Summary for 204381: Chapter 5

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📄 **Basic Information**
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★ **Calculator** 🧮 ☆ **Open Book** 🙌🏼

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System of Linear Equations → Matrix

Given a system of n equations with n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n \end{aligned}$$

We can transform the problem into a linear equation, as

$$
\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ & & ... & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}
$$

$$
A\vec{x} = B
$$

Ways to find $\vec{x}$

Inversion

Because

$$
\begin{aligned}
A\vec{x} &= B \\
\vec{x} &= A^{-1}B
\end{aligned}
$$

This can be done easily on scientific calculators, or with any techniques you like. However, the computation cost for large matrices is considerable.

Gaussian Elimination

Perform elemental row operations onto A and B until A is in the form of

$$
A' = \begin{bmatrix} \times & \times & \times & ... & \times \\ 0 & \times & \times & ... & \times \\ 0 & 0 & \times & ... & \times \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \times \end{bmatrix}
$$

where $\times$ is any non-zero number. Note that it does not always required to be perfectly align diagonally, but the zero terms should form like that.

then suppose B is transformed into

$$
B' = \begin{bmatrix} b'_1 \\ b'_2 \\ ... \\ b'_n \end{bmatrix}
$$

and a'_{ij} is a term corresponding to A above, then the solution for $\vec{x}$ is

$$\vec{x} = \begin{bmatrix} \frac{a'_{11}}{b'_1} \\ \frac{a'_{22}}{b'_2} \\ \dots \\ \frac{a'_{nn}}{b'_n} \end{bmatrix}$$

To sum up, a'_{ij} and b'_i can be evaluated for any pivot row k as follows:

$$a'_{ij} = \begin{cases} a_{ij} - \frac{a_{ik}}{a_{kk}} a_{kj} & \text{for } k < i, j \leq n \\ 0 & \text{for } k < i \leq n, j = k \\ a_{ij} & \text{else} \end{cases}$$

$$b'_i = \begin{cases} b_i - \frac{a_{ik}}{a_{kk}} b_k & \text{for } k < i \leq n \\ b_i & \text{else} \end{cases}$$

Best way to select pivot rows

Always choose the row with greatest ratio, for example:

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Calculate a ratio r_i for each row i, for pivot row k we get

$$r_{k,i} = \frac{a_{i,k}}{\max_{1 \leq j \leq n} a_{i,j}}$$

$$r_{1,1} = \frac{1}{2}, r_{1,2} = \frac{3}{4}, r_{1,3} = \frac{5}{8}, r_{1,4} = \frac{4}{5}$$

In this case, row 4 has the greatest ratio, so we should perform the elimination using row 4 as a pivot row.

Repeat this step then we will get A' as intended.

Gauss-Seidel Method

1. Initiate a **random** real vector $\vec{x}$
2. For each equation i in the system of equations,

$$a_{i,1}x_1 + a_{i,2}x_2 + ... + a_{i,i}x_i + ... + a_{i,n}x_n = b_i$$

$$x_i^{\text{new}} = \frac{b_i - a_{i,1}x_1 - a_{i,2}x_2 - ... - a_{i,i-1}x_{i-1} - a_{i,i+1}x_{i+1} - ... - a_{i,n}x_n}{a_{i,i}}$$

3. Repeat until the relative change is below a threshold ϵ that

$$\left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100 < \epsilon$$