E9 241 Time Frequency Analysis
Assignment 1

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1. Fundamentals of Fourier

Ans:

(i) $f \in L^2(\mathbb{R}) \iff \int_{-\infty}^{\infty} f^2 dx < \infty$.

So, $\int_{-\infty}^{\infty} f_1^2(t)\, dt = \int_{-\infty}^{\infty} \left(e^{-\alpha|t|}\right)^2 dt$

$= \int_{-\infty}^{\infty} e^{-2\alpha|t|} dt.$ $= 2\int_{0}^{\infty} e^{-2\alpha t} dt$ [Even function]

$= 2 \cdot \left[\frac{e^{-2\alpha t}}{-2\alpha}\right]_0^\infty = 2 \cdot \frac{1}{2\alpha} = \frac{1}{\alpha} < \infty$

$\forall \alpha > 0.$

So, $f_1 \in L^2(\mathbb{R})$

(ii) $\hat{f}_1(\omega) = \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-j\omega t} dt.$

$= \int_{-\infty}^{0} e^{(\alpha - j\omega)t} dt + \int_{0}^{\infty} e^{-(\alpha + j\omega)t} dt$

$$= \left[\frac{e^{(\alpha - j\omega)t}}{\alpha - j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha + j\omega)t}}{-(\alpha + j\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega} = \frac{\alpha + j\omega + \alpha - j\omega}{\alpha^2 + \omega^2}$$

$$= \frac{2\alpha}{\alpha^2 + \omega^2}$$

2.iii) $\hat{f}_2(\omega)$ is basically the same function as $f_1(t)$ just in frequency domain.

$$\int_{-\infty}^{\infty} \left(e^{-\alpha |t|}\right)^2 dt = \int_{-\infty}^{\infty} \left(e^{-\alpha |\omega|}\right)^2 d\omega.$$

$$= \hat{f}_2(\omega) < \infty.$$

So, $\hat{f}_2(\omega) \in L_2(\mathbb{R})$

2. iv)

a) Duality Property :- $f(t) \leftrightarrow \hat{f}(\omega)$

$$\hat{f}(t) \leftrightarrow 2\pi f(-\omega)$$

$$e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2+\omega^2} \quad \text{[Shown in 1(ii)]}$$

$$\frac{2\alpha}{\alpha^2+t^2} \longleftrightarrow 2\pi \cdot e^{-\alpha|-\omega|} = 2\pi e^{-\alpha|\omega|}$$

So, $$\frac{1}{2\pi} \cdot \frac{2\alpha}{\alpha^2+t^2} \longleftrightarrow e^{-\alpha|\omega|}$$

[By linearity]

so, $$\frac{\alpha}{\pi(\alpha^2+t^2)} \longleftrightarrow e^{-\alpha|\omega|}$$

b) Properties of Fourier transform.

We can apply Inverse Fourier Transform

$$f_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}_2(\omega)\, e^{j\omega t} d\omega.$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(\alpha-jt)\omega} d\omega + \int_{0}^{\infty} e^{-(\alpha-jt)\omega} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left.\frac{e^{(\alpha-jt)\omega}}{(\alpha-jt)}\right|_{-\infty}^{0} + \left.\frac{e^{-(\alpha-jt)\omega}}{-(\alpha-jt)}\right|_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{\alpha-jt} + \frac{1}{\alpha+jt}\right]$$

$$= \frac{\alpha}{\pi(\alpha^2+t^2)}.$$

1.v)

a) For it to be a density, $\int f_2(x)\,dx < \infty$

If $\int f_2(x)\,dx = 1$, then it can be a pdf.

Now,

$$\int_{-\infty}^{\infty} f_2(x)\,dx = \int_{-\infty}^{\infty} \frac{\alpha}{\pi(\alpha^2+x^2)}\,dx.$$

$$= \frac{\alpha}{\pi} \cdot \frac{1}{\alpha} \tan^{-1} \frac{x}{\alpha} \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi} \tan^{-1} \frac{x}{\alpha} \Big|_{-\infty}^{\infty}.$$

$$= \frac{1}{\pi} \times \pi = 1.$$

Also $\forall x$. $\frac{\alpha}{\alpha^2+x^2} \cdot \frac{1}{\pi} > 0$. $[\alpha > 0]$.

So, we can say it is a density, more specifically, probability density function.

2. Simulation of Plots

Ans:

i> sinc. $(2\pi t)$

Consider a box function which has a value 1 between $[-t_0, t_0]$ $(f(x))$

$$\int_{-t_0}^{t_0} e^{-j\omega t}\,dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-t_0}^{t_0}$$

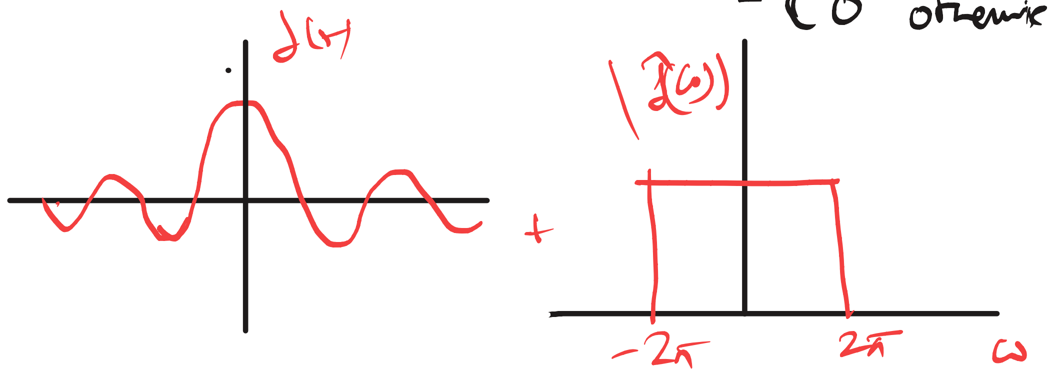
$$= \frac{e^{-j\omega t_0} - e^{j\omega t_0}}{-j\omega}$$

$$= \frac{\sin \omega t_0}{\omega'} = 2t_0 \operatorname{sinc} \omega t_0$$

By Duality, $f(x) \leftrightarrow 2t_0 \operatorname{sinc} \omega t_0$

$$\operatorname{Sin} 2\pi t \leftrightarrow \frac{1}{2} f(\omega)$$

$$= \frac{1}{2} \begin{cases} 1 & |\omega| \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$



ii) $\left(\delta(t-1) + \delta(t+1)\right) * e^{-at^2}$

$$= \int_{-\infty}^{\infty} \left(\delta(\tau-1) + \delta(\tau+1)\right) e^{-a(t-\tau)^2} d\tau$$

$$= e^{-a(t+1)^2} + e^{-a(t-1)^2}.$$

Summation of 2 Gaussian.

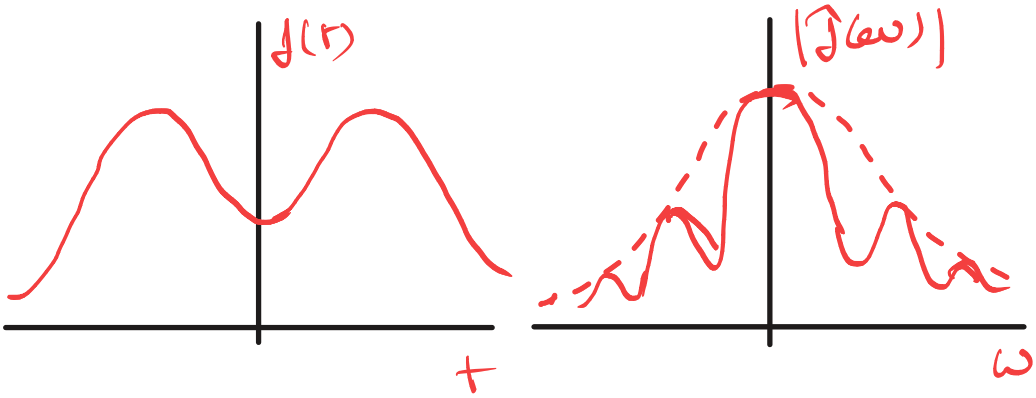
Fourier transform will also be two.

modulated Gaussian

$$= e^{-b\omega^2} \cdot e^{-j\omega} + e^{-b\omega^2} e^{j\omega}.$$

$$= e^{-b\omega^2} \cdot \cos.\omega.$$

$= Re($ Gabor function$)$



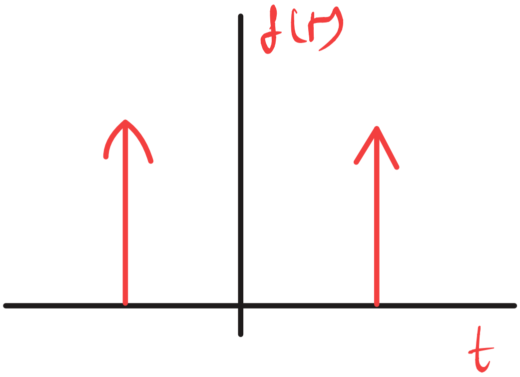
iii) In time domain it has value

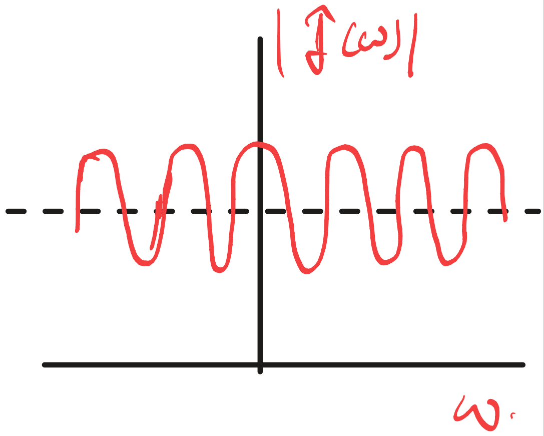
$$\delta(t-1).e^{-a} + \delta(t+1)e^{-a}$$

∵ It is two impulse. It will give modulated constant.

$$e^{-a}.e^{-j\omega.} + e^{-a}.e^{j\omega.}$$

$$= 2e^{-a}.\cos.\omega.$$





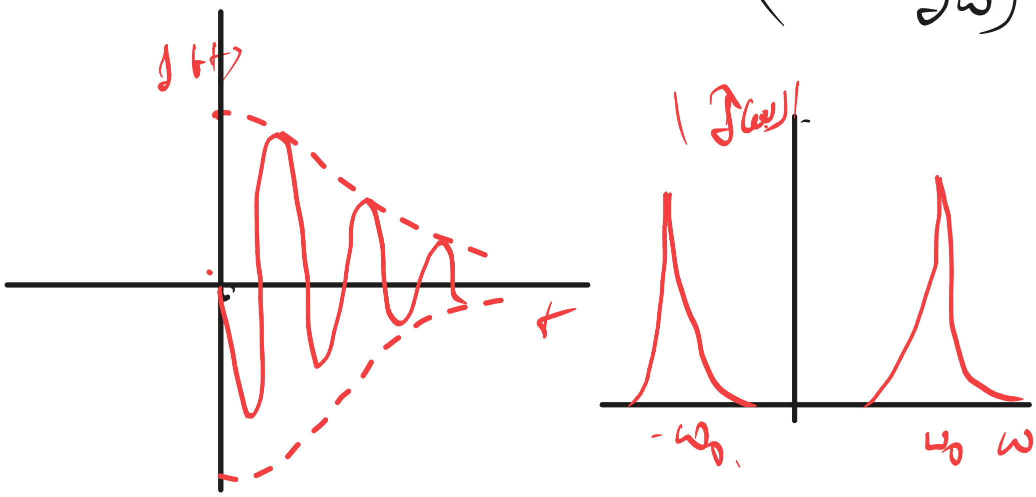
iv) $e^{-at^2} \longleftrightarrow e^{-b\omega^2}$.

$$e^{-at^2}\cdot\cos(\omega_0 t) \longleftrightarrow \frac{1}{2} e^{-b\omega^2} * \left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

So in fourier transform there is two impulse at $\omega = \omega_0$.

Also there is $u(t)$ so,

$$e^{-at^2}\cdot\cos(\omega_0 t)\cdot u(t) \Leftrightarrow \frac{1}{2} e^{-b\omega^2} * \left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right) * \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$



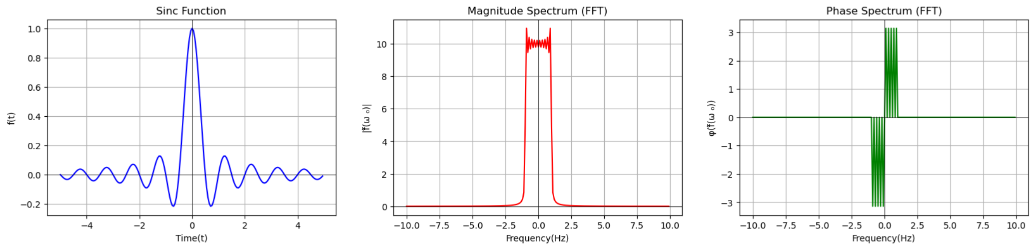


Figure 1: Time Domain and Frequency Domain representation of $\mathrm{sinc}(2\pi t)$

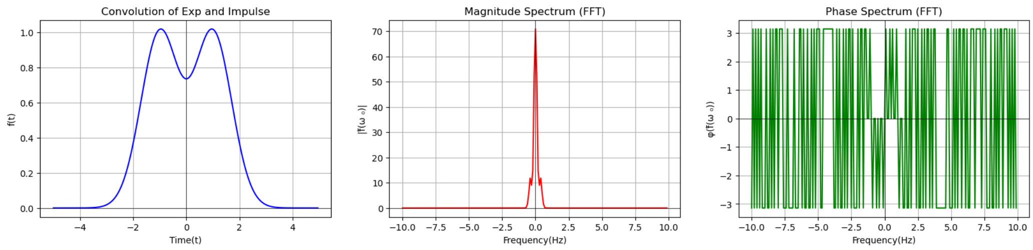


Figure 2: Time Domain and Frequency Domain representation of $(\delta(t-1)+\delta(t+1)) * e^{-at^2}$

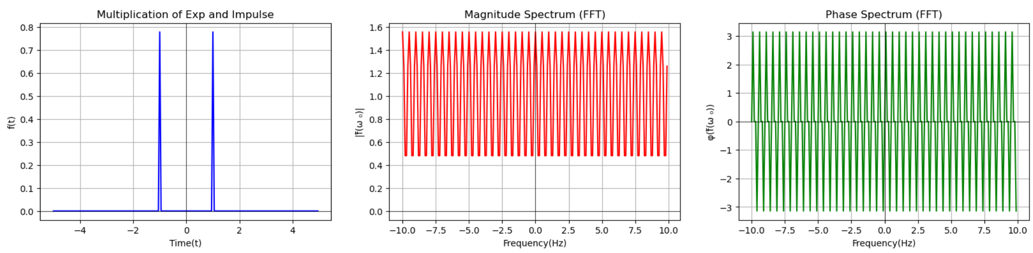


Figure 3: Time Domain and Frequency Domain representation of $(\delta(t-1)+\delta(t+1)) \times e^{-at^2}$

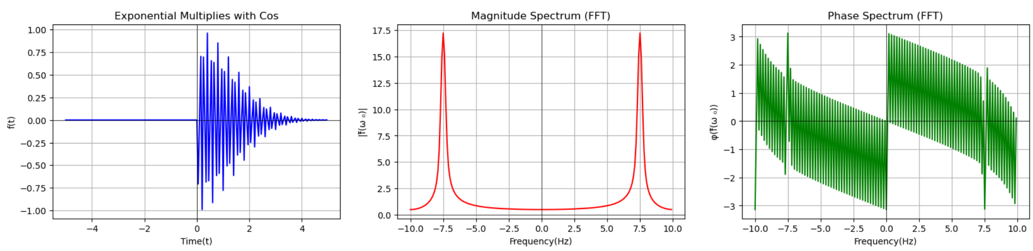


Figure 4: Time Domain and Frequency Domain representation of $e^{-at^2}\cos(\omega_0 t)u(t),\ \text{for } a>0, a\in\mathbb{R}$

3. Discrete Fourier Transform(DFT)

Ans:

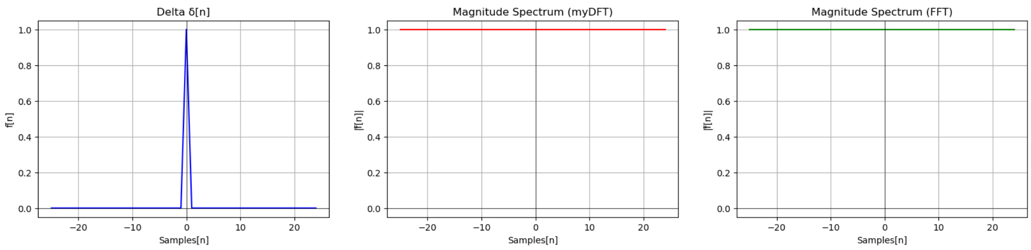


Figure 5: Fourier Transform of $\delta[n]$ with $myDFT$ function and comparison with fft library function

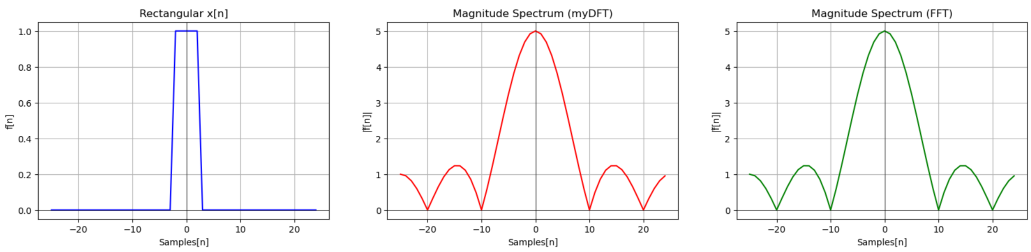


Figure 6: Fourier Transform of $box[n]$ with $myDFT$ function and comparison with fft library function

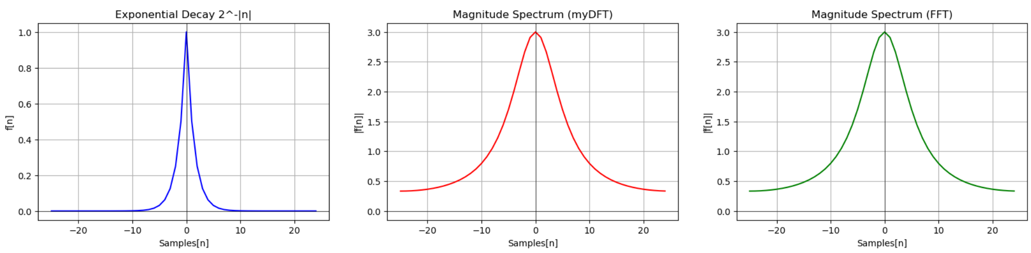


Figure 7: Fourier Transform of $2^{-|n|}$ with $myDFT$ function and comparison with fft library function

4. Learning Learning

Ans:

- Cauchy Density forms a Fourier pair with $e^{-|t|}$.
- In practical cases, we would never get a perfect box function from the *sinc*. It will have the Gibbs Phenomenon.
- The DFT of the box function will not give the *sinc* function. It is give something like $|sinc|$ function.