \text{constant}$$

$$\frac{\partial g(\mathbf{z}^{(k)})}{\partial z_i^{(k)}} = p(\varepsilon + z_i^{(k)})^{p-1} = p(\varepsilon + (DCT\mathbf{x}^{(k)})_i^2)^{p-1} =: w_i^{(k)} \text{(By Definition)}$$

$$\implies \nabla g(\mathbf{z}^{(k)}) = \left[w_1^{(k)} w_2^{(k)} \cdots w_N^{(k)}\right]^T \implies g(\mathbf{z}) \leq \left[w_1^{(k)} w_2^{(k)} \cdots w_N^{(k)}\right]^T \odot \mathbf{z} + \text{const}$$

$$\implies g(\mathbf{z}) \leq \left[w_1^{(k)} w_2^{(k)} \cdots w_N^{(k)}\right]^T \odot (DCT\mathbf{x})^2 + \text{const} = \sum_{i=i}^{N} w_i^{(k)} (DCT\mathbf{x})_i^2 + \text{const}$$

So, the quadratic surrogate of $J(\mathbf{x})$ is:

$$J(\mathbf{x}) \leq \|W\mathbf{x} - m\|_2^2 + \lambda \sum_{i=1}^{N} w_i^{(k)} (DCT\mathbf{x})_i^2 + \text{Constant}$$

$DCT\mathbf{x}$ can be replaced by $C\mathbf{x}$ as it is a linear transform of $\mathbf{x}$. And $IDCT\mathbf{x}$ can be replaced by $C^T\mathbf{x}$. Let $W_k = diag(w_1^{(k)} w_2^{(k)} \cdots w_N^{(k)})$. Doing these replacements, the equation can be rewritten as:

$$J(\mathbf{x}) \leq \|W\mathbf{x} - m\|_2^2 + \lambda (C\mathbf{x})^T W_k (C\mathbf{x}) + \text{Constant} = Q(\mathbf{x})(\text{let})$$

Making the gradient equals to 0 to get the minimum value.

$$\nabla Q(\mathbf{x}) = (2W^T W\mathbf{x} - 2W^T\mathbf{m}) + (2\lambda C^T W_k C\mathbf{x}) = \mathbf{0}$$

The equation can be rewritten as

$$(W^T W + \lambda C^T W_k C)(\mathbf{x}) = W^T\mathbf{m}$$

which is basically the equation as:

$$(W^T W + \lambda \text{IDCT}(\text{diag}(w^{(k)})\ \text{DCT}))\mathbf{x} = W^T\mathbf{m}$$

$\blacksquare$

2 Description of Experimental setup