APPROXIMATING FRACTIONAL DERIVATIVE OF THE GAUSSIAN FUNCTION AND DAWSON'S INTEGRAL

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ABSTRACT. A new method for approximating fractional derivatives of the Gaussian function and Dawson's integral are presented. Unlike previous approaches, which are dominantly based on some discretization of Riemann-Liouville integral using polynomial or discrete Fourier basis, we take an alternative approach which is based on expressing the Riemann-Liouville definition of the fractional integral for the semi-infinite axis in terms of a moment problem. As a result, fractional derivatives of the Gaussian function and Dawson's integral are expressed as a weighted sum of complex scaled Gaussian and Dawson's integral. Error bounds for the approximation are provided. Another distinct feature of the proposed method compared to the previous approaches, it can be extended to approximate partial derivative with respect to the order of the fractional derivative which may be used in PDE constraint optimization problems.

1. INTRODUCTION

Fractional derivatives find applications in many branches of physics and engineering ranging from quantum optics to astrophysics and cosmology, dynamics of materials to biophysics and medicine, dynamical chaos to control, signal processing to communications and more. Increase in the application areas within the last decades [4] fueled investigation of numerical methods to compute fractional derivatives. For recent comprehensive reviews on fractional derivatives and their applications we refer the reader to [6, 14, 4] and references within.

Majority of the numerical methods for computation of fractional derivatives are based on polynomial expansions, finite difference operators or discretization of Riemann-Liouville integral over a finite interval [14, 3]. For the semi-infinite case, finite or infinite impulse response filters are designed to approximate the fractional derivative provided that the functions can be approximated using discrete convolution or discrete Fourier transforms[13, 10, 2, 1, 11, 12]. In this work we took an alternative approach to compute the fractional derivative of Gaussian function and its Hilbert transform, Dawson's integral. Our method is based on expressing the Riemann-Liouville definition of the fractional integral for the semi-infinite axis in terms of a moment problem. Error bounds for the approximation are derived. Another advantage of the proposed method to previous approaches, it can be extended to approximate partial derivatives with respect to the order of the fractional derivative which may be used in PDE constraint optimization.

The outline of the paper is as follows. Section 2 provides background on fractional integral and derivative operators. In Section 3, the method for approximating fractional derivatives of Gaussian function and Dawson's integral and corresponding error are derived. Finally in Section 4, approximating partial derivative with respect

to the order of fractional derivative is presented. Appendix A provides a rational approximation to the Dawson's integral which is used in our numerical results presented in Sections 3 and 4.

2. Fractional integral and derivative operators

Fractional derivative of an analytic function $f(t)$ can be thought as a generalization of Cauchy's integral formula

$$D^n f(z) = f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \int_C \frac{f(t)}{(t-z)^{n+1}} dt,$$

from integer n to a real number ν using generalization of factorial by Gamma function, $n! = \Gamma(n+1)$,

$$f^{(\nu)}(z) = \frac{\Gamma(\nu+1)}{2\pi\mathrm{i}} \int_C \frac{f(t)}{(t-z)^{\nu+1}} dt,$$

for some appropriate contour C in the complex plane. Selection of the contour is important because for non integer ν the integrand would contain a branch point rather than a pole. Instead of a closed contour, which was considered by Sonin and Letnikov, choosing the contour to be open, Laurent produced the Riemann-Liouville definition of the fractional integral [8]

$$_c D_x^{-v} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt, \quad \mathrm{Re}\{\nu\} > 0, (c,x) \subset \mathbb{R}.$$

Formally, one cannot replace $-\nu$ with ν to obtain the fractional derivative operator $_c D_x^\nu$ from the fractional integral operator because the integral would be divergent. However, writing $\nu = n - \mu$, where $n = \lceil \nu \rceil$ is the smallest integer larger than ν, by analytic continuation, differentiation for arbitrary order can be defined by $_c D_x^\nu = {}_c D_x^n {}_c D_x^{-\mu}$.

For $c = 0$, the fractional derivative and integral operator satisfies

$$_0 D_x^\nu x^a = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)} x^{a-\nu}, \quad a \geq 0, \nu \in \mathbb{R},$$

and for $c = -\infty$

(2.1) $$_{-\infty} D_x^\nu \mathrm{e}^{ax} = a^\nu \mathrm{e}^{ax}.$$

(2.1) is Leibniz's definition of fractional derivative (see 4.8.1. in [14]). When a is imaginary it is also referred to as the Fourier definition of fractional derivative (see 4.8.3. in [14]). In computation of the fractional derivative of the Gaussian function and Dawson's integral, we will consider the case for $c = -\infty$ and, for simplicity of notation, use d_x^ν to denote $_{-\infty} D_x^\nu$.

In the literature d_x^ν has been implemented using discrete convolutions and discrete Fourier transforms through design of finite or infinite impulse response filters [13, 10, 2, 1, 11, 12]. We take an alternative approach and get an analytic approximation to the fractional derivative of the Gaussian and Dawson's integral by exploiting their Hilbert transform relationship . Because fractional derivative is a linear operator, if a function can be approximated as a sum of Gaussian functions (for example as in [7]) and/or Dawson's integrals, then our method can be used to approximate its fractional derivative. The proposed method also enables analytic approximation of derivative with respect to the order of the fractional derivative of the functions.

3. Gaussian function, Dawson's integral and approximating their fractional derivatives

3.1. Gaussian function and Dawson's integral.

Dawson's integral $F(x)$ is defined by (see 7.2(ii) of [9])

$$F(x) = \frac{1}{2} \int_0^\infty e^{-t^2/4} \sin(xt)\, dt. \tag{3.1}$$

It is related to the Gaussian function e^{-t^2} through Hilbert transform (see pg 465, (3.5) in [5]):

$$\mathcal{H}\left[e^{-t^2}\right] = \frac{2}{\sqrt{\pi}} F(t).$$

We define the function $g(t) = e^{-t^2} + i\, 2\pi^{-1/2} F(t)$, whose Taylor series expansion at $t = 0$ is given by

$$g(t) = e^{-t^2} + i\frac{2}{\sqrt{\pi}} F(t) = \sum_{n=0}^{\infty} \frac{i^n}{\Gamma\left(\frac{n+2}{2}\right)} t^n. \tag{3.2}$$

We approximate the fractional derivative $f_{a,\sigma}(t) = d_t^a f_{0,\sigma}(t)$ of $f_{0,\sigma}(t) = G_{0,\sigma}(t) + i\mathcal{H}[G_{0,\sigma}](t)$, where

$$G_{0,\sigma}(t) = e^{-\frac{t^2}{2\sigma^2}} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\omega^2}{2}} \cos\left(\frac{\omega}{\sigma}t\right) d\omega$$

$$\mathcal{H}[G_{0,\sigma}](t) = \frac{2}{\sqrt{\pi}} F\left(\frac{t}{\sqrt{2}\sigma}\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\omega^2}{2}} \sin\left(\frac{\omega}{\sigma}t\right) d\omega,$$

as a sum of $g(t)$:

$$f_{a,\sigma}(t) = \sum_{m=1}^{M} \alpha_m g(\gamma_m t) + \epsilon(t),$$

for some $(\alpha_m, \gamma_m) \in \mathbb{C}^2$.

Consequently, $f_{a,\sigma}(t) = G_{a,\sigma}(t) + i\mathcal{H}[G_{a,\sigma}](t)$, where $G_{a,\sigma}(t) = \mathrm{Re}\{f_{a,\sigma}\}(t) = d_t^a G_{0,\sigma}(t)$ and $\mathcal{H}[G_{a,\sigma}](t) = \mathrm{Im}\{f_{a,\sigma}\}(t) = d_t^a \mathcal{H}[G_{0,\sigma}](t)$ are related to the ath fractional derivative of the Gaussian function and Dawson's integral, respectively.

3.2. Approximating fractional derivatives of Gaussian function and Dawson's integral.

Starting with

$$f_{0,\sigma}(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\omega^2}{2}} \exp\left(i\frac{\omega}{\sigma}t\right) d\omega,$$

Taylor series expansion of $f_{a,\sigma}(t)$ at $t = 0$ is given by

$$(3.3) \qquad f_{a,\sigma}(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\omega^2}{2}} \left(\frac{\omega}{\sigma}\right)^a \exp\left(i\left[\frac{\omega}{\sigma}t + a\frac{\pi}{2}\right]\right) d\omega$$

$$= \sqrt{\frac{2}{\pi}} \exp\left(ia\frac{\pi}{2}\right) \int_0^\infty e^{-\frac{\omega^2}{2}} \left(\frac{\omega}{\sigma}\right)^a \exp\left(i\frac{\omega}{\sigma}t\right) d\omega$$

$$= \sqrt{\frac{2}{\pi}} \exp\left(ia\frac{\pi}{2}\right) \sum_{n=0}^\infty \left[\int_0^\infty e^{-\frac{\omega^2}{2}} \left(\frac{\omega}{\sigma}\right)^a \frac{\left(i\frac{\omega}{\sigma}\right)^n}{n!} d\omega\right] t^n$$

$$= \sqrt{\frac{2}{\pi}} \exp\left(ia\frac{\pi}{2}\right) \sum_{n=0}^\infty \frac{1}{n!} \frac{i^n}{\sigma^{n+a}} \left[\int_0^\infty e^{-\frac{\omega^2}{2}} \omega^{a+n} d\omega\right] t^n$$

$$(3.4) \qquad = \sqrt{\frac{2}{\pi}} \exp\left(ia\frac{\pi}{2}\right) \sum_{n=0}^\infty \frac{1}{n!} \frac{i^n}{\sigma^{n+a}} 2^{(a+n-1)/2} \Gamma\left(\frac{a+n+1}{2}\right) t^n.$$

Equating coefficients of the Taylor series of left and right hand side of

$$f_{a,\sigma}(t) = \sum_{m=1}^M \alpha_m g(\gamma_m t) + \epsilon(t),$$

we obtain the following moment problem for (α_m, γ_m):

$$(3.5) \qquad \frac{1}{\sqrt{\pi}} \exp\left(ia\frac{\pi}{2}\right) \left(\frac{\sqrt{2}}{\sigma}\right)^{a+n} \frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{a+n+1}{2}\right)}{n!} = \sum_m \alpha_m \gamma_m^n + \varepsilon_n,$$

which can be solved using the method in [15]. Here $\epsilon(t) = \sum_{n=0}^\infty \frac{\epsilon_n}{n!} t^n$ and $\varepsilon_n = i^{-n} \Gamma\left(\frac{n+2}{2}\right) \frac{\epsilon_n}{n!}$. For $a = 2^{-1}$ and $\sigma = \sqrt{2}^{-1}$, a solution for the moment problem (3.5) is presented in Table 1.

We will denote the approximation of the fractional derivative of $f_{0,\sigma}(t)$ obtained through solution of the moment problem (3.5) by

$$\tilde{f}_{a,\sigma}(t) = \sum_{m=1}^M \alpha_m g(\gamma_m t).$$

$\tilde{f}_{a,\sigma}(t)$ can also be written as

$$\tilde{f}_{a,\sigma}(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{\hat{f}}_{a,\sigma}(\omega) \exp\left(i\frac{\omega}{\sigma}t\right) d\omega = \tilde{G}_{a,\sigma}(t) + i\mathcal{H}\left[\tilde{G}_{a,\sigma}\right](t),$$

for some $\tilde{\hat{f}}_{a,\sigma}(\omega)$ and $\tilde{G}_{a,\sigma}(t) = \text{Re}\left\{\tilde{f}_{a,\sigma}(t)\right\}$, $\mathcal{H}\left[\tilde{G}_{a,\sigma}\right](t) = \text{Im}\left\{\tilde{f}_{a,\sigma}(t)\right\}$. Comparing with (3.3), $\tilde{\hat{f}}_{a,\sigma}(\omega)$ approximates $\hat{f}_{a,\sigma}(\omega) = e^{-\frac{\omega^2}{2}} \left(\frac{\omega}{\sigma}\right)^a \exp\left(ia\frac{\pi}{2}\right)$, $\omega \geq 0$. We present plots of $\tilde{G}_{a,\sigma}(t)$, $\mathcal{H}\left[\tilde{G}_{a,\sigma}\right](t)$, $\left|\tilde{\hat{f}}_{a,\sigma}(\omega)\right|$ and $\left|\hat{f}_{a,\sigma}(\omega)\right|$ for various values of a in Figure 3.1. In our computations, we used a rational approximation of the Dawson's integral which is provided in Appendix A.

Our approach can be generalized to approximate partial derivative with respect to the order of fractional derivative which will be discussed in Section 4.

3.3. **Approximation error.** Recall that

$$g(t) = f_{0,\sigma}\left(\sqrt{2}t\right) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\omega^2}{2}} \exp\left(i\frac{\omega}{\sqrt{2}}t\right) d\omega$$
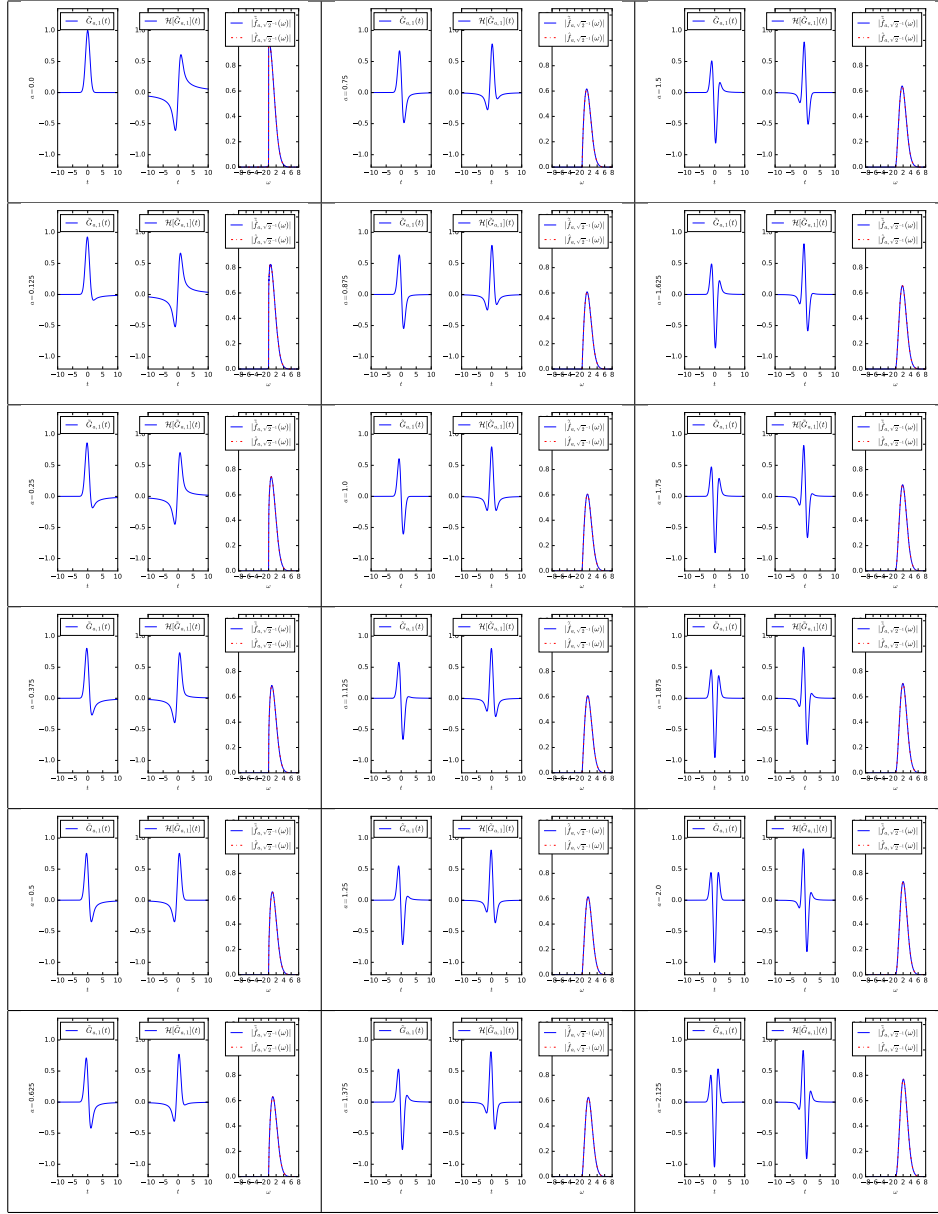
FIGURE 3.1. Table of plots of $\tilde{G}_{a,\sigma}(t)$ (left), $\mathcal{H}\left[\tilde{G}_{a,\sigma}\right](t)$ (middle), $\left|\tilde{\hat{f}}_{a,\sigma}(\omega)\right|$ (right, solid blue) and $\left|\hat{f}_{a,\sigma}(\omega)\right|$ (right, dashed-dotted red) for $\sigma = \sqrt{2}^{-1}$ and various a starting from 0 to 2.125 with increments of 0.125. Along each column of the table, a increases top to bottom by 0.125. Along each row of the table, a increases left to right by 0.75.

m	α_m	γ_m
1	-0.002327462216272 - 0.0023231170382811	0.022707194026268 - 0.000014485869526i
2	-0.008686498484022 - 0.008672831744140i	0.088058420558651 - 0.0000050483776843i
3	-0.017746649906246 - 0.017725233118614i	0.188199334944285 - 0.0000090762508709i
4	-0.028479502359746 - 0.028455359343906i	0.311644959787523 - 0.0001183019489411
5	-0.040781085680949 - 0.040758739735612i	0.445550822358306 - 0.0001242047772781
6	-0.055691041043028 - 0.055672403045923i	0.578033122632225 - 0.0001101608929711
7	-0.075799236577609 - 0.075783432858215i	0.699719488265476 - 0.000084621403249i
8	-0.106825493665143 - 0.106809444597271i	0.804243109619030 - 0.000056714249345i
9	2.224002927119579 + 2.223687081665376i	1.001024232778366 + 0.000000290415923i
10	-0.846706907066594 - 0.846589952992167i	0.986812368161672 - 0.0000003755178001i
11	-0.296187315352343 - 0.296147582995713i	0.948969382632748 - 0.000014668011082i
12	-0.163403417748510 - 0.163381667176418i	0.887911302390365 - 0.0000032513183747i

TABLE 1.   Table of (α_m, γ_m) obtained by solving the moment problem (3.5) for $a = 2^{-1}$ and $\sigma = \sqrt{2}^{-1}$. Approximations $\tilde{f}_{2^{-1},\sqrt{2}^{-1}}(t)$ and $\tilde{f}_{a,\sqrt{2}^{-1}}(t)$ for other values of a are presented in Figure 3.1

and

$$f_{a,\sigma}(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\omega^2}{2}} \left(\frac{\omega}{\sigma}\right)^a \exp\left(i\left[\frac{\omega}{\sigma}t + a\frac{\pi}{2}\right]\right) d\omega.$$

Then

$$\tilde{f}_{a,\sigma}(t) = \sum_{m=1}^M \alpha_m g(\gamma_m t)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\omega^2}{2}} \sum_{m=1}^M \alpha_m \exp\left(i\frac{\gamma_m}{\sqrt{2}}\omega t\right) d\omega.$$

Consequently, the error and its L2 norm are given by

$$\left[\tilde{f}_{a,\sigma} - f_{a,\sigma}\right](t)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\omega^2}{2}} \left[\sum_{m=1}^M \alpha_m \exp\left(i\frac{\gamma_m}{\sqrt{2}}\omega t\right) - \left(\frac{\omega}{\sigma}\right)^a \exp\left(i\left[\frac{\omega}{\sigma}t + a\frac{\pi}{2}\right]\right)\right] d\omega$$

and

$$\int_{-\infty}^\infty \left|\left[\tilde{f}_{a,\sigma} - f_{a,\sigma}\right](t)\right|^2 dt$$

$$= 4\left[\begin{array}{c} \sum_{m'=1}^M \sum_{m=1}^M \alpha_{m'}^* \alpha_m \left[\frac{\gamma_m \gamma_{m'}^*}{2}\right] \sqrt{\frac{\pi}{\gamma_{m'}^{*2}+\gamma_m^2}} + \frac{\sigma^{1-2a}}{\pi}\Gamma\left(a+\frac{1}{2}\right) \\ -2\mathrm{Re}\left\{\frac{\sigma}{\sqrt{2}}\exp\left(-ia\frac{\pi}{2}\right)\sum_{m=1}^M \alpha_m \gamma_m^a \left(1 + \frac{\sigma^2 \gamma_m^2}{2}\right)^{-\frac{a+1}{2}}\Gamma\left(\frac{a+1}{2}\right)\right\} \end{array}\right],$$

for $a > -\frac{1}{2}$, respectively.

4. Approximating derivative with respect to the order of the fractional derivative

We can generalize the approach presented in Section 3.2 to approximate partial derivative with respect to the order of fractional derivative. Consider (3.4):

$$f_{a,\sigma}(t) = \frac{1}{\sqrt{\pi}} \exp\left(ia\frac{\pi}{2}\right) \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\frac{\sqrt{2}}{\sigma}\right)^{n+a} \Gamma\left(\frac{a+n+1}{2}\right) t^n.$$

Then

$$\partial_a[f_{a,\sigma}](t)$$

$$= \frac{1}{\sqrt{\pi}} \exp\left(ia\frac{\pi}{2}\right) \sum_{n=0}^{\infty} \frac{i^n}{n!} \Gamma\left(\frac{a+n+1}{2}\right) \left(\frac{\sqrt{2}}{\sigma}\right)^{n+a} \left[\log\left(\frac{\sqrt{2}}{\sigma}\right) + \frac{1}{2}\psi\left(\frac{a+n+1}{2}\right) + i\frac{\pi}{2}\right] t^n.$$

Here we used the identity $\Gamma'(z) = \Gamma(z)\psi(z)$ to write the derivative of Gamma function where $\psi(z)$ is the Psi function (see 5.2(i), 5.7(ii) and 5.9(ii) of [9]):

$$\psi(x) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z}\right).$$

Then, partial derivative $\partial_a[f_{a,\sigma}](t)$ of $f_{a,\sigma}(t)$ with respect to the order of fractional derivative a can be approximated by

$$\widetilde{\partial_a[f_{a,\sigma}]}(t) = \widetilde{\partial_a[G_{a,\sigma}]}(t) + i\widetilde{\partial_a\mathcal{H}[G_{a,\sigma}]}(t) = \sum_{m=1}^{M} \alpha_m g(\gamma_m t).$$

where $g(t)$ was defined in (3.2) and (α_m, γ_m) is obtained by solving the following moment problem:

(4.1)

$$\frac{1}{\sqrt{\pi}} \exp\left(ia\frac{\pi}{2}\right) \left(\frac{\sqrt{2}}{\sigma}\right)^{a+n} \frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{a+n+1}{2}\right)\left[\log\left(\frac{\sqrt{2}}{\sigma}\right) + \frac{1}{2}\psi\left(\frac{a+n+1}{2}\right) + i\frac{\pi}{2}\right]}{n!}$$

$$= \sum_m \alpha_m \gamma_m^n + \varepsilon_n.$$

We compared $\widetilde{\partial_a[f_{a,\sigma}]}(t)$ with the two sided finite difference derivative approximation $\delta_{a,\Delta a}[\tilde{f}_{a,\sigma}](t)$ of $\tilde{f}_{a,\sigma}(t) = \tilde{G}_{a,\sigma}(t) + i\mathcal{H}[\tilde{G}_{a,\sigma}](t)$ with respect to a:

$$\delta_{a,\Delta a}[\tilde{f}_{a,\sigma}](t) = \frac{\tilde{f}_{a+\Delta a,\sigma}(t) - \tilde{f}_{a-\Delta a,\sigma}(t)}{2\Delta a},$$

for some small positive Δa. For $\Delta a = 10^{-3}$, we present plots of real and imaginary parts of $\widetilde{\partial_a[f_{a,\sigma}]}(t)$ and $\delta_{a,\Delta a}[\tilde{f}_{a,\sigma}](t)$ for various values of a in Figure 4.1.

Appendix A. A rational approximation of Dawson's integral

Consider $\operatorname{sinc}(x) = x^{-1}\sin(x)$, $\operatorname{cosinc}(x) = x^{-1}(1-\cos(x))$ and the approximation

$$(A.1)\quad \operatorname{sinc}(x) + i\operatorname{cosinc}(x) = \sum_m \alpha_m \left[\exp\left(-[\gamma_m x]^2\right) + i\frac{2}{\sqrt{\pi}}F(\gamma_m x)\right] + \epsilon_B(x)$$
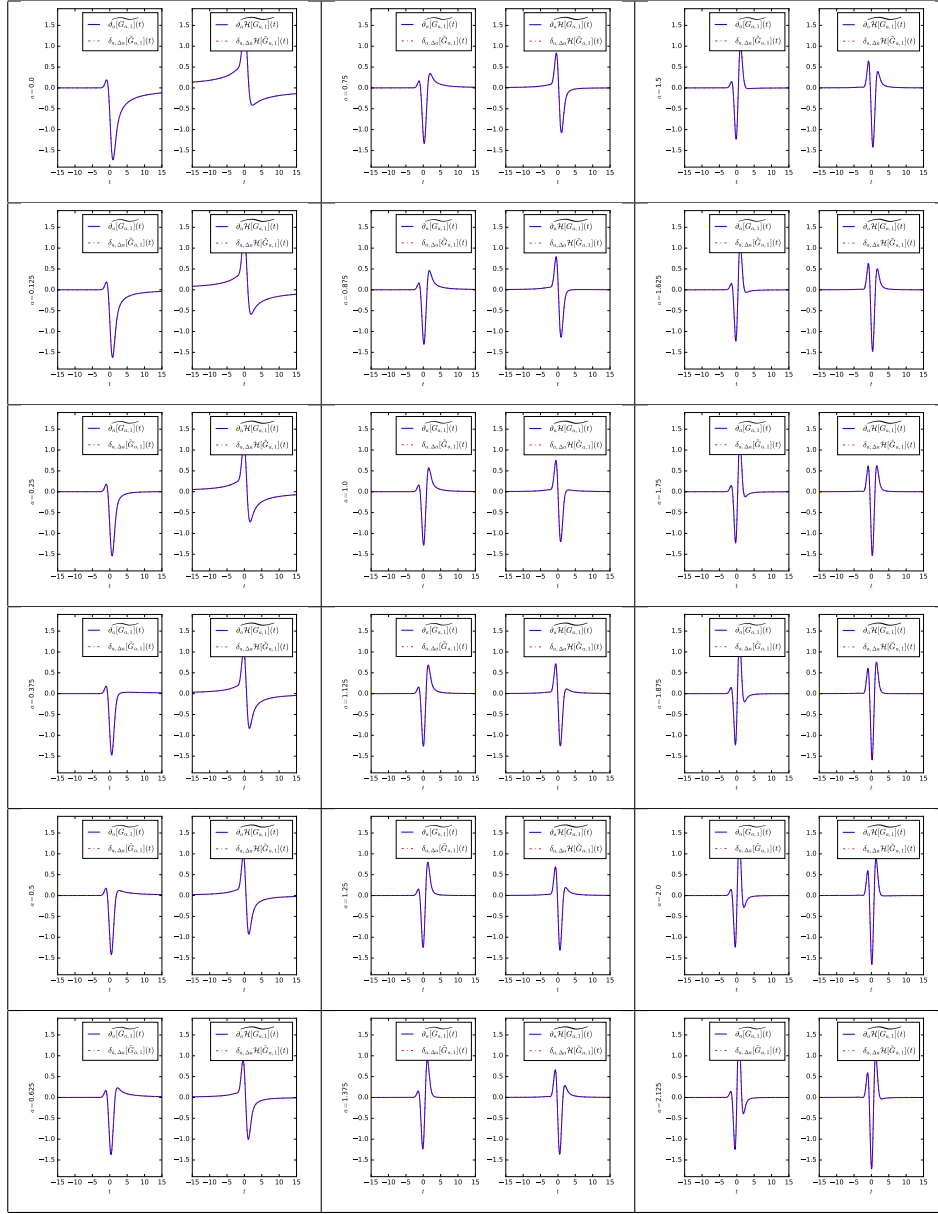
FIGURE 4.1.   Table of plots of $\widetilde{\partial_a\left[G_{a,\sigma}\right]}(t)$ (left, solid blue), $\delta_{a,\Delta a}\left[\tilde{G}_{a,\sigma}\right](t)$ (left, dashed-dotted red), and $\widetilde{\partial_a\mathcal{H}\left[G_{a,\sigma}\right]}(t)$ (right, solid blue), $\delta_{a,\Delta a}\mathcal{H}\left[\tilde{G}_{a,\sigma}\right](t)$ (right,dashed-dotted red), for $\sigma = \sqrt{2}^{-1}$, $\Delta a = 10^{-3}$ and various a starting from 0 to 2.125 with increments of 0.125. Along each column of the table, a increases top to bottom by 0.125. Along each row of the table, a increases left to right by 0.75.
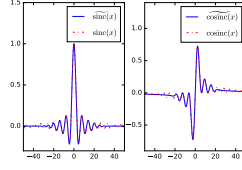
FIGURE A.1.   Plots of approximations of $\widetilde{\text{sinc}}\,(x)$ (left, solid blue) and $\widetilde{\text{cosinc}}\,(x)$ (right, solid blue), and $\text{sinc}\,(x)$ (left, dashed-dotted red) and $\text{cosinc}\,(x)$ (right, dashed-dotted red).

m	α_m	γ_m
1	-0.117532571756027 - 0.003575367485193i	0.003599251866768 + 0.023031988828263i
2	-0.117532571756027 + 0.003575367485193i	0.003599251866768 - 0.023031988828263i
3	2.658250413824904 - 0.872108421067261i	0.015512727403264 + 0.023419328488677i
4	2.658250413824904 + 0.872108421067261i	0.015512727403264 - 0.023419328488677i
5	9.141742362072840 - 34.331838882055756i	0.032137996250724 + 0.006297276091330i
6	9.141742362072840 + 34.331838882055756i	0.032137996250724 - 0.006297276091330i
7	-11.182460204141808 + 11.443034213692357i	0.026033590315800 + 0.017203623738202i
8	-11.182460204141808 - 11.443034213692357i	0.026033590315800 - 0.017203623738202i

TABLE 2.   Table of (α_m, γ_m) obtained by solving the moment problem (A.2) using the method of (A.2). These (α_m, γ_m) are used in (A.3) to approximate $\text{sinc}\,(x) + \mathrm{i}\,\text{cosinc}\,(x)$ and in (A.4) to approximate Dawson's integral.

with error $\epsilon_B\,(x)$, such that $\text{Re}\,\{\gamma_m\} > 0$. Here (α_m, γ_m) is a solution to the moment problem

$$(A.2) \qquad \frac{\Gamma\left(\frac{n+2}{2}\right)}{(n+1)!} = \sum_m \alpha_m \gamma_m^n + \epsilon_n,$$

for some ϵ_n, which can be solved using the method in A.2. A solution for the moment problem A.2 is presented in Table 2 and plots of the corresponding approximations

$$\widetilde{\text{sinc}}\,(x) = \text{Re}\left\{\sum_m \alpha_m \left[\exp\left(-[\gamma_m x]^2\right) + \mathrm{i}\frac{2}{\sqrt{\pi}} F\,(\gamma_m x)\right]\right\}$$

$$(A.3) \qquad \widetilde{\text{cosinc}}\,(x) = \text{Imag}\left\{\sum_m \alpha_m \left[\exp\left(-[\gamma_m x]^2\right) + \mathrm{i}\frac{2}{\sqrt{\pi}} F\,(\gamma_m x)\right]\right\}$$

are presented in Figure A.1.

Rewriting Dawson's integral (3.1),

$$F\,(x) = \frac{x}{2} \int_0^\infty \mathrm{e}^{-t^2/4} \text{sinc}\,(xt)\, t\, dt$$

and substituting the approximation (A.1) for $\mathrm{sinc}\,(x)$, we approximate Dawson's integral by

$$(A.4) \qquad \tilde{F}(x) = \frac{x}{2} \sum_m \alpha_m \int_0^\infty \mathrm{e}^{-\left(1/4 + (\gamma_m x)^2\right)t^2} t\, dt = x \sum_m \frac{\alpha_m}{\left(1 + [2\gamma_m x]^2\right)}.$$

Approximation error. For x within the vicinity of zero we have the error

$$\epsilon_F(x) = \left[F - \tilde{F}\right](x) = \frac{x}{2} \int_0^\infty \mathrm{e}^{-t^2/4} \left[\mathrm{sinc}\,(xt) - \sum_m \alpha_m \mathrm{e}^{-(\gamma_m xt)^2}\right] t\, dt$$

absolutely bounded by

$$|\epsilon_F(x)| \le \frac{1}{2} \int_0^\infty \mathrm{e}^{-(t/x)^2/4} \frac{t}{|x|} \left|\mathrm{sinc}\,(t) - \sum_m \alpha_m \mathrm{e}^{-(\gamma_m t)^2}\right| dt$$

$$\le \frac{\epsilon_1}{2} \int_0^\infty \mathrm{e}^{-(t/x)^2/4} \frac{t}{|x|} dt = \frac{\epsilon_1}{2} |x|,$$

where

$$\epsilon_1 = \max_{x \in \mathbb{R}} |\mathrm{Re}\,\{\epsilon_B(x)\}|,$$

for $B = 1$.

For x away from zero

$$|\epsilon_F(x)| \le \frac{1}{2x} \int_0^\infty \mathrm{e}^{-(t/x)^2/4} \left|\sin t - t \sum_m \alpha_m \mathrm{e}^{-(\gamma_m t)^2}\right| dt$$

$$\le \frac{1}{2x} \int_0^\infty \mathrm{e}^{-(t/x)^2/4} dt \left[1 + \max_{t \in \mathbb{R}^+} \left|t \sum_m \alpha_m \mathrm{e}^{-(\gamma_m t)^2}\right|\right]$$

$$\le \frac{1}{2} \sqrt{\frac{\pi}{x}} \left[1 + \sum_m \frac{|\alpha_m|}{\sqrt{2\mathrm{Re}\,\{\gamma_m^2\}}} \mathrm{e}^{-1/2}\right].$$

Here, we used the inequality

$$\max_{t \in \mathbb{R}^+} \left\{\left|t \sum_m \alpha_m \mathrm{e}^{-(\gamma_m t)^2}\right|\right\} \le \sum_m |\alpha_m| \max_{t \in \mathbb{R}^+} \left\{t \mathrm{e}^{-\mathrm{Re}\{\gamma_m^2\}t^2}\right\}$$

$$= \sum_m \frac{|\alpha_m|}{\sqrt{2\mathrm{Re}\,\{\gamma_m^2\}}} \mathrm{e}^{-1/2}$$

and obtained using $\max_{t \in \mathbb{R}^+} t\mathrm{e}^{-at^2} = \sqrt{(2a)^{-1}}\mathrm{e}^{-1/2}$.

References

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