Summary for 204381: Chapter 4

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📄 **Basic Information**
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★ **Calculator** 🧮 ☆ **Open Book** 🙌🏼

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Review: The Fundamental Theorem of Calculus

The First Theorem: Given $f(x)$ be a continuous on an open interval I containing a, then the area function is:

The Second Theorem: And $F'(x)$ is an antiderivative of $f(x)$ on I, that is, $F'(x) = f(x)$, equivalently:

$$F(x) = \int_a^x f(t)dt \qquad \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Review: Integral Rules

Given a, c be constants of real, and n be an integer, $f(x)$ and $g(x)$ be a continuous function of real x, then there are the following theorems we should know.

Basic Rules

$$\int a\ dx = ax + c$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + c \ \ ; n \neq -1$$

$$\int \frac{1}{x}\ dx = \ln|x| + c$$

$$\int e^x\ dx = e^x + c$$

$$\int a^x\ dx = a^x \ln(a) + c$$

$$\int \ln(x)\ dx = x\ln(x) - x + c$$

$$\int \sin(x)\ dx = -\cos(x) + c$$

$$\int \cos(x)\ dx = \sin(x) + c$$

$$\int \sec^2(x)\ dx = \tan(x) + c$$

$$\int c \cdot f(x)\ dx = c\int f(x)\ dx$$

$$\int f(x) \pm g(x)\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

Integration by Substitution

If $f(x)$ and $g(x)$, *or* u *given* $u = g(x)$, are composed, then the following integrals are the same as $\int f(x)\ dx$

$$\int f(g(x))\ g'(x)\ dx; \ \text{ or } \int f(u)\ du$$

Integration by Part

If $f(x)$ consists of 2 composite functions called u and v, in a way that $f(x) = uv$,

Closed Integral Rule

If $F(x) = \int f(x)\ dx$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

If $b > a$ and $\forall x_i \in [a,b] : f(x_i) \geq 0$,
or $b < a$ and $\forall x_i \in [a,b] : f(x_i) \leq 0$, then

$$\int_a^b f(x)\ dx \geq 0$$

If $b > a$ and $\forall x_i \in [a,b] : f(x_i) \leq 0$,
or $b < a$ and $\forall x_i \in [a,b] : f(x_i) \geq 0$, then

$$\int_a^b f(x)\ dx \leq 0$$

If b = a, then

$$\int_a^b \ f(x)\ dx = 0$$

For any $[a,b] \subseteq \mathbb{R}$ on which $f(x)$ be differentiable,

$$\int_a^b f(x)\ dx = -\int_b^a \ f(x)\ dx$$

we can evaluate $\int f(x)\ dx = \int uv\ dx$ using

$$u \int v\ dx - \int u' \left(\int v\ dx \right)\ dx$$

or

$$uv - \int v\ du$$

To calculate an area within an enclosed boundary of 2 curves f(x) and g(x) such that $\forall x_i \in [a, b]$:

$$\text{Area} = \int_a^b f(x) - g(x)\ dx$$

Numerical Error Analysis

Given y be an **actual** value, and $\hat{y}$ be an **estimated** one, the absolute error ϵ of estimated $\hat{y}$ is

$$\epsilon = |y - \hat{y}|$$

In case of integrals, we can estimate the numerical integration error of **one integrated portion** using such the method above.

$$y = \int_a^b f(x)\ dx \quad \text{and} \quad \hat{y} \text{ is the chosen method over } [a, b]$$

Here $f(x)$ can be generalized as a Taylor series of $n = 1$ around $x = a$, along with the residual $R_n(x)$,

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + R_n(x) \\ &= f(a) + f'(a)(x-a) + \frac{f''(\xi_1)}{2}(x-a)^2 \quad ; \xi_1 \in [a, x] \end{aligned}$$

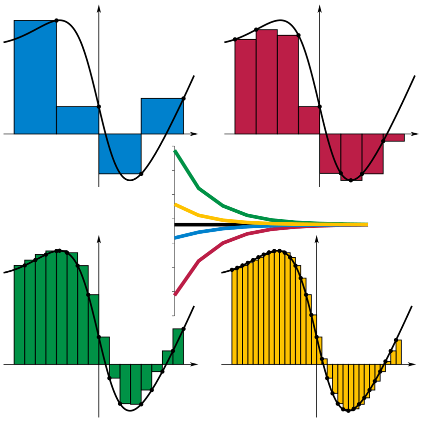
Sometimes, it is better to use $n = 0$ for the convenience in calculation.

$$f(x) = f(a) + f'(\xi_1)(x - a)$$

The example of error analysis will be given later.

Variations on Riemann Sum

📄 Reference:
https://en.wikipedia.org/wiki/Riemann_sum



The most basic aspect of integration is to calculate the area under the curve over a plane, which is what Riemann Sum is about: calculating a small area of rectangles representing each portion of area under the curve.

$$\begin{aligned}\text{All Area} = \text{height}_1 &\times \text{width}_1 \\ + \text{height}_2 &\times \text{width}_2 \\ + \text{height}_3 &\times \text{width}_3 \\ + \dots\end{aligned}$$

Review: Riemann Sum

Let $f : [a, b] \to \mathbb{R}$ be a function defined on real that is continuous and differentiable on $[a, b] \in \mathbb{R}$, and $P = (x_0, x_1, x_2, \dots, x_n)$ be a partition of $[a, b]$ that is $a = x_0 < x_1 < x_2 < \dots < x_n = b$.

A **Riemann Sum** S of f over $[a, b]$ with partition P is defined as

$$S = \sum_{i=0}^{n} f(x_i^*)\Delta x_i$$

Where $\Delta x_i = x_i - x_{i-1}$ and $x_i^* \in [x_{i-1}, x_i]$. Note that x_i^* can be different from one to another, but for the sake of simplicity, we give all $h = x_i - x_{i-1} = x_{i+1} - x_i = x_{i+2} - x_{i+1} = \dots$

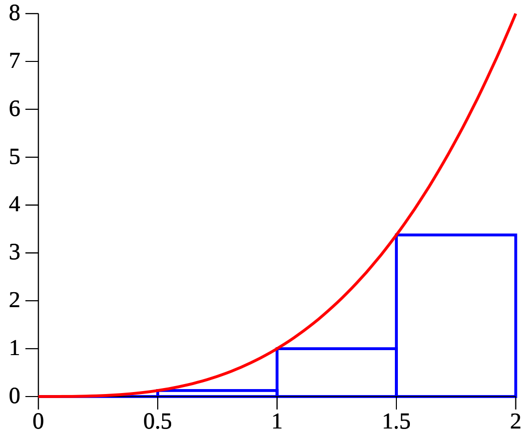
Riemann Sum is considered a perfect estimation of an integration within $x \in [a, b]$ if

$$\int_a^b f(x)\, dx = \lim_{||\Delta x|| \to \infty} \sum_{i=0}^{n} f(x_i^*)\Delta x_i$$

However, there is no way to actually compute $||\Delta x|| \to \infty$, so the only approximation is usable, or

$$\int_a^b f(x)\, dx \approx \sum_{i=0}^{n} f(x_i^*)\Delta x_i$$

Left Rule



Always choose $x_i^* = x_{i-1}$ as a representative of the whole segment.

Therefore,

$$S_{\text{left}} = \sum_{i=1}^{n} f(x_{i-1})h$$

Local Error Analysis

$$\epsilon_{\text{local,left}} = \left| \int_a^b f(x)\,\mathrm{dx} - f(a)(b-a) \right| \qquad \_\_\_\_(\alpha)$$

Consider the Taylor expression of $f(x)$ around $x = a$ (because we focus on the left) given $n = 0$:

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b \Big(f(a) + f'(\xi_1)(x-a)\Big)\,\mathrm{dx} \;\; ;\xi_1 \in [a, x] \\ &= f(a)(b-a) + \int_a^b (x-a)f'(\xi_1)\,\mathrm{dx} \qquad \_\_\_\_(1)\end{aligned}$$

Consider

$$\begin{aligned}f(a)(b-a) &= \Big[f(a) + \cancel{f'(\xi_2)(a-a)}\Big](b-a) \;\; ;\xi_2 = a \\ &= f(a)(b-a) \qquad \_\_\_\_(2)\end{aligned}$$

Then $|(1) - (2)|$;

$$\begin{aligned}\epsilon_{\text{local,left}} &= \left| \int_a^b f(x)\,\mathrm{dx} - f(a)(b-a) \right| \\ &= \left| \cancel{f(a)(b-a)} + \int_a^b (x-a)f'(\xi_1)\,\mathrm{dx} - \cancel{f(a)(b-a)} \right| \\ &= \left| \int_a^b (x-a)f'(\xi_1)\,\mathrm{dx} \right|\end{aligned}$$

Let M_1 be the maximum value of $|f'(x)|$ within $x \in [a, b]$, or $M_1 = \max_{x\in[a,b]} |f'(x)|$, then

$$
\begin{aligned}
\epsilon_{\text{local,left}} &\leq M_1 \left| \int_a^b (x-a)\ dx \right| \\
&= M_1 \frac{|b-a|^2}{2}
\end{aligned}
$$

Given $b > a$,

$$
\therefore \epsilon_{\text{local,left}} \leq \frac{M_1 (b-a)^2}{2} \qquad \text{\_\_\_\_\_}(3)
$$

Global Error Analysis

The error from estimating a single large rectangle of the maximum height within $x \in [a, b]$ is shown in (3). If we split such the rectangle into n smaller pieces, we get

$$
\therefore \epsilon_{\text{global,left}} = \frac{\epsilon_{\text{local,left}}}{n} \leq \frac{M_1 (b-a)^2}{2n}
$$

Code Implementation

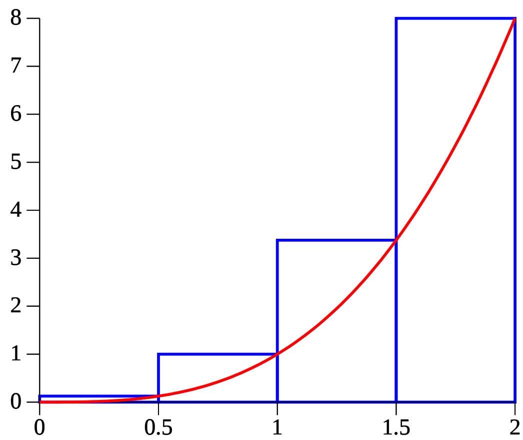
```
function integrate_left(
        f::Function, a::Number, b::Number, n::Number = 100
    )::Number
    width::Number = (b - a) / n
    area::Number = 0.0
    for i in 0:n-1
        area += f(a + i * width) * width
    end
    return area
end
```

Right Rule

Always choose $x_i^* = x_i$ as a representative of the whole segment.

Therefore,

$$
S_{\text{right}} = \sum_{i=1}^{n} f(x_i) h
$$



Local Error Analysis

$$\epsilon_{\text{local,right}} = \left| \int_a^b f(x)\ \mathrm{dx} - f(b)(b-a) \right| \qquad \_\_\_\_(\beta)$$

Consider the Taylor expression of $f(x)$ around $x = b$ (because we focus on the right) given $n = 0$:

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^b \Big(f(b) + f'(\xi_3)(x-b) \Big)\ \mathrm{dx}\ \ ; \xi_1 \in [x, b] \\ &= f(b)(b-a) + \int_a^b (x-b) f'(\xi_3)\ \mathrm{dx} \qquad \_\_\_\_(4) \end{aligned}$$

Consider

$$\begin{aligned} f(b)(b-a) &= \Big[f(b) + \cancel{f'(\xi_4)(b-b)} \Big] (b-a)\ \ ; \xi_4 = b \\ &= f(b)(b-a) \qquad \_\_\_\_(5) \end{aligned}$$

Then $|(4) - (5)|$;

$$\begin{aligned} \epsilon_{\text{local,right}} &= \left| \int_a^b f(x)\ \mathrm{dx} - f(b)(b-a)\ \mathrm{dx} \right| \\ &= \left| \cancel{f(b)(b-a)} + \int_a^b (x-b) f'(\xi_3)\ dx - \cancel{f(b)(b-a)} \right| \\ &= \left| \int_a^b (x-b) f'(\xi_3)\ \mathrm{dx} \right| \end{aligned}$$

Let M_1 be the maximum value of $|f'(x)|$ within $x \in [a, b]$, or $M_1 = \max_{x \in [a,b]} |f'(x)|$, then

$$\begin{aligned} \epsilon_{\text{local,right}} &\leq M_1 \left| \int_a^b (x-b)\ \mathrm{dx} \right| \\ &= M_1 \left| -\frac{(a-b)^2}{2} \right| = M_1 \frac{(b-a)^2}{2} \end{aligned}$$

Given $b > a$,

$$\therefore \epsilon_{\text{local,right}} \leq \frac{M_1 (b-a)^2}{2} \qquad \_\_\_\_(6)$$

Global Error Analysis

The error from estimating a **<u>single large</u>** rectangle $\epsilon_{\text{local,right}}$ of the maximum height times the distance between a and b (within $x \in [a, b]$) is shown in (6). If we split such the rectangle into n smaller pieces, then

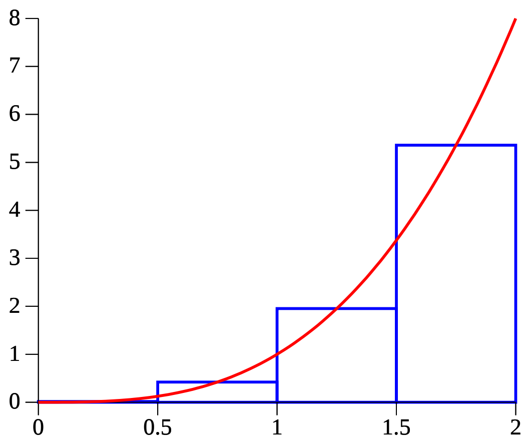
$$\therefore \epsilon_{\text{global,right}} = \frac{\epsilon_{\text{local, right}}}{n} \leq \frac{M_1 (b-a)^2}{2n}$$

💡 **Observation** $\epsilon_{\text{local,left}} = \epsilon_{\text{local,right}}$ **and** $\epsilon_{\text{global,left}} = \epsilon_{\text{global,right}}$

Code Implementation

```
function integrate_right(
        f::Function, a::Number, b::Number, n::Number = 100
    )::Number
    width::Number = (b - a) / n
    area::Number = 0.0
    for i in 1:n
        area += f(a + i * width) * width
    end
    return area
end
```

Midpoint Rule



Instead of choosing one end, we can average both ends to calculate the average result, so we choose

$$x^*_{i/2} = \frac{x_{i-1} + x_i}{2}$$

Therefore,

$$S_{\text{mid}} = \sum_{i=1}^{n} f(x^*_{i/2})h$$

Local Error Analysis

$$\epsilon_{\text{local,mid}} = \left| \int_a^b f(x)\, dx - f(a_{1/2})(b-a) \right| ; a_{1/2} = \frac{a+b}{2} \qquad \_\_\_\_(\gamma)$$

Consider the Taylor expression of $f(x)$ around $x = a$ (I don't know why, maybe it is easier to prove)

Given $n = 1$:

$$
\begin{aligned}
f(x) &= f(a) + f'(a)(x-a) + \frac{f''(\xi_1)}{2}(x-a)^2 \quad ; \xi_1 \in [a,x] \\
\int_a^b f(x)\,dx &= \int_a^b \left(f(a) + f'(a)(x-a) + \frac{f''(\xi_1)}{2}(x-a)^2\right) \mathrm{dx} \\
&= f(a)(b-a) + \frac{f'(a)}{2}(b-a)^2 + \frac{1}{2}\int_a^b f''(\xi_1)(x-a)^2 \,\mathrm{dx} \qquad \_\_\_\_(7)
\end{aligned}
$$

Consider

$$
\begin{aligned}
f(a_{1/2})(b-a) &= \left[f(a) + f'(a)(a_{1/2}-a) + \frac{f''(\xi_1)}{2}(a_{1/2}-a)^2\right](b-a) \quad ; \xi_4 \in [a_{1/2}, x] \\
&= \left[f(a) + \frac{f'(a)}{2}(b-a) + \frac{f''(\xi_1)}{8}(b-a)^2\right](b-a) \\
&= f(a)(b-a) + \frac{f'(a)}{2}(b-a)^2 + \frac{f''(\xi_1)}{8}(b-a)^3 \qquad \_\_\_\_(8)
\end{aligned}
$$

Then $|(7)-(8)|$;

$$
\begin{aligned}
\epsilon_{\text{local,mid}} &= \left|\int_a^b f(x)\,dx - f(b)(b-a)\right| \\
&= \left|\frac{1}{2}\int_a^b f''(\xi_1)(x-a)^2 \,\mathrm{dx} - \frac{f''(\xi_1)}{8}(b-a)^3\right|
\end{aligned}
$$

Let M_2 be the maximum value of $|f''(x)|$ within $x \in [a,b]$, or $M_2 = \max_{x\in[a,b]} |f''(x)|$, then

$$
\begin{aligned}
\epsilon_{\text{local,mid}} &\le \left|\frac{1}{2}\int_a^b f''(\xi_1)(x-a)^2 \,\mathrm{dx}\right| + \left|\frac{f''(\xi_1)}{8}(b-a)^3\right| \\
&= \frac{M_2}{2}\int_a^b (x-a)^2 \,\mathrm{dx} + \frac{M_2}{8}(b-a)^3 \\
&= \frac{M_2}{2} \cdot \frac{(b-a)^3}{3} + \frac{M_2}{8}(b-a)^3 \\
&= \frac{M_2}{6}(b-a)^3 + \frac{M_2}{8}(b-a)^3 \\
&= \frac{7M_2}{24}(b-a)^3
\end{aligned}
$$

Given $b > a$,

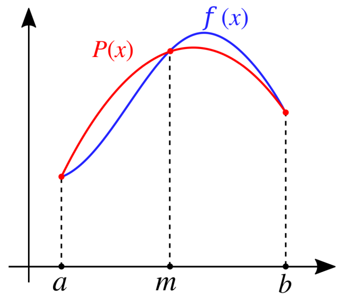
$$
\therefore \epsilon_{\text{local,mid}} \le \frac{7M_2}{24}(b-a)^3 \qquad \_\_\_\_(9)
$$

💡 **Observation** $\epsilon_{\text{global,mid}} \le \epsilon_{\text{local,right}}\epsilon_{\text{local,left}} = \epsilon_{\text{local,right}}$ **and** $\epsilon_{\text{global,left}} = \epsilon_{\text{global,right}}$

Code Implementation

```
function integrate_mid(
        f::Function, a::Number, b::Number, n::Number = 100
    )::Number
    width::Number = (b - a) / n
    area::Number = 0.0
    for i in 1:n
        area += f(a + i * width / 2) * width
    end
    return area
end
```

Simpson's Rules



Instead of splitting area under curves into multiple trapezoids like in Riemann's sum, we can achieve the target by creating a new curve from polynomial degree 2 then finding the area under it.

Starting from deriving Lagrange Polynomial degree 2 over $x \in \{-1, 0, 1\}$, we get:

$$p_2(x) = f(-1)\frac{x(x-1)}{2} - f(0)(x+1)(x-1) + f(1)\frac{(x+1)}{2}$$

Then find the integral within $x \in [-1, 1]$, we get:

$$\int_{-1}^{1} p_2(x)\,dx = \frac{1}{3}\left(f(-1) + 4f(0) + f(1)\right)$$

For any interval $x \in [a, b]$, we may do the same way by let $x = (b-a)\frac{y+1}{2} + a$ and $dx = (b-a)\frac{dy}{2}$, we get:

$$
\begin{aligned}
\int_a^b f(x)\,dx &= \int_{-1}^1 f\Big((b-a)\frac{y+1}{2}+a\Big)dx \\
&= \frac{b-a}{2}\int_{-1}^1 h(y)\,dy \\
&= \frac{b-a}{6}\Big(f(a)+4f\big(\frac{a+b}{2}\big)+f(b)\Big)
\end{aligned}
$$

where $h(y) = f\Big((b-a)\frac{y+1}{2}+a\Big)$

Error Analysis for Simpson's Method

would be

$$
\epsilon_{Simpson} = \frac{49}{2880}(b-a)^5 \max_{x\in[a,b]} |f''''(x)|
$$

Optimal Code Implementation

Practically, the result of integration may contain significant errors if we did not divide parts enough (the number of n is too few). So, if the error is above the epsilon threshold ϵ, we may double the value of n until it satisfy the condition. The following code is the example on how to implement Simpson's method optimally on computers.

```
function integrate_simpson(
        f::Function,
        a::Number,
        b::Number,
        n::Number,
        epsilon::Number = 1e-8,
        M::Number = 10000,
    )
    h::Number = (b - a) / n
    I::Number = 0.0  # area
    x::Number = a
    for i in 1:n
        xh = x + h
        I += f(xh - x) / 6 * (f(x) + 4 * f((x + xh) / 2) + f(xh))  # sum all sub-area
        x += h
    end
    I *= h
    error::Number = abs(I)

    j::Number = 1
    while j < M && error < epsilon * abs(I)
        n *= 2
        h = (b - a) / n
        Itemp::Number = I
        x = a
        for j in 1:n
            xh = x + h
            Itemp += f(h) / 6 * (f(x) + 4 * f((x + xh) / 2) + f(xh))  # sum all sub-area
            x += h
```

```
        end
        I *= h
        error = abs(I - Itemp)
    end

    return I
end
```