A generalised Padé approach to approximating the spectra for a class of highly oscillatory integral operators

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Abstract

Some abstract here.

1 Introduction

Highly oscillatory integral operators $\mathcal{F}_\omega : L^2(-1,1) \to L^2(-1,1)$ of the form

$$\mathcal{F}_\omega f(x) := \int_{-1}^{1} f(y) K_\omega(x,y) dy, \qquad \omega \gg 1, \tag{1.1}$$

with complex-symmetric kernels

$$K_\omega(x,y) := e^{\mathrm{i}\omega\eta(x,y)}, \quad \eta \in C(-1,1)^2, \quad \mathrm{i} = \sqrt{-1} \tag{1.2}$$

are well-established tools for modelling the image of a field distribution f in laser and maser interferometers [9, 10, 15]. The geometry of the reflective mirrors in the system is described by the real-valued *oscillator function* η. For example, the Fox-Li oscillator function

$$\eta(x,y) := (x-y)^2 \tag{1.3}$$

models a configuration in which rectangular plane-parallel mirrors are placed at a fixed distance opposite one another [10], while

$$\eta(x,y) := xy \tag{1.4}$$

models a confocal spherical system in which the mirrors are positioned so that the surface of each reflector is located at the center of its counterpart [10].

Of both practical and mathematical interest are the field distributions φ — termed *modes* in laser engineering literature [2, 10, 14] — that replicate themselves up to some constant λ. If λ is known, the energy loss associated to φ as it passes through the interferometer can then be quantified via the loss function $E(\lambda;\varphi) := 1 - |\lambda|^2$. If, in addition, a generic field distribution f can be expressed as a linear combination of a collection of

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modes, then it can be shown that its repeated passage through the system results in f converging to the mode with lowest energy loss [14].

Mathematically, determination of λ and φ gives rise to the well-known eigenvalue problem

$$\mathcal{F}_\omega \varphi = \lambda \varphi, \qquad \varphi \neq 0, \tag{1.5}$$

in which λ is the *eigenvalue* of the corresponding *eigenfunction* φ. Since (1.1) is compact, its spectrum is at most countable and has the origin as its only possible point of accumulation. This does not alone ensure existence of any eigenvalues; indeed the existence of infinitely many eigenvalues for all but countably-many ω was only first proved for (1.3) in [12].

Matters of determining the spectrum of (1.1)–(1.2) are significantly complicated by the fact that (1.2) is non-Hermitian; that is, $K_\omega(x,y) \neq \overline{K_\omega(x,y)}$. This feature excludes the possibility of applying stationary-point methods like the Rayleigh-Ritz procedure to approximate the eigenvalues [15] or making any a priori statements about whether they are real or complex valued (though our own results and the extensive numerical work in [4] suggests that, more often than not, the eigenvalues are indeed complex). In a few cases some analytical results have been obtained on specific choices of $\eta(x,y)$ or on relaxed problems. The spectral problem for (1.4) was fully solved in [7], while a convergence result as $\omega \to \infty$ on the ϵ-pseudo-spectrum $\{\lambda \in \mathbb{C} : \|(\mathcal{F}_\omega - \lambda I)^{-1}\|_{L^2} \geq 1/\epsilon\}$ was proved in [14]. More recently, in [5] an asymptotic result was proven for the eigenvalues of (1.1) with $\eta(x,y) := |x-y|$ and a Wiener-Hopf theoretical approach was used in [3] to obtain a topological result on the singular values of $\mathcal{F}_\omega$.

Taking a numerical approach to the problem is also not without its challenges. The highly-oscillatory nature of (1.2) dictates that an accurate quadrature scheme is too costly and worsens for large values of ω [4]. A typical alternative is to use the finite-section method, in which (1.5) is linearised by expressing an eigenfunction in terms of an orthonormal basis of the underlying space. This linearisation induces matrix entries that depend on the choice of basis and require computation to eigendecompose the resulting matrix. The accuracy of the eigenvalue approximations is then dictated by the conflicting principles of rapid matrix entry decay and the complexity of their computation [4].

In this paper we perform a similar linearisation of (1.5) but based on the Fourier series decomposition of φ. As was the case in [4], this formulation leads to matrix entries that are double integrals over $(-1,1)$ that require computation to obtain the eigenvalue approximations. The key component of our method is that we rewrite these integrals with domain $(-\infty,\infty)$ using the indicator function $\chi_{(-1,1)}(x)$ and then approximate $\chi_{(-1,1)}(x)$ by a linear combination of Gaussian functions. Under the assumption that $\eta(x,y)$ is a 2nd order polynomial

$$\eta(x,y) = A + Bx + Cxy + Dy + Ex^2 + Fy^2 \tag{1.6}$$

for some real-valued coefficients A, B, C, D, E, F, these integrals may be computed exactly and rapidly. At the price of Gibbs-type effects when approximating $\chi_{(-1,1)}(x)$ at $x = \pm 1$, this formulation eradicates the issue in the finite section approach of offsetting the complexity of the matrix entry computations against their decay. Such a choice of oscillator function has sufficient generality to include the cases (1.3) and (1.4), and in some applications (1.6) may also provide a reasonable two or three point Taylor series approximation to the function of interest on $[-1,1]^2$. Furthermore, (1.6) does not assume complex symmetry and therefore offers a form of generality beyond the conventional physical assumptions of the problem.

This paper is organised as follows. In Section 2 we describe the construction of the Gaussian approximation of $\chi_{(-1,1)}(x)$ using the generalised Padé approximation of [20] and derive the formula and asymptotic results for the matrix entries resulting from (1.6). In Section 3 we run our method for several choices of oscillator function included by (1.6) and, where applicable, compare the results against those obtained in previous works. In Section 4 we comment on the strengths and weaknesses of the method and whether it can be regarded as a credible addition to the current set of approaches.

2 Method

2.1 Generalised Padé approximation

The $[M/N]$ Padé approximant of a smooth function $f(x)$ is a rational function $R(x) = P_M(x)/Q_N(x)$ — where $P_M(x), Q_N(x)$ are polynomials of degrees M and N respectively — that is computed by equating the coefficients of $R(x)$ to the power series coefficients of $f(x)$. Matching the coefficients in this way gives rise to a linear system $\boldsymbol{A}\boldsymbol{q} = \boldsymbol{p}$, where $\boldsymbol{A} \in \mathbb{R}^{M\times N}$ is a known matrix, that can be solved to uniquely determine the coefficients $\boldsymbol{p} \in \mathbb{R}^M$ of $P_M(x)$ and $\boldsymbol{q} \in \mathbb{R}^N$ of $Q_N(x)$ whenever a solution exists. The resulting approximant often converges to $f(x)$ when the Taylor series of $f(x)$ does not [11].

In the generalised approach of [20], the approximant $G_J(x)$ of a function $f(x)$ that is analytic in a neighbourhood of the origin is constructed by computing complex-valued parameters $\{(\alpha_j, \gamma_j)\}_{j=1}^J$ and forming the sum

$$G_J(x) := \sum_{j=1}^{J} \alpha_j g(\gamma_j x), \tag{2.1}$$

where $g(x)$ is also analytic in a neighbourhood of the origin. The approximation of $f(x)$ by $G_J(x)$ holds in the sense that

$$\left| f^{(k)}(0) - G_J^{(k)}(0) \right| < \epsilon, \quad k = 0, 1, \ldots, 2K,\ J \leq K \tag{2.2}$$

for some chosen error tolerance $\epsilon > 0$ and matching level K. Such a matching argument is analogous to the computation of the polynomial coefficients in the standard Padé approximation, with $G_J(x)$ exhibiting comparable asymptotic growth to $f(x)$ if $g(x)$ is chosen appropriately. Furthermore, $G_J(x)$ possesses additional useful properties over the standard Padé method, such as preserving band-limitedness of $f(x)$ [20].

By definition, the functions $f(x)$ and $g(x)$ are given by the pointwise-convergent Maclaurin series $f(x) = \sum_{k=0}^{\infty} f_k x^k$ and $g(x) = \sum_{k=0}^{\infty} g_k x^k$ at every point in the neighbourhood of the origin for which they are analytic. Assuming that $g_k = 0 \Rightarrow f_k = 0$, the parameters $\{(\alpha_j, \gamma_j)\}_{j=1}^J$ are obtained by forming the quotients $h_k = f_k/g_k$ and solving the relaxed moment-matching problem

$$\left| h_k - \sum_{j=1}^{J} \alpha_j \gamma_j^k \right| < \epsilon, \quad k = 0, 1, \ldots, 2K,\ J \leq K. \tag{2.3}$$

One such solution to (2.3) is obtained via the algorithm of Kung [13], which is described in Algorithm 1 and is our choice of method in this paper. The number of terms J required to satisfy (2.3) depends on ϵ and on the singular value decay of the $(2K + 1) \times (2K + 1)$-dimensional Hankel matrix formed from the moments h_k; see [20] for a more in-depth discussion on the computation complexity of the algorithm. With $\{(\alpha_j,\gamma_j)\}_{j=1}^{J}$ determined, the approximation (2.2) is an immediate consequence of (2.3) [13, 20].

Algorithm 1: Kung's algorithm

Input: Generalised Padé tolerance $\epsilon > 0$; Maclaurin series coefficient level $K \in \mathbb{N}$; target function Maclaurin series coefficents $\{f_k\}_{k=0}^{2K}$; approximant Maclaurin series coefficents $\{g_k\}_{k=0}^{2K}$.

Output: α_j, γ_j for $j = 1,\dots,J$, $J \le K$.

for $k = 0,\dots,2K$ **do**
  $h_k = f_k/g_k$. `// Set the moments`

for $k = 0,\dots,K$ **do**
  for $l = 0,\dots,K$ **do**
    Set $\boldsymbol{H}[k+1,l+1] = h_{k+l}$. `// Form the Hankel matrix`

$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$. `// Compute the singular value decomposition`

Order the singular values σ_k into descending order if necessary and choose $J \le K$ such that $\sqrt{J}\sigma_{J+1} < \epsilon^2$.

Form the matrices $\tilde{\boldsymbol{U}} = \boldsymbol{U}[1:K-1,1:J]$, $\widehat{\boldsymbol{U}} = \boldsymbol{U}[2:K,1:J]$ and $\boldsymbol{\Sigma}_J = \boldsymbol{\Sigma}[1:J,1:J]$, where $\boldsymbol{A}[R_1:R_2,C_1:C_2]$ denotes the matrix formed by extracting rows R_1 to R_2 (inclusive) and columns C_1 to C_2 (inclusive) of $\boldsymbol{A}$.

Form $\boldsymbol{A} = (\tilde{\boldsymbol{U}}\boldsymbol{\Sigma}_J^{1/2})^\dagger(\widehat{\boldsymbol{U}}\boldsymbol{\Sigma}_J^{1/2})$, $\boldsymbol{b} = \boldsymbol{\Sigma}_J^{1/2}\boldsymbol{V}^*[1:J,1]$, $\boldsymbol{c} = \boldsymbol{U}[1,1:J]\boldsymbol{\Sigma}_J^{1/2}$, where $\boldsymbol{\Sigma}_J^{1/2}[k,l] = \sqrt{\boldsymbol{\Sigma}_J[k,l]}$ and † is the pseudo-inverse operator.

$\boldsymbol{A} = \boldsymbol{W}\boldsymbol{\Lambda}\boldsymbol{W}^{-1}$ `// Compute the eigendecomposition`

for $j = 1,\dots,J$ **do**
  $\alpha_j = (\boldsymbol{W}^T\boldsymbol{c}^T)\circ(\boldsymbol{W}^{-1}\boldsymbol{b})$, where $\boldsymbol{a}\circ\boldsymbol{b}$ denotes the element-wise (Hadamard) product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
  $\gamma_j = \boldsymbol{\Lambda}[j,j]$.

One of the benefits of generalised Padé approximation is its ability to be combined with the Fourier transform $\mathscr{F}$ to approximate one function $\mathscr{F}f$ by another $\mathscr{F}G_J$ with favourable analytic properties. How $\mathscr{F}f$ and $\mathscr{F}G_J$ differ analytically depends entirely on how $\mathscr{F}$ transforms f and G_J, for which a broad set of outcomes is possible. Irrespective of this difference, Plancherel's theorem [8, Chapter 1] ensures that the L^2 accuracy of the generalised Padé approximation is preserved under the action of $\mathscr{F}$.

We define the Fourier transform $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as

$$(\mathscr{F}f)(x) := \int_{-\infty}^{\infty} f(\xi)e^{\mathrm{i}\pi x\xi}d\xi \tag{2.4}$$

and consider the normalised cardinal sine function

$$\operatorname{sinc}(x) := \begin{cases} (\pi x)^{-1}\sin(\pi x) & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases} \tag{2.5}$$

By defining $\operatorname{sinc}(x) = 1$ at $x = 0$, it can be seen that $\operatorname{sinc}(x)$ coincides with the Maclaurin series of $x^{-1}\sin(x)$ at every $x \in \mathbb{R}$ and hence is analytic on $\mathbb{R}$; thus enabling its use for generalised Padé approximation. Equation (2.5) is termed the normalised cardinal sine function for the fact that its integral over $\mathbb{R}$ is equal to 1. It is a function that arises regularly in signal processing due to the following result.

Lemma 2.1. *For $f(\xi) = \operatorname{sinc}(\xi)$, the Fourier transform $\hat{f}$ of f is*

$$\hat{f}(x) = \chi_{(-1,1)}(x), \quad x \in \mathbb{R}. \tag{2.6}$$

In addition, $f \in L^2(\mathbb{R})$.

Proof. Let $g(x) = \chi_{(-1,1)}(x)$. Clearly $g \in L^2(\mathbb{R})$. Then

$$\mathscr{F}^{-1}[g](\xi) = \frac{1}{2}\int_{-\infty}^{\infty} g(x)e^{-\mathrm{i}\pi\xi x}dx = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{(-1,1)}(x)e^{-\mathrm{i}\pi\xi x}dx = \frac{1}{2}\int_{-1}^{1} e^{-\mathrm{i}\pi\xi x}dx = \operatorname{sinc}(\xi).$$

Hence $\mathscr{F}^{-1}[g](\xi) = f(\xi)$ and $g = \hat{f}$. The fact that $f \in L^2(\mathbb{R})$ follows from Plancherel's theorem. □

We conclude this subsection with two results that unify the previous results to construct the Gaussian sum approximation of $\chi_{(-1,1)}(x)$, denoted by

$$\tilde{\chi}_{(-1,1)}(x) := \sum_{j=1}^{J} \alpha_j e^{-\gamma_j x^2}, \quad \alpha_j \in \mathbb{C},\ \Re\{\gamma_j\} > 0,\ \Im\{\gamma_j\} \in \mathbb{R}, \tag{2.7}$$

where $\Re\{\gamma_j\}$ and $\Im\{\gamma_j\}$ are the real and imaginary components of γ_j respectively. A key property of this construction is that each Gaussian function in (2.7) is an eigenfunction of $\mathscr{F}$. By linearity, the effect of applying $\mathscr{F}$ to (2.7) is then fully described by transforming each α_j and γ_j using the Fourier transform formula for a Gaussian function. While this particular application is central to our method, it should be emphasised that this is just one use of generalised Padé approximation and numerous others are possible; cf. [19, 20].

Proposition 2.2. *Let*

$$f(\xi) = \operatorname{sinc}(\xi) + \mathrm{i}\mathcal{H}[\operatorname{sinc}(t)](\xi) = \frac{e^{\mathrm{i}\pi\xi} - 1}{\mathrm{i}\pi\xi}, \tag{2.8}$$

$$g(\xi) = e^{-\xi^2} + \mathrm{i}\mathcal{H}[e^{-t^2}](\xi) = e^{-\xi^2}\left(1 + \frac{2\mathrm{i}}{\sqrt{\pi}}\int_0^{\xi} e^{t^2}dt\right), \tag{2.9}$$

where $\mathcal{H}$ denotes the Hilbert transform, and let $\epsilon > 0$, $K > 0$. Then there exists $\{(\tilde{\alpha}_j, \tilde{\gamma}_j)\}_{j=1}^J$ such that f and g satisfy (2.3)*. In addition, if $|\Re\{\tilde{\gamma}_j\}| > |\Im\{\tilde{\gamma}_j\}|$ for $j = 1, \ldots, J$, then*

$$\chi_{(-1,1)}(x) = \sum_{j=1}^{J} \alpha_j e^{-\gamma_j x^2} + \hat{r}_J(x), \qquad \alpha_j \in \mathbb{C},\ \Re\{\gamma_j\} > 0,\ \Im\{\gamma_j\} \in \mathbb{R}, \tag{2.10}$$

where

$$\alpha_j = \sqrt{\pi}\frac{\tilde{\alpha}_j}{\tilde{\gamma}_j}, \quad \gamma_j = \frac{\pi^2}{4\tilde{\gamma}_j^2} \tag{2.11}$$

and

$$\|\hat{r}_J\|_{L^2} = \|r_J\|_{L^2}. \tag{2.12}$$

Proof. In addition to $\operatorname{sinc}(\xi)$ and $e^{-\xi^2}$ begin analytic on $\mathbb{R}$, the imaginary parts of (2.8) and (2.9) also share the same property, thus implying that the signals $f(\xi)$, $g(\xi)$ are analytic on $\mathbb{R}$. Hence Algorithm 1 can be applied to find $\{(\tilde{\alpha}_j, \tilde{\gamma}_j)\}_{j=1}^J$ that satisfy (2.2).

Since $\mathcal{H}$ is a multiplier operator with multiplier $m(x) = -\mathrm{i} \cdot \operatorname{sgn}(x)$ [8, Chapter 3], i.e.

$$\mathscr{F}(\mathcal{H}u)(\xi) = -\mathrm{i} \cdot \operatorname{sgn}(\xi)\mathscr{F}(u)(\xi), \quad u \in L^2(\mathbb{R}), \tag{2.13}$$

it follows from Plancherel's theorem that the analytic signal $u+\mathrm{i}\mathcal{H}[u]$ is in $L^2(\mathbb{R})$ if and only if $u \in L^2(\mathbb{R})$. Then by Lemma 2.1 and the assumption $|\Re\{\tilde{\gamma}_j\}| > |\Im\{\tilde{\gamma}_j\}|$ it follows that the analytic signals $f(\xi)$ and $g(\tilde{\gamma}_j\xi)$ are both in $L^2(\mathbb{R})$, and hence the residual $r_J(\xi) := f(\xi) - \sum_{j=1}^{J} \tilde{\alpha}_j g(\tilde{\gamma}_j\xi) \in L^2(\mathbb{R})$. By the multiplier property (2.13), to take the Fourier transform of $f(\xi)$ and $g(\tilde{\gamma}_j\xi)$ it suffices to consider the original signals $\operatorname{sinc}(\xi)$ and $e^{-\xi^2}$. The Fourier transform of $\operatorname{sinc}(\xi)$ was shown in Lemma 2.1, while the Fourier transform of $e^{-\gamma_j\xi^2}$ for each γ_j is

$$\mathscr{F}[e^{-\gamma_j^2\xi^2}](x) = \frac{\sqrt{\pi}}{\gamma_j} \exp\left[-\frac{\pi^2 x^2}{4\gamma_j^2}\right]. \tag{2.14}$$

Hence by (2.2) and (2.14),

$$\begin{aligned}
f(\xi) &= \sum_{j=0}^{J} \tilde{\alpha}_j e^{-\tilde{\gamma}_j\xi^2} + r_J(\xi) \\
\Rightarrow \mathscr{F}[\operatorname{sinc}(\xi)](x) &= \sum_{j=1}^{J} \alpha_j \mathscr{F}[e^{-\tilde{\gamma}_j^2\xi^2}](x) + \hat{r}_J(x), \\
\Rightarrow \chi_{(-1,1)}(x) &= \sum_{j=1}^{J} \alpha_j e^{-\gamma_j x^2} + \hat{r}_J(x), \quad \Re\{\gamma_j\} > 0,
\end{aligned}$$

where $\hat{r}_J(x) \in L^2(\mathbb{R})$, $\|\hat{r}_J\|_{L^2} = \|r_J\|_{L^2}$ and α_j, γ_j are given by (2.11). $\square$

Corollary 2.3. *For the choice* (2.8)–(2.9) *of f and g, the $L^2(\mathbb{R})$ norm of $\hat{r}_J(x)$ in* (2.10) *is*

$$\|\hat{r}_L\|^2_{L^2(\mathbb{R})} = \frac{1}{4} - \frac{1}{2}\Re\left\{\sum_{j=1}^{L} \alpha_j \operatorname{erf}\left(\frac{\pi}{\sqrt{\gamma_j}}\right)\right\} + \frac{\sqrt{\pi}}{2} \sum_{j,k=1}^{L} \frac{\alpha_j\alpha_k}{\sqrt{\gamma_j+\gamma_k}},$$

where $\operatorname{erf}(x)$ *is the error function.*

Proof. See Sine integral function paper. $\square$

2.2 Derivation of the spectral approximation formulas

To discretise the eigenvalue problem for (1.1)–(1.2) with oscillator function (1.6) we consider the Fourier coefficients of the eigenfunction $\varphi(x) \in L^2(-1,1)$, defined as

$$\hat{\varphi}[m] := \frac{1}{2}\int_{-1}^{1} \varphi(x) e^{\mathrm{i}\pi m x} dx, \quad m \in \mathbb{Z}. \tag{2.15}$$

By Hölder's inequality it follows that $L^1(U) \subset L^2(\mathbb{R})$ whenever U is a bounded subset of $\mathbb{R}$, so that (2.15) is well-defined for all m. Moreover, the smoothness of the kernel (1.2) with (1.6) ensures that every eigenfunction is smooth and hence has a uniformly convergent Fourier series on $(-1,1)$ [17, Chapter 2] given by

$$\varphi(x) = \sum_{m=-\infty}^{\infty} \hat{\varphi}[m] e^{-\mathrm{i}\pi m x} \quad x \in (-1,1). \tag{2.16}$$

For the periodic extension of φ from $(-1,1)$ to $\mathbb{R}$, the Fourier series (2.16) is not necessarily uniformly convergent on $\mathbb{R}$, as $\varphi(x)$ may not even be continuous at $x = 2n$, $n \in \mathbb{Z}$. However, since $\mathbb{Z}$ has Lebesgue measure 0 and we only require uniform convergence of (2.16) on $\mathbb{R}$ for the interchange of integration and summation, such points may be ignored. To this end, we denote the periodic extension of φ to $\mathbb{R}$ also by φ, and use the two functions interchangeably.

Letting λ be an eigenvalue of the eigenfunction φ and considering the m-th Fourier coefficient (2.15), it follows from (1.5) and (1.1) that for each $m \in \mathbb{Z}$,

$$\lambda\hat{\varphi}[m] = \frac{\lambda}{2}\int_{-1}^{1}\varphi(x)e^{i\pi mx}dx = \frac{1}{2}\int_{-1}^{1}\mathcal{F}_\omega(x)e^{i\pi mx}dx = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\varphi(y)e^{\mathrm{i}\omega\eta(x,y)}e^{i\pi mx}dxdy.$$

Using $\chi_{(-1,1)}$ to extend the domain of integration in each integral from $(-1,1)$ to $\mathbb{R}$, we then decompose $\varphi(y)$ into its Fourier series (2.16) and exploit its uniform convergence to induce the linearisation

$$\begin{aligned}
\lambda\hat{\varphi}[m] &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{(-1,1)}(x)\chi_{(-1,1)}(y)\varphi(y)e^{-\mathrm{i}\pi my}e^{\mathrm{i}\omega\eta(x,y)}dxdy, \quad m\in\mathbb{Z}\\
&= \frac{1}{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{(-1,1)}(x)\chi_{(-1,1)}(y)e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{i\pi mx}dxdy\hat{\varphi}[n] \qquad (2.17)\\
&= \sum_{n=-\infty}^{\infty}\boldsymbol{A}_\omega[m,n]\hat{\varphi}[n] \qquad (2.18)\\
\Rightarrow \lambda\hat{\boldsymbol{\varphi}} &= \boldsymbol{A}_\omega\hat{\boldsymbol{\varphi}},
\end{aligned}$$

where $\boldsymbol{A}_\omega := (\boldsymbol{A}_\omega[m,n])_{m,n=-\infty}^{\infty}$ is an infinite matrix and $\hat{\boldsymbol{\varphi}} := (\hat{\varphi}[m])_{m=-\infty}^{\infty}$ is the infinite vector of Fourier coefficients of φ. Under this formulation, the eigenvalue problem (1.5) reduces to one of determining the entries $\boldsymbol{A}_\omega[m,n]$ and performing an eigendecomposition on the resulting matrix $\boldsymbol{A}_\omega$.

Theorem 2.4. *Let* $\{(\alpha_j,\gamma_j)\}_{j=1}^{L}$ *parametrise the Gaussian approximation* (2.10) *to* $\chi_{(-1,1)}(x)$. *The eigenvalues of* $\mathcal{F}_\omega$ *with oscillator function* (1.6) *are the eigenvalues of* $\boldsymbol{A}+\tilde{\boldsymbol{A}}$, *where*

$$\boldsymbol{A}_\omega[m,n] = \sum_{j=1}^{L}\sum_{k=1}^{L}P_{j,k,\omega}\exp\left[\rho_{j,k,\omega}(m,n)\right], \qquad (2.19)$$

$$\tilde{\boldsymbol{A}}_\omega[m,n] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_L(x,y)\exp[-\mathrm{i}\pi ny]\exp[\mathrm{i}\omega\eta(x,y)]\exp[\mathrm{i}\pi mx]dxdy, \qquad (2.20)$$

$$\rho_{j,k,\omega} = \frac{Q_{j,k,\omega}(B\omega+\pi m)^2 - R_{j,\omega}(B\omega+\pi m)(D\omega-\pi n) + S_{j,\omega}(D\omega-\pi n)^2}{T_{j,k,\omega}}, \qquad (2.21)$$

$$\psi_L(x,y) = \frac{1}{2}\left(\hat{r}_L(y)\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2} + \hat{r}_L(x)\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2} + \hat{r}_L(x)\hat{r}_L(y)\right), \qquad (2.22)$$

and the coefficients in (2.21) *are known and given by*

$$P_{j,k,\omega} = \frac{\pi\exp[\mathrm{i}A\omega]\alpha_j\alpha_k}{\sqrt{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega}},$$

$$Q_{j,k,\omega} = 4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k), \quad R_{j,\omega} = 4\mathrm{i}C\omega(\mathrm{i}E\omega-\gamma_j), \quad S_{j,\omega} = 4(\mathrm{i}E\omega-\gamma_j)^2.$$

The rapid computation times associated to the approximation matrix entries (2.19) is somewhat comparable to the preference of modified Fourier basis over the faster-convergent Legendre basis in [4], in the sense that slower entry decay can be a worthwhile compromise for faster entry computation.

By Theorem 2.4 given an oscillator function (1.6) and indicator approximation (2.10), $2N+1$ eigenvalues are readily obtained by performing an eigendecomposition on the $(2N+1) \times (2N+1)$-dimensional truncated matrix

$$(\boldsymbol{A}_\omega + \tilde{\boldsymbol{A}}_\omega)|_N[m,n] := \boldsymbol{A}_\omega[m,n] + \tilde{\boldsymbol{A}}_\omega[m,n], \quad -N \leq m,n \leq N. \tag{2.23}$$

The symmetry of the truncation (2.23) about 0 is chosen to maintain consistency with the way in which the limit of the Fourier series in (2.17) is taken.

If, in addition to $\boldsymbol{A}_\omega[m,n]$, the error matrix entries $\tilde{\boldsymbol{A}}_\omega[m,n]$ were known, this truncation would be the only source of error in the spectral approximations and as many desired eigenvalues could be approximated to any desired degree of accuracy by considering large enough N.

Corollary 2.5. *Let* $\|\boldsymbol{\alpha}\|_{\ell^1} = \sum_{j=1}^{L} |\alpha_j|$ *and* $\gamma^\vee = \min_{j=1,\ldots,L} \Re\{\gamma_j\}$. *For every* $m,n \in \mathbb{Z}$,

$$|\tilde{A}_{m,n,\omega}| \leq \left[\left(\frac{5}{2\gamma^\vee}\sqrt{\frac{\pi}{2}} + \frac{1}{2}\right)\|\boldsymbol{\alpha}\|_{\ell^1} + \frac{1}{2}\right]\left[\left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}} + 1\right)\|\boldsymbol{\alpha}\|_{\ell^1} + 1\right]. \tag{2.24}$$

An immediate observation to be made about (2.24) is that, aside from the trivial case in which $\|\boldsymbol{\alpha}\|_{\ell^1} = 0$, the bound is strictly greater than 1/2 and can therefore in no way be used to send $|\tilde{A}_{m,n,\omega}|$ to 0. While it may seem tempting to let $\gamma^\vee \to \infty$ in order to minimise (2.24), this would eventually violate the solution of the moment-matching problem (2.3) and render the indicator approximation meaningless.

In practice, (2.24) is likely a conservative upper bound, since the only notable discrepancies between the Gaussian approximation and $\chi_{(-1,1)}$ arise as a result of the Gibbs effect at $x = \pm 1$. While these effects can never be eradicated [18, Chapter 4], the contribution they have to (2.24) via $\widehat{r}_L$ in (2.22) is negligible apart from in small neighbourhoods of $x = \pm 1$. It is also a consequence of the Gibbs effect that $\hat{r}_L$, $\psi_L(x,y)$ and hence $\tilde{A}_{m,n,\omega}$ can never be 0, and that some error will always be present by taking only the eigenvalues of $\boldsymbol{A}$ as the approximations to the spectrum of $\mathcal{F}_\omega$.

In Figure 1, the right plot shows the residual $|r_L(x)|$ of the indicator function and its Gaussian approximation in the left plot from running () with $\epsilon \sim 10^{-7.5}$ and $K = 30$. To gain some insight into the size of $|\tilde{A}_{m,n,\omega}|$, $|r_L(x)|$ has been approximately enveloped by two scaled Gaussian density functions centred at $x = \pm 1$, chosen for their integral over $\mathbb{R}$ being equal to their scaling factor – in this case 0.4. While still being fairly conservative, substitution of this empirical bound on $|r_L(x)|$ into the formula for $\tilde{A}_{m,n,\omega}$ leads to $|\tilde{A}_{m,n,\omega}| \leq 123.75$ to 5 significant figures, compared to $|\tilde{A}_{m,n,\omega}| \leq 69345$ offered by (2.24).

Before providing the proofs for Theorem 2.4 and Corollary 2.5, we specify the complete algorithm for our method.

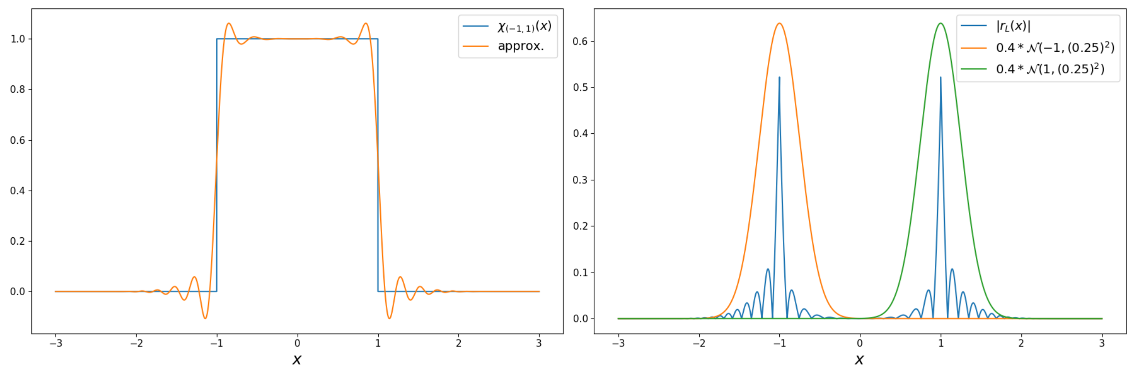

Figure 1: Approximate Gaussian envelopes of $|r_L(x)|$ for $L = 7$ resulting from Algorithm 2 with the Maclaurin coefficients of f and g in Proposition 2.2 for $K = 30$ and $\epsilon = 10^{-7.5}$.

Algorithm 2: Generalised Padé approximation of the spectrum of $\mathcal{F}_\omega$

Input: Oscillator function parameters (1.6); oscillation parameter $\omega \gg 1$; generalised Padé tolerance $\epsilon > 0$; Maclaurin series coefficient level $K \in \mathbb{N}$; Maclaurin series coefficients $f_k = \frac{\mathrm{i}^k}{(k+1)!}$ of (2.8), $g_k = \frac{\mathrm{i}^k}{\Gamma(\frac{k+2}{2})}$ of (2.9) for $k = 0, \ldots, 2K$; spectral truncation level $N \in \mathbb{N}$.

Output: Approximate eigenvalues $\{\widehat{\lambda}_n\}_{n=-N}^{N}$.

Produce $\{(\tilde{\alpha}_j, \tilde{\gamma}_j)\}_{j=1}^{L}$, $L \leq K$ using Algorithm 1 with tolerance $\epsilon > 0$, coefficient level $K \in \mathbb{N}$ and Maclaurin coefficients f_k, g_k for $k = 0, \ldots, 2K$.

for $j = 1, \ldots, L$ **do**

  $\alpha_j = \sqrt{\pi}\frac{\tilde{\alpha}_j}{\tilde{\gamma}_j}$, $\gamma_j = \frac{\pi^2}{4\tilde{\gamma}_j^2}$ `/* Transform the parameters using (2.11) */`

for $m = -N, \ldots, N$ **do**

  for $n = -N, \ldots, N$ **do**

    $\boldsymbol{A}[m, n] = \sum_{j=1}^{L}\sum_{k=1}^{L} P_{j,k,\omega} \exp[\rho_{j,k,\omega}(m, n)]$ using the formulas in Theorem 2.4.

$\boldsymbol{A} = \boldsymbol{\Phi}\hat{\boldsymbol{\Lambda}}\boldsymbol{\Phi}$. `/* Compute the eigendecomposition */`

for $n = -N, \ldots, N$ **do**

  $\widehat{\boldsymbol{\lambda}}_n = \widehat{\Lambda}_{n,n}$ `/* Compute the approximate eigenvalues */`

2.3 Approximation matrix asymptotics

2.4 Proofs

Proof of Theorem 2.4. Letting $\tilde{\chi}_{(-1,1)}(x) := \sum_{j=1}^{L} \alpha_j \exp[-\gamma_j x^2]$, the product of the two Gaussian indicator approximations given by (2.10) is

$$\chi_{(-1,1)}(x)\chi_{(-1,1)}(y) = \tilde{\chi}_{(-1,1)}(x)\tilde{\chi}_{(-1,1)}(y) + 2\psi_L(x,y), \tag{2.25}$$

where $\psi_L(x,y)$ is given by (2.22). Substitution of (2.25) into (**??**) gives

$$\begin{aligned}
\lambda\hat{\varphi}[m] &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\tilde{\chi}_{(-1,1)}(x)\tilde{\chi}_{(-1,1)}(y) + 2\psi_L(x,y)\right\} e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{i\pi m x} dxdy \\
&= \sum_{n=-\infty}^{\infty} (A_{m,n,\omega} + \tilde{A}_{m,n,\omega})\hat{\varphi}[n], \quad m \in \mathbb{Z},
\end{aligned}$$

which implies the linear system (**??**) with entries

$$A_{m,n,\omega} = \frac{1}{2}\sum_{j,k=1}^{L} \alpha_j\alpha_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\gamma_j x^2} e^{-\gamma_k y^2} e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{\mathrm{i}\pi m x} dxdy, \tag{2.26}$$

$$\tilde{A}_{m,n,\omega} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_L(x,y) e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{\mathrm{i}\pi m x} dxdy. \tag{2.27}$$

To calculate $A_{m,n,\omega}$ we apply the identity

$$\int_{-\infty}^{\infty} \exp[-a\xi^2]\exp[\pm(u+\mathrm{i}v)\xi]d\xi = \sqrt{\frac{\pi}{a}}\exp\left[\frac{(u+\mathrm{i}v)^2}{4a}\right], \quad \Re\{a\} > 0 \tag{2.28}$$

to each integral in x and y in (2.26). Let

$$\begin{aligned}
I &:= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\gamma_j x^2} e^{-\gamma_k y^2} e^{\mathrm{i}\omega(A+Bx+Cxy+Dy+Ex^2+Fy^2)} e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\pi m x} dxdy \\
&= \frac{e^{\mathrm{i}A\omega}}{2}\int_{-\infty}^{\infty} e^{(\mathrm{i}F\omega-\gamma_k)y^2} e^{\mathrm{i}(D\omega-\pi n)y}\int_{-\infty}^{\infty} e^{(\mathrm{i}E\omega-\gamma_j)x^2} e^{\mathrm{i}(B\omega+C\omega y+\pi m)x} dxdy.
\end{aligned}$$

Denote the integral in x by I_y. By construction of the γ_j, it follows that $\Re\{\mathrm{i}E - \gamma_j\} < 0$ and (2.28) may be applied to I_y to obtain

$$\begin{aligned}
I_y &= \int_{-\infty}^{\infty} \exp[(\mathrm{i}E\omega - \gamma_j)x^2]\exp[\mathrm{i}(B\omega + C\omega y + \pi m)x]dx \\
&= \sqrt{\frac{\pi}{\gamma_j - \mathrm{i}E\omega}}\exp\left[\frac{(B\omega + C\omega y + \pi m)^2}{4(\mathrm{i}E\omega - \gamma_j)}\right] \\
&= \sqrt{\frac{\pi}{\gamma_j - \mathrm{i}E\omega}}\exp\left[\frac{(B\omega + \pi m)^2}{4(\mathrm{i}E\omega - \gamma_j)}\right]\exp\left[\frac{C\omega(B\omega+\pi m)}{2(\mathrm{i}E\omega - \gamma_j)}y\right]\exp\left[\frac{C^2\omega^2}{4(\mathrm{i}E\omega - \gamma_j)}y^2\right].
\end{aligned}$$

Hence

$$\begin{aligned}
I &= P'_{j,m,\omega}\int_{-\infty}^{\infty} \exp[(\mathrm{i}F\omega - \gamma_k)y^2]\exp\left[\frac{C^2\omega^2}{4(\mathrm{i}E\omega - \gamma_j)}y^2\right]\exp[\mathrm{i}(D\omega - \pi n)y]\times \\
&\qquad\qquad \exp\left[\frac{C\omega(B\omega+\pi m)}{2(\mathrm{i}E\omega - \gamma_j)}y\right]dy \\
&= P'_{j,m,\omega}\int_{-\infty}^{\infty} \exp\left[S_{j,k,\omega}y^2\right]\exp\left[Q_{j,m,n,\omega}y\right]dy
\end{aligned} \tag{2.29}$$

where

$$\begin{aligned}
P'_{j,m,\omega} &= \frac{1}{2}\sqrt{\frac{\pi}{\gamma_j - \mathrm{i}E\omega}}\exp[\mathrm{i}A\omega]\exp\left[\frac{(B\omega+\pi m)^2}{4(\mathrm{i}E\omega-\gamma_j)}\right],\\
S_{j,k,\omega} &= \frac{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2}{4(\mathrm{i}E\omega-\gamma_j)},\\
Q_{j,m,n,\omega} &= \frac{2\mathrm{i}(D\omega-\pi n)(\mathrm{i}E\omega-\gamma_j)+C\omega(B\omega+\pi m)}{2(\mathrm{i}E\omega-\gamma_j)}.
\end{aligned}$$

Note that by virtue of $\Re\{\gamma_j\} > 0$ and $\omega \gg 1$,

$$\begin{aligned}
S_{j,k,\omega} &= \frac{4|\mathrm{i}E\omega-\gamma_j|^2(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2\overline{(\mathrm{i}E\omega-\gamma_j)}}{4|\mathrm{i}E\omega-\gamma_j|^2}\\
&= \frac{4|\mathrm{i}E\omega-\gamma_j|^2(\mathrm{i}F\omega-\gamma_k)-C^2\omega^2(\mathrm{i}E\omega+\overline{\gamma}_j)}{4|\mathrm{i}E\omega-\gamma_j|^2}\\
\Rightarrow \Re\{S_{j,k,\omega}\} &= \frac{-\beta_1^2\omega^2-\beta_2^2\omega^2+\mathcal{O}(\omega)+\mathcal{O}(1)}{\beta_3^2\omega^2}
\end{aligned}$$

for some real constants $\beta_1, \beta_2, \beta_3 \neq 0$, and hence $\Re\{S'_{j,k,\omega}\} < 0$ for all choices of oscillator polynomial $\eta(x,y)$. Therefore (2.28) may be applied to (2.29) to give

$$\begin{aligned}
I &= P'_{j,m,\omega}\sqrt{\frac{4\pi(\gamma_j-\mathrm{i}E\omega)}{T'_{j,k,\omega}}}\exp\left[-\frac{Q^2_{j,m,n\omega}}{4S_{j,k,\omega}}\right]\\
&= P_{j,k,\omega}\exp\left[\frac{(B\omega+\pi m)^2}{4(\mathrm{i}E\omega-\gamma_j)}\right]\exp\left[-\frac{\{2\mathrm{i}(D\omega-\pi n)(\mathrm{i}E\omega-\gamma_j)+C\omega(B\omega+\pi m)\}^2}{4(\mathrm{i}E\omega-\gamma_j)\{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2\}}\right]\\
&= P_{j,k,\omega}\exp\left[\frac{(B\omega+\pi m)^2T'_{j,k,\omega}-\{2\mathrm{i}(D\omega-\pi n)(\mathrm{i}E\omega-\gamma_j)+C\omega(B\omega+\pi m)\}^2}{T_{j,k,\omega}}\right]\\
&= P_{j,k,\omega}\exp\left[\frac{Q_{j,k,\omega}(B\omega+\pi m)^2-R_{j,\omega}(B\omega+\pi m)(D\omega-\pi n)+S_{j,\omega}(D\omega-\pi n)^2}{T_{j,k,\omega}}\right]
\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
P_{j,k,\omega} &= \frac{\pi\exp[\mathrm{i}A\omega]}{\sqrt{T'_{j,k,\omega}}},\\
T'_{j,k,\omega} &= 4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2,\\
T_{j,k,\omega} &= 4(\mathrm{i}E\omega-\gamma_j)T'_{j,k,\omega},\\
Q_{j,k,\omega} &= 4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k),\\
R_{j,\omega} &= 4\mathrm{i}C\omega(\mathrm{i}E\omega-\gamma_j),\\
S_{j,\omega} &= 4(\mathrm{i}E\omega-\gamma_j)^2,
\end{aligned}$$

and the entries $A_{m,n,\omega}$ follow from substituting (2.30) into (2.26). Approximations to the eigenvalues of $\mathcal{F}_\omega$ are then found via the diagonal entries of $\tilde{\boldsymbol{\Lambda}}$ in the eigendecomposition of $\boldsymbol{A}+\tilde{\boldsymbol{A}}$, given by

$$\boldsymbol{A}+\tilde{\boldsymbol{A}} = \hat{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Lambda}}\hat{\boldsymbol{\Phi}}^{-1},$$

where $\hat{\boldsymbol{\Phi}}$ is the infinite matrix of right eigenvectors of the Fourier coefficients of φ. $\square$

Proof of Corollary 2.5. Let $Z_{m,n,\omega}(x,y) = \exp[-\mathrm{i}\pi ny]\exp[\mathrm{i}\omega\eta(x,y)]\exp[\mathrm{i}\pi mx]$ and $\hat{r}(x)$ the indicator approximation remainder term given by (2.10). By definition of $\psi_L(x,y)$ and linearity, each entry $\tilde{A}_{m,n,\omega}$ may be written as

$$\tilde{A}_{m,n,\omega} = \mathcal{I}_{m,n,\omega} + \mathcal{J}_{m,n,\omega} + \mathcal{K}_{m,n,\omega},$$

where

$$\begin{aligned}
\mathcal{I}_{m,n,\omega} &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{r}_L(y)\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2} Z_{m,n,\omega}(x,y)dxdy,\\
\mathcal{J}_{m,n,\omega} &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{r}_L(x)\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2} Z_{m,n,\omega}(x,y)dxdy,\\
\mathcal{K}_{m,n,\omega} &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{r}(x)\hat{r}(y) Z_{m,n,\omega}(x,y)dxdy,
\end{aligned}$$

and the task becomes one of bounding these integrals. For $\mathcal{I}_{m,n,\omega}$,

$$\begin{aligned}
|\mathcal{I}_{m,n,\omega}| &\le \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\hat{r}_L(y)\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}\right| dxdy\\
&\le \frac{1}{2}\sum_{j=1}^{L}\int_{-\infty}^{\infty}\left|\alpha_j e^{-\gamma_j x^2}\right| dx\int_{-\infty}^{\infty}|\hat{r}_L(y)|\,dy\\
&\le \frac{\|\boldsymbol{\alpha}\|_{\ell^1}}{2}\int_{-\infty}^{\infty} e^{-\gamma^\vee x^2}dx\int_{-\infty}^{\infty}\left|\chi_{(-1,1)}(y) - \sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right| dy\\
&= \frac{\|\boldsymbol{\alpha}\|_{\ell^1}}{\gamma^\vee}\sqrt{\frac{\pi}{2}}\int_{0}^{\infty}\left|\chi_{(-1,1)}(y) - \sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right| dy\\
&= \frac{\|\boldsymbol{\alpha}\|_{\ell^1}}{\gamma^\vee}\sqrt{\frac{\pi}{2}}\left(\int_0^1\left|1 - \sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right| dy + \int_1^\infty\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}dy\right). \qquad (2.31)
\end{aligned}$$

The first integral in (2.31) may be bounded as

$$\int_0^1\left|1 - \sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right| dy \le 1 + \sum_{k=1}^{l}|\alpha_k|\int_0^1 e^{-\Re\{\gamma_k y^2\}}dy \le 1 + \|\boldsymbol{\alpha}\|_{\ell^1},$$

and the second integral as

$$\int_1^\infty\left|\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right| dy \le \|\boldsymbol{\alpha}\|_{\ell^1}\int_0^\infty e^{-\gamma^\vee y^2}dy = \frac{\|\boldsymbol{\alpha}\|_{\ell^1}}{\gamma^\vee}\sqrt{\frac{\pi}{2}}.$$

Hence

$$\int_{-\infty}^{\infty}|\hat{r}(y)|dy \le \left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}} + 1\right)\|\boldsymbol{\alpha}\|_{\ell^1} + 1 \qquad (2.32)$$

and

$$|\mathcal{I}_{m,n,\omega}| \le \frac{\|\boldsymbol{\alpha}\|_{\ell^1}}{\gamma^\vee}\sqrt{\frac{\pi}{2}}\left[\left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}} + 1\right)\|\boldsymbol{\alpha}\|_{\ell^1} + 1\right]^2. \qquad (2.33)$$

By the same argument (2.33) is also true for $\mathcal{J}_{m,n,\omega}$. For $\mathcal{K}_{m,n,\omega}$ we use (2.32) twice to deduce that

$$|\mathcal{K}_{m,n,\omega}| \leq \frac{1}{2}\left[\left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}}+1\right)\|\boldsymbol{\alpha}\|_{\ell^1}+1\right]^2$$

and hence

$$\begin{aligned}
|\tilde{A}_{m,n,\omega}| &\leq \frac{2\|\boldsymbol{\alpha}\|_{\ell^1}}{\gamma^\vee}\sqrt{\frac{\pi}{2}}\left[\left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}}+1\right)\|\boldsymbol{\alpha}\|_{\ell^1}+1\right]+\frac{1}{2}\left[\left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}}+1\right)\|\boldsymbol{\alpha}\|_{\ell^1}+1\right]^2 \\
&= \left[\frac{2\|\boldsymbol{\alpha}\|_{\ell^1}}{\gamma^\vee}\sqrt{\frac{\pi}{2}}+\left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}}+1\right)\frac{\|\boldsymbol{\alpha}\|_{\ell^1}}{2}+\frac{1}{2}\right]\left[\left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}}+1\right)\|\boldsymbol{\alpha}\|_{\ell^1}+1\right] \\
&= \left[\left(\frac{5}{2\gamma^\vee}\sqrt{\frac{\pi}{2}}+\frac{1}{2}\right)\|\boldsymbol{\alpha}\|_{\ell^1}+\frac{1}{2}\right]\left[\left(\frac{1}{\gamma^\vee}\sqrt{\frac{\pi}{2}}+1\right)\|\boldsymbol{\alpha}\|_{\ell^1}+1\right]. \qquad \square
\end{aligned}$$

3 Numerical results

4 Conclusions

References

[1] Milton Abramowitz, Irene A Stegun, and Robert H Romer. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* 1988.

[2] Michael Berry, Cornelis Storm, and Wim Van Saarloos. "Theory of unstable laser modes: edge waves and fractality". In: *Optics communications* 197.4-6 (2001), pp. 393–402.

[3] Albrecht Böttcher et al. "On the singular values and eigenvalues of the Fox-Li and related operators." In: *The New York Journal of Mathematics [electronic only]* 16 (2010), pp. 539–561.

[4] Hermann Brunner, Arieh Iserles, and Syvert P Nørsett. "The computation of the spectra of highly oscillatory Fredholm integral operators". In: *The Journal of Integral Equations and Applications* (2011), pp. 467–519.

[5] Hermann Brunner, Arieh Iserles, and Syvert P Nørsett. "The spectral problem for a class of highly oscillatory Fredholm integral operators". In: *IMA journal of numerical analysis* 30.1 (2010), pp. 108–130.

[6] J Alan Cochran. "The existence of eigenvalues for the integral equations of laser theory". In: *The Bell System Technical Journal* 44.1 (1965), pp. 77–88.

[7] James Alan Cochran and Erold W Hinds. "Eigensystems associated with the complex-symmetric kernels of laser theory". In: *SIAM Journal on Applied Mathematics* 26.4 (1974), pp. 776–786.

[8] Javier Duoandikoetxea and Javier Duoandikoetxea Zuazo. *Fourier analysis.* Vol. 29. American Mathematical Soc., 2001.

[9] AG Fox and Tingye Li. "Modes in a maser interferometer with curved and tilted mirrors". In: *Proceedings of the IEEE* 51.1 (1963), pp. 80–89.

[10] Arthur Fox and Tingye Li. "Resonant modes in a maser interferometer". In: *Bell System Technical Journal* 40.2 (1961), pp. 453–488.

[11] A George Jr et al. *Essentials of Padé approximants.* Elsevier, 1975.

[12] Harry Hochstadt. "On the eigenvalues of a class of integral equations arising in laser theory". In: *SIAM Review* 8.1 (1966), pp. 62–65.

[13] Sun-Yuan Kung. "A new identification and model reduction algorithm via singular value decomposition". In: *Proc. 12th Asilomar Conf. on Circuits, Systems and Computers, Pacific Grove, CA, November, 1978.* 1978.

[14] HJ Landau. "The notion of approximate eigenvalues applied to an integral equation of laser theory". In: *Quarterly of Applied Mathematics* 35.1 (1977), pp. 165–172.

[15] SoPc Morgan. "On the integral equations of laser theory". In: *IEEE Transactions on Microwave Theory and Techniques* 11.3 (1963), pp. 191–193.

[16] DJ Newman and SP Morgan. "Existence of eigenvalues of a class of integral equations arising in laser theory". In: *Bell System Technical Journal* 43.1 (1964), pp. 113–126.

[17] Elias M Stein and Rami Shakarchi. *Fourier analysis: an introduction.* Vol. 1. Princeton University Press, 2011.

[18] Anders Vretblad. *Fourier analysis and its applications.* Vol. 223. Springer Science & Business Media, 2003.

[19] Can Evren Yarman. "Approximating fractional derivative of the Gaussian function and Dawson's integral". In: *arXiv preprint arXiv:1709.02089* (2017).

[20] Can Yarman and Garret Flagg. "Generalization of Padé approximation from rational functions to arbitrary analytic functions—Theory". In: *Mathematics of Computation* 84.294 (2015), pp. 1835–1860.

5 Derivations

5.1 Linearisation

Let g be a 2nd-order polynomial in x and y denoted by

$$g(x,y) = A + Bx + Cxy + Dy + Ex^2 + Fy^2$$

Let $\hat{\varphi}[m]$ be the mth Fourier coefficient of the eigenfunction φ that corresponds to the eigenvalue λ. Then

$$\begin{aligned}
\lambda\hat{\varphi}[m] &= \frac{\lambda}{2}\int_{-1}^{1}\varphi(x)e^{\mathrm{i}\pi mx}dx \\
&= \frac{1}{2}\int_{-1}^{1}(\mathcal{F}_\omega\varphi)(x)e^{\mathrm{i}\pi mx}dx \\
&= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\varphi(y)e^{\mathrm{i}\omega g(x,y)}dye^{\mathrm{i}\pi mx}dx \\
&\approx \sum_{n=-\infty}^{\infty}\tilde{A}_{m,n}\hat{\varphi}[n], \qquad m\in\mathbb{Z}
\end{aligned}$$

where

$$\tilde{A}_{m,n} = \frac{1}{2}\sum_j\sum_k\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma_jx^2}e^{-\gamma_ky^2}e^{\mathrm{i}\omega g(x,y)}e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\pi mx}dxdy$$

and $\Re\{\gamma_j\} > 0$ for all j.

Denote the integral over x and y by I. Then

$$\begin{aligned}
I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp[-\gamma_jx^2]\exp[-\gamma_ky^2]\exp[\mathrm{i}\omega(A+Bx+Cxy+Dy+Ex^2+Fy^2)]\exp[-\mathrm{i}\pi ny]\exp[\mathrm{i}\pi mx]dxdy \\
&= \exp[\mathrm{i}A\omega]\int_{-\infty}^{\infty}\exp[(\mathrm{i}F\omega-\gamma_k)y^2]\exp[\mathrm{i}(D\omega-\pi n)y]\int_{-\infty}^{\infty}\exp[(\mathrm{i}E\omega-\gamma_j)x^2]\exp[\mathrm{i}(B\omega+C\omega y+\pi m)x]dxdy.
\end{aligned}$$

Denoting the integral over x by $I_0(y)$:

$$\begin{aligned}
I_0(y) &= \int_{-\infty}^{\infty}\exp[(\mathrm{i}E\omega-\gamma_j)x^2]\exp[\mathrm{i}(B\omega+C\omega y+\pi m)x]dx \\
&= \sqrt{\frac{\pi}{\gamma_j-\mathrm{i}E\omega}}\exp\left[\frac{(B\omega+C\omega y+\pi m)^2}{4(\mathrm{i}E\omega-\gamma_j)}\right] \\
&= \sqrt{\frac{\pi}{\gamma_j-\mathrm{i}E\omega}}\exp\left[\frac{(B\omega+\pi m)^2}{4(\mathrm{i}E\omega-\gamma_j)}\right]\exp\left[\frac{C\omega(B\omega+\pi m)}{2(\mathrm{i}E\omega-\gamma_j)}y\right]\exp\left[\frac{C^2\omega^2}{4(\mathrm{i}E\omega-\gamma_j)}y^2\right]
\end{aligned}$$

Hence

$$\begin{aligned}
I &= P_{j,m,\omega}\int_{-\infty}^{\infty}\exp[(\mathrm{i}F\omega-\gamma_k)y^2]\exp\left[\frac{C^2\omega^2}{4(\mathrm{i}E\omega-\gamma_j)}y^2\right]\exp[\mathrm{i}(D\omega-\pi n)y]\exp\left[\frac{C\omega(B\omega+\pi m)}{2(\mathrm{i}E\omega-\gamma_j)}y\right]dy \\
&= P_{j,m,\omega}\int_{-\infty}^{\infty}\exp\left[\frac{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2}{4(\mathrm{i}E\omega-\gamma_j)}y^2\right]\exp\left[\frac{2\mathrm{i}(D\omega-\pi n)(\mathrm{i}E\omega-\gamma_j)+C\omega(B\omega+\pi m)}{2(\mathrm{i}E\omega-\gamma_j)}y\right]dy
\end{aligned}$$

where

$$P_{j,m,\omega} = \sqrt{\frac{\pi}{\gamma_j-\mathrm{i}E\omega}}\exp[\mathrm{i}A\omega]\exp\left[\frac{(B\omega+\pi m)^2}{4(\mathrm{i}E\omega-\gamma_j)}\right].$$

By the Fourier transform result

$$\int_{-\infty}^{\infty} e^{at^2} e^{\pm(b+\mathrm{i}k)t} dt = \sqrt{\frac{\pi}{-a}} \exp\left[-\frac{(b+\mathrm{i}k)^2}{4a}\right]$$

the integral I evaluates to

$$I = P_{j,m,\omega}\sqrt{\frac{4\pi(\gamma_j - \mathrm{i}E\omega)}{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2}} \exp\left[-\frac{\{2\mathrm{i}(D\omega-\pi n)(\mathrm{i}E\omega-\gamma_j)+C\omega(B\omega+\pi m)\}^2}{4(\mathrm{i}E\omega-\gamma_j)\{4(\mathrm{i}F\omega-\gamma_k)(\mathrm{i}E\omega-\gamma_j)+C^2\omega^2\}}\right]$$

$$= \sqrt{\frac{4\pi^2}{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2}} \exp[\mathrm{i}A\omega]\times$$

$$\exp\left[\frac{\{2\mathrm{i}(D\omega-\pi n)(\mathrm{i}E\omega-\gamma_j)+C\omega(B\omega+\pi m)\}^2+(B\omega+\pi m)^2(4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2)}{4(\mathrm{i}E\omega-\gamma_j)\{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2\}}\right]$$

5.2 Adjusted Fourier transform

If for Fox-Li analysis we want the Fourier coefficients defined as

$$\hat{f}[m] := \frac{1}{2}\int_{-1}^{1} f(t)e^{\mathrm{i}\pi mt}dt$$

$$\Rightarrow f(x) = \sum_{m=-\infty}^{\infty} \hat{f}[m]e^{-\mathrm{i}\pi mt},$$

then we should have the Fourier transform defined as

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{\mathrm{i}\pi xt}dt$$

$$\Rightarrow f(t) = \frac{1}{2}\int_{-\infty}^{\infty} \hat{f}(x)e^{-\mathrm{i}\pi tx}dx.$$

This leads to some different results:

Lemma 5.1. *The Fourier transform of* $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$ *is*

$$F[\mathrm{sinc}(x)](\xi) = \chi_{(-1,1)}(\xi), \quad \xi \in \mathbb{R}. \tag{5.1}$$

In addition, $\mathrm{sinc}(x) \in L^2(\mathbb{R})$.

Proof. Clearly, $\|\chi_{(-1,1)}\|_{L^2} = \sqrt{2}$ and hence $\chi_{(-1,1)}(\xi) \in L^2(\mathbb{R})$. Taking the inverse Fourier transform F^{-1} of $\chi_{(-1,1)}(\xi)$,

$$F^{-1}[\chi_{(-1,1)}(\xi)](x) = \frac{1}{2}\int_{-\infty}^{\infty} \chi_{(-1,1)}(\xi)e^{-\mathrm{i}\pi x\xi}d\xi = \frac{1}{2}\int_{-1}^{1} e^{-\mathrm{i}\pi x\xi}d\xi = \mathrm{sinc}(x)$$

from which () follows. The fact that $\mathrm{sinc}(x) \in L^2(\mathbb{R})$ is another application of Plancherel's theorem. □

Corollary 5.2. *Let*

$$f(x) = \mathrm{sinc}(x) + \mathrm{i}\mathcal{H}[\mathrm{sinc}(t)](x) = \frac{e^{\mathrm{i}\pi x}-1}{\mathrm{i}\pi x}, \tag{5.2}$$

$$g(x) = e^{-x^2} + \mathrm{i}\mathcal{H}[e^{-t^2}](x) = e^{-x^2}\left(1 + \frac{2\mathrm{i}}{\sqrt{\pi}}\int_0^x e^{t^2}dt\right), \tag{5.3}$$

where $\mathcal{H}$ *denotes the Hilbert transform and let* $\{(\tilde{\alpha}_j, \tilde{\gamma}_j)\}_{j=1}^J$ *satisfy (). If* $|\Re\{\tilde{\gamma}_j\}| > |\Im\{\tilde{\gamma}_j\}|$ *for* $j = 1, \ldots, J$*, then*

$$\chi_{(-1,1)}(\xi) = \sum_{j=1}^{J} \alpha_j e^{-\gamma_j\xi^2} + \hat{r}_J(\xi), \tag{5.4}$$

where $\alpha_j = \alpha(\tilde{\alpha}_j)$*,* $\gamma_j = \gamma(\tilde{\gamma}_j)$*,* $\Re\{\gamma_j\} > 0$ *and* $\|\hat{r}_J\|_{L^2} = \|r_J\|_{L^2}$*.*

Proof. With the adjustment the power series coefficients are now

$$f_n = \frac{(\mathrm{i}\pi)^n}{(n+1)!}, \quad g_n = \frac{\mathrm{i}^n}{\Gamma\left(\frac{n+2}{2}\right)}.$$

$\square$

Based on these coefficients we form h_n and find $\{(\tilde{\alpha}_j, \tilde{\gamma}_j)\}_{j=1}^{J}$ such that

$$f(x) = \sum_{j=1}^{J} \alpha_j g(\tilde{\gamma}_j x) + r_J(x).$$

We then use the Fourier transform result as

$$\chi_{(-1,1)}(\xi) = \sum_{j=1}^{J} \alpha_j e^{-\gamma_j \xi^2} + \hat{r}_J(\xi),$$

where

$$\alpha_j = \tilde{\alpha}_j \sqrt{\frac{\pi}{\tilde{\gamma}_j^2}}, \quad \gamma_j = \frac{\pi^2}{4\tilde{\gamma}_j^2}.$$

5.3 Comparison of matrix coefficients to Fox-Li case

In the previous manuscript we had the Fox-Li case, in which $C = -2, E = F = 1$ and all other coefficients equal to 0. With these coefficients we have

$$\begin{aligned}
T'_{j,k,\omega} &= 4(\mathrm{i}\omega - \gamma_j)(\mathrm{i}\omega - \gamma_k) + 4\omega^2 = 4(\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k)) \\
P_{j,k,\omega} &= \frac{\pi}{2\sqrt{\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k)}} \\
T_{j,k,\omega} &= 4(\mathrm{i}\omega - \gamma_j)T'_{j,k,\omega} \\
Q_{j,k,\omega} &= 4(\mathrm{i}\omega - \gamma_j)(\mathrm{i}\omega - \gamma_k) \\
R_{j,\omega} &= -8\mathrm{i}\omega(\mathrm{i}\omega - \gamma_j) \\
S_{j,\omega} &= 4(\mathrm{i}\omega - \gamma_j)^2.
\end{aligned}$$

Substitution into (2.30) gives

$$\begin{aligned}
I &= \frac{\pi}{2\sqrt{\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k)}} \exp\left[\frac{4(\mathrm{i}\omega - \gamma_j)(\mathrm{i}\omega - \gamma_k)(\pi m)^2 + 8\mathrm{i}\omega(\mathrm{i}\omega - \gamma_j)(\pi m)(-\pi n) + 4(\mathrm{i}\omega - \gamma_j)^2(-\pi n)^2}{4(\mathrm{i}\omega - \gamma_j)T'_{j,k,\omega}}\right] \\
&= \frac{\pi}{2\sqrt{\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k)}} \exp\left[\frac{(\mathrm{i}\omega - \gamma_k)(\pi m)^2 + 2\mathrm{i}\omega(\pi m)(-\pi n) + (\mathrm{i}\omega - \gamma_j)(-\pi n)^2}{4(\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k))}\right] \\
&= \frac{\pi}{2\sqrt{\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k)}} \exp\left[\frac{\pi^2}{4} \cdot \frac{(\mathrm{i}\omega - \gamma_k)m^2 - 2\mathrm{i}\omega mn + (\mathrm{i}\omega - \gamma_j)n^2}{\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k)}\right] \\
&= \frac{\pi}{2\sqrt{\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k)}} \exp\left[\frac{\pi^2}{4} \cdot \frac{\mathrm{i}\omega(m^2 - 2mn + n^2) - (\gamma_k m^2 + \gamma_j n^2)}{\gamma_j\gamma_k - \mathrm{i}\omega(\gamma_j + \gamma_k)}\right],
\end{aligned}$$

which is exactly what we got in the first manuscript.

5.4 Matrix entry decay

To study the decay of (2.30) we first need to rationalise the denominator $T_{j,k,\omega}$. We have

$$\overline{T}_{j,k,\omega} = 4(\overline{\mathrm{i}E\omega - \gamma_j})\{4(\overline{\mathrm{i}E\omega - \gamma_j})(\overline{\mathrm{i}F\omega - \gamma_k}) + C^2\omega^2\}$$

and hence

$$
\begin{aligned}
Q_{j,k,\omega}\overline{T}_{j,k,\omega} &= 16|\mathrm{i}E\omega-\gamma_j|^2(\mathrm{i}F\omega-\gamma_k)\{4(\overline{\mathrm{i}E\omega-\gamma_j})(\overline{\mathrm{i}F\omega-\gamma_k})+C^2\omega^2\}\\
&= 64|\mathrm{i}E\omega-\gamma_j|^2|\mathrm{i}F\omega-\gamma_k|^2(\overline{\mathrm{i}E\omega-\gamma_j})+16C^2\omega^2|\mathrm{i}E\omega-\gamma_j|^2(\mathrm{i}F\omega-\gamma_k)\\
R_{j,\omega}\overline{T}_{j,k,\omega} &= 16\mathrm{i}C\omega|\mathrm{i}E\omega-\gamma_j|^2\{4(\overline{\mathrm{i}E\omega-\gamma_j})(\overline{\mathrm{i}F\omega-\gamma_k})+C^2\omega^2\}\\
S_{j,\omega}\overline{T}_{j,k,\omega} &= 16|\mathrm{i}E\omega-\gamma_j|^2(\mathrm{i}E\omega-\gamma_j)\{4(\overline{\mathrm{i}E\omega-\gamma_j})(\overline{\mathrm{i}F\omega-\gamma_k})+C^2\omega^2\}\\
&= 64|\mathrm{i}E\omega-\gamma_j|^4(\overline{\mathrm{i}F\omega-\gamma_k})+16C^2\omega^2|\mathrm{i}E\omega-\gamma_j|^2(\mathrm{i}E\omega-\gamma_j).
\end{aligned}
$$

Hence

$$
\begin{aligned}
Q\overline{T}(B\omega+\pi m)^2 &= \{64|\mathrm{i}E\omega-\gamma_j|^2|\mathrm{i}F\omega-\gamma_k|^2(\overline{\mathrm{i}E\omega-\gamma_j})+16C^2\omega^2|\mathrm{i}E\omega-\gamma_j|^2(\mathrm{i}F\omega-\gamma_k)\}(B\omega+\pi m)^2\\
R\overline{T}(B\omega+\pi m)(D\omega-\pi n) &= 16\mathrm{i}C\omega|\mathrm{i}E\omega-\gamma_j|^2\{4(\overline{\mathrm{i}E\omega-\gamma_j})(\overline{\mathrm{i}F\omega-\gamma_k})+C^2\omega^2\}(B\omega+\pi m)(D\omega-\pi n)\\
S\overline{T}(D\omega-\pi n)^2 &= \{64|\mathrm{i}E\omega-\gamma_j|^4(\overline{\mathrm{i}F\omega-\gamma_k})+16C^2\omega^2|\mathrm{i}E\omega-\gamma_j|^2(\mathrm{i}E\omega-\gamma_j)\}(D\omega-\pi n)^2.
\end{aligned}
$$

Fix m and ω. Clearly $S_{j,\omega}\overline{T}_{j,k,\omega}(D\omega-\pi n)^2\to-\infty$ as $n\to\pm\infty$. The same cannot be said for $R_{j,\omega}\overline{T}_{j,k,\omega}(B\omega+\pi m)(D\omega-\pi n)$ but this term is only $\mathcal{O}(n)$ compared to $\mathcal{O}(n^2)$ for the other term. Note that, under our assumptions, $S\overline{T}(D\omega-\pi n)^2=0$ if and only if $n=D\omega/\pi$.

Now fix n and ω. Since $\Re\{Q\overline{T}\}<0$, we have $Q\overline{T}(B\omega+\pi m)^2\to-\infty$ as $m\to\pm\infty$.

5.5 Revised error matrix entry derivations

In addition to $\boldsymbol{A}_\omega[m,n]$ in (2.19), take

$$
\tilde{\boldsymbol{A}}_\omega[m,n]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_L(x,y)e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy
$$

with

$$
\psi_L(x,y)=\frac{1}{2}\left(r_L(y)\sum_{j=1}^{L}\alpha_je^{-\gamma_jx^2}+r_L(x)\sum_{k=1}^{L}\alpha_ke^{-\gamma_ky^2}+r_L(x)r_L(y)\right).
$$

We decompose $\tilde{\boldsymbol{A}}_\omega[m,n]$ into the three integrals $\tilde{\boldsymbol{A}}_\omega[m,n]=\mathcal{I}_\omega[m,n]+\mathcal{J}_\omega[m,n]+\mathcal{K}_\omega[m,n]$, where

$$
\begin{aligned}
\mathcal{I}_\omega[m,n] &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}r_L(y)\sum_{j=1}^{L}\alpha_je^{-\gamma_jx^2}e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy,\\
\mathcal{J}_\omega[m,n] &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}r_L(x)\sum_{k=1}^{L}\alpha_ke^{-\gamma_ky^2}e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy,\\
\mathcal{K}_\omega[m,n] &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}r_L(x)r_L(y)e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy.
\end{aligned}
$$

Using the definition of r_L:

$$
\begin{aligned}
\mathcal{I}_\omega[m,n] &= \frac{1}{2}\sum_{j=1}^{L}\alpha_j\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\chi_{(-1,1)}(y)-\sum_{k=1}^{L}\alpha_ke^{-\gamma_ky^2}\right)e^{-\gamma_jx^2}e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy\\
\mathcal{J}_\omega[m,n] &= \frac{1}{2}\sum_{k=1}^{L}\alpha_k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\chi_{(-1,1)}(x)-\sum_{j=1}^{L}\alpha_je^{-\gamma_jx^2}\right)e^{-\gamma_ky^2}e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy\\
\mathcal{K}_\omega[m,n] &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\chi_{(-1,1)}(x)-\sum_{j=1}^{L}\alpha_je^{-\gamma_jx^2}\right)\left(\chi_{(-1,1)}(y)-\sum_{k=1}^{L}\alpha_ke^{-\gamma_ky^2}\right)e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathcal{K}_\omega[m,n] &= \frac{1}{2}\Bigg(-\sum_{j=1}^{L}\alpha_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\chi_{(-1,1)}(y) - \sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right) e^{-\gamma_j x^2} e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{\mathrm{i}\pi m x} dxdy \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{(-1,1)}(x)\left(\chi_{(-1,1)}(y) - \sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right) e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{\mathrm{i}\pi m x} dxdy\Bigg) \\
&= -\mathcal{I}_\omega[m,n] + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{(-1,1)}(x)\chi_{(-1,1)}(y) e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{\mathrm{i}\pi m x} dxdy \\
&\quad - \frac{1}{2}\sum_{k=1}^{L}\alpha_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{(-1,1)}(x) e^{-\gamma_k y^2} e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{\mathrm{i}\pi m x} dxdy.
\end{aligned}
$$

Hence altogether we have

$$
\tilde{\boldsymbol{A}}_\omega[m,n] = \tilde{\boldsymbol{A}}_\omega^{\chi}[m,n] - \tilde{\boldsymbol{A}}_\omega^{\mathcal{G}}[m,n],
$$

where

$$
\begin{aligned}
\tilde{\boldsymbol{A}}_\omega^{\chi}[m,n] &= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{\mathrm{i}\pi m x} dxdy, \\
\tilde{\boldsymbol{A}}_\omega^{\mathcal{G}}[m,n] &= \frac{1}{2}\sum_{j=1}^{L}\sum_{k=1}^{L}\alpha_j\alpha_k \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\gamma_j x^2} e^{-\gamma_k y^2} e^{-\mathrm{i}\pi n y} e^{\mathrm{i}\omega\eta(x,y)} e^{\mathrm{i}\pi m x} dxdy.
\end{aligned}
$$

Assume first that $E \neq 0$ and $F \neq 0$. The Fourier coefficients for $e^{\mathrm{i}E\omega x^2}$ are

$$
\begin{aligned}
a_{r,\omega} &= \frac{1}{2}\int_{-1}^{1}\exp[\mathrm{i}E\omega t^2]\exp[\mathrm{i}\pi r t]dt \\
&= \frac{1}{2}\int_{-1}^{1}\exp\left[\mathrm{i}E\omega\left(t^2 + \frac{\pi r t}{E\omega}\right)\right]dt \\
&= \frac{1}{2}\exp\left[-\frac{\mathrm{i}\pi^2 r^2}{4E\omega}\right]\int_{-1}^{1}\exp\left[\mathrm{i}E\omega\left(t + \frac{\pi r}{2E\omega}\right)^2\right]dt \\
&= \frac{1}{2\mathrm{i}\sqrt{\mathrm{i}E\omega}}\exp\left[-\frac{\mathrm{i}\pi^2 r^2}{4E\omega}\right]\int_{t_0(r)}^{t_1(r)}\exp(-t^2)dt \\
&= \frac{1}{4\mathrm{i}}\sqrt{\frac{\pi}{\mathrm{i}E\omega}}\exp\left[-\frac{\mathrm{i}\pi^2 r^2}{4E\omega}\right](\mathrm{erf}(t_1(r)) - \mathrm{erf}(t_0(r)))
\end{aligned}
$$

and hence

$$
e^{\mathrm{i}E\omega x^2} = \sum_{r=-\infty}^{\infty} a_{r,\omega} e^{-\mathrm{i}\pi r x}
$$

where

$$
t_0(r;E) = \mathrm{i}\sqrt{\mathrm{i}E\omega}\left(\frac{\pi r}{2E\omega} - 1\right), \quad t_1(r;E) = \mathrm{i}\sqrt{\mathrm{i}E\omega}\left(\frac{\pi r}{2E\omega} + 1\right).
$$

Similarly for $e^{\mathrm{i}F\omega y^2}$ we have

$$
e^{\mathrm{i}F\omega y^2} = \sum_{s=-\infty}^{\infty} b_{s,\omega} e^{-\mathrm{i}\pi s y}
$$

where

$$
b_{s,\omega} = \frac{1}{4\mathrm{i}}\sqrt{\frac{\pi}{\mathrm{i}F\omega}}\exp\left[-\frac{\mathrm{i}\pi^2 s^2}{4F\omega}\right](\mathrm{erf}(t_1(s;F)) - \mathrm{erf}(t_0(s;F))).
$$

This gives

$$\tilde{\boldsymbol{A}}^{\chi}_{\omega}[m,n] = \frac{e^{\mathrm{i}A\omega}}{2}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} a_{r,\omega}b_{s,\omega}\int_{-1}^{1}\int_{-1}^{1} e^{-\mathrm{i}\pi ny}e^{\mathrm{i}B\omega x}e^{\mathrm{i}C\omega xy}e^{\mathrm{i}D\omega y}e^{-\mathrm{i}\pi rx}e^{-\mathrm{i}\pi sy}e^{\mathrm{i}\pi mx}dxdy$$
$$= \frac{e^{\mathrm{i}A\omega}}{2}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} a_{r,\omega}b_{s,\omega}\int_{-1}^{1}\int_{-1}^{1} e^{\mathrm{i}[(B\omega+\pi m-\pi r)x+(D\omega-\pi n-\pi s)y]+C\omega xy}dxdy$$
$$= \frac{e^{\mathrm{i}A\omega}}{2C}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} a_{r,\omega}b_{s,\omega}\theta_{r,s,\omega}[m,n],$$

where

$$\theta_{r,s,\omega}[m,n] = \int_{-1}^{1}\int_{-C}^{C} e^{\mathrm{i}(p_{r,m,\omega}x+q_{s,n,\omega}y+\omega xy)}dxdy,$$
$$p_{r,m,\omega} = (B\omega+\pi m-\pi r)/C, \quad q_{s,n,\omega} = D\omega-\pi n-\pi s.$$

For sake of clarity in the following derivations we temporarily drop the dependence on the parameters in p and q. Now

$$\theta_{r,s,\omega}[m,n] = \int_{-1}^{1}\int_{-C}^{C} e^{\mathrm{i}(px+qy+\omega xy)}dxdy$$
$$= \frac{1}{\mathrm{i}}\int_{-1}^{1}\frac{1}{p+\omega y}\left(e^{\mathrm{i}(Cp+qy+\omega Cy)} - e^{\mathrm{i}(-Cp+qy-\omega Cy)}\right)dy.$$

Taking the first integral,

$$\frac{1}{\mathrm{i}}\int_{-1}^{1}\frac{e^{\mathrm{i}(Cp+qy+\omega Cy)}}{p+\omega y}dy = \frac{e^{\frac{pq}{\mathrm{i}\omega}}}{\mathrm{i}\omega}\int_{p-\omega}^{p+\omega}\frac{e^{\mathrm{i}\left(\frac{C\omega+q}{\omega}\right)y}}{y}dy$$
$$= \frac{e^{\frac{pq}{\mathrm{i}\omega}}}{\mathrm{i}\omega}\left(\int_{p-\omega}^{\infty}\frac{e^{\mathrm{i}\left(\frac{C\omega+q}{\omega}\right)y}}{y}dy - \int_{p+\omega}^{\infty}\frac{e^{\mathrm{i}\left(\frac{C\omega+q}{\omega}\right)y}}{y}dy\right)$$
$$= \frac{e^{\frac{pq}{\mathrm{i}\omega}}}{\mathrm{i}\omega}\left(\int_{1}^{\infty}\frac{e^{\mathrm{i}\left(\frac{(C\omega+q)(p-\omega)}{\omega}\right)y}}{y}dy - \int_{1}^{\infty}\frac{e^{\mathrm{i}\left(\frac{(C\omega+q)(p+\omega)}{\omega}\right)y}}{y}dy\right)$$
$$= \frac{e^{\frac{pq}{\mathrm{i}\omega}}}{\mathrm{i}\omega}\left[E_1\left(\frac{(C\omega+q)(p-\omega)}{\mathrm{i}\omega}\right) - E_1\left(\frac{(C\omega+q)(p+\omega)}{\mathrm{i}\omega}\right)\right],$$

where $E_1(z)$ is the exponential integral [1]

$$E_1(z) = \int_{1}^{\infty}\frac{e^{-zt}}{t}dt, \quad \Re\{z\} \geq 0.$$

The second integral follows from changing the sign of C. Therefore altogether we have

$$\tilde{\boldsymbol{A}}^{\chi}_{\omega}[m,n] = \frac{e^{\mathrm{i}A\omega}}{2\mathrm{i}\omega}\sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} a_{r,\omega}b_{s,\omega}\exp\left[\frac{p_r q_s}{\mathrm{i}\omega}\right]\left\{E_1\left(\frac{(C\omega+q_s)(p_r-\omega)}{\mathrm{i}\omega}\right)\right.$$
$$\left. - E_1\left(\frac{(C\omega+q_s)(p_r+\omega)}{\mathrm{i}\omega}\right) + E_1\left(\frac{(q_s-C\omega)(p_r+\omega)}{\mathrm{i}\omega}\right) - E_1\left(\frac{(q_s-C\omega)(p_r-\omega)}{\mathrm{i}\omega}\right)\right\}.$$

A special case in which $E \cdot F \neq 0$ and Fourier series is not required is when $C/\sqrt{EF} = -2$. In this case we can use Lemma 5 in [4] to compute $\tilde{\boldsymbol{A}}^{\chi}_{\omega}[m,n]$ exactly.

In the case for which $E = F = 0$, we don't need to perform the Fourier sum decomposition and can instead use the above to compute

$$
\begin{aligned}
\tilde{\boldsymbol{A}}^{\mathcal{X}}_{\omega}[m,n] &= \frac{e^{\mathrm{i}A\omega}}{2}\int_{-1}^{1}\int_{-1}^{1} e^{\mathrm{i}[(B\omega+\pi m)x+(D\omega-\pi n)y+C\omega xy]}dxdy \\
&= \frac{e^{\mathrm{i}A\omega}}{2\mathrm{i}\omega}\exp\left[\frac{(B\omega+\pi m)(D\omega-\pi n)}{\mathrm{i}C\omega}\right]\left\{E_1\left(\frac{(C\omega+D\omega-\pi n)(B\omega+\pi m-C\omega)}{\mathrm{i}C\omega}\right)\right. \\
&- E_1\left(\frac{(C\omega+D\omega-\pi n)(B\omega+\pi m+C\omega)}{\mathrm{i}C\omega}\right) + E_1\left(\frac{(D\omega-\pi n-C\omega)(B\omega+\pi m+C\omega)}{\mathrm{i}C\omega}\right) \\
&\left.- E_1\left(\frac{(D\omega-\pi n-C\omega)(B\omega+\pi m-C\omega)}{\mathrm{i}C\omega}\right)\right\}.
\end{aligned}
$$

Finally, we compute $\tilde{\boldsymbol{A}}^{\mathcal{G}}_{\omega}[m,n]$. Taking first the integral

$$
A_x = \int_{-\infty}^{\infty} e^{(\mathrm{i}E-\gamma_j)x^2} e^{\mathrm{i}(B+Cy+\pi m)x}dx
$$

we have

$$
A_x = \sqrt{\frac{\pi}{\gamma_j-\mathrm{i}E}}\exp\left[\frac{(B+\pi m)^2}{4(\mathrm{i}E-\gamma_j)}\right]\exp\left[\frac{C(B+\pi m)}{2(\mathrm{i}E-\gamma_j)}y\right]\exp\left[\frac{C^2}{4(\mathrm{i}E-\gamma_j)}y^2\right].
$$

Forgetting the constant terms:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\exp\left[\frac{4(\mathrm{i}E-\gamma_j)(\mathrm{i}F-\gamma_k)+C^2}{4(\mathrm{i}E-\gamma_j)}y^2\right]\exp\left[\frac{2\mathrm{i}(D-\pi n)(\mathrm{i}E-\gamma_j)+C(B+\pi m)}{2(\mathrm{i}E-\gamma_j)}y\right]dy \\
&= \sqrt{\frac{4\pi(\gamma_j-\mathrm{i}E)}{4(\mathrm{i}E-\gamma_j)(\mathrm{i}F-\gamma_k)+C^2}}\exp\left[-\frac{\{2\mathrm{i}(D-\pi n)(\mathrm{i}E-\gamma_j)+C(B+\pi m)\}^2}{4(\mathrm{i}E-\gamma_j)\{4(\mathrm{i}E-\gamma_j)(\mathrm{i}F-\gamma_k)+C^2\}}\right].
\end{aligned}
$$

Altogether:

$$
\begin{aligned}
\tilde{\boldsymbol{A}}^{\mathcal{G}}_{\omega}[m,n] = \pi e^{\mathrm{i}A\omega}\sum_{j=1}^{L}\sum_{k=1}^{L}\frac{\alpha_j\alpha_k}{\sqrt{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2}}\exp\left[\frac{(B\omega+\pi m)^2}{4(\mathrm{i}E\omega-\gamma_j)}\right]\times \\
\exp\left[-\frac{\{2\mathrm{i}(D\omega-\pi n)(\mathrm{i}E\omega-\gamma_j)+C\omega(B\omega+\pi m)\}^2}{4(\mathrm{i}E\omega-\gamma_j)\{4(\mathrm{i}E\omega-\gamma_j)(\mathrm{i}F\omega-\gamma_k)+C^2\omega^2\}}\right].
\end{aligned}
$$

5.6 Exact computation attempt

Drop the dependence on ω, since we can just absorb this into the constants for now. Then

$$
\begin{aligned}
2\tilde{\boldsymbol{A}}^{\mathcal{X}}_{\omega}[m,n] &= \int_{-1}^{1}\int_{-1}^{1} e^{-\mathrm{i}ny}e^{\mathrm{i}\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy \\
&= \int_{-1}^{1}\int_{-1}^{1} e^{\mathrm{i}[(B+\pi m)x+(D-\pi n)]y}e^{\mathrm{i}(Ex^2+Cxy+Fy^2)}dxdy \\
&= \int_{-1}^{1}\int_{-1}^{1} e^{\mathrm{i}(ax+by)}e^{\mathrm{i}(c(x-y)^2+fx^2+gy^2)}dxdy, \\
&= \int_{-1}^{1}\int_{-1}^{1} e^{\mathrm{i}(ax+by)}e^{\mathrm{i}[(1-c)(x-y)^2+c(x+y)^2]}dxdy,
\end{aligned}
$$

Note that the constants change.

Let

$$
\theta(a,b) = \int_{-1}^{1}\int_{-1}^{1} e^{\mathrm{i}(ax+by)}e^{\mathrm{i}cxy}e^{\mathrm{i}(x-y)^2}dxdy
$$

We will try to write the integrand of $\theta(a,b)$ as

$$e^{\mathrm{i}(ax+by+cxy)}e^{\mathrm{i}(x-y)^2}=e^{\mathrm{i}(ax+by+cxy)}+e^{\mathrm{i}(ax+by)}e^{\mathrm{i}(x-y)^2}+\text{something else useful}$$

or as

$$e^{\mathrm{i}(ax+by+cxy)}e^{\mathrm{i}(x-y)^2}=e^{\mathrm{i}(ax+by)}e^{\mathrm{i}c(x-y)^2}+e^{\mathrm{i}(ax+by)}e^{\mathrm{i}(x-y)^2}+\text{something else useful}$$

5.7 Complete Linearisation

Let η the 2nd-order polynomial in x and y

$$\eta(x,y)=A+Bx+Cxy+Dy+Ex^2+Fy^2,\quad C\neq 0$$

Let $\hat{\varphi}[m]$ be the mth Fourier coefficient of the eigenfunction φ that corresponds to the eigenvalue λ. Then for each $m\in\mathbb{Z}$,

$$\begin{aligned}
\lambda\hat{\boldsymbol{\varphi}}[m]&=\frac{\lambda}{2}\int_{-1}^{1}\varphi(x)e^{\mathrm{i}\pi mx}dx\\
&=\frac{1}{2}\int_{-1}^{1}(\mathcal{F}_\omega\varphi)(x)e^{\mathrm{i}\pi mx}dx\\
&=\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\varphi(y)e^{\mathrm{i}\omega\eta(x,y)}dye^{\mathrm{i}\pi mx}dx\\
&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_{(-1,1)}(x)\chi_{(-1,1)}(y)e^{\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy\hat{\varphi}[n]\\
&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{L}\alpha_je^{-\gamma_jx^2}+r_L(x)\right)\left(\sum_{k=1}^{L}\alpha_ke^{-\gamma_ky^2}+r_L(y)\right)e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy\hat{\varphi}[n]\\
&=\sum_{n=-\infty}^{\infty}\left(\boldsymbol{A}_\omega[m,n]+\tilde{\boldsymbol{A}}_\omega[m,n]\right)\hat{\boldsymbol{\varphi}}[n],
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{A}_\omega[m,n]&=\frac{1}{2}\sum_{j=1}^{L}\sum_{k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma_jx^2}e^{-\gamma_ky^2}e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy,\\
\tilde{\boldsymbol{A}}_\omega[m,n]&=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(r_L(y)\sum_{j=1}^{L}\alpha_je^{-\gamma_jx^2}+r_L(x)\sum_{k=1}^{L}\alpha_ke^{-\gamma_ky^2}+r_L(x)r_L(y)\right)e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy
\end{aligned}$$

and $\Re\{\gamma_j\}>0$ for all j.

For $\boldsymbol{A}_\omega[m,n]$, denote the integral by

$$\theta_{j,k}(m,n)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma_jx^2}e^{-\gamma_ky^2}e^{-\mathrm{i}\pi ny}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi mx}dxdy.$$

Since the dependence on ω is only in the coefficients of the polynomial, we drop the dependence on ω and assume this is accounted for in the polynomial coefficients. Then

$$\begin{aligned}
\theta_{j,k}(m,n)&=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma_jx^2}e^{-\gamma_ky^2}e^{-\mathrm{i}\pi ny}e^{\mathrm{i}(Ax+Bxy+Cy+Dx^2+Ey^2)}e^{\mathrm{i}\pi mx}dxdy\\
&=\frac{1}{2}\int_{-\infty}^{\infty}e^{(\mathrm{i}E-\gamma_k)y^2}e^{\mathrm{i}(C-\pi n)y}\int_{-\infty}^{\infty}e^{(\mathrm{i}D-\gamma_j)x^2}e^{\mathrm{i}(A+By+\pi m)x}dxdy.
\end{aligned}$$

Integrating with respect to x gives

$$\begin{aligned}\theta_{j,k}(m,n) &= \frac{1}{2}\sqrt{\frac{\pi}{\gamma_j - \mathrm{i}D}}\int_{-\infty}^{\infty}\exp[(\mathrm{i}E-\gamma_k)y^2]\exp[\mathrm{i}(C-\pi n)y]\exp\left[\frac{(A+By+\pi m)^2}{4(\mathrm{i}D-\gamma_j)}\right]dy\\ &= P'_{j,m}\int_{-\infty}^{\infty}\exp\left[\frac{4(\mathrm{i}D-\gamma_j)(\mathrm{i}E-\gamma_k)+B^2}{4(\mathrm{i}D-\gamma_j)}y^2\right]\exp\left[\frac{2\mathrm{i}(C-\pi n)(\mathrm{i}D-\gamma_j)+B(A+\pi m)}{2(\mathrm{i}D-\gamma_j)}y\right]dy,\end{aligned}$$

where

$$P'_{j,m} = \frac{1}{2}\sqrt{\frac{\pi}{\gamma_j-\mathrm{i}D}}\exp\left[\frac{(A+\pi m)^2}{4(\mathrm{i}D-\gamma_j)}\right].$$

The integral

$$\rho_{j,k}(m,n) = \int_{-\infty}^{\infty}\exp\left[\frac{4(\mathrm{i}D-\gamma_j)(\mathrm{i}E-\gamma_k)+B^2}{4(\mathrm{i}D-\gamma_j)}y^2\right]\exp\left[\frac{2\mathrm{i}(C-\pi n)(\mathrm{i}D-\gamma_j)+B(A+\pi m)}{2(\mathrm{i}D-\gamma_j)}y\right]dy$$

is

$$\rho_{j,k}(m,n) = \sqrt{\frac{4\pi(\gamma_j-\mathrm{i}D)}{4(\mathrm{i}D-\gamma_j)(\mathrm{i}E-\gamma_k)+B^2}}\exp\left[-\frac{\{2\mathrm{i}(C-\pi n)(\mathrm{i}D-\gamma_j)+B(A+\pi m)\}^2}{4(\mathrm{i}D-\gamma_j)(4(\mathrm{i}D-\gamma_j)(\mathrm{i}E-\gamma_k)+B^2)}\right].$$

Hence

$$\theta_{j,k}[m,n] = P_{j,k}[m]\exp\left[\frac{(A+\pi m)^2}{4(\mathrm{i}D-\gamma_j)}\right]\exp\left[-\frac{\{2\mathrm{i}(C-\pi n)(\mathrm{i}D-\gamma_j)+B(A+\pi m)\}^2}{4(\mathrm{i}D-\gamma_j)(4(\mathrm{i}D-\gamma_j)(\mathrm{i}E-\gamma_k)+B^2)}\right],$$

where

$$P_{j,k} = \frac{\pi}{\sqrt{4(\mathrm{i}D-\gamma_j)(\mathrm{i}E-\gamma_k)+B^2}}.$$

For $\tilde{\boldsymbol{A}}_\omega[m,n]$, we decompose into the three integrals $\tilde{\boldsymbol{A}}_\omega[m,n] = \mathcal{I}_\omega[m,n]+\mathcal{J}_\omega[m,n]+\mathcal{K}_\omega[m,n]$, where

$$\begin{aligned}\mathcal{I}_\omega[m,n] &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_L(y)\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}e^{-\mathrm{i}\pi n y}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi m x}dxdy,\\ \mathcal{J}_\omega[m,n] &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_L(x)\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}e^{-\mathrm{i}\pi n y}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi m x}dxdy,\\ \mathcal{K}_\omega[m,n] &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_L(x)r_L(y)e^{-\mathrm{i}\pi n y}e^{\mathrm{i}\omega\eta(x,y)}e^{\mathrm{i}\pi m x}dxdy.\end{aligned}$$

By definition of r_L:

$$\begin{aligned}2\mathcal{I}_\omega[m,n] &= \sum_{j=1}^{L}\alpha_j\int_{-\infty}^{\infty}\int_{-1}^{1}\left(1-\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right)e^{-\gamma_j x^2}F(x,y)dydx - \sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{|y|>1}e^{-\gamma_k y^2}e^{-\gamma_j x^2}F(x,y)dydx\\ &= -\sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}e^{-\gamma_k y^2}F(x,y)dydx + \sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{-1}^{1}e^{-\gamma_j x^2}F(x,y)dydx,\\ 2\mathcal{J}_\omega[m,n] &= \sum_{k=1}^{L}\alpha_k\int_{-\infty}^{\infty}\int_{-1}^{1}\left(1-\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}\right)e^{-\gamma_k y^2}F(x,y)dxdy - \sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{|x|>1}e^{-\gamma_j x^2}e^{-\gamma_k y^2}F(x,y)dxdy\\ &= -\sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}e^{-\gamma_k y^2}F(x,y)dxdy + \sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{-1}^{1}e^{-\gamma_k y^2}F(x,y)dxdy.\end{aligned}$$

For $\mathcal{K}_\omega[m,n]$ we have

$$2\mathcal{K}_\omega[m,n] = \int_{-1}^{1}\left(1-\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right)\int_{-1}^{1}\left(1-\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}\right)F(x,y)dxdy$$
$$-\int_{-1}^{1}\left(1-\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\right)\int_{|x|>1}\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}F(x,y)dxdy$$
$$-\int_{|y|>1}\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\int_{-1}^{1}\left(1-\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}\right)F(x,y)dxdy$$
$$+\int_{|y|>1}\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\int_{|x|>1}\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}F(x,y)dxdy$$

which can be expanded and collected to give

$$2\mathcal{K}_\omega[m,n] = -\int_{-\infty}^{\infty}\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\int_{-1}^{1}\left(1-\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}\right)F(x,y)dxdy$$
$$+\int_{-1}^{1}\int_{-1}^{1}\left(1-\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}\right)F(x,y)dxdy$$
$$+\int_{-\infty}^{\infty}\sum_{k=1}^{L}\alpha_k e^{-\gamma_k y^2}\int_{|x|>1}\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}F(x,y)dxdy$$
$$-\int_{-1}^{1}\int_{|x|>1}\sum_{j=1}^{L}\alpha_j e^{-\gamma_j x^2}F(x,y)dxdy$$

and once more to get

$$2\mathcal{K}_\omega[m,n] = \sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}e^{-\gamma_k y^2}F_{m,n}(x,y)dxdy - \sum_{k=1}^{L}\alpha_k\int_{-\infty}^{\infty}\int_{-1}^{1}e^{-\gamma_k y^2}F_{m,n}(x,y)dxdy$$
$$-\sum_{j=1}^{L}\alpha_j\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}F_{m,n}(x,y)dxdy + \int_{-1}^{1}\int_{-1}^{1}F_{m,n}(x,y)dxdy.$$

Summing all of these integrals gives

$$2\tilde{\boldsymbol{A}}_\omega[m,n] = -\sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}e^{-\gamma_k y^2}F_{m,n}(x,y)dxdy$$
$$+\sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}F_{m,n}(x,y)dxdy + \sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_k y^2}F_{m,n}(x,y)dydx$$
$$-\sum_{k=1}^{L}\alpha_k\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_k y^2}F_{m,n}(x,y)dydx - \sum_{j=1}^{L}\alpha_j\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}F_{m,n}(x,y)dxdy + \int_{-1}^{1}\int_{-1}^{1}F_{m,n}(x,y)dxdy.$$

We note that the first double integral cancels out $\theta_{j,k}(m,n)$ that we calculated earlier. So we only need to calculate

$$2\tilde{\boldsymbol{A}}_\omega[m,n] = \sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}F_{m,n}(x,y)dxdy+\sum_{j,k=1}^{L}\alpha_j\alpha_k\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_k y^2}F_{m,n}(x,y)dydx$$

$$-\sum_{k=1}^{L}\alpha_k\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_k y^2}F_{m,n}(x,y)dydx-\sum_{j=1}^{L}\alpha_j\int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma_j x^2}F_{m,n}(x,y)dxdy+\int_{-1}^{1}\int_{-1}^{1}F_{m,n}(x,y)dxdy$$

$$= \sum_{j,k=1}^{L}\alpha_j\alpha_k\theta_j^1[m,n]+\sum_{j,k=1}^{L}\alpha_j\alpha_k\theta_k^2[m,n]-\sum_{k=1}^{L}\alpha_k\theta_k^3[m,n]-\sum_{j=1}^{L}\alpha_j\theta_j^4[m,n]+\theta_5[m,n].$$

Lemma 5.3. *Let*

$$\theta[p,q] = \int_{-1}^{1}\int_{-\infty}^{\infty}e^{-\gamma s^2}F_{p,q}(s,t)dsdt, \quad \Re\{\gamma\}>0$$

where

$$F_{p,q}(s,t) = e^{-\mathrm{i}\pi pt}e^{\mathrm{i}(as+bst+ct+ds^2+wt^2)}e^{\mathrm{i}\pi qs}.$$

Then

$$\theta[p,q] = \frac{\pi}{2\mathrm{i}\sqrt{u(\gamma-\mathrm{i}d)}}\exp\left[\frac{(\pi q+a)^2}{4(\mathrm{i}d-\gamma)}-\frac{v^2}{4u}\right]\Big(\mathrm{erf}(t^+)-\mathrm{erf}(t^-)\Big),$$

where

$$u = \frac{4\mathrm{i}w(\mathrm{i}d-\gamma)+b^2}{4(\mathrm{i}d-\gamma)}, \quad v = \frac{2\mathrm{i}(\mathrm{i}d-\gamma)(c-\pi p)+b(a+\pi q)}{2(\mathrm{i}d-\gamma)}, \quad t^{\pm} = \mathrm{i}\sqrt{u}\Big(\pm 1+\frac{v}{2u}\Big). \tag{5.5}$$

Proof. Expanding $\eta(s,t)$ we have

$$\theta[p,q] = \int_{-1}^{1}e^{\mathrm{i}t^2}e^{\mathrm{i}(c-\pi p)t}\int_{-\infty}^{\infty}e^{(\mathrm{i}d-\gamma)s^2}e^{\mathrm{i}(a+bt+\pi q)s}dsdt.$$

Denote by $\rho[q;t]$ the integral in s. Since $\Re\{\gamma\}>0$,

$$\begin{aligned}\rho[q;t] &= \sqrt{\frac{\pi}{\gamma-\mathrm{i}d}}\exp\left[\frac{(a+bt+\pi q)}{4(\mathrm{i}d-\gamma)}\right]\\ &= \sqrt{\frac{\pi}{\gamma-\mathrm{i}d}}\exp\left[\frac{(\pi q+a)^2}{4(\mathrm{i}d-\gamma)}\right]\exp\left[\frac{b^2}{4(\mathrm{i}d-\gamma)}t^2\right]\exp\left[\frac{b(\pi q+a)}{2(\mathrm{i}d-\gamma)}t\right]\end{aligned}$$

and hence

$$\begin{aligned}\theta[p,q] &= P'[q]\int_{-1}^{1}\exp\left[\frac{4\mathrm{i}(\mathrm{i}d-\gamma)+b^2}{4(\mathrm{i}d-\gamma)}t^2\right]\exp\left[\frac{2\mathrm{i}(c-\pi p)(\mathrm{i}d-\gamma)+b(\pi q+a)}{2(\mathrm{i}d-\gamma)}t\right]dt\\ &= P'[q]\int_{-1}^{1}\exp(ut^2+vt)dt,\end{aligned} \tag{5.6}$$

where u, t are given by (5.5) and

$$P'[q] = \sqrt{\frac{\pi}{\gamma-\mathrm{i}d}}\exp\left[\frac{(\pi q+a)^2}{4(\mathrm{i}d-\gamma)}\right].$$

Completing the square in the integrand in (5.6) and changing variables:

$$\begin{aligned}\theta[p,q] &= P'[q]\exp\Big(-\frac{v^2}{4u}\Big)\int_{-1}^{1}\exp\left[u\Big(t+\frac{v}{2u}\Big)^2\right]dt\\ &= \frac{P'[q]}{2\mathrm{i}}\sqrt{\frac{\pi}{u}}\exp\Big(-\frac{v^2}{4u}\Big)\Big(\mathrm{erf}(t^+)-\mathrm{erf}(t^-)\Big)\\ &= \frac{\pi}{2\mathrm{i}\sqrt{u(\gamma-\mathrm{i}d)}}\exp\left[\frac{(\pi q+a)^2}{4(\mathrm{i}d-\gamma)}-\frac{v^2}{4u}\right]\Big(\mathrm{erf}(t^+)-\mathrm{erf}(t^-)\Big).\end{aligned}$$

□

Corollary 5.4. *Let*

$$\widehat{\theta}[p,q] = \int_{-1}^{1}\int_{-\infty}^{\infty} e^{-\gamma t^2} F_{p,q}(s,t)dtds, \quad \Re\{\gamma\} > 0.$$

Then

$$\widehat{\theta}[p,q] = \frac{\pi}{2\mathrm{i}\sqrt{\widehat{u}(\gamma - \mathrm{i}w)}} \exp\left[\frac{(c-\pi p)^2}{4(\mathrm{i}w-\gamma)} - \frac{\widehat{v}^2}{4\widehat{u}}\right]\Big(\operatorname{erf}(\widehat{t}^+) - \operatorname{erf}(\widehat{t}^-)\Big),$$

where

$$\widehat{u} = \frac{4\mathrm{i}d(\mathrm{i}w-\gamma)+b^2}{4(\mathrm{i}w-\gamma)}, \quad \widehat{v} = \frac{2\mathrm{i}(\mathrm{i}w-\gamma)(a+\pi q)+b(c-\pi p)}{2(\mathrm{i}w-\gamma)}, \quad \widehat{t}^{\pm} = \mathrm{i}\sqrt{\widehat{u}}\Big(\pm 1 + \frac{\widehat{v}}{2\widehat{u}}\Big).$$

Proof. Since we are integrating with respect to the second variable of F first, this has the effect of interchanging the parameters of η in (5.5) for their counterpart with the same order in the other variable. □