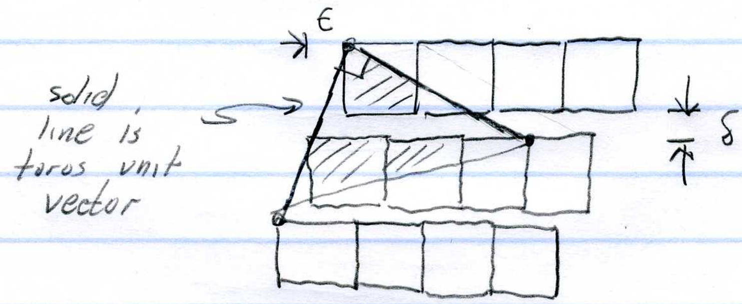
Notes on squares on a torus:

1) Perfect squares and sums of two squares fill the torus with density one. These case cover $N = 1, 2, 4, 5, 8, 9, 10, 13, 16, 18, 20, \ldots$

Here are some conjectures for other cases

$N = 6$ "Gapped bricklayer"



The two lines are the unit vectors of the torus and must be equal length and at right angles yielding the equations. Gap is δ, shift is ϵ.

$$3 - \epsilon = 2 + 2\delta$$
$$2\epsilon = 1 + \delta$$

The soln of these equations is

$$\epsilon = 3/5 \quad \delta = 1/5$$

Thus, the area of the torus in units of the squares is

$$A = (2\epsilon)^2 + (3-\epsilon)^2 = \left(\tfrac{6}{5}\right)^2 + \left(\tfrac{12}{5}\right)^2$$

The number of squares on the torus is 6 (shaded area) so the density is

$$d = \frac{6}{A} = \frac{6}{5} = 0.833\overline{3}$$

N = 7 "Trivial"
Same as 8 with a hole

$$d = 7/8 = .875$$

Note that both of these configurations have continuous degeneracies. For N=6 there are two sublattices that can be shifted relative to each other in the y direction (direction $\perp$ to the gap
For N=7, the hole can be arbitrarily distributed in either its row or its column.

N=11 probably also a gapped bricklayer

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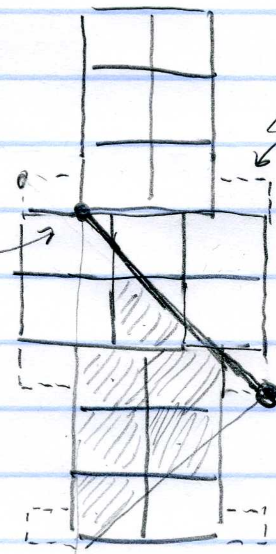
$N=12$ "Square array of holes



dotted line bounds a hole

holes are $\frac{1}{2} \times \frac{1}{2}$

solid line is torus unit vector

The length of the unit vector is $\sqrt{2\left(\frac{5}{2}\right)^2}$

$A = \frac{25}{2}$ $N=12$ (shaded area)

$d = \frac{24}{25} = 0.96$ (Slightly better than the trivial one hole, $d = 12/13$)

All of the above conjectured densities are very close to the numerical densities found by annealing.