MATH 170A HOMEWORK 5

Detailed hints to these problems will be discussed in the discussion sections and will be uploaded under Discussion notes of the relevant section. Also, please visit your TA's office hours for any questions, discussions or clarifications.

MATERIAL COVERED BY HOMEWORK 5: This homework covers roughly the following sections of [1]: 7.2, 7.3.

§1: (Section 7.2: Jacobi, Gauss-Seidel, and SOR; Related to Exercises 7.2.4/7.2.12/7.2.24) Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ be given, with $a_{ii} \neq 0, \forall i$. Define splitting $A = D - L - U$, where D is the diagonal of A, L is its strict lower-triangle, and U is its strict upper-triangle. (NOTE: L and U have nothing to do with LU factorization of A.) We wish to solve $Ax = b$ for $x \in \mathbb{R}^n$; this system can be written as:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, \ldots, n \tag{1.1}$$

(a) The Jacobi Method (7.2.1 in [1]) involves solving (1.1) one equation at a time for x_i, taking $i = 1, \ldots, n$, while treating each of the other variables $x_j, j \neq i$, as fixed:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)} \right), \quad i = 1, \ldots, n \tag{1.2}$$

Show that the Jacobi Method (1.2) can be written in matrix form as:

$$x^{(k+1)} = [I - BA]x^{(k)} + Bb, \qquad \text{where} \qquad B = D^{-1}$$

(b) The Gauss-Seidel Method (7.2.9 in [1]) involves solving (1.1) one equation at a time for x_i, taking $i = 1, \ldots, n$, while treating each of the other variables $x_j, j \neq i$, as fixed as in Jacobi, but using the most recent information:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right), \quad i = 1, \ldots, n. \tag{1.3}$$

Show that the Gauss-Seidel Method (1.3) can be written in matrix form as:

$$x^{(k+1)} = [I - BA]x^{(k)} + Bb, \quad \text{where} \quad B = (D - L)^{-1}$$

(c) Show that the SOR Method (7.2.19 in [1]) can be written in matrix form as:

$$x^{(k+1)} = [I - BA]x^{(k)} + Bb, \quad \text{where} \quad B = \left(\frac{1}{\omega} D - L\right)^{-1}$$

§2: (Section 7.3: Classical Linear Methods: Convergence; Related to Exercise 7.3.10) Consider the basic linear (classical, stationary) method:

$$x^{(k+1)} = [I - BA]x^{(k)} + Bb, \tag{1.4}$$

where $B \approx A^{-1}$, and $x^{(0)}$ is chosen as an initial guess. This method generates an infinite sequence of approximate solutions $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ to: $Ax = b$.

(a) Theorem 7.3.7 in [1] tells us that a necessary and sufficient condition for convergence of (1.4) from any initial guess is:

$$\rho(I - BA) < 1 \tag{1.5}$$

This means that for any $x^{(0)}$, if (1.5) holds, then:

$$\lim_{k \to \infty} x^{(k)} = x$$

Show that for any matrix norm, a sufficient condition for convergence is:

$$\|I - BA\| < 1$$

(b) Show that if:

$$\|I - BA\| \leqslant \sigma < 1$$

then a lower bound on the number of iterations needed to ensure:
is given by:

$$\frac{\|x - x^{(k)}\|}{\|x - x^{(0)}\|} \leqslant \epsilon < 1$$

$$k \geqslant \frac{|\ln \epsilon|}{|\ln \sigma|}$$

§3: (Section 7.2/7.1: Linear Methods: Implementation Related to Exercises 7.2.6/7.2.14) Let A and b be defined as:

$$A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{1.6}$$

(a) Implement the Jacobi Method as a function in MATLAB, and use it to produce an approximate solution $x^{(k)}$ to the system $Ax = b$, stopping the iteration at whatever step k the residual $\left\|r^{(k)}\right\|_2 = \left\|b - Ax^{(k)}\right\|_2$ first falls below $\epsilon = 1e-10$.

(b) Implement the Gauss-Seidel Method as a function in MATLAB, and use it to produce an approximate solution $x^{(k)}$ to the system $Ax = b$, stopping the iteration at whatever step k the residual $\left\|r^{(k)}\right\|_2 = \left\|b - Ax^{(k)}\right\|_2$ first falls below $\epsilon = 1e-10$.

(c) Compare the number of iterations required in parts (a) and (b).

(d) Use MATLAB to confirm the following to help explain what you found in part (c):

$$\rho\left(I - B_G A\right) \leqslant \left\|I - B_G A\right\|_2 \leqslant \left\|I - B_J A\right\|_2 = \rho\left(I - B_J A\right),$$

where $B_J = D^{-1}$ and $B_G = (D - L)^{-1}$ for the two methods ($A = D - L - U$).

(Hint: Implement (1.4), then pass in B for each method; you need B for part (d) anyway. Debug your codes using Example 7.2.3 and 7.2.11 before trying the system (1.6).)

References

[1] **D.S. Watkins. Fundamentals of Matrix Computations. Second. Wiley-Interscience. New York, NY: John Wiley & Sons, Inc., 2002.**