HW 9

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1)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = M^T M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$u = 1$

$2v = 1 \Rightarrow v = \frac{1}{2}$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

2

let $\vec{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$$b_i = (A\vec{x} - \vec{c})_i$$

$$E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$$

$$= (A\vec{x}-\vec{c})_1^2 + 4(A\vec{x}-\vec{c})_2^2 + 25(A\vec{x}-\vec{c})_3^2 + 9(A\vec{x}-\vec{c})_4^2$$

like weighted norm squared

$$\Rightarrow E^2 = (A\vec{x}-\vec{c})^T D (A\vec{x}-\vec{c})$$

where $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$

$$= \vec{x}^T A^T D A \vec{x} - 2\vec{c}^T D A x + \vec{c}^T D \vec{c}$$

$$\nabla E^2 = 2A^T D A \vec{x} - 2A^T D \vec{c} + 0$$

$$= 0$$

$$\Rightarrow A^T D A \vec{x} - A^T D \vec{c} = A^T D (A\vec{x} - \vec{c}) = \vec{0}$$

(normal eqns)

$$\Rightarrow A^T D A \vec{x} = A^T D \vec{c}$$

$$\Rightarrow \vec{x} = (A^T D A)^{-1} A^T D \vec{c}$$

$$= \left(\begin{bmatrix} 1 & 6 & 4 & 2 \\ 3 & -1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 4 & & \\ & & 25 & \\ 0 & & & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 6 & 4 & 2 \\ 3 & -1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 4 & & \\ & & 25 & \\ 0 & & & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 6 & 4 & 2 \\ 3 & -1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 24 & -4 \\ 100 & 0 \\ 18 & 63 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 6 & 4 & 2 \\ 3 & -1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 75 \\ 36 \end{bmatrix}$$

$$= \begin{bmatrix} 581 & 105 \\ 105 & 454 \end{bmatrix}^{-1} \begin{bmatrix} 421 \\ 247 \end{bmatrix}$$

I checked, is invertible

3

(a):

$$X^j = c_1 X^{j-1} + c_2 x^{j-2} + \dots + c_{j-1} x + c_j$$

$$\frac{d}{dx}[x^j] = j x^{j-1} = 0 \text{ if } x = 0$$

but $c_{j-1} = 0$

$$\frac{d^2}{dx^2}[x^j] = j(j-1)x^{j-2} = 0 \text{ if } x = 0$$

but $c_{j-2} = 0$

$\vdots$

$$\frac{d^j}{dx^j}[x^j] = 0 \text{ if } x = 0$$

but $c_1 = 0$

$\Rightarrow$ all c's $= 0$

$\Rightarrow \{1, x, \dots, x^n\}$ lin. indep.

(b):

$$c_0 + \sum_{k=1}^{n} c_k \cos(kx) + \sum_{k=1}^{n} d_k \sin(kx) = 0$$

$$\frac{d}{dx} \rightarrow \sum_{k=1}^{n} -k c_k \sin(kx) + \sum_{k=1}^{n} d_k k \cos(kx) = 0$$

$$\Rightarrow \sum_{k=1}^{n} k d_k \cos(kx) = \sum_{k=1}^{n} k c_k \sin(kx)$$

$$\Rightarrow \sum_{k=1}^{n} d_k \cos(kx) = \sum_{k=1}^{n} c_k \sin(kx)$$

$$\frac{d}{dx} \rightarrow \sum_{k=1}^{n} d_k \sin(kx) = \sum_{k=1}^{n} c_k \cos(kx)$$

$$\Rightarrow c_0 + 2 \sum_{k=1}^{n} c_k \cos(kx) = 0$$

$$c_0 + 2 \sum_{k=1}^{n} d_k \sin(kx) = 0$$

$\Rightarrow$ all c's $= 0$

$\Rightarrow$ lin. indep.

5)

$$x = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\Rightarrow z + \frac{1}{z} = \frac{z^2+1}{z} = 2x$$

$$\Rightarrow z^2 - 2xz + 1 = 0$$

$$\Rightarrow z = \frac{2x}{2} \pm \frac{1}{2}\sqrt{4x^2-4}$$

$$= x \pm \sqrt{x^2-1}$$

$$T_0 = \frac{1}{2}\left(z^0 + \frac{1}{z^0}\right) = 1$$

$$T_1 = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$= \frac{1}{2}\left(\frac{z^2+1}{z}\right)$$

$$= \frac{1}{2}(2x) \quad \text{(from above)}$$

$$= x$$

$$2xT_1 - T_0 = 2x(x) - 1 = 2x^2 - 1$$

$$T_2 = \frac{1}{2}\left(\frac{z^4+1}{z^2}\right)$$

$$x^2 = \frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right)$$

$$= \frac{1}{4}\left(\frac{z^4+1}{z^2} + 2\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\,\frac{z^4+1}{z^2} + 1\right)$$

$$= \frac{1}{2}\left(T_2 + T_0\right)$$

$$\Rightarrow 2x^2 - 1 = T_2 + T_0 - T_0 = T_2$$

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