APPROXIMATING SPECTRUM OF FOX-LI OPERATOR

Scope

Aim if the project is to approximate the spectrum of Fox-Li operator.

Project Tasks

In order to achieve the project's objective, here are the tasks that needs to be done:

(1) Write a python code that outputs (α_m, γ_m) given h_n, $n = 0, 1, \ldots, 2N$ for some integer N (See Algorithm 1 and corresponding reference below Section B) where $h_n = \sum_{m=1}^{M} \alpha_m \gamma_m^n$.

(2) Write a python code that computes $\tilde{\chi}_{[-1,1]}(t)$ (See Section B)

(3) Derive a closed form expression for the elements of the matrix $\tilde{A}_{mn}$ (See Section 2.1)

$$\tilde{A}_{mn} = \sum_k \sum_l \alpha_k \alpha_l \mathrm{e}^{\frac{(\pi n)^2}{4(\mathrm{i}\omega+\gamma_k)}} \sqrt{\frac{\pi}{-(\mathrm{i}\omega+\gamma_k)}} \sqrt{\frac{\pi}{-\left(\mathrm{i}\omega+\gamma_l+\frac{\omega^2}{(\mathrm{i}\omega+\gamma_k)}\right)}} \mathrm{e}^{-\frac{\left(\mathrm{i}\pi m+\frac{\pi n\omega}{(\mathrm{i}\omega+\gamma_k)}\right)^2}{4\left(\mathrm{i}\omega+\gamma_l+\frac{\omega 2}{(\mathrm{i}\omega+\gamma_k)}\right)}}$$

(4) Write a python code that computes $\tilde{A}_{mn}$ using the analytic expression obtained in Task 1.

(5) Write a python code that computes the eigenvalues of the matrix $\tilde{A}_{mn}$ for a given N, where $-N \leq m, n \leq N$.

(6) Use the python code in Task 1 to compute (α_m, γ_m) in equation (2.1) to approximate $\tilde{\lambda}_n$, $n = 0, 1, \ldots, 2N$ for some integer N (See Section 2.2).

(7) Write a python code that computes $\tilde{\lambda}_C(t) = \sum_m \alpha_m \gamma_m^t$ from (α_m, γ_m) of Task 6.

(8) Derive a closed form expression for $\tilde{k}(\xi) = \int \mathrm{e}^{\mathrm{i}\omega x^2} \tilde{\chi}_{[-1,1]}(x)\, \mathrm{e}^{\mathrm{i}\pi\xi x} dx$

$$\tilde{k}(\xi) = \sum_m \alpha_m \sqrt{-\frac{\pi}{\mathrm{i}\omega+\gamma_m}} \mathrm{e}^{\frac{(\pi\xi)^2}{4(\mathrm{i}\omega+\gamma_m)}} dx$$

(9) Write a python code that compute the closed form expression of $\tilde{k}(\xi)$

(10) Write a python code that compute $\hat{k}(\xi)$ using equation (3.5) of Example 3.7 in [1].

(11) Plot $\tilde{\lambda}_n$, $\tilde{\lambda}_C(\xi)$, $\tilde{k}(\xi)$, $\hat{k}(\xi)$ using the parameters of Figure 3 in [1] (i.e. ω equal to 50, 100, 200 and 400. In evaluation of $\hat{k}$ choose $\varepsilon = 1/4$.)

(12) Compare $\tilde{k}(\xi)$ for various values of ω(i.e. ω equal to 50, 100, 200 and 400). Denoting $\tilde{k}(\xi)$ computed for ω by $\tilde{k}_\omega(\xi)$, observe if there is a relationship between $\tilde{k}_{B\omega}(\xi)$ and $\tilde{k}_\omega(\xi)$. (I give my suspicion in Section 3)

(13) Compare $\tilde{\lambda}_C(\xi)$ for various values of ω. See if observations in 10 can be extended to this case.
(14) When in doubt, have a question or need advise contact Evren and Hartmut.

1. Problem Outline

Fox-Li operator is defined by

$$\mathcal{F}[f](x) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(x-y)^2} f(y)\, dy, \qquad |x| \leq 1$$

which can be rewritten as

$$\mathcal{F}[f](x) = \int \mathrm{e}^{\mathrm{i}\omega(x-y)^2} \chi_{[-1,1]}(y) f(y)\, dy, \qquad |x| \leq 1$$

where $\chi_{[-1,1]}(y)$ is the characteristic function over the internal $[-1,1]$. We consider the following approximation of the characteristic function:

$$\begin{aligned}\chi_{[-1,1]}(y) &= \underbrace{\sum_m \alpha_m \mathrm{e}^{\gamma_m y^2}}_{\tilde{\chi}_{[-1,1]}(y)} + \epsilon_\chi(y) \\ &= \tilde{\chi}_{[-1,1]}(y) + \epsilon_\chi(y)\end{aligned}$$

for some $\alpha_m, \gamma_m \in \mathbb{C}$, Real $\{\gamma_m\} < 0$, where $\max_{y\in[-1,1]} |\epsilon_\chi(y)| = \epsilon \leq .1$. This approximation can be constructed using the Fourier transform of the complex Gaussian approximation of the sine cardinal function, sinc, as discussed in Appendix B. See (B.8).

Then

$$\begin{aligned}\mathcal{F}[f](x) &= \int \mathrm{e}^{\mathrm{i}\omega(x-y)^2} \left[\sum_m \alpha_m \mathrm{e}^{\gamma_m y^2} + \epsilon_\chi(y)\right] f(y)\, dy, \qquad |x| \leq 1 \\ &= \tilde{\mathcal{F}}_\infty[f](x) + \mathcal{F}_\epsilon[f](x)\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathcal{F}}_\infty[f](x) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega(x-y)^2} \left[\sum_m \alpha_m \mathrm{e}^{\gamma_m y^2}\right] f(y)\, dy \\ \mathcal{F}_\epsilon[f](x) &= \int \mathrm{e}^{\mathrm{i}\omega(x-y)^2} \epsilon_\chi(y) f(y)\, dy \\ |\mathcal{F}_\epsilon[f](x)| &\leq \int |\epsilon_\chi(y) f(y)|\, dy\end{aligned}$$

We will focus on the approximation $\tilde{\mathcal{F}}_\infty[f](x)$ of the Fox-Li operator $\mathcal{F}[f](x)$ and perform its spectral decomposition:

$$\tilde{\mathcal{F}}_\infty[\varphi](x) = \lambda\varphi(x)$$

for eigenvalue λ and eigenfunction $\varphi(x)$ which are to be numerically determined.

2. Approximate matrix representation of Fox-Li operator

Note that compactly supported functions can be represented in terms of Fourier series,

$$\chi_{[-1,1]}(y)\, f(y) = \chi_{[-1,1]}(y) \sum_{n=-\infty}^{\infty} \hat{f}[n]\, \mathrm{e}^{\mathrm{i}\pi n y}$$

where

$$\hat{f}[n] = \int \chi_{[-1,1]}(y)\, f(y)\, \mathrm{e}^{-\mathrm{i}\pi n y} dy.$$

Because we are interested in the Fox-Li operator's action restricted to $[-1,1]$, consider the eigenfunctions φ of Fox-Li operator, the Fourier series expansion

$$\chi_{[-1,1]}(y)\, \varphi(y) = \chi_{[-1,1]}(y) \sum_{n=-\infty}^{\infty} \hat{\varphi}[n]\, \mathrm{e}^{\mathrm{i}\pi n y}$$

and

$$\chi_{[-1,1]}(x)\, \mathcal{F}[\varphi](x) = \lambda \chi_{[-1,1]}(x)\, \varphi(x) = \chi_{[-1,1]}(x) \int \mathrm{e}^{\mathrm{i}\omega (x-y)^2} \chi_{[-1,1]}(y)\, \varphi(y)\, dy.$$

Then

$$\begin{aligned}
\lambda \hat{\varphi}[m] &= \int \chi_{[-1,1]}(x)\, \mathcal{F}[\varphi](x)\, \mathrm{e}^{-\mathrm{i}\pi m x} dx \\
&= \int \chi_{[-1,1]}(x) \int \mathrm{e}^{\mathrm{i}\omega (x-y)^2} \chi_{[-1,1]}(y) \sum_{n=-\infty}^{\infty} \hat{\varphi}[n]\, \mathrm{e}^{\mathrm{i}\pi n y} dy \mathrm{e}^{-\mathrm{i}\pi m x} dx \\
&= \sum_{n=-\infty}^{\infty} \int\int \mathrm{e}^{-\mathrm{i}\pi m x} \chi_{[-1,1]}(x)\, \mathrm{e}^{\mathrm{i}\omega (x-y)^2} \chi_{[-1,1]}(y)\, \hat{\varphi}[n]\, \mathrm{e}^{\mathrm{i}\pi n y} dy\, dx \\
&= \sum_{n=-\infty}^{\infty} \left[\int\int \mathrm{e}^{-\mathrm{i}\pi m x} \chi_{[-1,1]}(x)\, \mathrm{e}^{\mathrm{i}\omega (x-y)^2} \chi_{[-1,1]}(y)\, \mathrm{e}^{\mathrm{i}\pi n y} dy\, dx \right] \hat{\varphi}[n] \\
&= \sum_{n=-\infty}^{\infty} A_{mn} \hat{\varphi}[n] \\
&\approx \sum_{n=-\infty}^{\infty} \tilde{A}_{mn} \hat{\varphi}[n]
\end{aligned}$$

where

$$\begin{aligned}
A_{mn} &= \int\int \mathrm{e}^{-\mathrm{i}\pi m x} \chi_{[-1,1]}(x)\, \mathrm{e}^{\mathrm{i}\omega (x-y)^2} \chi_{[-1,1]}(y)\, \mathrm{e}^{\mathrm{i}\pi n y} dy\, dx \\
\tilde{A}_{mn} &= \int\int \mathrm{e}^{-\mathrm{i}\pi m x} \tilde{\chi}_{[-1,1]}(x)\, \mathrm{e}^{\mathrm{i}\omega (x-y)^2} \tilde{\chi}_{[-1,1]}(y)\, \mathrm{e}^{\mathrm{i}\pi n y} dy\, dx.
\end{aligned}$$

Consider the integral and its approximation

$$\begin{aligned}
\int \mathrm{e}^{\mathrm{i}\omega(x-y)^2}\chi_{[-1,1]}(y)\,\mathrm{e}^{\mathrm{i}\pi n y}dy &= \int \mathrm{e}^{\mathrm{i}\omega(x-y)^2}\left[\tilde{\chi}_{[-1,1]}(y)+\epsilon_\chi(y)\right]\mathrm{e}^{\mathrm{i}\pi n y}dy \\
&\approx \int \mathrm{e}^{\mathrm{i}\omega(x-y)^2}\tilde{\chi}_{[-1,1]}(y)\,\mathrm{e}^{\mathrm{i}\pi n y}dy \\
&= \int \mathrm{e}^{\mathrm{i}\omega(x-y)^2}\sum_m \alpha_m \mathrm{e}^{\gamma_m y^2}\mathrm{e}^{\mathrm{i}\pi n y}dy \\
&= \sum_m \alpha_m\left[\int \mathrm{e}^{\mathrm{i}\omega(x-y)^2}\mathrm{e}^{\gamma_m y^2}\mathrm{e}^{\mathrm{i}\pi n y}dy\right] \\
&= \mathrm{e}^{\mathrm{i}\omega x^2}\sum_m \alpha_m\left[\int \mathrm{e}^{(\mathrm{i}\omega+\gamma_m)y^2}\mathrm{e}^{\mathrm{i}(\pi n-2\omega x)y}dy\right] \\
&= \mathrm{e}^{\mathrm{i}\omega x^2}\sum_m \alpha_m\left[\sqrt{\frac{\pi}{-(\mathrm{i}\omega+\gamma_m)}}\,\mathrm{e}^{\frac{(\pi n-2\omega x)^2}{4(\mathrm{i}\omega+\gamma_m)}}\right] \\
&= \mathrm{e}^{\mathrm{i}\omega x^2}\sum_m \alpha_m\left[\sqrt{\frac{\pi(\mathrm{i}\omega-\gamma_m^*)}{|\mathrm{i}\omega+\gamma_m|^2}}\,\mathrm{e}^{\frac{(\gamma_m^*-\mathrm{i}\omega)(\pi n-2\omega x)^2}{4|\mathrm{i}\omega+\gamma_m|^2}}\right]
\end{aligned}$$

It is bounded by

$$\begin{aligned}
\left|\int \mathrm{e}^{\mathrm{i}\omega(x-y)^2}\tilde{\chi}_{[-1,1]}(y)\,\mathrm{e}^{\mathrm{i}\pi n y}dy\right| &\le \sum_m |\alpha_m|\left|\sqrt{\frac{\pi(\mathrm{i}\omega-\gamma_m^*)}{|\mathrm{i}\omega+\gamma_m|^2}}\,\mathrm{e}^{\frac{(\gamma_m^*-\mathrm{i}\omega)(\pi n-2\omega x)^2}{4|\mathrm{i}\omega+\gamma_m|^2}}\right| \\
&\le \sum_m \left|\alpha_m\sqrt{\frac{\pi(\mathrm{i}\omega-\gamma_m^*)}{|\mathrm{i}\omega+\gamma_m|^2}}\right|\left|\mathrm{e}^{-\frac{\Re\{-\gamma_m^*\}(\pi n-2\omega x)^2}{4\omega^2|\mathrm{i}+\omega^{-1}\gamma_m|^2}}\right|
\end{aligned}$$

For some constant C sufficiently large, i.e. $\mathrm{e}^{-C^2}\ll 1$, the integral will be significant for n satisfying

$$\begin{aligned}
&\frac{\Re\{-\gamma_m^*\}(\pi n-2\omega x)^2}{4\omega^2|\mathrm{i}+\omega^{-1}\gamma_m|^2}\le C^2 \\
&\implies -\frac{2\omega}{\pi}\max_m\left[C\frac{|\mathrm{i}+\omega^{-1}\gamma_m|}{\Re\{-\gamma_m^*\}}+1\right]\le n\le \max_m\frac{2\omega}{\pi}\left[C\frac{|\mathrm{i}+\omega^{-1}\gamma_m|}{\Re\{-\gamma_m^*\}}+1\right]
\end{aligned}$$

Similar argument can be made for the integral $\int \mathrm{e}^{\mathrm{i}\omega(x-y)^2}\chi_{[-1,1]}(x)\,\mathrm{e}^{-\mathrm{i}\pi m x}dx$. Thus, we can consider a finite submatrix of matrix $\tilde{A}_{mn}$ for approximating the eigenvalues of the Fox-Li operator.

Alternatively, if we consider $\sum_{n=-\infty}^{\infty}\hat{\varphi}[n]\,\mathrm{e}^{\mathrm{i}\pi n x}$ as an approximation of the Fourier representation of the eigenfunction $\varphi(x)=\int\hat{\varphi}(k)\,\mathrm{e}^{\mathrm{i}\pi k x}dk$, then truncation of the sum can be effectively achieved by considering bandlimited projection of the eigenfunctions and approximating them as

$$\varphi(x)\approx\sum_{n=-\infty}^{\infty}\varphi\left(\frac{n}{B}\right)\operatorname{sinc}\left(\pi B\left[x-\frac{n}{B}\right]\right)$$

where $B\ge\max_m 2\omega\left[C\frac{|\mathrm{i}+\omega^{-1}\gamma_m|}{\Re\{-\gamma_m^*\}}+1\right]$. It will be interesting to compute the matrix representation of the operator using this representation, however, we will leave it to future research.

2.1. **Eigendecomposition of the matrix.** Because $\tilde{\chi}_{[-1,1]}(x)$ is a sum of complex decaying exponentials, the matrix elements

$$\tilde{A}_{mn} = \int\int \mathrm{e}^{-\mathrm{i}\pi m x}\tilde{\chi}_{[-1,1]}(x)\,\mathrm{e}^{\mathrm{i}\omega(x-y)^2}\tilde{\chi}_{[-1,1]}(y)\,\mathrm{e}^{\mathrm{i}\pi n y}dy\,dx$$

can be analytically computed. Once the matrix $\tilde{A}$ is formed, then the eigenvalues $\tilde{\lambda}$and eigenvectors $\tilde{\hat{\varphi}}$

$$\tilde{\lambda}\tilde{\hat{\varphi}} = \tilde{A}\tilde{\hat{\varphi}}$$

will be approximations of the eigenvalue of the Fox-Li operator λ the Fourier coefficients of the corresponding eigenvectors $\hat{\varphi}$.

2.2. **Interpolating the eigenvalues.** Once the eigenvalues of the matrix $\tilde{A}$ are computed, sort the eigenvalues according to their norm, i.e. $\left|\tilde{\lambda}_n\right| < \left|\tilde{\lambda}_m\right|$ for $n < m$. We would like to see if we can interpolate the trajectory of the eigenvalues by

$$\tilde{\lambda}_n = \sum_m \alpha_m \gamma_m^n \tag{2.1}$$

which can be tackled solving the appropriate moment problem. See [4] for an algorithm that solves the moment problem.

References

[1] Albrecht Böttcher, Hermann Brunner, Arieh Iserles, and Syvert P Nørsett. On the singular values and eigenvalues of the fox–li and related operators. *New York J. Math*, 16:539–561, 2010.

[2] Emmanuel J Candes. Multiscale chirplets and near-optimal recovery of chirps. Technical report, Technical Report, Stanford University, 2002.

[3] Steve Mann and Simon Haykin. The chirplet transform: Physical considerations. *Signal Processing, IEEE Transactions on*, 43(11):2745–2761, 1995.

[4] Can Evren Yarman and Garret Flagg. Generalization of Padé approximation from rational functions to arbitrary analytic functions - Theory. *Math. Comp.*, 84:1835–1860, 2015.

3. Conjecture on the relationship between spectrum of Fox-Li operators with different ω

Consider the asymptotic analysis of the Fox-Li operator studied in Example 3.7 of [1]. Let $\hat{k}_\omega(\xi)$ be defined by

$$\hat{k}_\omega(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{(\mathrm{i}-\varepsilon)\omega t^2} \mathrm{e}^{\mathrm{i}\xi t} dt = \sqrt{\frac{\pi}{\omega(\varepsilon - \mathrm{i})}} \exp\left(-\frac{\xi^2}{4\omega(\varepsilon-\mathrm{i})}\right)$$

Then

$$\begin{aligned}
\hat{k}_{B\omega}(\xi) &= \int_{-\infty}^{\infty} \mathrm{e}^{(\mathrm{i}-\varepsilon)\omega t^2} \mathrm{e}^{\mathrm{i}\xi t} dt \\
&= \sqrt{\frac{\pi}{B\omega(\varepsilon-\mathrm{i})}} \exp\left(-\frac{\xi^2}{4B\omega(\varepsilon-\mathrm{i})}\right) \\
\left[\hat{k}_{B\omega}(\xi)\right]^B &= \left(\frac{\pi}{B\omega(\varepsilon-\mathrm{i})}\right)^{B/2} \exp\left(-\frac{\xi^2}{4\omega(\varepsilon-\mathrm{i})}\right) \\
&= \left(\frac{1}{B}\right)^{B/2} \left(\frac{\pi}{\omega(\varepsilon-\mathrm{i})}\right)^{B/2} \exp\left(-\frac{\xi^2}{4\omega(\varepsilon-\mathrm{i})}\right) \\
&= \left(\frac{1}{B}\right)^{B/2} \left(\frac{\pi}{\omega(\varepsilon-\mathrm{i})}\right)^{\frac{B-1}{2}} \hat{k}_\omega(\xi) \\
\hat{k}_\omega(\xi) &= B^{1/2} \left(\frac{B\omega(\varepsilon-\mathrm{i})}{\pi}\right)^{\frac{B-1}{2}} \left[\hat{k}_{B\omega}(\xi)\right]^B
\end{aligned}$$

$$\left|\hat{k}_\omega(\xi)\right| = B^{1/2} \left(\frac{B\omega\sqrt{1+\varepsilon^2}}{\pi}\right)^{\frac{B-1}{2}} \left|\hat{k}_{B\omega}(\xi)\right|^B$$

for some positive ε. We see that $\frac{\hat{k}_\omega(\xi)}{\left[\hat{k}_{B\omega}(\xi)\right]^B}$ is not dependent on ξ, i.e. $\frac{\hat{k}_\omega(\xi)}{\left[\hat{k}_{B\omega}(\xi)\right]^B} \approx a(B,\omega)$. I wonder if $\frac{\tilde{k}_\omega(\xi)}{\left[\tilde{k}_{B\omega}(\xi)\right]^B}$ will have a similar relationship, i.e. weak dependence on ξ...

Appendix A. Hilbert transform pairs and Taylor series expansion at zero

$f(t)$	$\mathcal{H}[f](t)$	$g(t) = f(t) + \mathrm{i}\mathcal{H}[f](t)$	g_n, s.t. $g(t) = \sum_{n=0}^{\infty} g_n t^n$
$\mathcal{H}[f](t)$	$\mathcal{H}[\mathcal{H}[f]](t) = -f(t)$	$\mathcal{H}[f](t) - \mathrm{i}f(t) = -\mathrm{i}\,g(t)$	$-\mathrm{i}\,g_n$
$\cos(t)$	$\sin(t)$	$\cos(t) + \mathrm{i}\sin(t) = \mathrm{e}^{\mathrm{i}t}$	$\frac{\mathrm{i}^n}{n!}$
$\sin(t)$	$-\cos(t)$	$\sin(t) - \mathrm{i}\cos(t) = -\mathrm{i}\mathrm{e}^{\mathrm{i}t}$	$\frac{\mathrm{i}^{n-1}}{n!}$
$\exp(\mathrm{i}t)$	$-\mathrm{i}\exp(\mathrm{i}t)$	$2\mathrm{e}^{\mathrm{i}t}$	$2\frac{\mathrm{i}^n}{n!}$
$\exp(-\mathrm{i}t)$	$\mathrm{i}\exp(-\mathrm{i}t)$	0	0
e^{-t^2}	$\frac{2}{\sqrt{\pi}}F(t)$ $F(t)$: Dawson fnc.	$\mathrm{e}^{-x^2} + \mathrm{i}\,2\pi^{-1/2}F(x)$	$\frac{\mathrm{i}^n}{\Gamma\left(\frac{n+2}{2}\right)}$
$\frac{1}{1+t^2}$	$\frac{t}{1+t^2}$	$\frac{1+\mathrm{i}t}{1+t^2}$	i^n
$\mathrm{sinc}(t)$	$\frac{1-\cos(t)}{t}$	$\mathrm{sinc}(t) + \mathrm{i}\,\mathrm{cosinc}(t) = \frac{\mathrm{e}^{\mathrm{i}t}-1}{\mathrm{i}t}$	$\frac{\mathrm{i}^n}{(n+1)!}$
$\delta(t)$	$\frac{1}{\pi t}$	$\delta(t) + \mathrm{i}\frac{1}{\pi t}$	N.A. / Not analytic
$\chi_{[a,b]}(t)$	$\frac{1}{\pi}\ln\left\lvert\frac{t-a}{t-b}\right\rvert$	$\chi_{[a,b]}(t) + \frac{1}{\pi}\ln\left\lvert\frac{t-a}{t-b}\right\rvert$	N.A. / Not analytic

Table 1. Hilbert transform pairs

Appendix B. Approximating sinc and characteristic function over an interval

Theorem 1. *Let*

$$\epsilon_B(x) = \mathrm{sinc}(Bx) - f(x). \tag{B.1}$$

Then

$$\left| \mathrm{sinc}(3^n Bx) - \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx) + 1\right] f(x) \right| \leq |\epsilon_B(x)|, \textit{ for } n \geq 0. \tag{B.2}$$

Example 2. Let

$$\epsilon_B(x) = \mathrm{sinc}(Bx) - \sum_m \alpha_m \exp\left(-\gamma_m x^2\right), \tag{B.3}$$

such that $\mathrm{Re}\{\gamma_m\} > 0$. Then

$$\left| \mathrm{sinc}(3^n Bx) - \frac{1}{3^n}\sum_m \sum_{l=-(3^n-1)/2}^{(3^n-1)/2} a_{m,l}\, g_{m,l}(x) \right| \quad \leq \quad |\epsilon_B(x)|, \text{ for } n \geq 0. \tag{B.4}$$

where $a_{m,l} = \alpha_m \exp\left(\mathrm{i}\frac{(Bl)^2}{\mathrm{Im}\{\gamma_m\}}\right)$ and

$$g_{m,l}(x) = \exp\left(-\mathrm{Re}\{\gamma_m\}x^2\right)\exp\left(-\mathrm{i}\mathrm{Im}\{\gamma_m\}\left(x - \frac{Bl}{\mathrm{Im}\{\gamma_m\}}\right)^2\right). \tag{B.5}$$

Where $a_{m,l}$ and $g_{m,l}(x)$ are obtained using the identities

(B.6)

$$\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx) + 1\right]\exp\left(-\gamma_m x^2\right) = \sum_{l=-(3^{n+1}-1)/2}^{(3^{n+1}-1)/2} \exp\left(-\gamma_m x^2 + \mathrm{i}2Blx\right)$$

Algorithm 1 Representation of $\mathrm{sinc}\left(B_0 x\right)$ as a sum of chirplets

Given $0 \leq B_0 \in \mathbb{R}$, consider $f\left(x\right) = \mathrm{sinc}\left(x\right) + \mathrm{i}\,\mathrm{cosinc}\left(x\right) = \frac{\mathrm{e}^{\mathrm{i}x}-1}{\mathrm{i}x}$ and $g\left(x\right) = \mathrm{e}^{-x^2} + \mathrm{i}\,2\pi^{-1/2}F\left(x\right)$, $F\left(x\right)$ being the Dawson's function.

(1) Compute $n = \log_3\lfloor B_0 \rfloor + 1$
(2) Set $B = B_0 3^{-n}$.
(3) Solve the moment problem corresponding to $f\left(Bx\right)$ and $g\left(x\right)$:

$$h_n = \frac{f_n}{g_n} = B^n \frac{\frac{\mathrm{i}^n}{(n+1)!}}{\frac{\mathrm{i}^n}{\Gamma\left(\frac{n+2}{2}\right)}} = B^n \frac{\Gamma\left(\frac{n+2}{2}\right)}{(n+1)!} = \sum_m \alpha_m \theta_m^n$$

for $\left(\alpha_m, \theta_m\right)$ using the generalization of Pade approximation in [4]
(4) Set $\left(\alpha_m, \gamma_m\right) = \left(\alpha_m, \theta_m^2\right)$
(5) Form the approximation

$$\mathrm{sinc}\left(B_0 x\right) \approx \frac{1}{3^n} \sum_m \sum_{l=-(3^n-1)/2}^{(3^n-1)/2} \alpha_m \left[\begin{array}{l} \exp\left(\mathrm{i}\frac{(Bl)^2}{\mathrm{Im}\{\gamma_m\}}\right) \exp\left(-\mathrm{Re}\left\{\gamma_m\right\} x^2\right) \\ \times \exp\left(\begin{array}{c} -\mathrm{i}\,\mathrm{Im}\left\{\gamma_m\right\} \\ \times \left(x - \frac{Bl}{\mathrm{Im}\{\gamma_m\}}\right)^2 \end{array}\right) \end{array}\right]$$

and

$$\text{(B.7)} \quad \exp\left(-\gamma_m x^2 + \mathrm{i}2Blx\right)$$
$$= \exp\left(-\mathrm{Re}\left\{\gamma_m\right\} x^2\right) \exp\left(-\mathrm{iIm}\left\{\gamma_m\right\} \left(x - \frac{Bl}{\mathrm{Im}\left\{\gamma_m\right\}}\right)^2\right) \exp\left(\mathrm{i}\frac{(Bl)^2}{\mathrm{Im}\left\{\gamma_m\right\}}\right).$$

Example2 says that the sinc function can be approximated as a sum of shifted, Gaussian tapered chirps. One can determine $\left(\alpha_m, \gamma_m\right)$ using the generalization of Pade approximation in [4] by solving the appropriate moment problem (see Step 3 of Algorithm 1). This type of approximations of $\mathrm{sinc}\left(x\right)$ can be used to construct a multiresolution scheme for band-limited function as an alternative to existing multiscale approaches. It is important to point out that unlike chirplet decomposition methods presented in [3, 2], the moment problem provides an explicit solution for $\left(\alpha_m, \gamma_m\right)$ while coupling the real and imaginary part of the complex Gaussian parameters γ_m. Algorithm 1 outlines approximating a sinc of arbitrary bandwidth as a sum of scaled cosines based on the moment problem and Example 2. A corresponding plot is presented in Figure B.1.

Corollary 3. *Let*

$$\epsilon_B\left(x\right) = \mathrm{sinc}\left(Bx\right) - f\left(x\right)$$

and

$$\epsilon_{3^n B}\left(x\right) = \mathrm{sinc}\left(3^n Bx\right) - \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos\left(2Blx\right) + 1\right] f\left(x\right) = \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos\left(2Blx\right) + 1\right] \epsilon_B\left(x\right)$$

Then

$$\chi_{[-B,B]}\left(k\right) = \frac{B}{\pi}\hat{f}\left(k\right) + \frac{B}{\pi}\hat{\epsilon}_B\left(k\right)$$

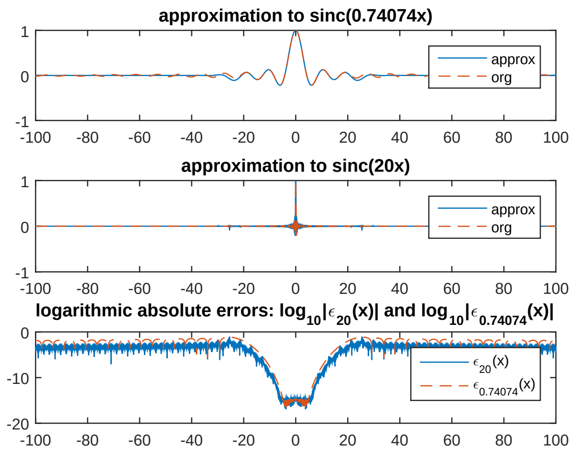

FIGURE B.1. Approximation of $\operatorname{sinc}(Bx)$ as a sum of chirplets (see Corollary 2) using Algorithm 1. On top and middle plots, $\operatorname{sinc}(B\,x)$ and $\operatorname{sinc}(B_0 x)$ (red dashed) along with their approximations (solid blue), for $B = 3^{-3}20$ and $B_0 = 20$, respectively. On the bottom plot, the logarithmic absolute errors for B (red dashed) and B_0(blue solid). As derived the error corresponding to B_0 is less than that of B.

and

$$\chi_{[-3^n B, 3^n B]}(k) = \frac{B}{\pi} \sum_{l=-(3^n-1)/2}^{(3^n-1)/2} \hat{f}(k - 2Bl) + \frac{B}{\pi} \hat{\epsilon}_{3^n B}(k)$$

where

$$\hat{\epsilon}_B(k) = \int_{-\infty}^{\infty} \epsilon_B(x)\, \mathrm{e}^{-\mathrm{i}kx} dk$$

and

$$\hat{\epsilon}_{3^n B}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \epsilon_{3^n B}(x)\, \mathrm{e}^{-\mathrm{i}kx} dk = \sum_{l=-(3^n-1)/2}^{(3^n-1)/2} \hat{\epsilon}_B(k - 2Bl)\,.$$

Proof. Direct consequence of Theorem 1. Taking the Fourier transform of (B.1)

$$2\pi \frac{\chi_{[-B,B]}(k)}{2B} = \int_{-\infty}^{\infty} \operatorname{sinc}(Bx)\, \mathrm{e}^{-\mathrm{i}kx} dx = \hat{f}(k) + \hat{\epsilon}_B(k)$$

$$\chi_{[-B,B]}(k) = \frac{B}{\pi} \hat{f}(k) + \frac{B}{\pi} \hat{\epsilon}_B(k)$$

Consequently,

$$\begin{aligned}\frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)+1\right]\operatorname{sinc}(Bx) = \operatorname{sinc}(3^n Bx)&\\ &= \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)+1\right][f(x)+\epsilon_B(x)]\\ &= \frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)+1\right] f(x)+\epsilon_{3^n B}(x)\end{aligned}$$

Taking Fourier transform of both sides

$$\begin{aligned}\int_{-\infty}^{\infty}\operatorname{sinc}(3^n Bx)\,\mathrm{e}^{-\mathrm{i}kx}dx &= \int_{-\infty}^{\infty}\frac{1}{3^n}\left[\sum_{l=1}^{(3^n-1)/2} 2\cos(2Blx)+1\right][f(x)-\epsilon_B(x)]\,\mathrm{e}^{-\mathrm{i}kx}dx\\ 2\pi\frac{\chi_{[-3^nB,3^nB]}(k)}{2\,3^n B} &= \frac{1}{3^n}\int_{-\infty}^{\infty}\left[\sum_{l=1}^{(3^n-1)/2}[\delta(k'-2Bl)+\delta(k'+2Bl)]+\delta(k')\right]\left[\hat{f}(k-k')+\hat{\epsilon}_B(k-k')\right]dk'\\ &= \frac{1}{3^n}\int_{-\infty}^{\infty}\left[\sum_{l=-(3^n-1)/2}^{(3^n-1)/2}\delta(k'-2Bl)\right]\left[\hat{f}(k-k')+\hat{\epsilon}_B(k-k')\right]dk'\\ &= \frac{1}{3^n}\sum_{l=-(3^n-1)/2}^{(3^n-1)/2}\left[\hat{f}(k-2Bl)+\hat{\epsilon}_B(k-2Bl)\right]\\ \chi_{[-3^nB,3^nB]}(k) &= \frac{B}{\pi}\sum_{l=-(3^n-1)/2}^{(3^n-1)/2}\hat{f}(k-2Bl)+\frac{B}{\pi}\hat{\epsilon}_{3^nB}(k)\end{aligned}$$

□

Example 4. By (B.3), and using the identity

$$\int_{-\infty}^{\infty}\mathrm{e}^{-pt^2}\mathrm{e}^{\mathrm{i}\omega t}dt = \sqrt{\frac{\pi}{p}}\mathrm{e}^{-\frac{\omega^2}{4p}},\quad \forall p\in\mathbb{C},\ \mathrm{Re}\{p\}>0,$$

we have

$$\chi_{[-B,B]}(k) = \frac{B}{\pi}\sum_m \alpha_m\sqrt{\frac{\pi}{\gamma_m}}\exp\left(-\frac{k^2}{4\gamma_m}\right)+\frac{B}{\pi}\hat{\epsilon}_B(k) \tag{B.8}$$

and

$$\chi_{[-3^nB,3^nB]}(k) = \frac{B}{\pi}\sum_{l=-(3^n-1)/2}^{(3^n-1)/2}\sum_m \alpha_m\sqrt{\frac{\pi}{\gamma_m}}\exp\left(-\frac{(k-2Bl)^2}{4\gamma_m}\right)+\frac{B}{\pi}\hat{\epsilon}_{3^nB}(k)$$

equivalently

$$\chi_{[-B,B]}(k) = \frac{B}{\pi}\sum_{l=-(3^n-1)/2}^{(3^n-1)/2}\sum_m \alpha_m\sqrt{\frac{\pi}{\gamma_m}}\exp\left(-\frac{(3^nk-2Bl)^2}{4\gamma_m}\right)+\frac{B}{\pi}\hat{\epsilon}_{3^nB}(3^nk)$$

$$
\begin{aligned}
\tilde{k}(\xi) &= \int \mathrm{e}^{\mathrm{i}\omega x^2} \tilde{\chi}_{[-1,1]}(x)\, \mathrm{e}^{\mathrm{i}\pi\xi x} dx \\
&= \int \mathrm{e}^{\mathrm{i}\omega x^2} \sum_m \alpha_m \mathrm{e}^{\gamma_m x^2} \mathrm{e}^{\mathrm{i}\pi\xi x} dx \\
&= \sum_m \alpha_m \int \mathrm{e}^{(\mathrm{i}\omega+\gamma_m)x^2} \mathrm{e}^{\mathrm{i}\pi\xi x} dx \\
&= \sum_m \alpha_m \sqrt{-\frac{\pi}{\mathrm{i}\omega+\gamma_m}}\, \mathrm{e}^{\frac{(\pi\xi)^2}{4(\mathrm{i}\omega+\gamma_m)}} dx
\end{aligned}
$$

$$
\begin{aligned}
\tilde{A}_{mn} &= \int\int \mathrm{e}^{-\mathrm{i}\pi m x} \tilde{\chi}_{[-1,1]}(x)\, \mathrm{e}^{\mathrm{i}\omega(x-y)^2} \tilde{\chi}_{[-1,1]}(y)\, \mathrm{e}^{\mathrm{i}\pi n y} dy\, dx \\
&= \int \mathrm{e}^{-\mathrm{i}\pi m x} \sum_l \alpha_l \mathrm{e}^{\gamma_l x^2} \mathrm{e}^{\mathrm{i}\omega x^2} \left[\sum_k \alpha_k \int \mathrm{e}^{(\mathrm{i}\omega+\gamma_k)y^2} \mathrm{e}^{\mathrm{i}(\pi n - 2\omega x)y} dy \right] dx \\
&= \int \mathrm{e}^{-\mathrm{i}\pi m x} \sum_l \alpha_l \mathrm{e}^{(\mathrm{i}\omega+\gamma_l)x^2} \left[\sum_k \alpha_k \sqrt{\frac{\pi}{-(\mathrm{i}\omega+\gamma_k)}}\, \mathrm{e}^{\frac{(\pi n - 2\omega x)^2}{4(\mathrm{i}\omega+\gamma_k)}} \right] dx \\
&= \sum_k \sum_l \alpha_k \alpha_l \sqrt{\frac{\pi}{-(\mathrm{i}\omega+\gamma_k)}} \int \mathrm{e}^{-\mathrm{i}\pi m x} \mathrm{e}^{(\mathrm{i}\omega+\gamma_l)x^2} \mathrm{e}^{\frac{(\pi n - 2\omega x)^2}{4(\mathrm{i}\omega+\gamma_k)}} dx \\
&= \sum_k \sum_l \alpha_k \alpha_l \sqrt{\frac{\pi}{-(\mathrm{i}\omega+\gamma_m)}} \int \mathrm{e}^{-\mathrm{i}\pi m x} \mathrm{e}^{\left[\frac{(\pi n)^2}{4(\mathrm{i}\omega+\gamma_m)} - \frac{\pi n \omega}{(\mathrm{i}\omega+\gamma_k)}x + \left(\mathrm{i}\omega+\gamma_l+\frac{\omega^2}{(\mathrm{i}\omega+\gamma_k)}\right)x^2\right]} dx \\
&= \sum_k \sum_l \alpha_k \alpha_l \mathrm{e}^{\frac{(\pi n)^2}{4(\mathrm{i}\omega+\gamma_k)}} \sqrt{\frac{\pi}{-(\mathrm{i}\omega+\gamma_k)}} \int \mathrm{e}^{-\left(\mathrm{i}\pi m + \frac{\pi n \omega}{(\mathrm{i}\omega+\gamma_k)}\right)x} \mathrm{e}^{\left(\mathrm{i}\omega+\gamma_l+\frac{\omega^2}{(\mathrm{i}\omega+\gamma_k)}\right)x^2} dx \\
&= \sum_k \sum_l \alpha_k \alpha_l \mathrm{e}^{\frac{(\pi n)^2}{4(\mathrm{i}\omega+\gamma_k)}} \sqrt{\frac{\pi}{-(\mathrm{i}\omega+\gamma_k)}} \sqrt{\frac{\pi}{-\left(\mathrm{i}\omega+\gamma_l+\frac{\omega^2}{(\mathrm{i}\omega+\gamma_k)}\right)}}\, \mathrm{e}^{-\frac{\left(\mathrm{i}\pi m+\frac{\pi n\omega}{(\mathrm{i}\omega+\gamma_k)}\right)^2}{4\left(\mathrm{i}\omega+\gamma_l+\frac{\omega^2}{(\mathrm{i}\omega+\gamma_k)}\right)}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{(\pi n - 2\omega x)^2}{4(\mathrm{i}\omega+\gamma_k)} + (\mathrm{i}\omega+\gamma_{m'})\, x^2 &= \frac{(\pi n)^2 + (2\omega)^2 x^2 - 4\pi n \omega x}{4(\mathrm{i}\omega+\gamma_k)} + (\mathrm{i}\omega+\gamma_l)\, x^2 \\
&= \frac{(\pi n)^2}{4(\mathrm{i}\omega+\gamma_k)} - \frac{\pi n \omega}{(\mathrm{i}\omega+\gamma_k)}x + \left(\mathrm{i}\omega+\gamma_l+\frac{\omega^2}{(\mathrm{i}\omega+\gamma_k)}\right)x^2
\end{aligned}
$$

$$\int \mathrm{e}^{ax^2} \mathrm{e}^{\pm(b+\mathrm{i}k)x} dx = \sqrt{\frac{\pi}{-a}} \mathrm{e}^{-\frac{(b+\mathrm{i}k)^2}{4a}}$$