

CS 112 – Introduction to Computing II

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Today:

Deletion in Binary Search Trees
Tree Traversals (recursive and non-recursive)

Next Time:

Efficiency of binary trees;
Balanced Trees
2-3 Trees



Deletion in BSTs



Deletion is somewhat more complicated than insertion or lookup. We will warm up by considering a simple case first: How do we delete the minimal element in a BST?

```
// reconstruct the tree r without its minimal element

public static Node deleteMin(Node r) {
    if(r.left == null)
        return r.right;
    else {
        r.left = deleteMin(r.left);
        return r;
    }
}
```

Deletion in BSTs



Note for later: If we want to keep track of the minimal node (say, to remove it and use it later), we could write a simple helper function to look up the minimal element:

```
public static Node findMin(Node r) {
    if(r.left == null)           // this is the minimal node
        return r;
    else
        return findMin(r.left);
}

public static Node deleteMin(Node r) {
    if(r.left == null)
        return r.right;
    else {
        r.left = deleteMin(r.left);
        return r;
    }
}

// Remove the node containing the minimal element and store it as p
Node p = findMin(root);
root = deleteMin(root);
```

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Deletion in BSTs



Ok, we know how to delete the minimal element; how to delete an arbitrary element?
As usual, the place to start with by enumerating all the cases, starting with null:

```
public static Node delete(int n, Node t) {
    if (t == null)           // Case 1: tree is null
        return t;
```

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Deletion in BSTs



```

public static Node delete(int n, Node t) {
    if (t == null)                // Case 1: tree is null
        return t;

    else if (n < t.key) {          // Case 2: key n is in left subtree
        t.left = delete(n, t.left);
        return t;
    } else if (n > t.key) {        // Case 3: key n is in right subtree
        t.right = delete(n, t.right);
        return t;
    }
}

```

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Deletion in BSTs



```

public static Node delete(int n, Node t) {
    if (t == null)                // Case 1: tree is null
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        return t;
    } else                        // Case 4: found key n at root;

```

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Deletion in BSTs



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    if (t == null)                // Case 1: tree is null
        return t;
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        return t;
    } else if (n > t.key) {       // Case 3: key n is in right subtree
        t.right = delete(n, t.right);
        return t;
    }

    // Case 4: found key n at root;
    // Case 4a: no left child, so reroute around
    else if (t.left == null)
        return t.right;
    // Case 4b: no right child, ditto
    else if (t.right == null)
        return t.left;
    else {

```

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Deletion in BSTs



```

public static Node delete(int n, Node t) {
    if (t == null)                // Case 1: tree is null
        return t;
    else if (n < t.key) {          // Case 2: key n is in left subtree
        t.left = delete(n, t.left);
        return t;
    } else if (n > t.key) {       // Case 3: key n is in right subtree
        t.right = delete(n, t.right);
        return t;
    } else
        // Case 4: found key n at root;
        if (t.left == null)       // Case 4a: no left child, so reroute around
            return t.right;
    else if (t.right == null)     // Case 4b: no right child, ditto
        return t.left;
    else {
        // Case 4c: both children exist, so replace
        //         root by minimal element in right subtree
        Node min = findMin(t.right); // Find minimal node
        t.right = deleteMin(t.right); // Reconstruct the right subtree without it
        min.left = t.left;           // Finally, replace root node with min node
        min.right = t.right;
        return min;
    }
}

```

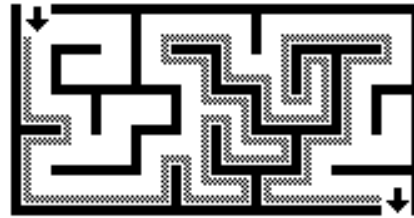
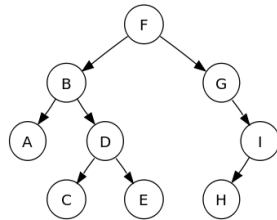
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Tree Traversals



A **Tree Traversal** is an algorithm for visiting each node of a Binary Tree in some order; all algorithms which perform some kind of operation on the tree as a whole usually follow one of these traversals. We will later generalize these to traversals of arbitrary Graphs, and many problems in computer science can be phrased as traversal of some kind of graph.

Let us begin by considering how you might “explore” a tree by walking around the links (for the moment considering them as two-way) to “visit” each node at least once; one way to do this is to use a version of the famous “**right-hand rule**” for solving a simple maze: **just start at the root, keep your right hand on the wall, and keep walking.**

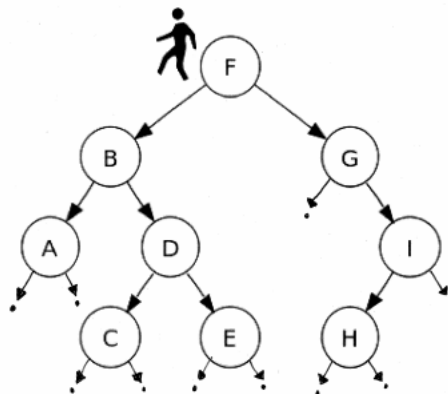


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Tree Traversals



Instead, we will keep our left hand on the outside “wall” of the tree, and keep walking.....

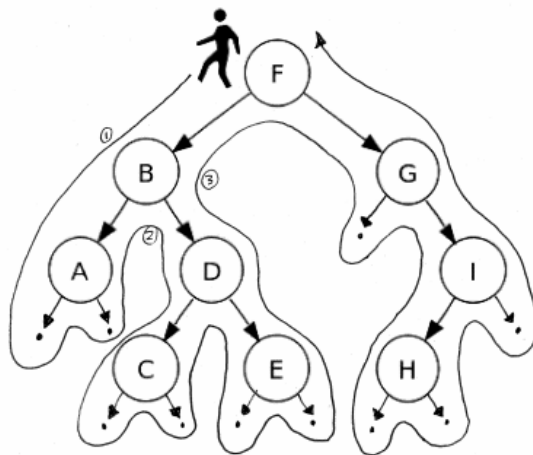


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Tree Traversals



Instead, we will keep our left-hand on the outside “wall” of the tree, and keep walking.....



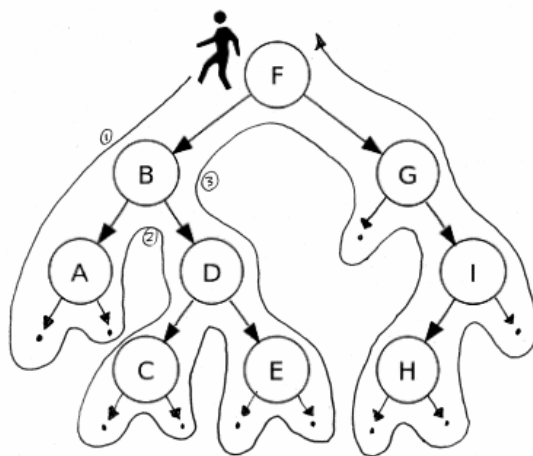
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Tree Traversals



NOTE: Each node is touched exactly three times:

The nodes are touched (for the **first** time) in this order: F, B, A, D, C, E, G, I, H



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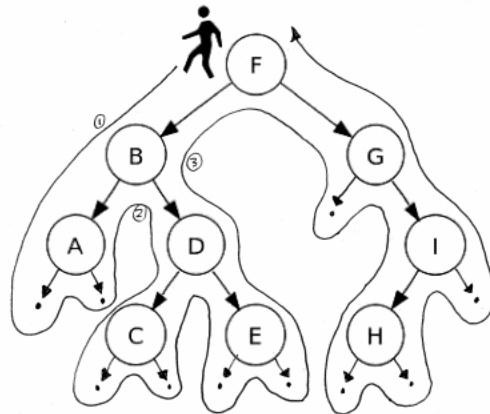
Tree Traversals



NOTE: Each node is touched exactly three times:

The nodes are touched (for the first time) in this order: F, B, A, D, C, E, G, I, H

The nodes are touched (for the **second** time) in this order: A, B, C, D, E, F, G, H, I



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Tree Traversals

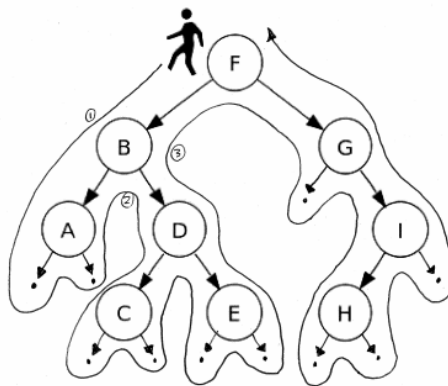


NOTE: Each node is touched exactly three times:

The nodes are touched (for the first time) in this order: F, B, A, D, C, E, G, I, H

The nodes are touched (for the second time) in this order: A, B, C, D, E, F, G, H, I

The nodes are touched (for the **third** time) in this order: A, C, E, D, B, H, I, G, F

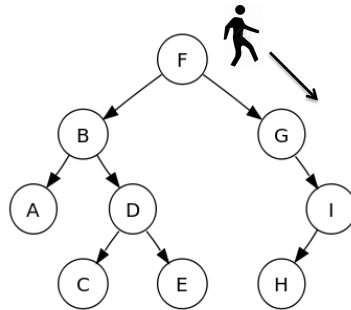


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Tree Traversals



Or, we could walk around “clock-wise” instead, by using our right hand:



The nodes are visited for the first time in this order: F, G, I, H, B, D, E, C, A

The nodes are visited for the second time in this order: I, H, G, F, E, D, C, B, A

The nodes are visited for the third time in this order: H, I, G, E, C, D, A, B, F

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Tree Traversals



How could we do this using an algorithm (which---no surprise!---will be recursive)?

As usual, we use the **recursive definition of a tree** as a basis for our algorithm:

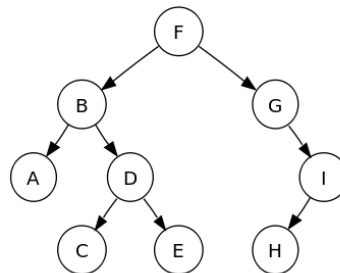
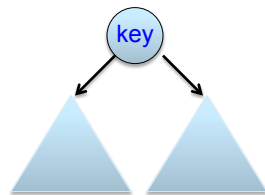
A **Binary Tree** is either

(1) null



or:

(2) A node pointing to two Binary Trees



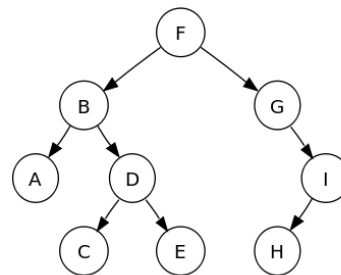
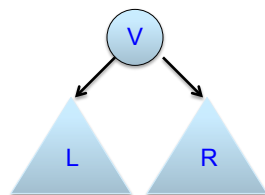
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Tree Traversals



The **base case** is easy (do nothing!);
 For the **recursive case**, we have to do three things:
 (V) Visit the root (say, by printing out the key);
 (L) Recursively traverse the left subtree; and
 (R) Recursively traverse the right subtree.

It does not matter which order we do these in,
 as long as we do all three.....



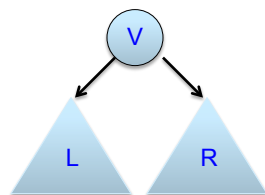
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Tree Traversals



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 For the **recursive case**, we have to do three things:
 (V) Visit the root (say, by printing out the key);
 (L) Recursively traverse the left subtree; and
 (R) Recursively traverse the right subtree.

It does not matter which order we do these in,
 as long as we do all three.....



There are $3! = 6$ possible orderings of these three lines, giving us 6 possible recursive traversals:

Preorder - Visit root first
Inorder - Visit root second
Postorder - Visit root last

Normally, you do L before R;
 If you add "**Reverse**" do R before L

Reverse Postorder:

R
 L
 V

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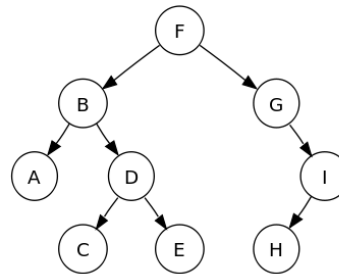
Tree Traversals



How could we do this using an algorithm? Here is a Preorder Traversal:

```
void traverse(Node t) {
    if( t != null ) {          // Base case is implicit
        visit(t);              // V
        traverse(t.left);      // L
        traverse(t.right);     // R
    }
}

void visit(Node t) {
    System.out.print(t.key + " ");
}
```



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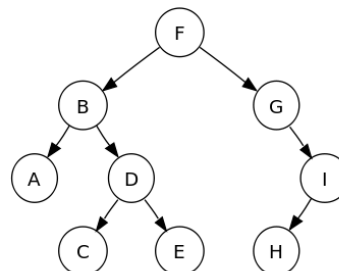
Tree Traversals



Which one is this?

```
void traverse(Node t) {
    if( t != null ) {          // Base case is implicit
        traverse(t.left);      // L
        visit(t);              // V
        traverse(t.right);     // R
    }
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void visit(Node t) {
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```



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Tree Traversals

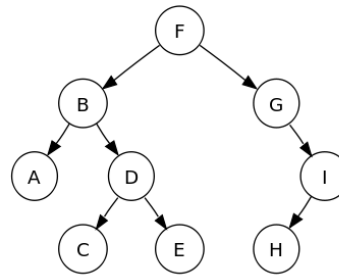


Which one is this?

Note: L before R, Visit root in middle: **Inorder**

```
void traverse(Node t) {
    if( t != null ) {        // Base case is implicit
        traverse(t.left);    // L
        visit(t);            // V
        traverse(t.right);   // R
    }
}

void visit(Node t) {
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}
```



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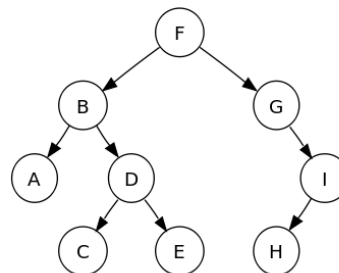
Tree Traversals



How about this?

```
void traverse(Node t) {
    if( t != null ) {        // Base case is implicit
        traverse(t.right);    // R
        traverse(t.left);     // L
        visit(t);            // V
    }
}

void visit(Node t) {
    System.out.print(t.key + " ");
}
```



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Tree Traversals

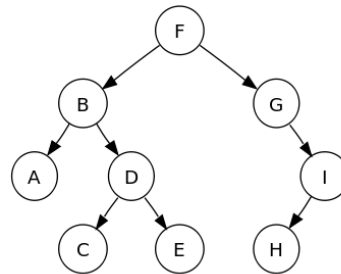


How about this?

R before S (so Reverse) and Visit root last: **Reverse Postorder**

```
void traverse(Node t) {
    if( t != null ) {          // Base case is implicit
        traverse(t.right);    // R
        traverse(t.left);     // L
        visit(t);             // V
    }
}

void visit(Node t) {
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```



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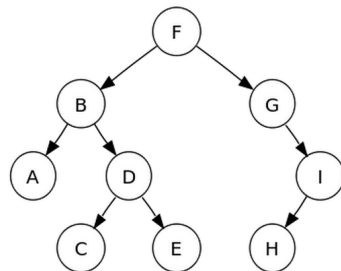
Tree Traversals: Non-Recursive Traversals



We can traverse a tree without recursion if we use an auxiliary data structure such as a stack or queue to keep track of the path traversed.

Let's try using a stack first:

```
void traverse(Node t) {
    Stack<Node> S = new Stack<Node>();
    S.push(t);
    while( !S.isEmpty() ) {
        Node p = S.pop();
        visit( p );
        if( p.left != null )
            S.push(p.left);
        if( p.right != null )
            S.push(p.right);
    }
}
```



What recursive traversal is this equivalent to?

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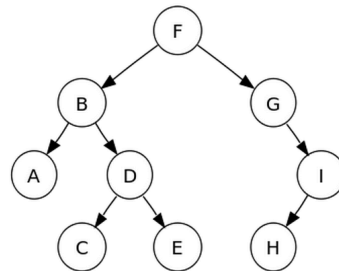
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    }
}
```



What recursive traversal is this equivalent to? **Reverse Preorder**

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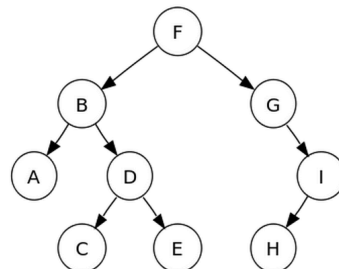
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```



How could we get a (normal) Preorder Traversal?

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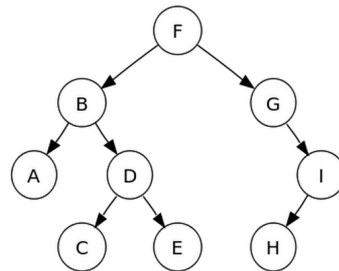
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        visit( p );
        if( p.right != null )
            S.push(p.right);
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}
```



How could we get a (normal) Preorder Traversal?

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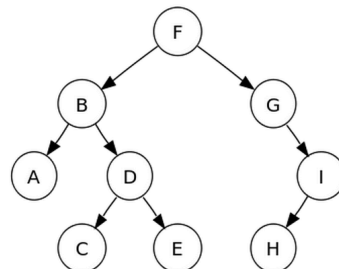
Tree Traversals: Non-Recursive Traversals



We can traverse a tree without recursion if we use an auxiliary data structure such as a stack or queue to keep track of the path traversed.

What happens if we use a **Queue** instead of a stack?

```
void traverse(Node t) {
    Queue<Node> Q = new Queue<Node>();
    Q.enqueue(t);
    while( !Q.isEmpty() ) {
        Node p = Q.dequeue();
        visit( p );
        if( p.left != null )
            Q.enqueue(p.left);
        if( p.right != null )
            Q.enqueue(p.right);
    }
}
```



Does this correspond to any of our recursive traversals?

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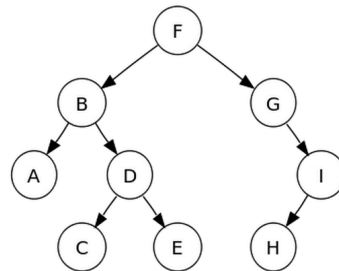
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            Q.enqueue(p.right);
    }
}
```



Does this correspond to any of our recursive traversals? **NO!**

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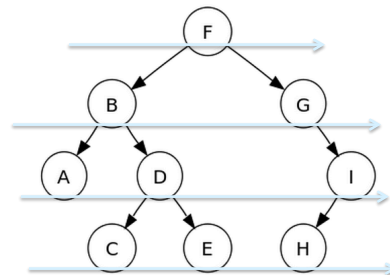
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            Q.enqueue(p.right);
    }
}
```



This is called a **Breadth-First or Level-Order Traversal**.

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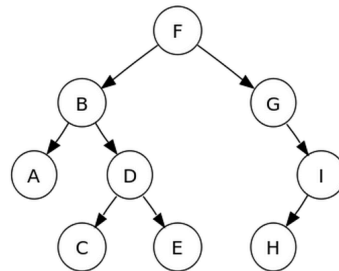
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        visit( p );
        if( p.right != null )
            Q.enqueue(p.right);
        if( p.left != null )
            Q.enqueue(p.left);
    }
}
```



What happens if we reverse the order we enqueue the children?

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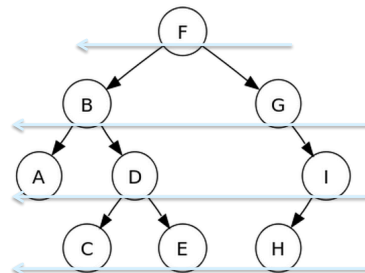
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```



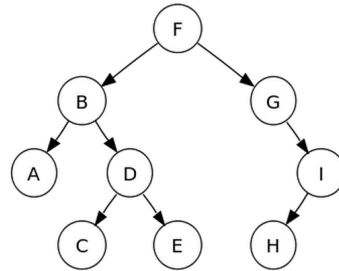
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Tree Traversals: Non-Recursive Traversals



In general, we can use any collection that supports adding and removing elements!
 Suppose we have an arbitrary collection (call it Unvisited) which stores the nodes in **some** order:

```
void traverse(Node t) {
    Unvisited<Node> Q = new Unvisited<Node>();
    Q.add(t);
    while( !Q.isEmpty() ) {
        Node p = Q.removeNext();
        visit( p );
        if( p.left != null )
            Q.add(p.left);
        if( p.right != null )
            Q.add(p.right);
    }
}
```



We will traverse ALL the nodes, in SOME order.....
 We will talk more about this when we study games...

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