Assignment - 2

3.1). $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, $x \in \mathbb{R}$,

$$\mathcal{X}(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \cdot e^{jtx} \, d.x = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2} + 2jtx\right)} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left\{\left(\frac{x^2}{\sigma} + jt\sigma\right)^2 + t^2\sigma^2\right\}} dx$$

$$= \frac{e^{-\frac{1}{2}t^2\sigma^2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma} + jt\sigma\right)^2} dx$$

Now, $\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma} + jt\sigma\right)^2} dx$ is just another shifted Gaussian,

so, $= 1$.

Then,

$$\mathcal{X}(t) = e^{-\frac{1}{2}t^2\sigma^2}$$

3.ii) $\int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-j\lambda t} \, dt. = \mathcal{F}\left(e^{-\alpha|t|}\right).$

$$= \int_{-\infty}^{0} e^{\alpha t} \cdot e^{-j\lambda t} \, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j\lambda t} \, dt$$

$$= \left. \frac{e^{(\alpha - j\lambda)t}}{\alpha - j\lambda} \right|_{-\infty}^{0} + \left. \frac{e^{-(\alpha + j\lambda)t}}{-(\alpha + j\lambda)} \right|_{0}^{\infty}$$

$$= \frac{1}{\alpha - j\lambda} + \frac{1}{\alpha + j\lambda} = \frac{2\alpha}{\alpha^2 + \lambda^2}$$

Now, doing IFFT.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + \lambda^2} \cdot e^{j\lambda t} \, d\lambda = e^{-\alpha|t|}$$

Putting $\alpha = 1$, we get

$$\int \frac{1}{\pi(1+x^2)} \cdot e^{jxt} \, dx = e^{-|t|}$$

So, characteristic function of $\frac{1}{\pi(1+x^2)}$ is $e^{-|t|}$

3 (iii)
$$\int_{0}^{\infty} k\lambda e^{-\lambda x} \cdot e^{jxt} \, dx$$

$$= k\lambda \int_{0}^{\infty} e^{-(\lambda - jt)x} \, dx$$

$$= k\lambda \left[\frac{e^{-(\lambda - jt)x}}{-(\lambda - jt)} \right]_0^{\infty}$$

$$= k\lambda \cdot \frac{1}{\lambda - jt} = \frac{k\lambda}{\lambda - jt}.$$

5)
$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \leftrightarrow \hat{f}(\omega)$$

$$\frac{df(x)}{dx} \leftrightarrow j\omega \hat{f}(\omega)$$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{-2x}{2\sigma^2} \leftrightarrow j\omega \hat{f}(\omega)$$

$$-\frac{x f(x)}{\sigma^2} \leftrightarrow j\omega \hat{f}(\omega)$$

$$\Updownarrow$$

$$-\frac{1}{\sigma^2} j \cdot \frac{d}{d\omega} \hat{f}(\omega) = j\omega \hat{f}(\omega)$$

$$\frac{d\hat{f}(\omega)}{\hat{f}(\omega)} = -\sigma^2 \omega \, d\omega$$

$$\ln \hat{f}(\omega) = -\frac{\sigma^2 \omega^2}{2} + C$$

$$\hat{f}(\omega) = k e^{-\frac{\sigma^2 \omega^2}{2}}$$

At $\omega = 0$, $\hat{f}(\omega) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = 1$. So, applying boundary conditions $k = 1$