HW 1

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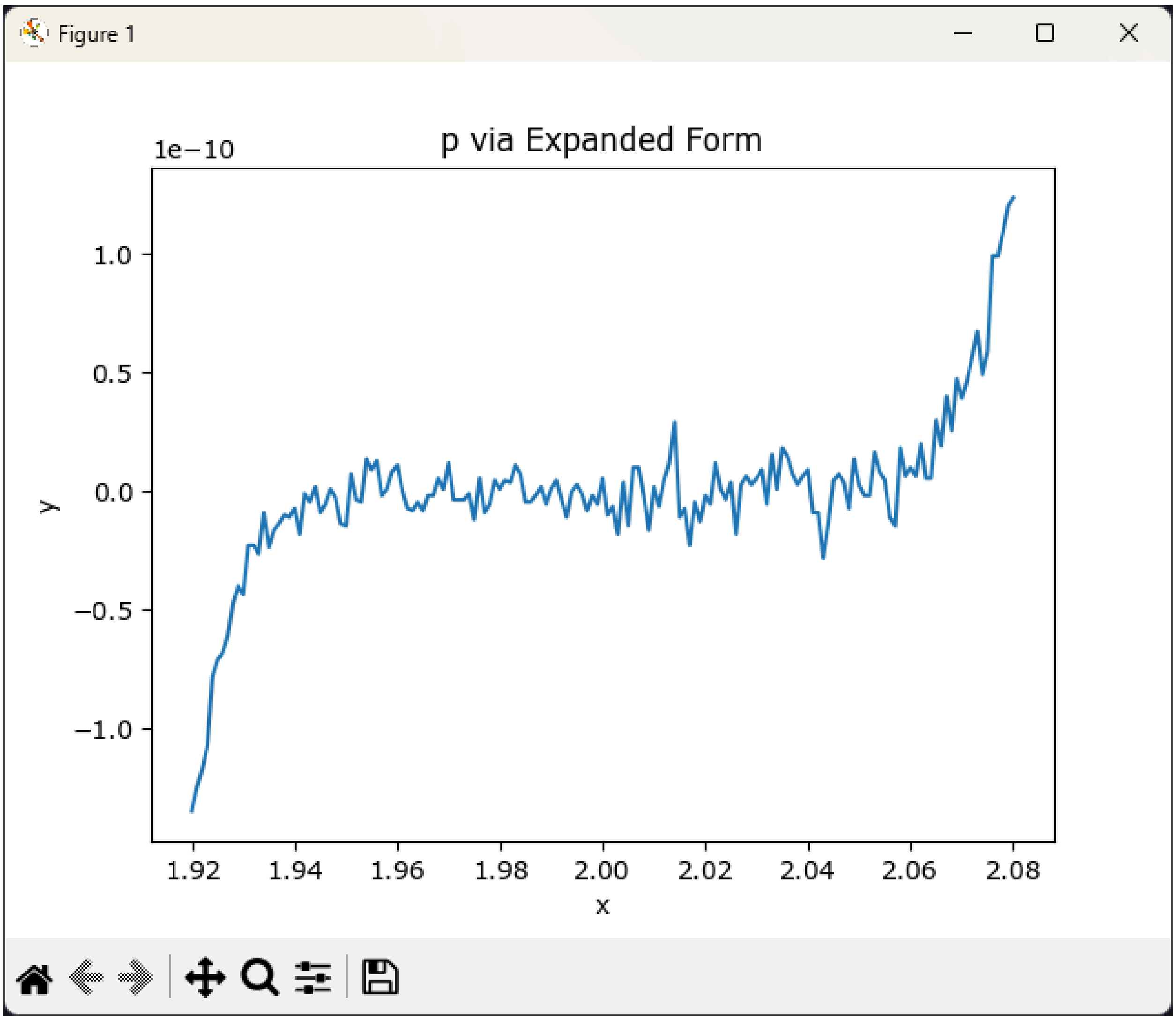
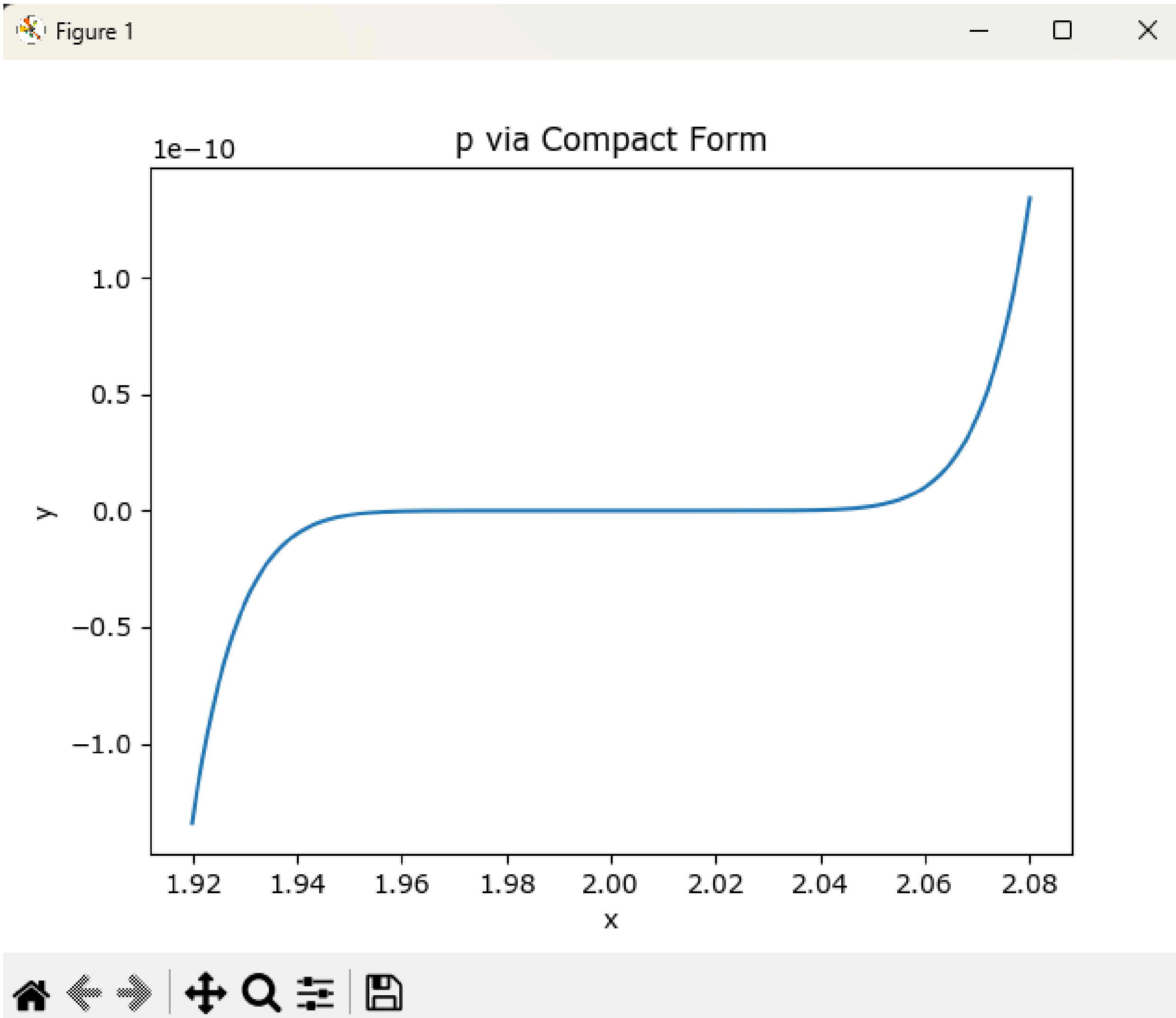
1.

i. Graph of the function p via its coefficient form



ii. Graph of the function p via its compact form (x-2)^9



iii.

Graph 2 would be the one I would expect to see. It’s more accurate because there is only 1 subtraction term for which there is loss of precision whereas the expanded form has more subtraction terms which further cause loss of precision. Therefore, the more correct graph would be Graph 2. The difference between the two is that Graph 1 is “noisier” due to inaccurate rounding and representation of the y-values. Also, according to condition number, exponentiating values over and over again will incur more losses on the order of n.

2.

i.

Make the expression $\sqrt{x + 1} - 1$ into a fraction by putting it over 1. Then, multiply by the conjugate of the numerator like this $\frac{\sqrt{x+1}-1}{1}(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1})$. Then by simplifying algebraically, you get $\frac{x}{\sqrt{x+1}+1}$ which contains no subtraction yet it is mathematically identical to the original expression.

ii.

Using trig identities, $sin(x) - sin(y) = 2cos(\frac{x+y}{2})sin(\frac{x-y}{2})$. x and y are close to each other so the calculation of the arg sin would cause loss of precision. However, the cancellation wouldn't happen unless x and y are so close together that there is less than machine epsilon precision for the difference; it would then round to 0 canceling out sin(x) - sin(y). If that isn't the case, the identity above should work for avoiding cancellation.

iii.

This expression is identically equal to $tan(x/2)$. So, I would simply use the tan function instead to avoid cancellation in the numerator.

3.

(a):

$$f(x) = (1 + x + x^3)\cos x$$

$$f'(x) = (1 + x + x^3)(-\sin x) + (1 + 3x^2)\cos x$$

$$= (1 + 3x^2)\cos x - (1 + x + x^3)\sin x$$

$$f''(x) = (1 + 3x^2)(-\sin x) + (6x)\cos x - \left[(1 + x + x^3)\cos x + (1 + 3x^2)\sin x\right]$$

$$= -2(1 + 3x^2)\sin x + (6x - 1 - x - x^3)\cos x$$

$$= -2(1 + 3x^2)\sin x - (1 - 5x + x^3)\cos x$$

$$P_2(x) = 1 + (1 - 0)x + \frac{1}{2}(-1)x^2$$

$$= 1 + x - \frac{x^2}{2}$$

$$P_2(0.5) = 1 + 0.5 + \frac{0.25}{2}$$

$$= 1.625$$

$$f(0.5) = (1 + 0.5 + (0.5)^3)\cos(1/2)$$

$$\cong 1.426$$

$$|f(0.5) - P_2(0.5)| = 0.1989$$

$$f'''(x) = -2(1+3x^2)\cos x - 2(6x)\sin x - (1-5x+x^3)(-\sin x) - (-5+3x^2)\cos x$$

$$= \cos(x)(-2-6x^2+5-3x^2) + \sin(x)(-12x+1-5x+x^3)$$

$$= (-9x^2+3)\cos(x) + (1-17x+x^3)\sin(x)$$

$$R_2(x) = \frac{1}{6}\left[(-9c^2+3)\cos(c) + (1-17c+c^3)\sin(c)\right]x^3 \quad \text{for some } c$$

$$|R_2(0.5)| = \left|0.125 \cdot \frac{1}{6}\left[(-9c^2+3)\cos(c) + (1-17c+c^3)\sin(c)\right]\right|$$

$$= 0.125 \cdot \frac{1}{6}\left[|(-9c^2+3)||\cos(c)| + |1-17c+c^3||\sin(c)|\right]$$

$$\leq 0.125 \cdot \frac{1}{6}\left[\underbrace{|-9c^2+3|}_{f_1} + \underbrace{|1-17c+c^3|}_{f_2}\right]$$

$$\left.\begin{aligned} f_1'(c) &= -18c = 0 \Rightarrow c = 0 \\ f_2'(c) &= -17+3c^2 \Rightarrow c = \pm\sqrt{17/3} \end{aligned}\right\} \text{maximize } |R_2(0.5)|$$

choose $c = -\sqrt{17/3}$

$$\Rightarrow |R_2(0.5)| \leq 0.125 \cdot \frac{1}{6}\left[9(17/3)+3 + 1+17(17/3)+(-\sqrt{17/3})^3\right]$$

$$\approx 2.87$$

$0.1989 < 2.87$

$\Rightarrow$ |actual error| $< |R_2(0.5)|$

(b):

$$|R_2(x)| \leq \frac{x^3}{6}\left[|-9c^2+3| + |1-17c+c^3|\right]$$

where $c = -\sqrt{17/3}$

then $|R_2(x)| \lessapprox \frac{x^3}{6} M$

(C):

$$\int_0^1 P_2(x)dx = \int_0^1 \left(1+x-\frac{x^2}{2}\right)dx = \left[x+\frac{x^2}{2}-\frac{x^3}{6}\right]_0^1$$

$$= 1+\frac{1}{2}-\frac{1}{6}$$

$$= \frac{6+3-1}{6}$$

$$= \frac{8}{6}$$

$$= \frac{4}{3}$$

$$\Rightarrow \int_0^1 f(x)dx \approx \frac{4}{3}$$

(d):

error in the integral : $\left| \int_0^1 f(x)\,dx - \int_0^1 P_2(x)\,dx \right|$

$= \left| \int_0^1 (f(x) - P_2(x))\,dx \right|$

$f(x) = P_2(x) + R_2(x)$

$\Rightarrow f(x) - P_2(x) = R_2(x)$

$\Rightarrow$ error $= \left| \int_0^1 R_2(x)\,dx \right|$

$\leq \frac{M}{6} \left| \int_0^1 x^3\,dx \right|$

$= \frac{M}{6} \left[\frac{x^4}{4} \right]_0^1$

$= \frac{M}{24}$

M on $[0,1]$: (when $c=1$)

$\Rightarrow M \leq |-9(1)^2+3| + |1-17(1)+(1)^3|$

$= 6 + 15$

$= 21$

$\Rightarrow$ error $\leq \frac{21}{24}$

to actually estimate the error:

$R_2(x) = \frac{1}{6}\left[(-9c^2+3)\cos(c) + (1-17c+c^3)\sin(c) \right] x^3$

where $c \in [x, 0]$

limits on integral: $[0,1]$

$\Rightarrow c = 0$

$\Rightarrow R_2(x) = \frac{x^3}{6}[3] = \frac{x^3}{2}$

err $\approx \left| \int_0^1 \frac{x^3}{2}\,dx \right| = \frac{1}{8}$

4.

(a):

estimate the actual roots:

$$r_1 = \frac{56 + \sqrt{56^2 - 4}}{2}$$

$$r_1 = 28 + \frac{1}{2}\sqrt{3132} \approx 55.98213715926644$$

$$r_2 = 28 - \frac{1}{2}\sqrt{3132} \approx 0.017862840733556595$$

estimate the roots using $\pm\frac{1}{2}10^{-3}$ sqrt accuracy:

$$r_1 \approx 55.982 \pm \frac{1}{2}10^{-3}$$

$$r_2 \approx 0018 \pm \frac{1}{2}10^{-3}$$

$$relerr_{r_1} = \frac{|55.982 - 55.98213715926644|}{|55.98213715926644|}$$

$$relerr_{r_1} \approx 2.45 \times 10^{-6}$$

$$relerr_{r_2} = \frac{|0.018 - 0.017862840733556595|}{|0.017862840733556595|}$$

$$relerr_{r_2} \approx 7.68 \times 10^{-3}$$

(b):

$$(x-r_1)(x-r_2)=0$$

$$x^2 - r_2 x - r_1 x + r_1 r_2 = 0$$

$$\Rightarrow x^2 - (r_1+r_2)x + r_1 r_2 = 0$$

$$\Rightarrow a=1,\ b=-(r_1+r_2),\ c=r_1 r_2$$

$$\Rightarrow r_1 = -b - r_2 = \frac{c}{r_2}$$

$$r_2 = -b - r_1 = \frac{c}{r_1}$$

Since r_1 is known to good accuracy,

$r_2 = \frac{c}{r_1}$ can be used (division is better than subtraction in terms of error)

$c = 1$

$$\Rightarrow r_2 = \frac{1}{r_1} \approx 0.017862840733554864$$

which is much better than 0.018

5.

(a).

The upper bounds to $|\Delta y|$ would be $|\Delta x_1|$ in the case when $\tilde{x}_2 = x_2$ or $|\Delta x_2|$ in the case when $\tilde{x}_1 = x_1$. This would maximize the upper bounds of Δy.

As for the relative error, in playing around with the values I found that the upper bounds is 1. The relative error is large when $|\Delta y|$ approaches $|y|$. This would be that the subtraction of the two estimated terms reaches closer to the difference in the errors of the terms. This can happen when x_1 and x_2 are small and close together which makes intuitive sense.

(b).

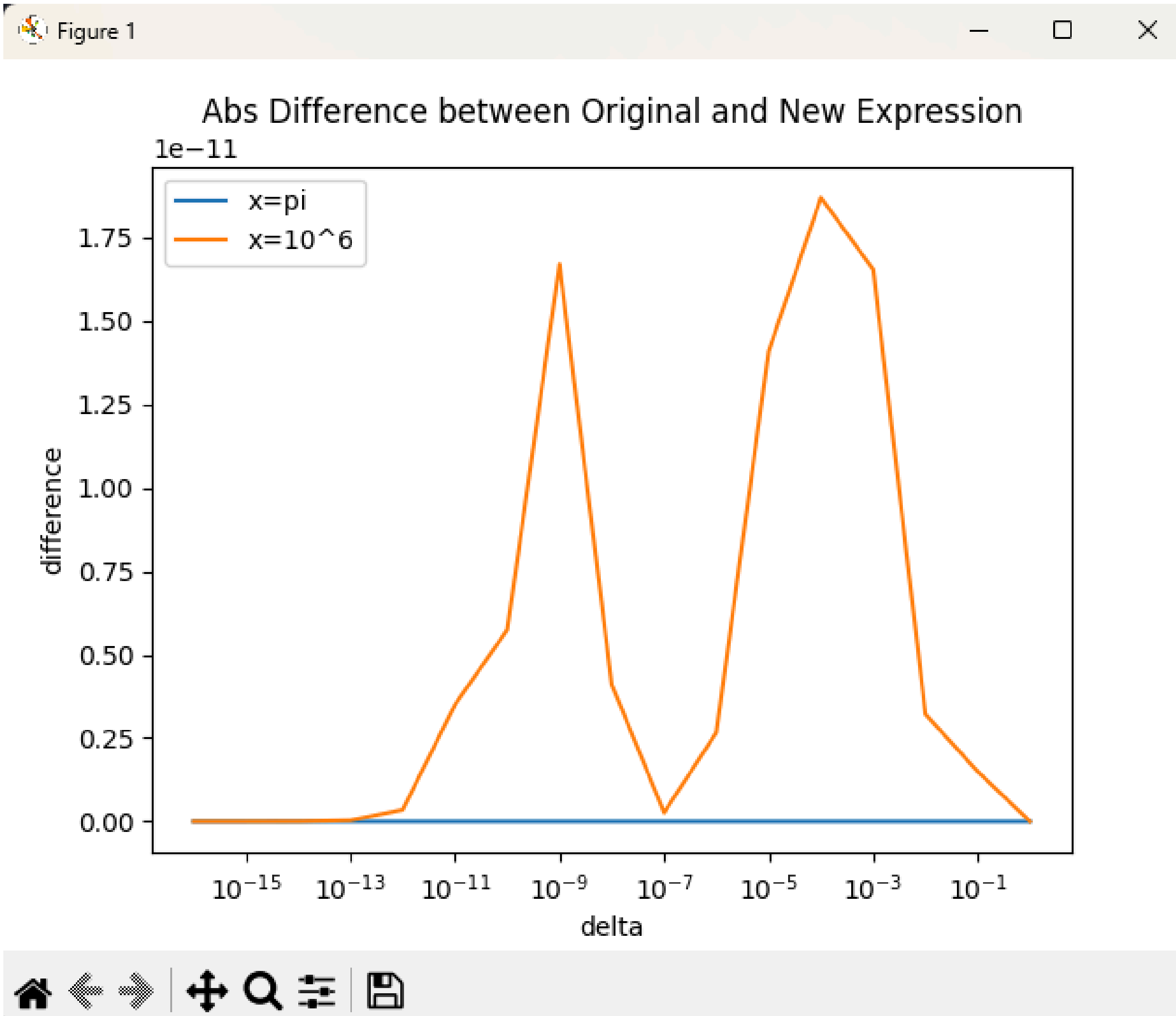
(b):

$\cos(x+\delta) - \cos(x)$

$= -2\sin\left(\frac{x+\delta+x}{2}\right)\sin\left(\frac{x+\delta-x}{2}\right)$

$= -2\sin\left(\frac{2x+\delta}{2}\right)\sin\left(\frac{\delta}{2}\right)$

by trig identity



There is almost no error for $x = \pi$, but there are two places where the error inflates to around 1.75 for $x = 10^6$. I'm honestly not sure why.

(c).

I found that no matter which η value I choose (in the interval $[x, x + \delta]$, $\eta = x$ gives a good enough answer such that there is no addition of relative error of the algorithm and the actual answer because the difference is less than machine epsilon (~ 10^{-17}).

Therefore, the following algorithm was chosen:

x = π

eta = x

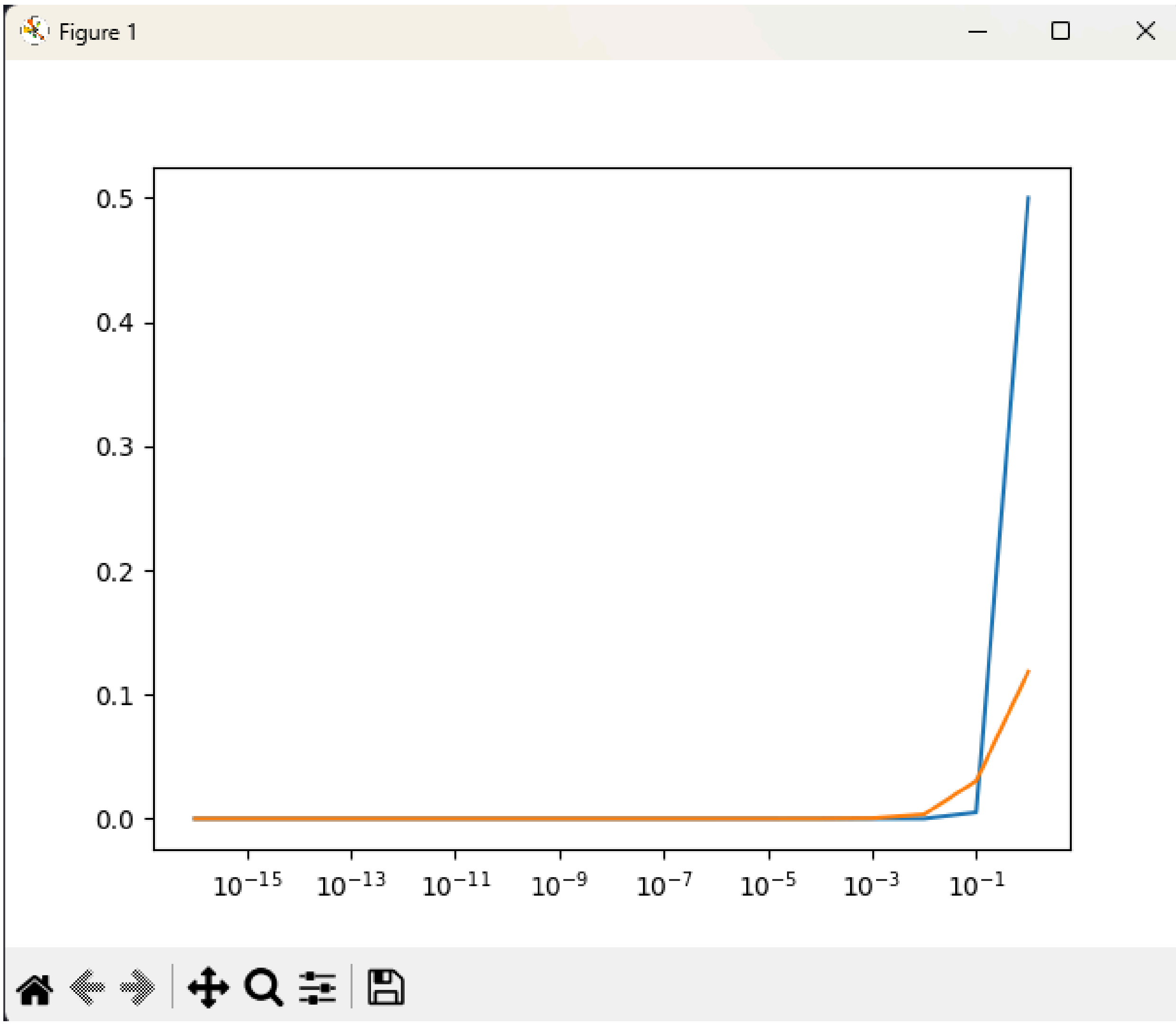
delta = $[10^{-16}, 10^{-15}, \ldots, 10^{0}]$

first_term = -1 * delta * sin(x)

second_term = -1 * delta * cos(eta)

difference = abs(first_term + second_term)

repeat for x = 10^{6}



There error is much larger overall at about 10^{-4} versus the previous method's 10^{-11}. However, relatively speaking the error of the large x-value was minimized to near 0 for the whole domain of delta. This algorithm would be better to use for large x-values. However, for large delta values and small x-values, using the previous method would be better.