HW 2

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1.

(a):

binomial theorem:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \text{higher order terms}$$

Let $g(x) = \frac{n(n-1)}{2}x^2$

$$\lim_{x\to 0}\left|\frac{g(x)}{x}\right| = \lim_{x\to 0}\left|\frac{n(n-1)}{2}x\right| = 0$$

$\Rightarrow\ g(x) = o(x)$

$\Rightarrow\ (1+x^n) = 1 + nx + o(x)$

(b):

$\sin\sqrt{x} \approx \sqrt{x}$ as $x\to 0$

$\Rightarrow\ x\sin\sqrt{x} \approx x\sqrt{x} = x^{3/2} = O(x^{3/2})$

$\Rightarrow\ x\sin\sqrt{x} = O(x^{3/2})$

(c):

$$\lim_{t\to 0}\left|\frac{e^{-t}}{1/t^2}\right| = \lim_{t\to 0}\left|\frac{t^2}{e^t}\right| = \lim_{t\to 0}\left|\frac{2t}{e^t}\right| = 0$$

$\Rightarrow\ e^{-t} = o(1/t^2)$

(d):

$$\int_0^{\varepsilon} e^{-x^2}dx = \int_0^{\varepsilon}\sum_{k=0}^{\infty}\frac{x^{2k}}{k!}dx$$

$$\approx \int_0^{\varepsilon}\left(1 + x^2 + \frac{x^4}{2}\right)dx$$

$$= \varepsilon + O(\varepsilon^3)$$

$$\lim_{\varepsilon\to 0}\frac{\varepsilon + O(\varepsilon^3)}{\varepsilon} \approx 1$$

$\Rightarrow\ \int_0^{\varepsilon}e^{-x^2}dx = O(\varepsilon)$

2.

(a):

$$A(\vec{x}+\Delta\vec{x}) = \vec{b}+\Delta\vec{b}$$
$$A\vec{x} + A\Delta\vec{x} = \vec{b}+\Delta\vec{b}$$
$$A\vec{x} = \vec{b} \Rightarrow A\Delta\vec{x} = \Delta\vec{b}$$
$$\Rightarrow \Delta\vec{x} = A^{-1}\Delta\vec{b}$$

(b):

$$\text{relerr}_x = \frac{\|\Delta x\|}{\|x\|} = \frac{\|A^{-1}\Delta b\|}{\|x\|}$$

$$\kappa_A(b) = \lim_{\|\Delta b\|\to 0} \frac{\|A^{-1}\Delta b\|/\|x\|}{\|\Delta b\|/\|b\|}$$
$$= \lim_{\|\Delta b\|\to 0} \frac{\|A^{-1}\Delta b\|\,\|b\|}{\|x\|\,\|\Delta b\|}$$
$$= \lim_{\|\Delta b\|} \frac{\|A^{-1}\Delta b\|\,\|Ax\|}{\|x\|\,\|\Delta b\|}$$
$$\leq \|A\|_2 \|A^{-1}\|_2 = \kappa_A$$

$$\Rightarrow \kappa_A = \|A\|\,\|A^{-1}\|$$

using different norm ($\|\cdot\|_\infty$) b/c easier

$$\kappa_A = \|A\|_\infty \|A^{-1}\|_\infty$$
$$= 1 \cdot 1$$
$$= 1$$

(c):

$$\text{relerr}_x = \frac{\|\Delta x\|}{\|x\|}$$

$$\|\Delta b\| = 10^{-5} = \|A\Delta x\| \leq \|A\|\,\|\Delta x\|$$

$$\Rightarrow \text{relerr}_x \leq \frac{\|\Delta b\|}{\|A\|\,\|x\|} = \frac{\|\Delta b\|}{\|x\|} = \frac{10^{-5}}{\sqrt{2}}$$

The behavior will be different for different Δb b/c $A\Delta x = \Delta b$ w/ A exact.
The more realistic situation is w/ different Δb.

larger perturbation $\Rightarrow$ larger relerr
κ_A stays the same

3.

3]

(a):

$$\kappa_f(x) = |f'(x)| \frac{|x|}{|f(x)|}$$

$$= e^x \frac{|x|}{|e^x - 1|}$$

$$= \frac{|x|e^x}{|e^x - 1|}$$

$x=0$ results in an asymptote

$\Rightarrow \kappa_f \to \infty \Rightarrow$ ill-conditioned

(b):

This algorithm is stable for $x > 0$ but unstable for $x \to 0$ due to the loss of precision w/ subtraction. for $x > 0$, κ goes down I think.

(C):

I get 1.0×10^{-9} up to 7 decimal digits. This is less than 16 which makes sense b/c subtraction is involved & x is close to 0. $\Rightarrow$ loss of precision (d):

$$f(x) = -1 + e^x$$
$$= -1 + \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Want $\text{relerr}_f(x) \leq 10^{-16}$

$$f(x) = -1 + 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$
$$= x + \frac{x^2}{2} + \frac{x^3}{6}$$

(e):

It does.

(f):

It seems to work w/ $f(x) = x + \frac{x^2}{2}$ as well to 16 digits.

(g):

Cool.

4.

(a):

See the code attachment for details, but I got -17.545259710757044 as the sum.

(b):

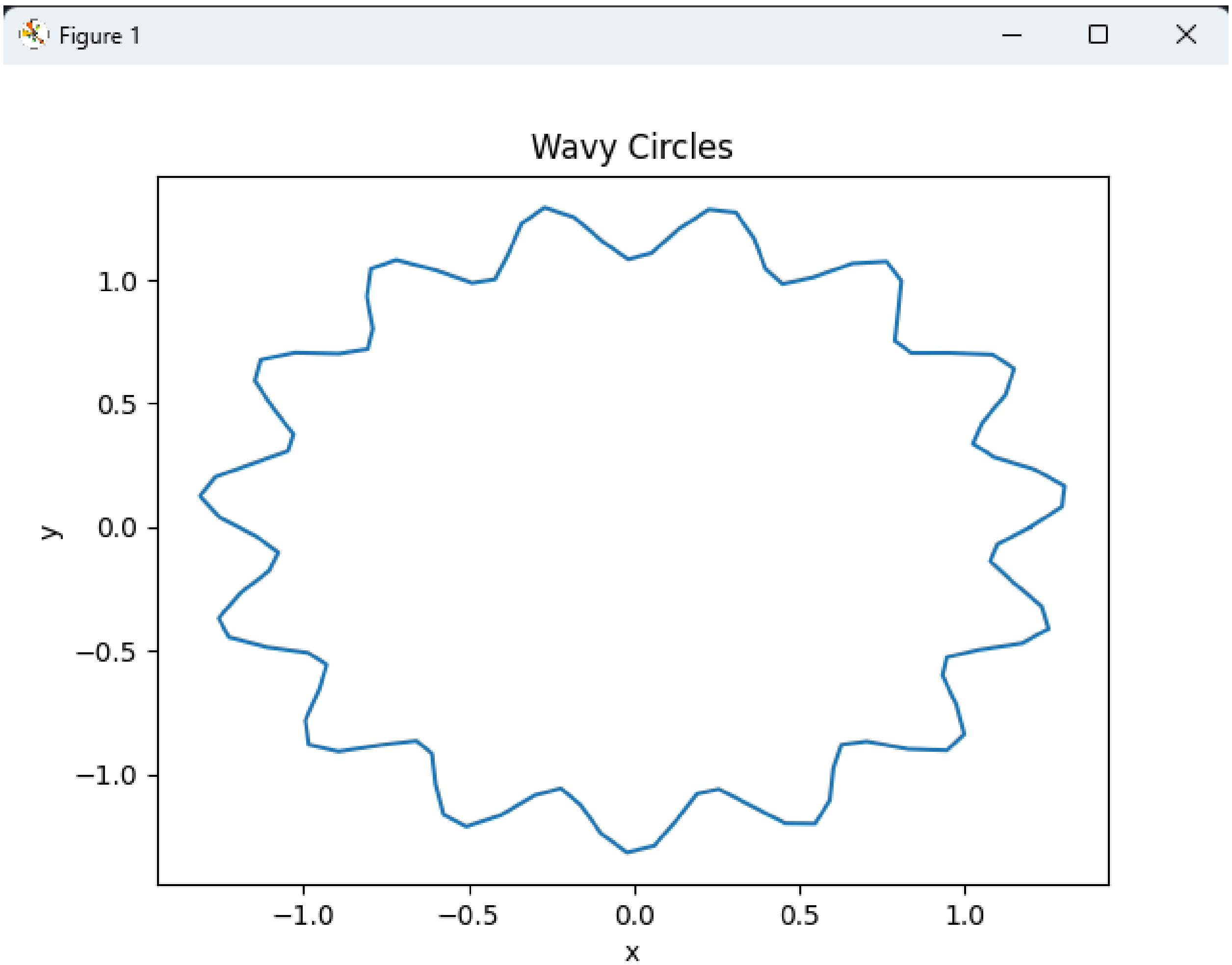

Figure 1: Parametric wavy circle curve

Figure 1 used one iteration of the x and y functions. See code attachment for details.

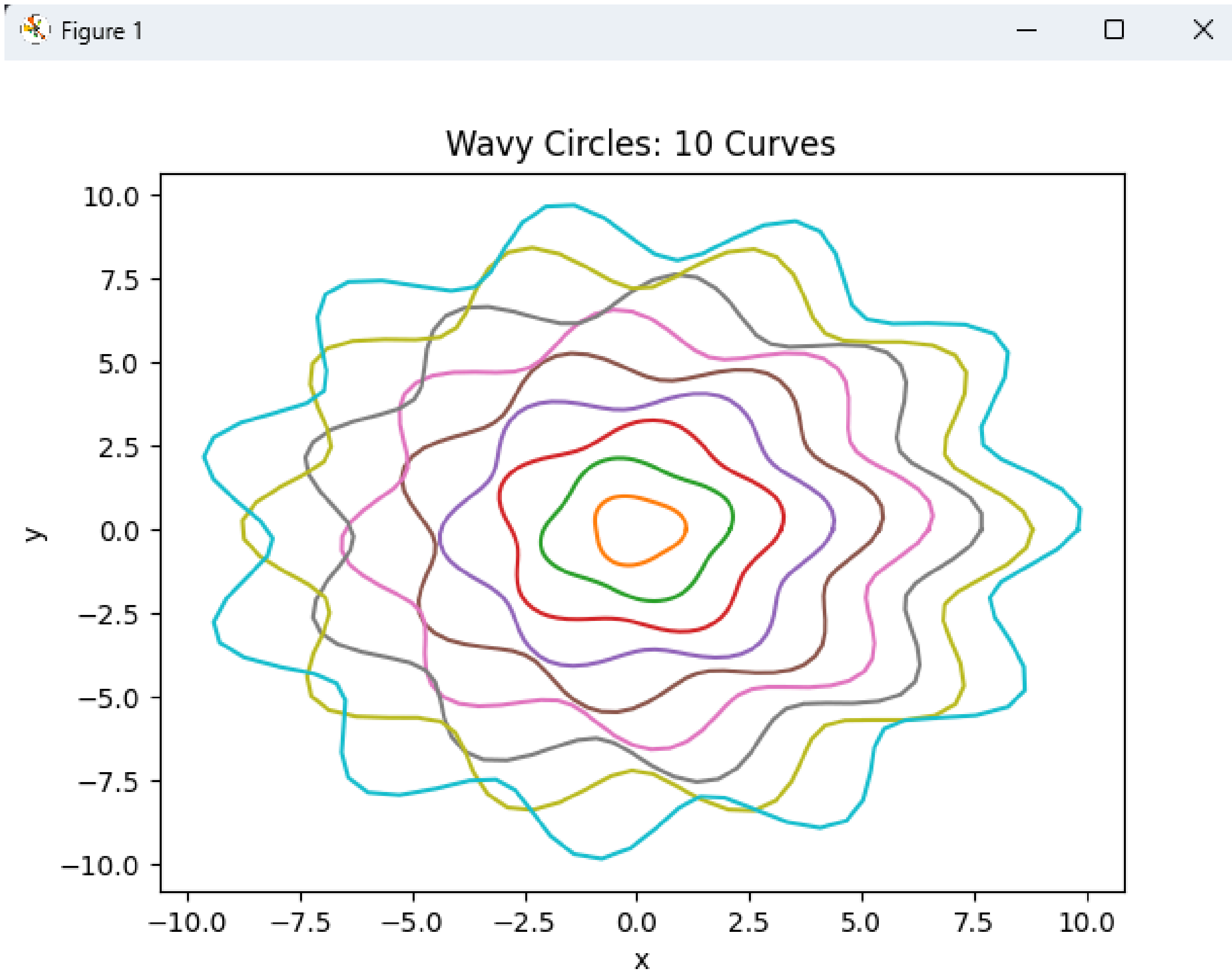

Figure 2: 10 wavy circles graphed using varied iterations of the parametric curve in Figure 1.

See code attachment for details.