E9 241 Time Frequency Analysis
Assignment 2

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1. **Cosine Signal Analysis**

Ans: Here the cosine signal is taken with a frequency of 50 Hz. Sampling time period T_s is taken to be 0.001 s. 5 cycles of cosine is sketched. Discrete Fourier Transform is done via the fft function in Scipy Package.

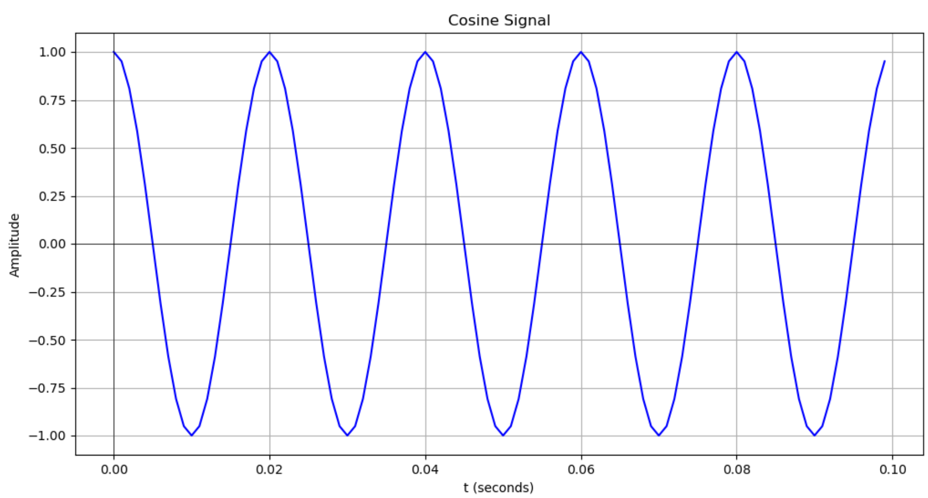


Figure 1: 5 cycles of 50Hz Cosine Signal

Here, the phase spectrum is coming out to be random due to numerical approximation. By the real and complex value are arbitrarily small but the angle by them is not. The phase spectrum should have been 0 for all of the frequencies but the error comes here due to numerical approximation.

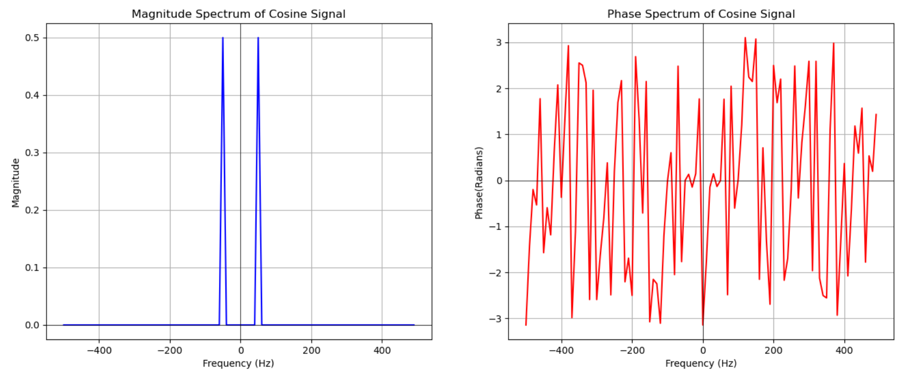


Figure 2: Magnitude and Phase spectrum of the cosine signal generated

2. **Plotting Densities**

Ans:

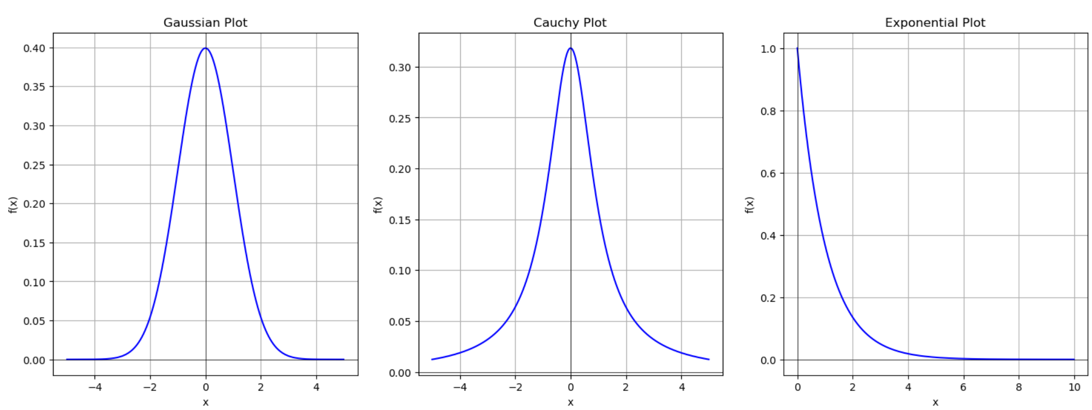


Figure 3: Different Densities plotted with respect to x

3. **Characteristics Functions**

Ans: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \quad x \in \mathbb{R}$

$$\chi(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} e^{jtx}\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2} + 2jtx\right)} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{1}{2}\left\{\left(\frac{x}{\sigma} + jt\sigma\right)^2 + t^2\sigma^2\right\}} dx$$

$$= e^{-\frac{1}{2}t^2\sigma^2} \cdot \underbrace{\frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{1}{2}\left(\frac{x}{\sigma} + jt\sigma\right)^2} dx}_{\text{Shifted Gaussian so } = 1}$$

So, $\chi(t) = e^{-\frac{1}{2}t^2\sigma^2}$.

3.ii) $$\int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-jxt} \, dt = \mathcal{F}\left(e^{-\alpha|t|}\right)$$

$$= \int_{-\infty}^{0} e^{\alpha t} e^{-jxt} \, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-jxt} \, dt$$

$$= \left. \frac{e^{(\alpha - jx)t}}{\alpha - jx} \right|_{-\infty}^{0} + \left. \frac{e^{-(\alpha + jx)t}}{-(\alpha + jx)} \right|_{0}^{\infty}$$

$$= \frac{1}{\alpha - jx} + \frac{1}{\alpha + jx} = \frac{2\alpha}{\alpha^2 + x^2}$$

Now, by IFFT

$$\frac{1}{2\pi} \int \frac{2\alpha}{\alpha^2 + x^2} e^{jxt} \, dx = e^{-\alpha|t|}$$

Putting $\alpha = 1$, we get

$$\int \frac{1}{\pi(1 + x^2)} e^{ixt} \, dx = e^{-|t|}$$

So, $x(t)$ for $\frac{1}{\pi(1+x^2)} = e^{-|t|}$

3. iii) $f(x) = \begin{cases} k\lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

$$x(t) = \int_0^\infty k\lambda e^{-\lambda x} \cdot e^{jxt} \, dx$$

$$= k\lambda \int_0^\infty e^{-(\lambda - jt)x} \, dx$$

$$= k\lambda \left[\frac{e^{-(\lambda - jt)x}}{-(\lambda - jt)} \right]_0^\infty$$

$$= k\lambda \cdot \frac{1}{\lambda - jt} = \frac{k\lambda}{\lambda - jt}.$$

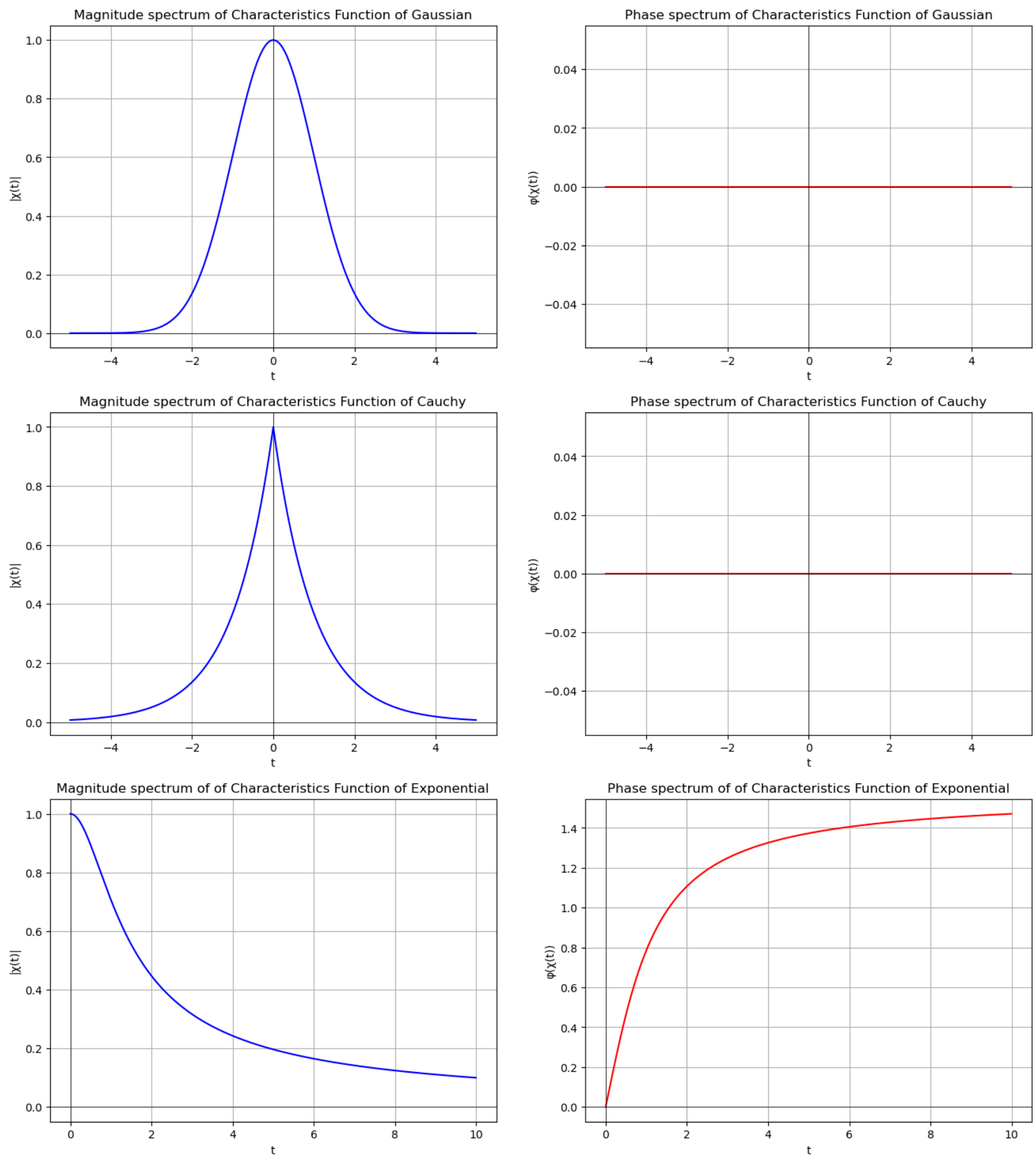


Figure 4: Characteristics Function of the densities with respect to variable t

4. **Displaying Complex Signals**

Ans:

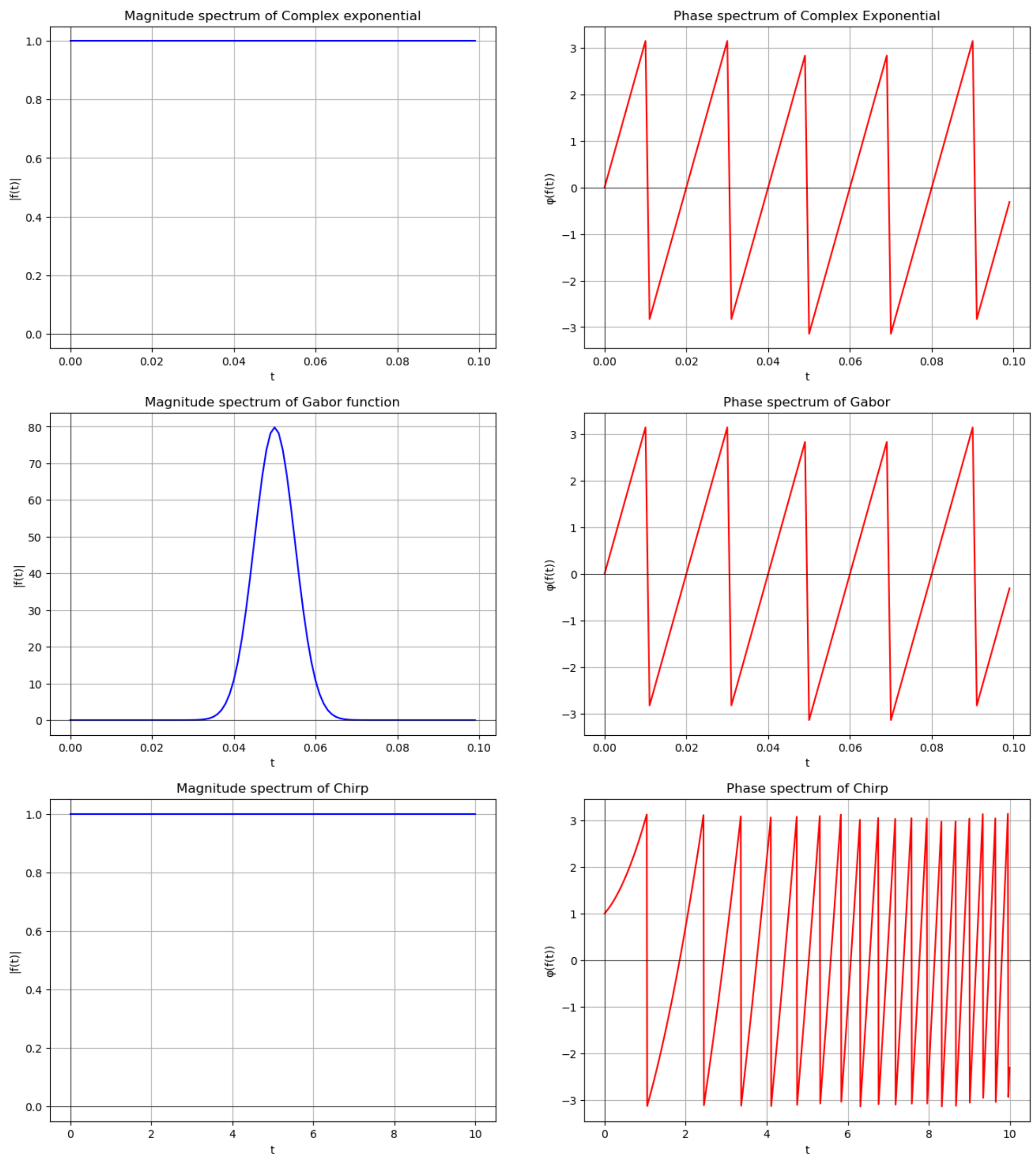


Figure 5: Complex Signals Magnitude and Phase Spectrum

5. **Fourier Transform of a Gaussian**

Ans:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \longleftrightarrow \hat{f}(\omega)$$

$$\frac{d f(x)}{dx} \longleftrightarrow j\omega \hat{f}(\omega)$$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \times \left(-\frac{\cancel{2}x}{\cancel{2}\sigma^2}\right) \longleftrightarrow j\omega \hat{f}(\omega)$$

$$-\frac{x f(x)}{\sigma^2} \longleftrightarrow j\omega \hat{f}(\omega)$$

Again

$$\frac{-x f(x)}{\sigma^2} \longleftrightarrow -\frac{1}{\sigma^2} j \frac{d}{d\omega} \hat{f}(\omega)$$

So,

$$-\frac{1}{\sigma^2} \cancel{j} \frac{d}{d\omega} \hat{f}(\omega) = \cancel{j}\omega \hat{f}(\omega)$$

$$\Rightarrow \quad \frac{d\hat{f}(\omega)}{\hat{f}(\omega)} = -\sigma^2 \omega \, d\omega$$

$$\Rightarrow \quad \ln \hat{f}(\omega) = -\frac{\sigma^2\omega^2}{2} + c \quad \left[\text{Assuming } \hat{f}(\omega) > 0\right]$$

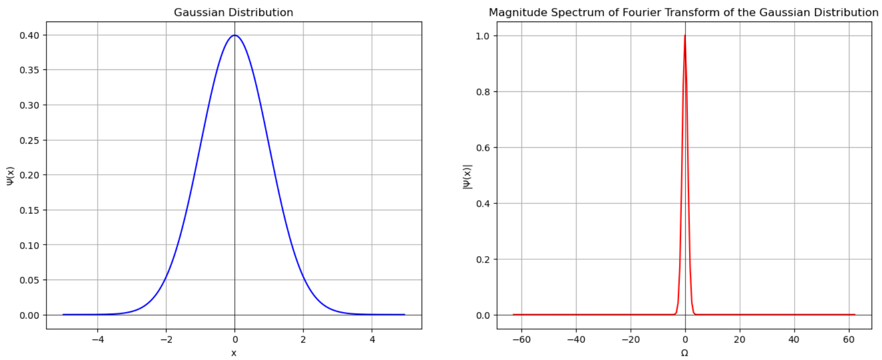
$$\Rightarrow \quad \hat{f}(\omega) = k e^{-\frac{\sigma^2\omega^2}{2}}$$

At $\omega = 0$, $\hat{f}(\omega) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \cdot e^{-j \cdot 0 \cdot x} dx = 1$. So applying boundary condition $k = 1$. So,

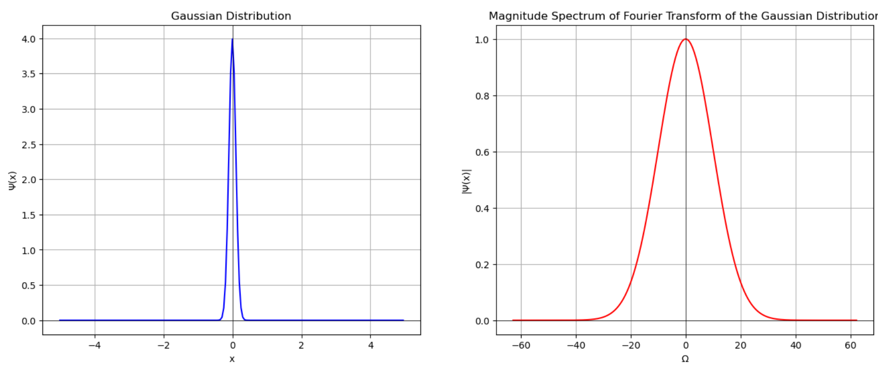
$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \longleftrightarrow e^{-\frac{\sigma^2\omega^2}{2}}$$

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \longleftrightarrow e^{-\frac{\sigma^2\omega^2}{2}} \cdot e^{j\mu\omega}$$

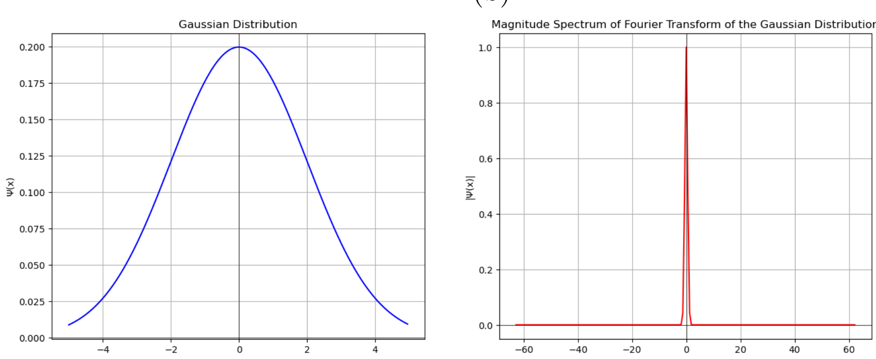
Here, I have used interactive sliders to dynamically change the parameters and observe the changes in the plots. The sketches given here are saved for particular values of μ and σ.



(a)

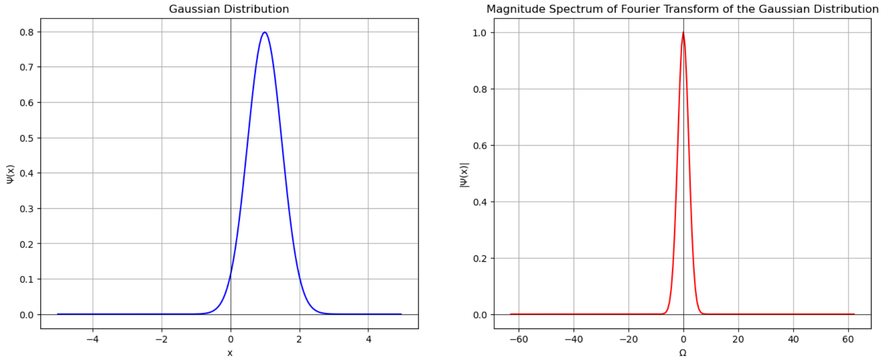


(b)

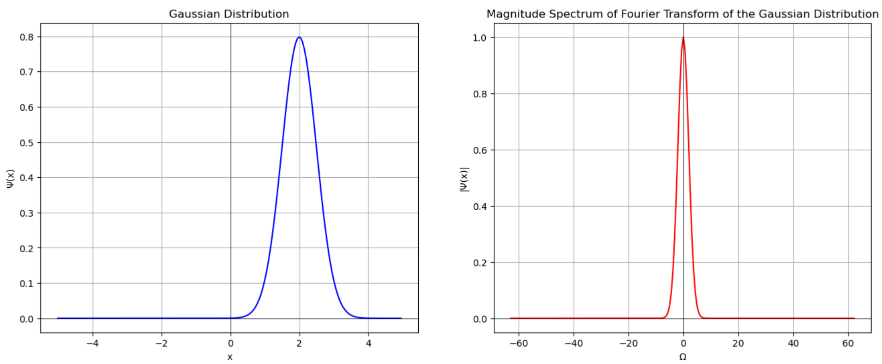


(c)

Figure 6: (6a) $\mu = 0, \sigma = 1$; (6b) $\mu = 0, \sigma = 0.1$; (6c) $\mu = 0, \sigma = 2$



(a)



(b)

Figure 7: (7a) $\mu = 1, \sigma = 0.5$; (7b) $\mu = 2, \sigma = 0.5$

6. **Modulated Gaussian Signal**

Ans:

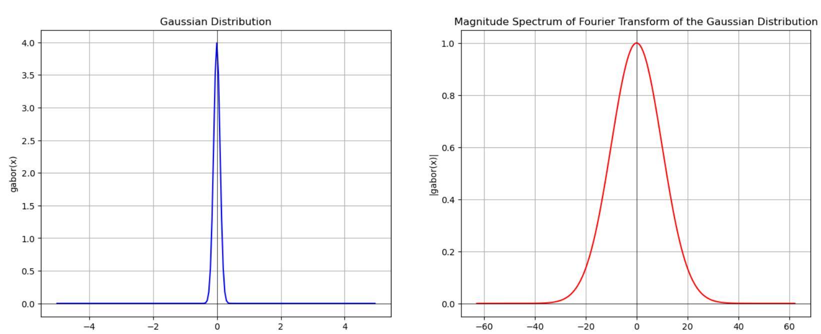
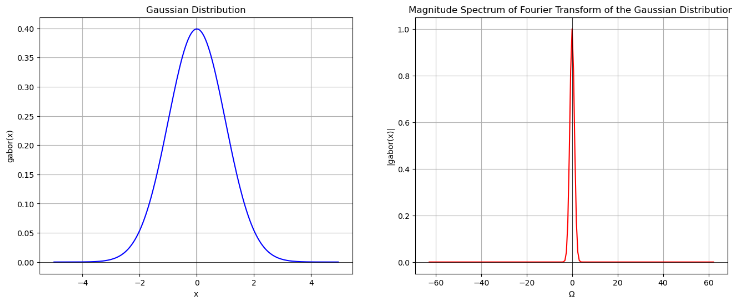
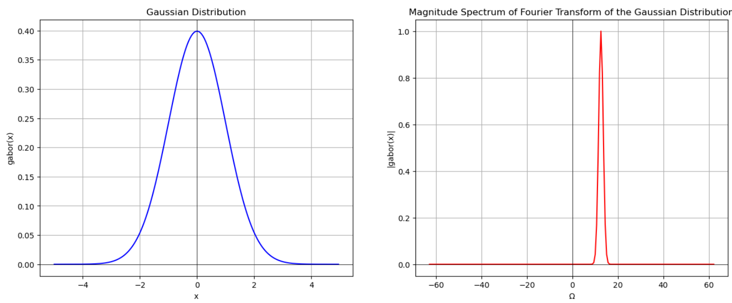


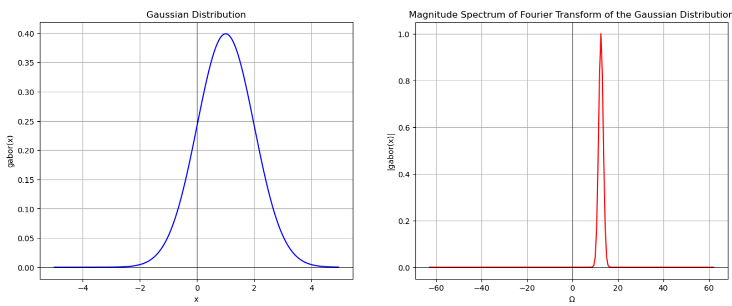
Figure 8: $\mu = 0, \sigma = 0.1, \omega_0 = 0$



(a)



(b)



(c)

Figure 9: (9a) $\mu = 0, \sigma = 1, \omega_0 = 0$; (9b) $\mu = 0, \sigma = 1, \omega_0 = 3\pi$; (9c) $\mu = 1, \sigma = 1, \omega_0 = 3\pi$