The Diameter and Chemical Distance of Random Clusters

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Abstract

A relatively unexplored geometric property of Potts models clusters is their "diameter", D – the longest shortest path between any two points on the cluster. We report numerical results for the fractal dimension of the diameter, D_{min} and the fractal dimension of the chemical distance, d_{min}, for 2D critical Potts clusters with $q = 1, 2, 3, 4, 5$. We find that $D_{min} = d_{min}$ within numerical error.

I. INTRODUCTION

Topics: Potts Model (history, applications); Chemical distance (history, applications) The Potts Model, initially introduced as a generalization of the 2-state Ising Model to q possible spin states, can in fact be mapped onto the Random Cluster Model for all $q \geq 0$: $q \to 1$ corresponds to the Percolation Model, and $q \to 2$ corresponds to the Ising Model. The Potts Model has found application in an impressively diverse range of contexts, including conformal field theory, percolation theory, knot theory, quantum groups, mathematical biology, and complex networks. Although easy to formulate, the Potts model exhibits rich phase behavior, and its study has yielded many significant insights into critical phenomena in statistical physics.

An important geometric property of Potts clusters is the "chemical distance", l – the length of the "chemical" or shortest path between two randomly chosen sites on a cluster. The average chemical distance on critical Potts clusters is known to scale as $\langle l \rangle \propto r^{d_{min}}$ at criticality, where r is the Euclidean distance between the endpoints of the chemical path l. Attempts to establish a relationship between d_{min} and other known critical exponents have as yet proved inconclusive. For the $q \to 1$ (Percolation) case, much work has already been done to determine d_{min} numerically [1, 2] and an exact solution has been found using results from conformal field theory [3].

In this paper we generalize previous studies of d_{min} for the 2D, $q = 1$ Potts Model by reporting results for the $q = 2, 3, 4$ and 5. We also study the critical scaling of a related quantity: the diameter, D, defined as the longest shortest path between any two points on a cluster. We show that D also exhibits scaling behavior at criticality: $\langle D \rangle \propto r^{D_{min}}$; and that $d_{min} = D_{min}$ to within the error in our numerical results.

II. POTTS MODEL SIMULATIONS

Topics: Swedsen Wang; Determining the chemical distance; determining the diameter; autocorrelation time; scaling

A. The Potts Model

The q-state Potts model consists of a lattice of Potts spin variables σ_i, each of which can have integer values $\sigma_i = 1 \ldots q$. Any two neighboring spins σ_i and σ_j contribute an amount $-K$ to the Hamiltonian if they have the same value, or zero otherwise; the Hamiltonian can thus be written as:

$$\mathcal{H} = -K \sum_{\langle i,j \rangle} \delta_{\sigma_i,\sigma_j}, \tag{1}$$

with K a dimensionless coupling constant.

B. Swendsen Wang Algorithm

We performed Monte Carlo simulations of critical q-state Potts model clusters in 2D and 3D using the Swendsen-Wang algorithm (SW) [4, 5].

The SW algorithm, which is itself based on the work of Fortuin and Kasteleyn [6], works by first introducing bonds between neighboring spins, with probability

$$p(\sigma_i, \sigma_j) = \delta_{\sigma_i,\sigma_j}(1 - e^{-K}), \tag{2}$$

thus creating clusters of bonded spins. All clusters thus formed are then, with probability $1/2$, flipped by choosing a random spin value from the q possible values, and assigning this value to all sites in the cluster. Such cluster-flipping algorithms dramatically reduce critical slowing down in computer simulations of spin models, as compared with algorithms that flip each spin individually [5] (e.g. the Metropolis algorithm [7]).

C. Algorithms for determining the chemical distance and the diameter

The average chemical distance $\langle l \rangle$ for each lattice size L was determined in the following manner. The largest cluster in the lattice was identified, and a randomly chosen site A on this cluster was used as the initial seed for a Leath growth (CITE) process, which amounts to a breadth-first growth along cluster bonds. Each iteration i of the Leath growth process covers all sites at chemical shell i; once the Leath process has covered all N sites in the cluster, the chemical distance l between the seed site, A, and any other site, B, is thus equal to the chemical shell on which site B resides. We chose B at random from the sites on the

largest possible chemical shell reached from site A on the cluster. The chemical distance between A and B chosen in this manner was then averaged over the largest cluster in all of the lattice realizations considered.

D. Details of simulation

1. autocorrelation time & independence

We measured the autocorrelation time for each L and q. We then chose intervals such that the samples were deemed to be statistically independent. (As a check, we also performed the block method technique.)

2. scaling methods, detail

We chose the fit with the lowest L (max no. of points) and highest Q value for each fit. (Alternative methods? See Sokal)

III. RESULTS AND DISCUSSION

A. Results

Using these methods we were able to determine through numerical simulations the scaling exponents for the chemical distance d_{min} and for the diameter D_{min} for system sizes $L \times L$, $4 \leq L \leq 128$ in 2D and $4 \leq L \leq X$ in 3D. The results of these simulations (see Tables I and **??**) indicate that d_{min} and D_{min} are equivalent to within error .

[Mean field limit?]

IV. BIBLIOGRAPHY

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q	1	2	3	4	5
d_{min}	1.127(3)	1.0911(2)	1.063(1)	1.023(7)	(1.000)
D_{min}	1.129(2)	1.087(8)	1.060(2)	1.025(2)	(1.000)

TABLE I: **Results for 2D Potts Model.** Scaling exponent for the chemical distance (d_{min}) and for the diameter (D_{min}) for the 2D Potts model with various values of q, with system size L=4, 8, 16, 32, 48, 64, 96, 128.

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