A NEW WAY TO COMPUTE SINE INTEGRAL FUNCTION

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Abstract. The sine integral function appears when working with bandlimited functions. In digital signal processing it is used for representing the Gibb's phenomenon, ideal low pass filters, and integration of bandlimited functions. It has found applications in the determination of radiation patterns, spectroscopy, fiber optic systems, quantum theory and image reconstruction in medical, radar and seismic imaging. Considering its wide application area, an efficient calculation of the sine integral function is essential. In this work, we present a new method to calculate the sine integral function. Our method is based on an accurate representation of the sinc function as a sum of scaled versions of a known analytic function over an interval centered around zero. Given the interval, the computational cost of the proposed method is linear with the respect to the number of evaluation points times number of terms of the sum used in representing the sinc function. We compare the accuracy of the proposed method with MATLAB's symbolic toolbox's sine integral function implementation, `sinint`.

keywords. sine integral function, sinc, generalization of Padé approximation

1. Introduction

The sine integral function is defined by

$$\mathrm{Si}\,(x) = \int_0^x \mathrm{sinc}\,(\omega)\,d\omega, \tag{1.1}$$

where $\mathrm{sinc}\,(x) = x^{-1}\sin(x)$ is the sinc function. The sine integral function has found visibility in signal processing [21], determination of radiation patterns and antenna design [8, 24, 20, 1], spectroscopy [4, 7], fiber optic systems [3] and quantum theory [12, 23, 14]. Another application of sine integral function is least squares slant stack, also referred to as $\tau-p$ transform. $\tau-p$ transform is related with Radon transform and is used for image reconstruction in medical, radar, geophysical and other imaging modalities [2, 18].

Computer implementations of the sine integral function are based on the spherical Bessel function expansion [1], polynomial (Taylor or Chebyshev) expansions, continued fraction expansions and asymptotic approximations [2] [5, 10, 22, 16, 13, 19].

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[1] `http://sourceforge.net/p/octave/specfun/ci/03e27dc96440686b9e5ae529a66d79b614559d39/tree/inst/Si.m`

[2] `http://technicalc.org/tiplist/en/files/pdf/tips/tip6_33.pdf`

MATLAB's implementation [3] of the sine integral function, `sinint`, claims to depend on these approaches. We used this implementation to benchmark the accuracy of our method.

In our work, we took an alternative approach to the aforementioned methods. For a chosen analytic even function $g(x)$, our method is based on both an approximation of the sinc by a sum of scaled versions of $g(x)$, over an interval centered around zero and a scaling property of the sinc function,

$$\operatorname{sinc}(3^n Bx) = \frac{1}{3}\left[2\cos\left(2\,3^{n-1}Bx\right)+1\right]\operatorname{sinc}\left(3^{n-1}Bx\right).$$

First, we accurately approximate the sinc function in terms of a sum of scaled $g(x)$s

$$\operatorname{sinc}(x) \approx f(x) = \sum_{m=1}^{M}\alpha_m g(\gamma_m x), \tag{1.2}$$

over an interval centered around zero, using a generalization of Padé approximation from rational functions to analytic functions [25]. Then, we use a scaling property of the sinc function to extend this approximation to another approximation of the sinc on a desired interval,

$$\operatorname{sinc}(x) \approx \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos\left(2Bl\frac{x}{3^n B}\right)+1\right] f\left(\frac{x}{3^n B}\right).$$

We analytically integrate the resulting approximation of the sinc in order to obtain an approximation to the sine integral function.

Building up the approximation for the sine integral function doesn't require any tabulation of its values, but only the knowledge of the Taylor series expansions of the sinc and $g(x)$. For example, if we choose $g(x)=\cos(x)$, approximation of the sinc in terms of a sum of cosines can be integrated explicitly as a sum of sincs, and can therefore be evaluated using standard methods. Thus, the computational cost of evaluating our approximation of the Sine integral function (see Proposition (1)),

$$\operatorname{Si}(x) \approx x\frac{1}{3^n}\sum_{m=1}^{M}\alpha_m \sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\operatorname{sinc}\left((\theta_m+2k)\,3^{-n}x\right), \qquad \exists \alpha_m, \theta_m \in [0,1]$$

is linear in the number of terms in the sum, $3^n M$, times the number of points to be evaluated.

The choice of $g(x)$ not only impacts construction of the approximation (1.2), but also approximation of the sine integral function. If the Fourier transform of $g(x)$is known analytically then the approximation of the sine integral can be expressed as a sum of Fourier transforms of $g(x)$. For example, if we choose $g(x)=\mathrm{e}^{-x^2}$,

$$\operatorname{Si}(x) \approx \frac{3^n B}{3^n}\sum_{m=1}^{M}\sum_{l=-(3^n-1)/2}^{(3^n-1)/2}\frac{\alpha_m\sqrt{\pi}}{2\sqrt{\gamma_m}}\mathrm{e}^{-\frac{[2Bl]^2}{4\gamma_m}}\left[\begin{array}{l}\operatorname{erf}\left(\sqrt{\gamma_m}\frac{x}{3^n B}-\mathrm{i}\frac{Bl}{\sqrt{\gamma_m}}\right)\\ -\operatorname{erf}\left(\frac{\mathrm{i}2B}{2\sqrt{\gamma_m}}l\right)\end{array}\right], \qquad \exists \alpha_m, \gamma_m \in \mathbb{C},$$

whose computational cost for evaluation is also linear in the number of terms in the sum, $3^n M$, times the number of points to be evaluated.

[3] `http://www.mathworks.com/help/symbolic/sinint.html`

One of the standard approaches to approximating $\mathrm{Si}(x)$ is to evaluate its truncated spherical Bessel function expansion

$$\mathrm{Si}\left(x\right)=x\sum_{n=0}^{\infty}\left(j_{n}\left(\frac{x}{2}\right)\right)^{2}\approx x\sum_{n=0}^{N(x)}\left(j_{n}\left(\frac{x}{2}\right)\right)^{2},$$

where

$$j_{n}\left(x\right)=\left(-x\right)^{n}\left(\frac{1}{x}\frac{d}{dx}\right)^{n}\mathrm{sinc}\left(x\right).$$

An efficient implementation of spherical Bessel functions can be achieved through the recursion

$$j_{n}\left(x\right)=\frac{\left(2n-1\right)}{x}j_{n-1}\left(x\right)-j_{n-2}\left(x\right).$$

The proposed method may be considered as a viable option to approximate the sine integral function when $3^{n}M<N\left(x\right)$.

We compared accuracy our approach to MATLAB's `sinint` function and the difference is within the machine precision.

Organization of the paper is as follows. We derive the proposed method to compute the sine integral function and compared with the existing method in Section 2. We summarize our discussion in Section 3.

2. Computation of the sine integral function

In this section we present computation of the sine integral function through integration of an accurate approximation of the sinc function.

2.1. **Approximation of the sinc function.** A good approximation to $\mathrm{sinc}\left(x\right)$ within the vicinity of zero can be achieved by building up quadratures for the integral representation of the sinc with bandlimit B, $\mathrm{sinc}\left(Bx\right)$,

$$\mathrm{sinc}\left(Bx\right)=\frac{1}{B}\int_{0}^{B}\cos\left(\omega x\right)d\omega, \tag{2.1}$$

and then rescaling the approximation by $1/B$. One way to do this is using the method summarized in Appendix A to obtain

$$\mathrm{sinc}\left(Bx\right)=\sum_{m=1}^{M}\alpha_{m}\cos\left(\omega_{m}x\right)+\epsilon_{B}\left(x\right), \tag{2.2}$$

where $\left(\alpha_{m},\omega_{m}\right)$ satisfies the moment problem

$$h_{n}=\frac{f_{n}}{g_{n}}=\sum_{m=1}^{M}\alpha_{m}\omega_{m}^{2n}+\epsilon_{n} \tag{2.3}$$

for some small $\left|\epsilon_{n}\right|$ (see Figure 2.1 for $\left(\alpha_{m},\theta_{m}=B^{-1}\omega_{m}\right)$). Here

$$f_{n}=\tfrac{(-1)^{n}B^{2n}}{(2n+1)!},\quad g_{n}=\tfrac{(-1)^{n}}{(2n)!} \tag{2.4}$$

are the Taylor series coefficients of $\operatorname{sinc}(x)$ and $\cos(x)$ at zero, respectively, and

$$\epsilon_B(x) = \sum_{n=0}^{\infty} \epsilon_n x^{2n}. \tag{2.5}$$

Solution to the moment problem is equivalent to computing Gauss-Legendre quadratures.

The approximation given in equation (2.2) yields a highly accurate approximation to the $\operatorname{sinc}(Bx)$ in a neighborhood of zero (see Figure 2.2). However, due to the rapid increase in the values of the moments h_n for large bandlimit B, the construction of this approximation suffers from numerical instabilities, and therefore requires B to be in the range $0 < B \leq 2$. To overcome these challenges, approximation (2.2) can be coupled with a scaling property of the sinc, for example

$$\operatorname{sinc}(3^n Bx) = \frac{1}{3}\left[2\cos\left(2\,3^{n-1}Bx\right)+1\right]\operatorname{sinc}\left(3^{n-1}Bx\right), \tag{2.6}$$

to derive an error bound on the approximation of $\operatorname{sinc}(3^n Bx)$ in terms of the error in the approximation to the lower bandlimit $\operatorname{sinc}(Bx)$ as a sum of scaled cosines:

Proposition 1. *Let*

$$\epsilon_B(x) = \operatorname{sinc}(Bx) - \sum_{m=1}^{M} \alpha_m \cos(B\theta_m x). \tag{2.7}$$

Then

$$\left|\operatorname{sinc}(3^n Bx) - \frac{1}{3^n}\sum_{m=1}^{M}\alpha_m \sum_{k=-(3^n-1)/2}^{(3^n-1)/2} \cos\left((\theta_m+2k)\,Bx\right)\right| \leq \left|\epsilon_B(x)\right|, \tag{2.8}$$

for $n \geq 0$.

Proof. For $n = 0$, this is trivial by assumption (2.7).

Let us define

$$\epsilon_{3^n B}(x) = \operatorname{sinc}(3^n Bx) - \frac{1}{3^n}\sum_{m=1}^{M}\alpha_m \sum_{k=-(3^n-1)/2}^{(3^n-1)/2} \cos\left((\theta_m+2k)\,Bx\right). \tag{2.9}$$

For $n = 1$, substituting (2.2) into the scaling property (2.6), we obtain (2.13)

(2.10)

$$\operatorname{sinc}(3Bx) = \frac{1}{3}\sum_{m=1}^{M}\alpha_m \sum_{k=-1}^{1}\cos\left(B\left(\theta_m - 2k\right)x\right) + \frac{1}{3}\left[2\cos(2Bx)+1\right]\epsilon_B(x),$$

implying

$$\left|\epsilon_{3B}(x)\right| = \left|\frac{1}{3}\left[2\cos(2Bx)+1\right]\epsilon_B(x)\right| \leq \left|\epsilon_B(x)\right|. \tag{2.11}$$

Multiplying $\epsilon_{3^n B}(x)$ by $\frac{1}{3}\left[2\cos\left(2\,3^n Bx\right)+1\right]$, we have

$$
\begin{aligned}
&\frac{1}{3}\left[2\cos\left(2\,3^n Bx\right)+1\right]\epsilon_{3^n B}(x)\\
&=\frac{1}{3}\left[2\cos\left(2\,3^n Bx\right)+1\right]\operatorname{sinc}\left(3^n Bx\right)\\
&\qquad-\frac{1}{3^{n+1}}\sum_m \alpha_m\left[2\cos\left(2\,3^n Bx\right)+1\right]\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\cos\left(\left(\theta_m+2k\right)Bx\right)\\
&=\operatorname{sinc}\left(3^{n+1}Bx\right)\\
&\qquad-\frac{1}{3^{n+1}}\sum_m \alpha_m\left[\begin{array}{c}\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\cos\left(\left(\theta_m+2k\right)Bx\right)\\+\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}2\cos\left(2\,3^n Bx\right)\cos\left(\left(\theta_m+2k\right)Bx\right)\end{array}\right]\\
&=\operatorname{sinc}\left(3^{n+1}Bx\right)\\
&\qquad-\frac{1}{3^{n+1}}\sum_m \alpha_m\left[\begin{array}{c}\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\cos\left(\left(\theta_m+2k\right)Bx\right)\\+\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\cos\left(\left(\theta_m+2\left(k+3^n\right)\right)Bx\right)\\+\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\cos\left(\left(\theta_m+2\left(k-3^n\right)\right)Bx\right)\end{array}\right]\\
&=\operatorname{sinc}\left(3^{n+1}Bx\right)\\
&\qquad-\frac{1}{3^{n+1}}\sum_m \alpha_m\left[\begin{array}{c}\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\cos\left(\left(\theta_m+2k\right)Bx\right)\\+\sum_{k=(3^n+1)/2}^{\left(3^{n+1}-1\right)/2}\cos\left(\left(\theta_m+2k\right)Bx\right)\\+\sum_{k=-(3^{n+1}-1)/2}^{-(3^n+1)/2}\cos\left(\left(\theta_m+2\left(k-3^n\right)\right)Bx\right)\end{array}\right]\\
&=\operatorname{sinc}\left(3^{n+1}Bx\right)-\frac{1}{3^{n+1}}\sum_m \alpha_m\left[\sum_{k=-(3^{n+1}-1)/2}^{\left(3^{n+1}-1\right)/2}\cos\left(\left(\theta_m+2k\right)Bx\right)\right]\\
&=\epsilon_{3^{n+1}B}(x)
\end{aligned}
$$

which implies

$$
\left|\epsilon_{3^{n+1}B}(x)\right|\leq\left|\epsilon_{3^n B}(x)\right|\leq\ldots\leq\left|\epsilon_{3B}(x)\right|\leq\left|\epsilon_B(x)\right|. \tag{2.12}
$$

□

The practical implications of Proposition 1 are as long as one has a good approximation $\operatorname{sinc}(x)\approx\sum_m \alpha_m\cos\left(\theta_m x\right)$ over an interval around zero, the $\left(\alpha_m,\theta_m\right)$ can be used to build up an approximation of (i) $\operatorname{sinc}(Bx)$, for any $B\in\mathbb{R}$, on the same interval, (ii) $\operatorname{sinc}(x)$ on any interval around zero or, equivalently, (iii) $\operatorname{sinc}(Bx)$, for any $B\in\mathbb{R}$, on any interval around zero, without compromising the accuracy as demonstrated in Figure 2.2.

In the case of an approximation computed in the form of (2.2), we set $\theta_m=B^{-1}\omega_m$ and approximate $\operatorname{sinc}\left(3^n Bx\right)$ by

$$
\operatorname{sinc}\left(3^n Bx\right)=\frac{1}{3^n}\sum_{m=1}^{M}\alpha_m\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\cos\left(\left(\theta_m+2k\right)Bx\right)+\epsilon_{3^n B}(x). \tag{2.13}
$$

Algorithm 1 outlines our approach to approximating the sinc with bandlimit $B_0=3^n B$, $\operatorname{sinc}\left(B_0 x\right)$, as a sum of scaled cosines. For $B_0=20$, we have $B=3^{-3}B_0\approx 0.\overline{740}$ and corresponding $\left(\alpha_m,\theta_m\right)$s obtained by following steps 3

m	α_m	θ_m
1	0.1365860988822261400	0.0684516770505271270
2	0.1328501583243817400	0.2034709824425706400
3	0.1257484892855010600	0.3330267845850311500
4	0.1159412602086237100	0.4540643786894249000
5	0.1042427058462060400	0.5642788494029894500
6	0.0914720434565549360	0.6621937832095096100
7	0.0783322249687766340	0.7471020397323778800
8	0.0653386483333428520	0.8189093330986442200
9	0.0528000037137510840	0.8779330633710573700
10	0.0408403075611107540	0.9247038760439186800
11	0.0294443474232695750	0.9598025622687430500
12	0.0185086245736669070	0.9837464401871046600
13	0.0078950874225888028	0.9969261496638826000

TABLE 1. For $B = 3^{-3}20$,(α_m, θ_m) obtained by following step 3 of Algorithm 1. The plot of $(\alpha_m, B\theta_m)$ is presented in Figure (2.1).

Algorithm 1 Representation of $\operatorname{sinc}(B_0 x)$ as a sum of scaled cosines

Given $0 \leq B_0 \in \mathbb{R}$

(1) Compute $n = \log_3 \lfloor B_0 \rfloor + 1$
(2) Set $B = B_0 3^{-n}$.
(3) Solve the moment problem

$$h_n = B^{2n} (2n+1)^{-1} = \sum_{m=1}^{M} \alpha_m \left(\omega_m^2\right)^n + \epsilon_n$$

for (α_m, ω_m) using the method of [25] (see Appendix A and Figure 2.1)
(4) Set $\theta_m = \omega_m B^{-1}$
(5) Form the approximation (see Figure 2.2)

$$\operatorname{sinc}(B_0 x) \approx \frac{1}{3^n} \sum_{m=1}^{M} \alpha_m \sum_{k=-(3^n-1)/2}^{(3^n-1)/2} \cos\left((\theta_m + 2k) Bx\right)$$

and 4 of Algorithm 1 are shown in Table 1. Plots for the approximations $\operatorname{sinc}(Bx)$ and $\operatorname{sinc}(B_0 x)$ are presented in Figure 2.2.

Remark on uniform sampling: Consider approximation of the integral

$$f(x) = \frac{1}{2B} \int_{-B}^{B} e^{ix\omega} d\omega = \operatorname{sinc}(Bx) \tag{2.14}$$

by discretization of the integral using uniform sampling over the interval $[-B, B]$:

$$\tilde{f}_{B,N}(x) = \frac{1}{2B} \frac{2B}{2N+1} \sum_{n=-N}^{N} e^{ix\frac{2B}{2N+1}n} = \frac{1}{2N+1} \frac{\sin(Bx)}{\sin\left(\frac{Bx}{2N+1}\right)} \tag{2.15}$$

Because $\tilde{f}_{B,N}(x)$ is periodic with period $\pi B^{-1}(2N+1)$, it is also referred to as periodic sinc function.

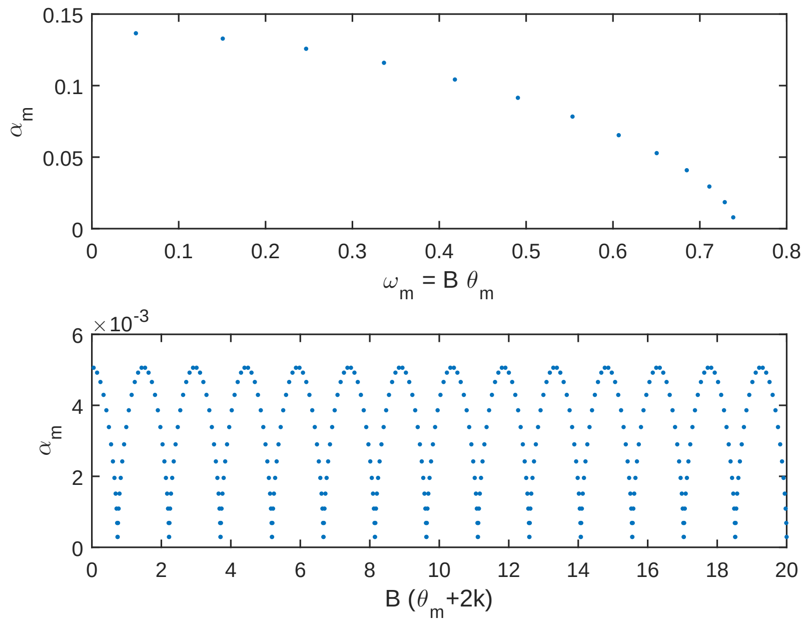

FIGURE 2.1. On top plot, solution (α_m, ω_m) to the moment problem (2.3) for $B = 3^{-3}B_0$ and $B_0 = 20$ which is used to approximate $\text{sinc}\,(Bx)$ (see top plot in Figure 2.2). On the bottom plot, $(\alpha_m, |\theta_m + 2k|\, B)_{k=-(3^3-1)/2}^{(3^3-1)/2}$ used to approximatesinc $(B_0 x)$ (see middle plot in Figure 2.2).

Let us consider the first period of $\tilde{f}_{B,N}(x)$ centered around zero, i.e. $|x| \leq \pi (2B)^{-1}(2N+1)$. Using the Taylor series expansion of $(1-x)^{-1}$ and $\text{sinc}\,(x)$ around zero [4] series representation of the error becomes

$$\begin{aligned}
\epsilon_B(x) &= f(x) - \tilde{f}_{B,N}(x) \\
&= \frac{\sin(Bx)}{Bx}\left(1 - \frac{1}{1 - \left[1 - \text{sinc}\left(\frac{Bx}{2N+1}\right)\right]}\right) \\
&= \text{sinc}\,(Bx) \sum_{m=1}^{\infty} (-1)^{m+1} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{B}{2N+1}x\right)^{2n}\right]^m
\end{aligned} \tag{2.16}$$

which decays like $\mathcal{O}\left((2N+1)^{-2}\right)$ within the vicinity of zero and increases away from zero for $|x| \leq \pi (2B)^{-1}(2N+1)$. Consequently, maximum absolute error for

[4] $(1-x)^{-1} = \sum_{m=0}^{\infty} x^m$ and $\text{sinc}\,(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$

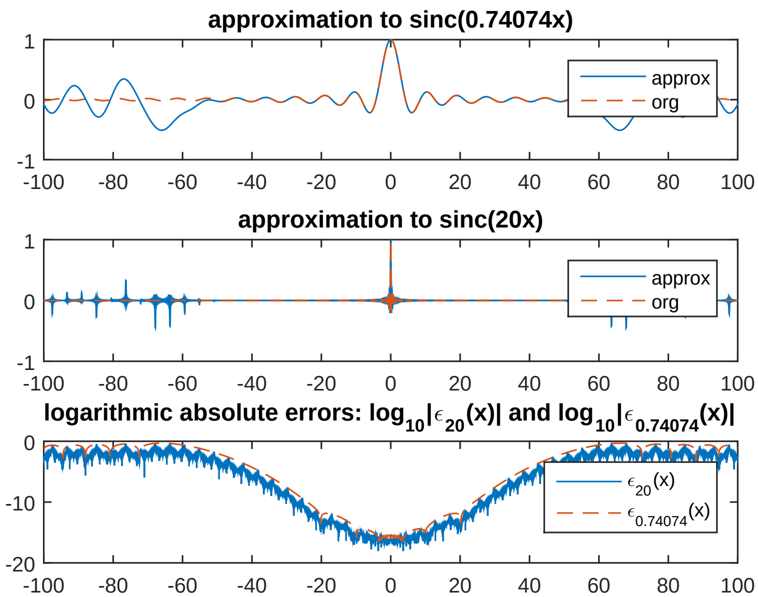

FIGURE 2.2. Approximation of sinc (Bx) by (2.2) using Algorithm 1. On top and middle plots, sinc (Bx) and sinc (B_0x) (red dashed) along with their approximations (solid blue), for $B = 3^{-3}20$ and $B_0 = 20$, respectively. On the bottom plot, the logarithmic absolute errors for B (red dashed) and B_0(blue solid). As derived, the error corresponding to B_0 is less than that of B.

for $|x| \leq \pi (2B)^{-1} (2N+1)$ is obtained at $|x| = \pi (2B)^{-1} (2N+1)$ which is

$$\begin{aligned} \left|\epsilon_B\left(\frac{(2N+1)\pi}{2B}\right)\right| &= \left|\operatorname{sinc}\left(\frac{(2N+1)\pi}{2}\right)\left(1-\operatorname{sinc}\left(\frac{\pi}{2}\right)^{-1}\right)\right| \\ &= \frac{2}{(2N+1)\pi}\left(\frac{\pi}{2}-1\right) \end{aligned} \tag{2.17}$$

and decays in the order of N. For $(N, B) = \left(13, 20 \times 3^{-3}\right)$ and $(N_0, B_0) = \left(13 \times 3^3, 20\right)$ we present $\tilde{f}_{B,N}(x)$ and $\tilde{f}_{B_0,N_0}(x)$ in Figure 2.3. (2.16) and (2.17) indicate that integral of (2.15) will suffer from accuracy, even in the vicinity of zero, unless N is very large. This makes uniform sampling inferior to Gauss-Legendre quadratures, which achieve machine precision within a larger vicinity of zero for the same number of quadrature points as demonstrated in Figures 2.2 and 2.3.

Proposition 1 can be generalized to an arbitrary function $f(x)$ with the same error behavior.

Theorem 2. *Let*

$$\epsilon_B(x) = \operatorname{sinc}(Bx) - f(x). \tag{2.18}$$

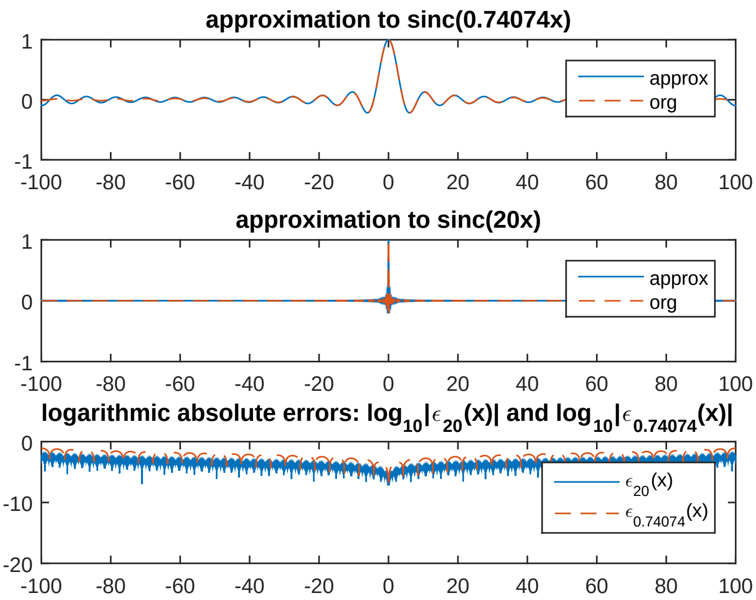

FIGURE 2.3. Approximation of $\mathrm{sinc}\,(Bx)$ by (2.15). On top and middle plots, $\mathrm{sinc}\,(Bx)$ and $\mathrm{sinc}\,(B_0x)$ (red dashed) along with their approximations $\tilde{f}_{B,N}\,(x)$ and $\tilde{f}_{B_0,N_0}\,(x)$ (solid blue) using uniform sampling, for $(B,N) = \left(3^{-3}20, 13\right)$ and $(B_0, N_0) = \left(20, 3^3 13\right)$, respectively. On the bottom plot, the logarithmic absolute errors for B (red dashed) and B_0(blue solid). As derived the error corresponding to B_0 is less than that of B.

Then

$$\left|\mathrm{sinc}\,(3^n Bx) - \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos\left(2Blx\right)+1\right] f\left(x\right)\right| \;\le\; \left|\epsilon_B\left(x\right)\right|, \quad \textit{for } n \ge 0. \tag{2.19}$$

Proof. For $n = 0$, this is trivial by assumption (2.18).

Let us define

$$\epsilon_{3^n B}\left(x\right) = \mathrm{sinc}\,(3^n Bx) - \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos\left(2Blx\right)+1\right] f\left(x\right). \tag{2.20}$$

We next show that the inequality holds for $n = 1$, and then proove that the sequence $\epsilon_{3^n B}(x)$ is monotonically nonincreasing.

For $n = 1$, substituting (2.18) into the scaling property (2.6), we obtain

$$\mathrm{sinc}\,(3Bx) \;=\; \frac{1}{3}\left[2\cos\left(2Bx\right)+1\right] f\left(x\right) \;+\; \frac{1}{3}\left[2\cos\left(2Bx\right)+1\right]\epsilon_B\left(x\right), \tag{2.21}$$

implying

$$|\epsilon_{3B}(x)| = \left|\frac{1}{3}\left[2\cos(2Bx)+1\right]\epsilon_B(x)\right| \le |\epsilon_B(x)|. \tag{2.22}$$

Multiplying $\epsilon_{3^nB}(x)$ by $\frac{1}{3}\left[2\cos\left(2\,3^nBx\right)+1\right]$ and using the identity

$$2\cos(a)\cos(b) = \cos(a+b)+\cos(a-b),$$

we have

(2.23)

$$\begin{aligned}
&\frac{1}{3}\left[2\cos\left(2\,3^nBx\right)+1\right]\epsilon_{3^nB}(x)\\
&= \operatorname{sinc}\left(3^{n+1}Bx\right) - \frac{1}{3^{n+1}}\left[2\cos\left(2\,3^nBx\right)+1\right]\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)+1\right] f(x)\\
&= \operatorname{sinc}\left(3^{n+1}Bx\right) - \frac{1}{3^{n+1}}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)\left[2\cos\left(2\,3^nBx\right)+1\right]+1\right] f(x)\\
&= \operatorname{sinc}\left(3^{n+1}Bx\right) - \frac{1}{3^{n+1}}\left[\begin{array}{l}\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)\,2\cos\left(2\,3^nBx\right)\\ +\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)+1\end{array}\right] f(x)\\
&= \operatorname{sinc}\left(3^{n+1}Bx\right) - \frac{1}{3^{n+1}}\left[\begin{array}{l}\sum_{l=1}^{(3^n-1)/2}\left(2\left[\begin{array}{l}\cos\left(2B\left[3^n+l\right]x\right)\\ +\cos\left(2B\left[3^n-l\right]x\right)\end{array}\right]\right)\\ +\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)+1\end{array}\right] f(x)\\
&= \operatorname{sinc}\left(3^{n+1}Bx\right) - \frac{1}{3^{n+1}}\left[\begin{array}{l}\sum_{l=(3^n+1)/2}^{\left(3^{n+1}-1\right)/2} 2\cos(2Blx)\\ +\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)+1\end{array}\right] f(x)\\
&= \operatorname{sinc}\left(3^{n+1}Bx\right) - \frac{1}{3^{n+1}}\left[\sum_{l=1}^{\left(3^{n+1}-1\right)/2} 2\cos(2Blx)+1\right] f(x)\\
&= \epsilon_{3^{n+1}B}(x).
\end{aligned}$$

which implies

$$|\epsilon_{3^{n+1}B}(x)| \le |\epsilon_{3^nB}(x)| \le \ldots \le |\epsilon_{3B}(x)| \le |\epsilon_B(x)|. \tag{2.24}$$

□

Corollary 3. *The error is given by*

$$\epsilon_{3^{n+1}B}(x) = \sum_{l=-\infty}^{\infty} \operatorname{sinc}\left(3^{n+1}B\left[x-\frac{\pi l}{B}\right]\right)\epsilon_B(x) \tag{2.25}$$

which for $x = m\pi/B$ becomes

$$\epsilon_{3^{n+1}B}\left(\frac{m\pi}{B}\right) = \epsilon_B\left(\frac{m\pi}{B}\right). \tag{2.26}$$

Furthermore,

$$\lim_{n\to\infty} 3^{n+1}2B\,\epsilon_{3^{n+1}B}(x) = \sum_{l=-\infty}^{\infty}\delta\left(x-\frac{\pi l}{B}\right)\epsilon_B(x). \tag{2.27}$$

Proof. We showed in Theorem 2 that the error satisfies the scaling property (2.23). Using this, we can write

$$\begin{aligned}\epsilon_{3^{n+1}B}(x) &= \frac{1}{3}\left[2\cos\left(2\,3^{n}Bx\right)+1\right]\epsilon_{3^{n}B}(x)\\ &= \frac{1}{3^{2}}\left[2\cos\left(2\,3^{n-1}Bx\right)+1\right]\left[2\cos\left(2\,3^{n}Bx\right)+1\right]\epsilon_{3^{n-1}B}(x)\\ &= \frac{1}{3^{n+1}}\prod_{l=1}^{n}\left[2\cos\left(2\,3^{l}Bx\right)+1\right]\epsilon_{B}(x) \qquad (2.28)\end{aligned}$$

which in the Fourier domain can be written as

$$\begin{aligned}\hat{\epsilon}_{3^{n+1}B}(k) &= \frac{1}{3^{n+1}}\left(*_{l=0}^{n}\left[\begin{array}{c}\delta\left(k-2\,3^{l}B\right)\\ +\delta\left(k+2\,3^{l}B\right)+\delta(k)\end{array}\right]\right)*\hat{\epsilon}_{B}(k)\\ &= \left(\frac{\chi_{[-3^{n+1}B,3^{n+1}B]}(k)}{3^{n+1}2B}2B\sum_{l=-\infty}^{\infty}\delta\left(k-2Bl\right)\right)*\hat{\epsilon}_{B}(k) \qquad (2.29)\end{aligned}$$

where

$$\hat{\epsilon}_{B}(k) = (2\pi)^{-1}\int\epsilon_{B}(x)\,\mathrm{e}^{\mathrm{i}kx}dx \qquad (2.30)$$

is the inverse Fourier transform of $\epsilon_{B}(x)$ and $*_{l=0}^{n}f_{l}(k) = \left(f_{0}*f_{1}*\cdots*f_{n}\right)(k)$ denotes a cascaded convolution operator. Taking the inverse Fourier transform, we obtain

$$\begin{aligned}\epsilon_{3^{n+1}B}(x) &= \left(\operatorname{sinc}\left(3^{n+1}Bx\right)*\sum_{l=-\infty}^{\infty}\delta\left(x-\frac{2\pi l}{2B}\right)\right)\epsilon_{B}(x)\\ &= \sum_{l=-\infty}^{\infty}\operatorname{sinc}\left(3^{n+1}B\left[x-\frac{\pi l}{B}\right]\right)\epsilon_{B}(x) \qquad (2.31)\end{aligned}$$

which for $x = m\pi/B$ is

$$\begin{aligned}\epsilon_{3^{n+1}B}\left(\frac{m\pi}{B}\right) &= \sum_{l=-\infty}^{\infty}\operatorname{sinc}\left(3^{n+1}\pi\left[m-l\right]\right)\epsilon_{B}\left(\frac{m\pi}{B}\right)\\ &= \sum_{l=-\infty}^{\infty}\delta_{ml}\epsilon_{B}\left(\frac{m\pi}{B}\right). \qquad (2.32)\end{aligned}$$

By using the identity

$$\lim_{a\to\infty}2a\operatorname{sinc}(ax) = \delta(x) \qquad (2.33)$$

and the dominated convergence theorem (see page 14 of [9]), we have

$$\begin{aligned}\lim_{n\to\infty}3^{n+1}2B\,\epsilon_{3^{n+1}B}(x) &= \lim_{n\to\infty}\sum_{l=-\infty}^{\infty}3^{n+1}2B\operatorname{sinc}\left(3^{n+1}B\left[x-\frac{\pi l}{B}\right]\right)\epsilon_{B}(x)\\ &= \sum_{l=-\infty}^{\infty}\delta\left(x-\frac{\pi l}{B}\right)\epsilon_{B}(x). \qquad (2.34)\end{aligned}$$

□

The practical implications of Theorem 2 and Corollary 3 are same as Proposition 1. As long as $f(x) \approx \operatorname{sinc}(x)$ accurately over an interval around zero, it can be used to build up an approximation $\operatorname{sinc}(Bx)$, for any $B \in \mathbb{R}$, on any interval around zero, at worst with the same accuracy obtained around zero.

Example 4. Let

$$\epsilon_B(x) = \operatorname{sinc}(Bx) - \sum_{m=1}^{M} \alpha_m \exp\left(-\gamma_m x^2\right), \tag{2.35}$$

such that $\operatorname{Re}\{\gamma_m\} > 0$. Then

$$\left| \operatorname{sinc}(3^n Bx) - \frac{1}{3^n} \sum_{m=1}^{M} \sum_{l=-(3^n-1)/2}^{(3^n-1)/2} a_{m,l} g_{m,l}(x) \right| \leq |\epsilon_B(x)|, \text{ for } n \geq 0. \tag{2.36}$$

where $a_{m,l} = \alpha_m \exp\left(\mathrm{i}\frac{(Bl)^2}{\operatorname{Im}\{\gamma_m\}}\right)$ and

$$g_{m,l}(x) = \exp\left(-\operatorname{Re}\{\gamma_m\} x^2\right) \exp\left(-\mathrm{i}\operatorname{Im}\{\gamma_m\}\left(x - \frac{Bl}{\operatorname{Im}\{\gamma_m\}}\right)^2\right). \tag{2.37}$$

Furthermore,

$$\|\epsilon_B\|^2 = \frac{1}{4B} - 2\operatorname{Re}\left\{\sum_{m=1}^{M} \alpha_m \frac{1}{4B} \operatorname{erf}\left(\frac{2B\pi}{2\sqrt{\gamma_m}}\right)\right\} + \frac{\sqrt{\pi}}{2} \sum_{m,m'=1}^{M} \frac{\alpha_m \alpha_{m'}}{\sqrt{\gamma_m + \gamma_{m'}}} \tag{2.38}$$

Proof. (2.36) is a direct consequence of Theorem 2 and identities

$$\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx) + 1\right] \exp\left(-\gamma_m x^2\right) = \sum_{l=-(3^{n+1}-1)/2}^{(3^{n+1}-1)/2} \exp\left(-\gamma_m x^2 + \mathrm{i}2Blx\right) \tag{2.39}$$

$$\begin{aligned}\exp\left(-\gamma_m x^2 + \mathrm{i}2Blx\right) \\ = \exp\left(-\operatorname{Re}\{\gamma_m\} x^2\right) \exp\left(-\mathrm{i}\operatorname{Im}\{\gamma_m\}\left(x - \frac{Bl}{\operatorname{Im}\{\gamma_m\}}\right)^2\right) \exp\left(\mathrm{i}\frac{(Bl)^2}{\operatorname{Im}\{\gamma_m\}}\right)\end{aligned} \tag{2.40}$$

For (2.38), using the identities

$$\int_0^\infty |\mathrm{sinc}\,(t)|^2\,dt = \frac{\pi}{2}$$

$$\int_0^\infty |\mathrm{sinc}\,(2B\pi t)|^2\,dt = \frac{1}{4B}$$

$$\int_0^\infty \mathrm{sinc}\,(t)\exp\left(-\gamma_m t^2\right)dt = \frac{\pi}{2}\mathrm{erf}\left(\frac{1}{2\sqrt{\gamma_m}}\right), \quad \mathrm{Re}\left\{\gamma_m\right\} > 0$$

$$\begin{aligned}\int_0^\infty \mathrm{sinc}\,(2B\pi t)\exp\left(-\gamma_m t^2\right)dt &= \frac{1}{2B\pi}\int_0^\infty \mathrm{sinc}\,(t)\exp\left(-\frac{\gamma_m}{(2B\pi)^2}t^2\right)dt \\ &= \frac{1}{4B}\mathrm{erf}\left(\frac{2B\pi}{2\sqrt{\gamma_m}}\right), \quad \mathrm{Re}\left\{\frac{\gamma_m}{(2B\pi)^2}\right\} > 0\end{aligned}$$

$$\int_0^\infty \exp\left(-\gamma_m t^2\right)\exp\left(-\gamma_n t^2\right)dt = \frac{\sqrt{\pi}}{2\sqrt{\gamma_m+\gamma_n}}, \quad \mathrm{Re}\left\{\gamma_m+\gamma_n\right\} > 0,$$

the least square error is derived to be

$$\begin{aligned}\|\epsilon_B\|^2 &= \int_0^\infty \left|\mathrm{sinc}\,(2B\pi t) - \sum_{m=1}^{M}\alpha_m \exp\left(-\gamma_m t^2\right)\right|^2 dt \\ &= \int_0^\infty |\mathrm{sinc}\,(2B\pi t)|^2\,dt \\ &\quad - 2\mathrm{Re}\left\{\sum_{m=1}^{M}\alpha_m\int_0^\infty \mathrm{sinc}\,(2B\pi t)\exp\left(-\gamma_m t^2\right)dt\right\} \\ &\quad + \sum_{m,m'=1}^{M}\alpha_m\left[\int_0^\infty \exp\left(-\gamma_m t^2\right)\exp\left(-\gamma_{m'} t^2\right)dt\right]\alpha_{m'} \\ &= \frac{1}{4B} - 2\mathrm{Re}\left\{\sum_{m=1}^{M}\alpha_m\frac{1}{4B}\mathrm{erf}\left(\frac{2B\pi}{2\sqrt{\gamma_m}}\right)\right\} + \frac{\sqrt{\pi}}{2}\sum_{m,m'=-M}^{M}\frac{\alpha_m\alpha_{m'}}{\sqrt{\gamma_m+\gamma_{m'}}}\end{aligned}$$

□

Example 4 considers an approximation of sinc as a sum of shifted, Gaussian tapered chirps also referred to as chirplets. One can determine (α_m, γ_m) using the method in Appendix A by solving the appropriate moment problem (see Step 3 of Algorithm 2). This type of approximation of $\mathrm{sinc}\,(x)$ can be used to construct a multiresolution scheme for bandlimited function as an alternative to existing multiscale approaches. It is important to point out that unlike chirplet decomposition methods presented in [17, 6], the moment problem provides an explicit solution for (α_m, γ_m) while coupling the real and imaginary part of the complex Gaussian parameters γ_m.

Algorithm 2 outlines approximating the sinc with arbitrary bandlimit as a sum of scaled chirplets based on the moment problem, Corollary 4 and scaling property (2.6). For $B_0 = 3^n B = 20$, we have $B = 3^{-3}B_0 \approx 0.\overline{740}$ and corresponding $(\alpha_{m,}\gamma_m)$s obtained by following step 3 of Algorithm 2 are shown in Table 2. Plots for the approximations $\mathrm{sinc}\,(Bx)$ and $\mathrm{sinc}\,(B_0 x)$ are presented in Figure 2.4.

Algorithm 2 Representation of $\operatorname{sinc}(B_0 x)$ as a sum of chirplets

Given $0 \leq B_0 \in \mathbb{R}$

(1) Compute $n = \log_3 \lfloor B_0 \rfloor + 1$
(2) Set $B = B_0 3^{-n}$.
(3) Solve the moment problem

$$h_n = B^{2n} \frac{n!}{(2n+1)!} = \sum_{m=1}^{M} \alpha_m \gamma_m^n + \epsilon_n$$

for (α_m, γ_m) using the method of [25] (see Appendix A)

(4) Form the approximation

$$\operatorname{sinc}(B_0 x) \approx \frac{1}{3^n} \sum_{m=1}^{M} \sum_{l=-(3^n-1)/2}^{(3^n-1)/2} \alpha_m \left[\begin{array}{l} \exp\left(\mathrm{i}\frac{(Bl)^2}{\operatorname{Im}\{\gamma_m\}}\right) \exp\left(-\operatorname{Re}\{\gamma_m\} x^2\right) \\ \times \exp\left(\begin{array}{l} -\mathrm{i}\operatorname{Im}\{\gamma_m\} \\ \times \left(x - \frac{Bl}{\operatorname{Im}\{\gamma_m\}}\right)^2 \end{array} \right) \end{array} \right]$$

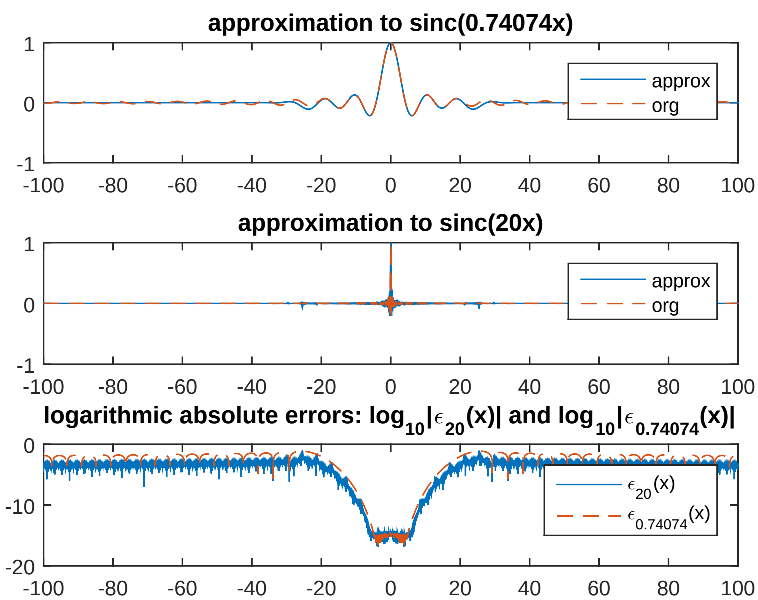

FIGURE 2.4. Approximation of $\operatorname{sinc}(Bx)$ as a sum of chirplets (see Corollary 4) using Algorithm 2. On top and middle plots, $\operatorname{sinc}(B\,x)$ and $\operatorname{sinc}(B_0 x)$ (red dashed) along with their approximations (solid blue), for $B = 3^{-3}20$ and $B_0 = 20$, respectively. On the bottom plot, the logarithmic absolute errors for B (red dashed) and B_0(blue solid). As derived the error corresponding to B_0 is less than that of B.

m	α_m	γ_m
1	0.157507376276801 - 2.229296828214125i	0.006498718817695 + 0.012418133575209i
2	0.157507376276801 + 2.229296828214125i	0.006498718817695 - 0.012418133575209i
3	3.531584324100828 - 60.900996460159689i	0.016429304202308 + 0.003449976157238i
4	3.531584324100828 + 60.900996460159689i	0.016429304202308 - 0.003449976157238i
5	-3.189091700377720 + 22.447610985235745i	0.012851888214297 + 0.009360692052044i
6	-3.189091700377720 - 22.447610985235745i	0.012851888214297 - 0.009360692052044i

TABLE 2. For $B = 3^{-2}20$,(α_m, θ_m) obtained by following steps 3 of Algorithm 2.

2.2. **Approximation of the sine integral function.**

Proposition 5. *Given the approximations for* $\operatorname{sinc}(Bx)$ *and* $\operatorname{sinc}(3^n Bx)$ *in Theorem 2, for* $n \geq 0$*, we can approximate the sine integral function,* $\operatorname{Si}(x)$ *by*

$$\operatorname{Si}(x) = x\frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2F_l\left(\frac{x}{3^n}\right) + F_0\left(\frac{x}{3^n}\right)\right] + \epsilon_{\mathrm{Si}}(x) \tag{2.41}$$

where

$$F_l(x) = \int_0^1 \cos(2l\omega x) f(x\omega)\, d\omega \tag{2.42}$$

and

$$\epsilon_{\mathrm{Si}}(x) = x\int_0^1 \epsilon_{3^n B}\left(\frac{x\omega}{3^n B}\right) d\omega \tag{2.43}$$

satisfies

$$\left|\epsilon_{Si}(y)\right|_{y\in[0,x]} \leq x \max_{y\in\left[0,(3^n B)^{-1}x\right]} |\epsilon_{3^n B}(y)| \leq x \max_{y\in\left[0,(3^n B)^{-1}x\right]} |\epsilon_B(y)|\,. \tag{2.44}$$

for $x \geq 0$.

Proof. By Theorem 2, we have

$$\operatorname{sinc}(x) = \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos\left(2l\frac{x}{3^n}\right) + 1\right] f\left(\frac{x}{3^n B}\right) + \epsilon_{3^n B}\left(\frac{x}{3^n B}\right). \tag{2.45}$$

Rewriting (1.1) as

$$\operatorname{Si}(x) = x\int_0^1 \operatorname{sinc}(x\omega)\, d\omega. \tag{2.46}$$

and substituting (2.45) into (2.46), we obtain (2.41). □

If $f(x)$ approximates $\operatorname{sinc}(x)$ within machine precision within the vicinity of zero, by choosing n such that $(3^n B)^{-1} x$ is small, we can control the error $\epsilon_{\mathrm{Si}}(x)$ to be sufficiently small for x that are smaller than a fraction of one over the machine precision. On the other hand, the larger the n, the more the number of cosines needed to approximate the sinc and the longer the computation time to evaluate the approximation.

Approximating sine integral function as a sum of sincs. Substituting the approximation (2.8) in (2.46), we obtain the approximation for the sine integral function

$$\begin{aligned}
\mathrm{Si}(x) &= x\frac{1}{3^n}\sum_{m=1}^{M}\alpha_m\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\left[\int_0^1\cos\left((\theta_m+2k)\frac{x\omega}{3^n}\right)d\omega\right]+\epsilon_{\mathrm{Si}}(x)\\
&= x\frac{1}{3^n}\sum_{m=1}^{M}\alpha_m\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\frac{\sin\left((\theta_m+2k)3^{-n}x\right)}{(\theta_m+2k)3^{-n}x}+\epsilon_{\mathrm{Si}}(x)\\
&= x\frac{1}{3^n}\sum_{m=1}^{M}\alpha_m\sum_{k=-(3^n-1)/2}^{(3^n-1)/2}\mathrm{sinc}\left((\theta_m+2k)3^{-n}x\right)+\epsilon_{\mathrm{Si}}(x) \qquad (2.47)
\end{aligned}$$

Here, $\epsilon_{\mathrm{Si}}(x)$ is given by (2.43) and satisfies the bounds (2.44).

In our implementation, we chose $3^nB=\max(x)/5$. We present the result of our approximation evaluated at ten thousand uniformly sampled points between zero and one hundred (i.e. $x\in[0,100]$ and, consequently, $3^nB=20$). We compared our result with MATLAB 2014a Symbolic Toolbox's `sinint` and saw that our approximation of sine integral function is as accurate as MATLAB's `sinint` function (see Figure 2.5).

Approximating sine integral function as a sum of complex error functions. Substituting the approximation (2.36) in (2.46), we obtain the approximation for the sine integral function

$$\begin{aligned}
\mathrm{Si}(x) &= x\int_0^1\mathrm{sinc}(x\omega)\,d\omega\\
&= x\int_0^1\left[\frac{1}{3^n}\sum_{m=1}^{M}\sum_{l=-(3^n-1)/2}^{(3^n-1)/2}a_{m,l}g_{m,l}\left([3^nB]^{-1}x\omega\right)d\omega+\epsilon_{3^nB}\left(\frac{x\omega}{3^nB}\right)\right],
\end{aligned}$$

where $a_{m,l}=\alpha_m\exp\left(\mathrm{i}\frac{(Bl)^2}{\mathrm{Im}\{\gamma_m\}}\right)$ and $g_{m,l}(x)$ is defined by (2.37), we obtain

(2.48)

$$\mathrm{Si}(x)=\frac{3^nB}{3^n}\sum_{m=1}^{M}\sum_{l=-(3^n-1)/2}^{(3^n-1)/2}\frac{\alpha_m\sqrt{\pi}}{2\sqrt{\gamma_m}}\exp\left(-\frac{[2Bl]^2}{4\gamma_m}\right)\left[\begin{array}{l}\mathrm{erf}\left(\sqrt{\gamma_m}\frac{x}{3^nB}-\mathrm{i}\frac{Bl}{\sqrt{\gamma_m}}\right)\\-\mathrm{erf}\left(\frac{\mathrm{i}2B}{2\sqrt{\gamma_m}}l\right)\end{array}\right]+\epsilon_{\mathrm{Si}}(x)$$

with $\mathrm{erf}(z)$ being the error function [19] defined by

$$\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z\mathrm{e}^{-t^2}dt,\quad z\in\mathbb{C}.$$

Here, $\epsilon_{\mathrm{Si}}(x)$ is given by (2.43) and satisfies the bounds (2.44).

In computation of error function of complex numbers, we used erfz function implemented by Marcel Leutenegger shared on MathWorks' File Exchange[5].

[5] Error function of complex numbers by Marcel Leutenegger. https://uk.mathworks.com/matlabcentral/fileexchange/18312-error-function-of-complex-numbers/content/@double/erfz.m

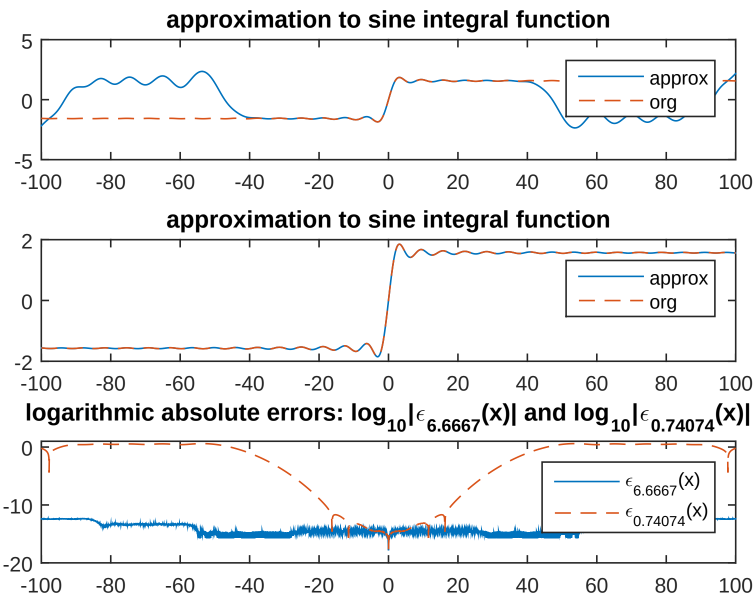

FIGURE 2.5. Approximation of Si (x) using (2.47) with (α_m, θ_m) of Table 1. On top and middle plots, Si (x) (red dashed) along with its approximations (solid blue) for $n = 0$ and $n = 2$, respectively. On the bottom plot, the logarithmic absolute errors for $n = 0$ (red dashed, labeled by $\epsilon_{0.74074}(x)$) and $n = 2$ (blue solid, labeled by $\epsilon_{6.6667}(x)$).

3. Conclusion

We derived a new method to compute the sine integral function. This method is based on approximation of the sinc function as a sum of scaled versions of an analytic function combined with a scaling property of the sinc. Approximation of sine integral function is obtained by analytically integrating the approximation of the sinc. Thus, evaluation of the sine integral function is linear with respect to the number of points to be evaluated, times the number of terms used in the sum that approximates the sinc function. We compared the accuracy of the proposed method with MATLAB's symbolic toolbox's sine integral function implementation, `sinint`. The proposed method is as accurate as `sinint`. The proposed method is similar to that of truncated spherical Bessel function expansion of the sine integral function, which can efficiently be implemented in a recursive fashion. It may be considered as an option to compute sine integral function when it acquires the comparable accuracy with less amount of computation.

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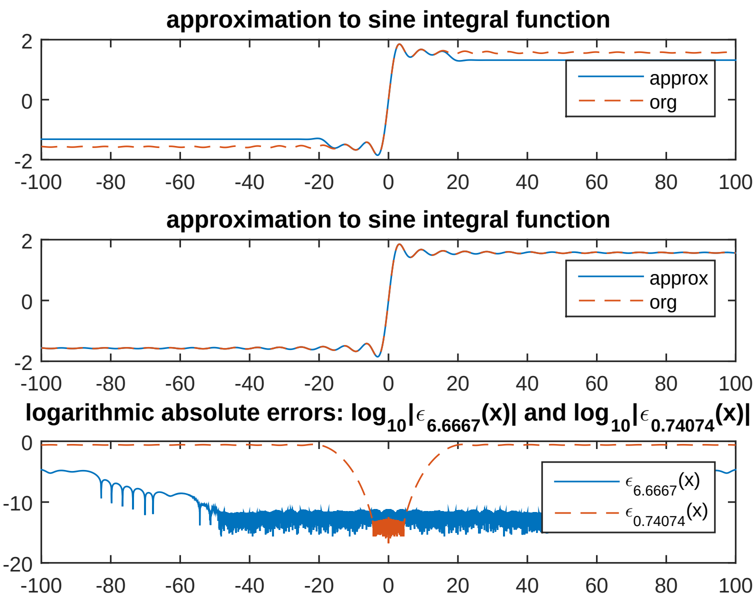

FIGURE 2.6. Approximation of $\mathrm{Si}\,(x)$ using (2.48) with (α_m, γ_m) of Table 2. On top and middle plots, $\mathrm{Si}\,(x)$ (red dashed) along with its approximations (solid blue) for $n = 0$ and $n = 2$, respectively. On the bottom plot, the logarithmic absolute errors for $n = 0$ (red dashed) and $n = 2$ (blue solid).

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Appendix A. Generalization of Padé approximation

Let $f(x)$ and $g(x)$ be two analytic functions related to each other by the Cauchy integral

$$f(x) = \int_\Gamma \rho(z)\, g(zx)\, dz \tag{A.1}$$

for some closed contour $\Gamma \in \mathbb{C}$ and a weighting function $\rho(z)$. A generalization of Padé approximation is achieved by finding a rational approximation to the weighting function

$$\rho(z) = \frac{1}{2\pi \mathrm{i}} \sum_m \frac{\alpha_m}{z - \gamma_m} + \epsilon_\rho(z), \quad z \in \mathbb{C} \tag{A.2}$$

for some distinct $\gamma_m \in \mathbb{C}$ and error $\epsilon_\rho(z)$. Then we refer to

$$f(x) = \sum_m \alpha_m g(\gamma_m x) + \epsilon(x) \tag{A.3}$$

as the generalization of Padé approximation from rational function to analytic functions, for some error function $\epsilon(x)$. Substituting the power series expansion of f and g at zero into (A.1)

$$f(x) = \sum_{m=0}^{\infty} f_n x^n = \int \rho(z) \left[\sum_{m=0}^{\infty} g_n (zx)^n \right] dz \tag{A.4}$$

$f(Bx)$	$g(x)$	f_{2n}	g_{2n}	Moment problem $h_n = \sum_m \alpha_m \gamma_m^{2n}$	Quadrature names
$\operatorname{sinc}(Bx)$	$\cos(x)$	$\frac{(-1)^n B^{2n}}{(2n+1)!}$	$\frac{(-1)^n}{(2n)!}$	$\frac{B^{2n}}{2n+1}$	Gauss-Legendre
$J_0(Bx)$	$\cos(x)$	$\frac{(-1)^n B^{2n}}{(2^n n!)^2}$	$\frac{(-1)^n}{(2n)!}$	$\frac{(2n)! B^{2n}}{(2^n n!)^2}$	Clenshaw-Curtis
$e^{-(Bx)^2}$	$\cos(x)$	$\frac{(-1)^n B^{2n}}{n!}$	$\frac{(-1)^n}{(2n)!}$	$\frac{(2n)! B^{2n}}{n!}$	Gauss-Hermite
$\operatorname{sinc}(Bx)$	$\exp(-x^2)$	$\frac{(-1)^n B^{2n}}{(2n+1)!}$	$\frac{(-1)^n}{n!}$	$\frac{(-1)^n B^{2n}}{(2n+1)!}$	
$J_0(Bx)$	$\operatorname{sinc}(x)$	$\frac{(-1)^n B^{2n}}{(2^n n!)^2}$	$\frac{(-1)^n}{(2n+1)!}$	$\frac{(2n+1)! B^{2n}}{(2^n n!)^2}$	
$(Bx)^{-1} J_1(Bx)$	$x^{-1}\operatorname{cosinc}(x)$	$\frac{(-1)^n}{2^{2n+1} n!(n+1)!}$	$\frac{(-1)^n}{(2n+2)!}$	$\frac{(2n+2)!}{2^{2n+1} n!\,(n+1)!}$	

TABLE 3. Example of moment problems corresponding to approximation of some even functions $f(Bx) \approx \sum_m \alpha_m g(\gamma_m x)$ in terms of other even functions $g(x)$, along with the known quadrature names (see [11]).

and equating the terms of the series, one obtains that the moments of $\rho(z)$ are given by the ratio of the power series coefficients, which we denote by h_n

$$\int \rho(z)\, z^n dz = h_n = \frac{f_n}{g_n} = \sum_{m=1}^{M} \alpha_m \gamma_m^n + \epsilon_n \tag{A.5}$$

for some error ϵ_n. Because (A.3) is a discrete approximation to the integral (A.1), (α_m, γ_m) are referred to as the quadratures. Individually, we refer to α_m and γ_m as weights and nodes, respectively. In [25], we presented the detailed theory of this generalization of Padé approximation and a method to compute the quadratures (α_m, γ_m) which is based on [15]. Some examples of moment problems are given in Table 3.

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