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The Gibbs phenomenon in generalized Padé approximation

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The Gibbs phenomenon in generalized Padé approximation is discussed, and with the aid of some rational approximants the Gibbs constants are determined. In addition, the steepest of the rational approximants is calculated.

I. INTRODUCTION

If one approximates a discontinuous function by polynomials (or by Fourier series) it leads to an unusual property—the Gibbs phenomenon. The polynomials do not converge to the function near the discontinuity. The maximal value of the error is called the Gibbs constant. For example, it is well known that when we approximate the function sgn (x) in $(-1, +1)$ by Fourier series the Gibbs constant is

$$G = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt - 1 = 0.178\,979\,7\,\ldots\,.$$

Another important property of the approximation is the steepness. We call the value of the derivative of the approximant at the discontinuity the steepness. For the function sgn (x) the steepness is $(4/\pi)(n+1)$, for an n-term Fourier approximation. It is noted that both properties in a multidimensional generalization can appear in more difficult analytical features (rapid behavior of the trajectory in nonlinear system, strange attractors, etc.). It is desirable to obtain an approximation for which the Gibbs constant is as small as possible and the steepness is as high as possible.

Zygmund[1] proved that one can decrease the Gibbs constant by Cesaro's method of summing series, but as experimentally shown by Arfken[2] this method halves the steepness.

In this paper we consider some rational functions and we show that in our case the generalized Padé approximants have Gibbs constants smaller than G and their steepness is higher than Cn.

The paper is arranged as follows. In Sec. II we consider the generalized Padé approximation in the sense of Cheney[3]; in Sec. III we treat the same problem using the method of Clenshaw and Lord.[4] We provide proofs of the results of the previous sections in Sec. IV, and in Sec. V we present some calculations of the steepness following Cesaro's method of summing series.

II. APPROXIMANTS FOR sgn (x) BY CHENEY'S METHOD

Here and further we apply to a series representation for the function sgn (x) in the form

$$\operatorname{sgn}(x) = \left\{\begin{matrix} -1, & -1 \le x < 0 \\ +1, & 0 < x \le 1 \end{matrix}\right\} = \sum_{n=0}^{\infty} C_n T_{2n+1}(x), \tag{1}$$

where $T_{2n+1}(x)$ is the Chebyshev polynomial and $C_n = (4/\pi)(-1)^n/(2n+1)$. The rationals

$$R_{n,m}(x) = \frac{\Sigma_{i=0}^m p_i T_{2i+1}(x)}{\Sigma_{i=0}^n q_i T_{2i}(x)}, \quad n,m = 0,1,2,\ldots, \tag{2}$$

which satisfy the relation

$$\left\{\sum_{i=0}^n q_i T_{2i}(x)\right\}\operatorname{sgn}(x) - \sum_{i=0}^m p_i T_{2i+1}(x) = O(T_{2n+2m+3}(x)), \tag{3}$$

are called the generalized Padé approximants.[3] The O term in (3) means a function for which the series in $T_i(x)$ begins with the term $T_{2n+2m+3}(x)$.

Next we shall list our main results. The solution of problem (3) in explicit form is

$$R_{n,m}(x) = A_{n,m}x\,\frac{{}_3F_2(-m, -n+\frac12, n+m+2; \frac32, \frac32; x^2)}{{}_3F_2(-n, -m-\frac12, n+m+\frac32; \frac12, 1; x^2)}, \tag{4}$$

where the steepness A is

$$A_{n,m} = \frac{4}{\sqrt{\pi}}\cdot\frac{n!}{m!}\cdot\frac{\Gamma(m+\frac32)}{\Gamma(n+\frac12)}\,\frac{\Gamma(n+m+2)}{\Gamma(n+m+\frac32)}. \tag{5}$$

For $n=0$ we can get the classic result. In this case the approximating polynomial is

$$R_{0,m}(x) = A_{0,m}x\,{}_3F_2(-m, m+2, \tfrac12; \tfrac32, \tfrac32; x^2). \tag{6}$$

Its error function takes the highest maximum at the point $x = \alpha/m$, $m\to\infty$. This value is the Gibbs constant

$$G = (4/\pi)\alpha\,{}_1F_2(\tfrac12; \tfrac32, \tfrac32; -\alpha^2) - 1.$$

Differentiating by α we get an equation for α:

$${}_0F_1(;\tfrac32; -\alpha^2) = \sin 2\alpha/2\alpha = 0.$$

Its first zero is $\alpha = \pi/2$. The previous series considered in integral form gives the classical result

$$G = \frac{4}{\pi}\int_0^1 \frac{\sin 2\alpha u}{2u}\,du - 1 = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt - 1.$$

The steepness is

$$A_{0,m} = (4/\pi)(m+1). \tag{7}$$

Second, we consider the case $m=0$, the reciprocal polynomial case. In this case the approximants are

$$R_{n,0}(x) = A_{n,0}x/{}_3F_2(-n, n+\tfrac32, -\tfrac12; \tfrac12, 1; x^2). \tag{8}$$

Its error function takes its maximum at the point $x = \beta/n$ $n\to\infty$. By elementary calculations one can prove that β is the root of the equation

$$J_0(2\beta) = 0,$$

where $J_0(x)$ is the Bessel function. From its first root we get

$$\beta = 1.202\,412\,778\,8\ldots,$$

therefore the Gibbs constant is

$$G_{0,1} = \left(\int_0^{2\beta}\frac{J_1(u)}{u}\,du\right)^{-1} - 1 = 0.051\,356\,067\ldots\,. \tag{9}$$

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That is, in this case the Gibbs constant is approximately 5%. The steepness is

$$A_{n,0} = 2n!(n+1)!/\Gamma(n+\tfrac{1}{2})\Gamma(n+\tfrac{3}{2}) = 2(n+1)a_n,$$

where $a_n \approx 1$ for moderate and large values of n. The most interesting case is $n=m$. The approximants are

$$R_{n,n}(x) = A_{n,n}\, x\, \frac{{}_3F_2(-n,-n+\frac{1}{2},2n+2;\frac{3}{2},\frac{3}{2};x^2)}{{}_3F_2(-n,-n-\frac{1}{2},2n+\frac{3}{2};\frac{1}{2},1;x^2)}, \tag{10}$$

where

$$A_{n,n} = \frac{2(2n+1)}{\sqrt{\pi}}\,\frac{\Gamma(2n+2)}{\Gamma(2n+\frac{3}{2})}.$$

The error function takes its maximal value at the point $x = \gamma/n^{3/2}$, $n \to \infty$. The constant γ is the root of the equation

$$\sum_{k=0}^{\infty} \frac{(-2\gamma^2)^k}{k!^2(\frac{3}{2})_k} = 0,$$

and its value $\gamma = 0.951\,020\,874\ldots$. The Gibbs constant is given by the formula

$$G_{1,1} = \frac{8\gamma}{\sqrt{2\pi}}\,\frac{{}_0F_2(;\frac{3}{2},\frac{3}{2};2\gamma^2)}{{}_0F_2(;\frac{1}{2},1;2\gamma^2)} - 1 = 0.008\,148\,902\ldots. \tag{11}$$

Semerdjiev and Nedelchev[5] performed a numerical experiment for determining $G_{1,1}$ enabling them to state that $G_{1,1}$ does not exceed 2%.

The steepest is

$$A_{n,n} = (4\sqrt{2}/\sqrt{\pi})\, n^{3/2} b_n,$$

where $b_n \approx 1$ for moderate ($n > 10$) and large values of n.

III. APPROXIMATIONS FOR sgn(x) BY THE CLENSHAW–LORD METHOD

Again, from series representation (1) we determine the rationals $S_{n,m}(x)$ by the method of Clenshaw and Lord[4]:

$$S_{n,m}(x) = \frac{\sum_{i=0}^{m} r_i T_{2i+1}(x)}{\sum_{i=0}^{n} s_i T_{2i}(x)}. \tag{12}$$

The coefficients r_i and s_i can be determined from the equality

$$\operatorname{sgn}(x) - S_{n,m}(x) = O(T_{2n+2m+3}(x)). \tag{13}$$

Our result is

$$S_{n,m}(x) = \frac{4}{\pi}(m+1)(2n+1)x \times \frac{{}_4F_3(-m,m+2,-n+\frac{1}{2},n+\frac{3}{2};\frac{3}{2},\frac{3}{2},\frac{3}{2};x^2)}{{}_4F_3(-n,n+1,-m-\frac{1}{2},m+\frac{3}{2};\frac{1}{2},1,1;x^2)}. \tag{14}$$

First we consider the reciprocal polynomial approximants ($m=0$)

$$S_{n,0}(x) = \frac{(4/\pi)(2n+1)x}{{}_4F_3(-n,n+1,-\frac{1}{2},\frac{3}{2};\frac{1}{2},1,1;x^2)}. \tag{15}$$

Its error function takes the maximal value at the point $x = \delta/n$, $n \to \infty$, where δ is the root of the equation,

$$J_0(\delta) = 2\delta J_1(\delta),$$

and $\delta = 0.940\,770\,564\ldots$. Its Gibbs constant is

$$G_{1,0} = 0.082\,417\,272\ldots. \tag{16}$$

The steepness is $4/\pi(2n+1)$.

The case $n=m$ presents powerful approximants. Here

$$S_{n,n}(x) = \frac{4}{\pi}(n+1)(2n+1)x \times \frac{{}_4F_3(-n,-n+\frac{1}{2},n+\frac{3}{2},n+2;\frac{3}{2},\frac{3}{2},\frac{3}{2};x^2)}{{}_4F_3(-n,-n-\frac{1}{2},n+1,n+\frac{3}{2};\frac{1}{2},1,1;x^2)}. \tag{17}$$

The error function takes its maximal value at the point $x = \eta/n^2$, $n \to \infty$. The value of η is the root of the equation

$$\operatorname{ber}(2\sqrt{2\eta}) = 0,$$

its first root is $\eta = 1.014\,541\,594\ldots$. The Gibbs constant is

$$G_{1,1} = \frac{8\eta}{\pi}\,\frac{{}_0F_3(;\frac{3}{2},\frac{3}{2},\frac{3}{2};\eta^2)}{{}_0F_3(;\frac{1}{2},1,1;\eta^2)} - 1 = 0.049\,325\,286\ldots. \tag{18}$$

The steepness is $(4/\pi)(n+1)(2n+1)$. This is the highest value in all cases.

IV. PROOFS

First we will prove formula (4). Let us consider a more generalized series expansion for sgn (x) like (1):

$$x^{2k}\operatorname{sgn}(x) = 2k!(\tfrac{1}{2})_k \sum_{j=0}^{\infty} \frac{T_{2j+1}(x)}{\Gamma(k+j+\frac{3}{2})\Gamma(k-j+\frac{1}{2})},$$

$$k = 0,1,2,\ldots.$$

Next, multiplying it by numbers q_k ($k = 0,1,\ldots,n$), then summing these equations, we get

$$\left(\sum_{k=0}^{n} q_k x^{2k}\right)\operatorname{sgn}(x) = \frac{4}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} T_{2j+1}(x) \sum_{k=0}^{n} q_k \frac{k!(\frac{1}{2})_k}{(\frac{3}{2}+j)_k(\frac{1}{2}-j)_k}.$$

We want to determine the coefficients in such a manner that the following equations are satisfied:

$$\sum_{k=0}^{n} q_k \frac{k!(\frac{1}{2})_k}{(\frac{3}{2}+j)_k(\frac{1}{2}-j)_k} = 0,$$

$$j = m+1, m+2, \ldots, m+n.$$

In this case the numerator polynomial will be

$$\frac{4}{\pi}\sum_{j=0}^{m} \frac{(-1)^j}{2j+1} T_{2j+1}(x) \sum_{k=0}^{n} q_k \frac{k!(\frac{1}{2})_k}{(\frac{3}{2}+j)_k(\frac{1}{2}-j)_k}.$$

To solve the previous equations let us suppose for a moment that

$$q_k = (-n)_k(-m-\tfrac{1}{2})_k(n+m+\tfrac{3}{2})_k/k!^2(\tfrac{1}{2})_k.$$

Consider now the sum

$$S = \sum_{k=0}^{n} \frac{(-n)_k(-m-\frac{1}{2})_k(n+m+\frac{3}{2})_k}{k!(\frac{3}{2}+j)_k(\frac{1}{2}-j)_k}$$

$$= {}_3F_2(-n,-m-\tfrac{1}{2},n+m+\tfrac{3}{2};\tfrac{3}{2}+j,\tfrac{1}{2}-j;1).$$

S is a Saalschütz-type hypergeometric sum and therefore it is summable by factorial functions. Really,[6]

$$S = (1+m-j)_n(-1-m-n-j)_n \times [(\tfrac{1}{2}-j)_n(-\tfrac{1}{2}-j-n)_n]^{-1}.$$

It is not difficult to see that all products differ from zero except the first one. Further, when j runs from $m+1$ to $m+n$ then $1+m-j$ runs from 0 to $-n+1$ by -1. Therefore, $S=0$ for all j ($j=m+1,...,m+n$). We have thus proved the form of the denominator polynomial. To get the explicit form of the numerator polynomial we apply the value of S for $j=0,1,...,m$:

$$Z=\frac{4}{\pi}\sum_{j=0}^{m}\frac{(-1)^j}{2j+1}T_{2j+1}(x)\times\frac{(1+m-j)_n(-1-m-n-j)_n}{(\frac{1}{2}-j)_n(-\frac{1}{2}-n-j)_n}.$$

Taking the power form of the Chebyshev polynomial

$$(1/x)T_{2j+1}(x)=(2j+1)(-1)^j\,{}_2F_1(-j,j+1;\tfrac{3}{2};x^2),$$

we get

$$Z=\frac{(n+m)!\Gamma(n+m+2)}{(\frac{1}{2})_n(\frac{3}{2})_n m!\Gamma(m+2)}\frac{4}{\pi}x\times\sum_{j=0}^{m}(2j+1)\,{}_2F_1(-j,j+1;\tfrac{3}{2};x^2)\times\frac{(-m)_j(n+m+2)_j(\frac{1}{2}-n)_j}{(-n-m)_j(m+2)_j(\frac{3}{2}+n)_j}.$$

Let us transform Z to the power form in x^2:

$$Z=\frac{4}{\pi}x\frac{(n+m)!(n+m+1)!}{(\frac{1}{2})_n(\frac{3}{2})_n m!(m+1)!}\times\sum_{i=0}^{m}(-4x^2)^i\frac{(-m)_i(n+m+2)_i(\frac{1}{2}-n)_i}{(-n-m)_i(m+2)_i(\frac{3}{2}+n)_i}W,$$

where

$$W=\sum_{j=0}^{m-i}\frac{(2i+1)_j}{j!}\times\frac{(-m+i)_j(n+m+i+2)_j(\frac{1}{2}-n+i)_j(\frac{3}{2}+i)_j}{(-n-m+i)_j(m+2+i)_j(\frac{3}{2}+n+i)_j(\frac{1}{2}+i)_j}.$$

The sum W is an ${}_5F_4$ hypergeometric function which one can sum by theorem of Dougall[7]:

$$W=(-1)^m\times\frac{\Gamma(i+m+2)\Gamma(-\frac{1}{2}-i)\Gamma(\frac{3}{2}+n+i)n!(-n-m)_i}{\Gamma(2i+2)\Gamma(-\frac{1}{2}-m)\Gamma(\frac{3}{2}+n+m)(n+m)!}.$$

By elementary calculations we get the required result:

$$Z=\frac{4}{\sqrt{\pi}}\frac{n!}{m!}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(n+\frac{1}{2})}\frac{\Gamma(n+m+2)}{\Gamma(n+m+\frac{3}{2})}x\times\sum_{i=0}^{m}\frac{(-m)_i(-n+\frac{1}{2})_i(n+m+2)_i}{i!(\frac{3}{2})_i(\frac{3}{2})_i}x^{2i}.$$

Proof of the form of Gibbs constants for the cases $m=0$ and $m=n$ can be obtained by elementary analysis. Here we omit the details. The proof of the results of Sec. III is analogous with the previous one.

V. CESARO'S METHOD OF SUMMING SERIES FOR sgn (x) AND THE STEEPNESS

It is well known that if we have a series

$$\sum_{\nu=0}^{\infty}a_\nu,$$

its Cesaro's sum is defined by the formula

$$C_n^\alpha=\sum_{\nu=0}^{n}a_\nu\frac{(-n)_\nu}{(-n-\alpha)_\nu}.$$

Here α is a positive parameter. It is well known that if $\alpha=1$, C_n^α is Fejér's arithmetic mean and in this case the Gibbs phenomenon does not occur. (The case $\alpha=0$ gives the original series.)

Next we will prove that if $\alpha>\alpha_0=0.439\,551\,2893...$, then, again, the Gibbs phenomenon does not occur.

Consider again the series (1), thus

$$a_\nu=(4/\pi)[(-1)^\nu/(2\nu+1)]T_{2\nu+1}(x).$$

By short, elementary calculation we get

$$C_n^\alpha(x)=\frac{4}{\pi}\frac{n+1+\alpha}{1+\alpha}x\sum_{j=0}^{n}\frac{(-n)_j(n+\alpha+2)_j(\frac{1}{2})_j}{(\frac{3}{2})_j(1+\alpha/2)_j(\frac{3}{2}+\alpha/2)_j}x^{2j}.$$

Its error function has the maximum at the point $x=s/n$, $n\to\infty$. The maximum is

$$G_\alpha(s)=\frac{4}{\pi}\frac{s}{1+\alpha}\sum_{k=0}^{\infty}\frac{1}{2k+1}\frac{(-s^2)^k}{(1+\alpha/2)_k(\frac{3}{2}+\alpha/2)_k}-1=\frac{2}{\pi}\int_0^1(1-t)^\alpha\frac{\sin 2st}{t}dt-1.$$

By determining the value s we get the equation

$$\sum_{k=0}^{\infty}\frac{(-s^2)^k}{(1+\alpha/2)_k(\frac{3}{2}+\alpha/2)_k}=0,$$

or in integral form

$$\int_0^1(1-t)^\alpha\cos 2st\,dt=0,$$

The solutions α and s of the equation $G_\alpha(s)$ and of the previous equation are

$$\alpha=0.439\,551\,2893...,$$

$$s=2.025\,782\,092....$$

Note: Gronwall[8] also determined the values α and s, but the stated precision of his results is incorrect. The steepness in Cesaro's method is $(4/\pi)(n+1+\alpha)/(1+\alpha)$. For $\alpha=1$, the steepness is $(4/\pi)(n+2)/2$. It is halved corresponding to

TABLE I. Gibbs constant and the steepness.

Fourier series	Cheney's method: Reciprocal polynomial	Cheney's method: Rational	Method of Clenshaw and Lord: Reciprocal polynomial	Method of Clenshaw and Lord: Rational	Cesaro's sum
18%	5.1%	0.8%	8.2%	4.9%	$18\%\leq G\leq 0\%$
$(4/\pi)\,n$	$2n$	$(4\sqrt{2}/\sqrt{\pi})\,n^{3/2}$	$(8/\pi)\,n$	$(8/\pi)\,n^2$	$(4/\pi)n/(1+\alpha)$

$\alpha = 0$. Thus, we have proved that Cesaro's method of summing series decreases the Gibbs constant, but it also decreases the steepness.

VI. CONCLUSIONS

As a means of summarizing our results, we have listed in Table I the Gibbs constants and their steepness corresponding to the methods used.

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