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Combinatorial cube packings in the cube and the torus

Mathieu Dutour Sikirić [a], Yoshiaki Itoh [b]

[a] Rudjer Bošković Institute, Bijenicka 54, 10000 Zagreb, Croatia

[b] The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan

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ABSTRACT

We consider sequential random packing of cubes $z + [0, 1]^n$ with $z \in \frac{1}{N}\mathbb{Z}^n$ into the cube $[0, 2]^n$ and the torus $\mathbb{R}^n/2\mathbb{Z}^n$ as $N \to \infty$. In the cube case, $[0, 2]^n$ as $N \to \infty$, the random cube packings thus obtained are reduced to a single cube with probability $1 - O(\frac{1}{N})$. In the torus case, the situation is different: for $n \le 2$, sequential random cube packing yields cube tilings, but for $n \ge 3$ with strictly positive probability, one obtains non-extensible cube packings.

So, we introduce the notion of combinatorial cube packing, which instead of depending on N depends on some parameters. We use them to derive an expansion of the packing density in powers of $\frac{1}{N}$. The explicit computation is done in the cube case. In the torus case, the situation is more complicated and we restrict ourselves to the case $N \to \infty$ of strictly positive probability. We prove the following results for torus combinatorial cube packings:

- We give a general Cartesian product construction.
- We prove that the number of parameters is at least $\frac{n(n+1)}{2}$ and we conjecture it to be at most $2^n - 1$.
- We prove that cube packings with at least $2^n - 3$ cubes are extensible.
- We find the minimal number of cubes in non-extensible cube packings for n odd and $n \le 6$.

1. Introduction

Two cubes $z + [0, 1]^n$ and $z' + [0, 1]^n$ are *non-overlapping* if the relative interiors $z +]0, 1[^n$ and $z' +]0, 1[^n$ are disjoint. A family of cubes $(z^i + [0, 1]^n)_{1 \le i \le m}$ with $z^i \in \frac{1}{N}\mathbb{Z}^n$ and $N \in \mathbb{Z}_{>0}$ is called a

E-mail addresses: mdsikir@irb.hr (M.D. Sikirić), itoh@ism.ac.jp (Y. Itoh).

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Table 1
Number (Nr) of packings and tilings for the cases $N = \infty$ and $N = 2$ (see [5]).

	n	1	2	3	4	5
$N = \infty$	Nr cube tilings	1	1	3	32	?
	Nr non-extensible cube packings	0	0	1	31	?
	$f^T_{>0,\infty}(n)$	2	4	4	6	6
	$\frac{1}{2^n}E(M^T_\infty(n))$	1	1	$\frac{35}{36}$	$\frac{15258791833}{16102195200}$	?
$N = 2$	Nr cube tilings	1	2	8	744	?
	Nr cube packings	0	0	1	139	?
	$f^T_2(n)$	2	4	4	8	$10 \le f^T_2(5) \le 12$

discrete cube packing if any two cubes are non-overlapping. We consider packing of cubes $z + [0, 1]^n$ with $z \in \frac{1}{N}\mathbb{Z}^n$ into the cube $[0, 2]^n$ and the torus $\mathbb{R}^n/2\mathbb{Z}^n$. In those two cases, two cubes $z + [0, 1]^n$ and $z' + [0, 1]^n$ are non-overlapping if and only if there exists an index $i \in \{1, \dots, n\}$ such that $z_i \equiv z'_i + 1 \pmod 2$. A discrete cube packing is a *tiling* if the number of cubes is 2^n and it is *non-extensible* if it is maximal by inclusion with less than 2^n cubes.

A *sequential random cube packing* consists of putting a cube $z + [0, 1]^n$ with $z \in \frac{1}{N}\mathbb{Z}^n$ uniformly at random in the cube $[0, 2]^n$ or the torus $\mathbb{R}^n/2\mathbb{Z}^n$ until a maximal packing is obtained. Let us denote by $M^C_N(n)$, $M^T_N(n)$ the random variables of the number of cubes of those non-extensible cube packings and by $E(M^C_N(n))$, $E(M^T_N(n))$ their expectation. We are interested in the limit $N \to \infty$ and we prove that if $N > 1$ then

$$E(M^U_N(n)) = \sum_{k=0}^{\infty} \frac{U_k(n)}{(N-1)^k} \quad \text{with } U \in \{C, T\} \text{ and } U_k(n) \in \mathbb{Q}. \tag{1}$$

In the cube case we prove that $C_k(n)$ are polynomials of degree k, which we compute for $k \le 6$ (see Theorem 3.3). In particular, $C_0 = 1$, since as $N \to \infty$ with probability $1 - O(\frac{1}{N})$, one cannot add any more cubes after the first one. In the torus case the coefficients $T_k(n)$ are no longer polynomials in the dimension n. The first coefficient $T_0(n) = \lim_{N\to\infty} E(M^T_N(n))$ is known only for $n \le 4$ (see Table 1). But we prove in Theorem 4.4 that if $n \ge 3$ then $T_0(n) < 2^n$. This upper bound is related to the existence in dimension $n \ge 3$ of non-extensible torus cube packings (see Fig. 2, Table 1, Theorem 4.4 and Section 5).

Those results are derived using the notion of *combinatorial cube packings* which is introduced in Section 2. A combinatorial cube packing does not depend on N but instead depends on some parameters t_i; with a cube or torus discrete cube packing $\mathcal{CP}$, one can associate a combinatorial cube packing $\mathcal{CP}' = \phi(\mathcal{CP})$. Given a combinatorial cube packing $\mathcal{CP}$ the probability $p(\mathcal{CP}, N)$ of obtaining a discrete cube packing $\mathcal{CP}'$ with $\phi(\mathcal{CP}') = \mathcal{CP}$ is a fractional function of N. We say that $\mathcal{CP}$ is obtained with *strictly positive probability* if the limit $\lim_{N\to\infty} p(\mathcal{CP}, N)$ is strictly positive.

In Section 3 the method of combinatorial cube packings is applied to the cube case and the polynomials C_k are computed for $k \le 6$. In the torus case, the situation is more complicated and we restrict ourselves to the case of strictly positive probability, i.e. the limit case $N \to \infty$. In Section 4 we consider a Cartesian product construction for continuous cube packings obtained with strictly positive probability. The related lamination construction is used to derive an upper bound on $E(M^T_\infty(n))$ in Theorem 4.4.

In Section 5, we consider properties of non-extensible combinatorial torus cube packings. Firstly, we prove in Theorem 5.1 that combinatorial cube packings with at least $2^n - 3$ cubes are extensible to tilings. In Propositions 5.3 and 5.5, we prove that non-extensible combinatorial torus cube packings obtained with strictly positive probability have at least $\frac{n(n+1)}{2}$ parameters and that this number is attained by a combinatorial cube packing with $n + 1$ cubes if n is odd. We conjecture that the number of parameters is at most $2^n - 1$ (see Conjecture 5.4). In Proposition 5.7 we prove that in dimension 6 the minimal number of cubes in non-extensible combinatorial cube packings is 8 and that none of those cube packings are attained with strictly positive probability. In Proposition 5.8 we show that in dimensions 3, 5, 7 and 9, there exist combinatorial cube tilings obtained with strictly positive probability and $\frac{n(n+1)}{2}$ parameters.

We now explain the origin of the model considered here. Palásti [11] considered maximal packings obtained from random packings of cubes $[0, 1]^n$ into $[0, x]^n$. She conjectured that the expectation $E(M_x(n))$ of the packing density $M_x(n)$ satisfies the limit

$$\lim_{x\to\infty} \frac{E(M_x(n))}{x^n} = \beta_n \tag{2}$$

with $\beta_n = \beta_1^n$. The value of β_1 has been known since the work of Rényi [15] and in Penrose [12] the limit (2) is proved to exist. Note that on the basis of simulations it is expected that $\beta_n > \beta_1^n$ and an experimental formula from simulations

$$\beta_n^{1/n} - \beta_1 \simeq (n-1)(\beta_2^{1/2} - \beta_1)$$

is known [2].

The Itoh–Ueda model [8] is a variant of the above: one considers packing of cubes $z + [0, 2]^n$ with $z \in \mathbb{Z}^n$ into $[0, 4]^n$. It is proved in [3,14,13] that the average number of cubes satisfies the inequality $E(M_2^C(n)) \geq (\frac{3}{2})^n$ and some computer estimates of the average density $\frac{1}{2^n}E(M_2^C(n))$ were obtained in [7]. In [5], we considered the torus case, similar questions to the one of this paper and a measure of regularity called the second moment, which has no equivalent here.

2. Combinatorial cube packings

If $z + [0, 1]^n \subset [0, 2]^n$ and $z = (z_1, \ldots, z_n) \in \frac{1}{N}\mathbb{Z}^n$ then $z_i \in \{0, \frac{1}{N}, \ldots, 1\}$. Take a discrete cube packing $\mathcal{CP} = (z^i + [0, 1]^n)_{1\leq i\leq m}$ of $[0, 2]^n$. For a given coordinate $1 \leq j \leq n$ we set $\phi(z_j^i) = t_{i,j}$ with $t_{i,j}$ a parameter if $0 < z_j^i < 1$ and $\phi(z_j^i) = z_j^i$ if $z_j^i = 0$ or 1. If $z^i = (z_1^i, z_2^i, \ldots, z_n^i)$ then we set $\phi(z^i) = (\phi(z_1^i), \ldots, \phi(z_n^i))$ and with $\mathcal{CP}$ we associate the combinatorial cube packing $\phi(\mathcal{CP}) = (\phi(z^i) + [0, 1]^n)_{1\leq i\leq m}$.

Take a torus discrete cube packing $\mathcal{CP} = (z^i + [0, 1]^n)_{1\leq i\leq m}$ with $z^i \in \frac{1}{N}\mathbb{Z}^n$. For a given coordinate $1 \leq j \leq n$ we set $\phi(z_j^i) = t_{k,j}$ if $z_j^i \equiv \frac{k}{N} \pmod 2$ and $\phi(z_j^i) = t_{k,j} + 1$ if $z_j^i \equiv \frac{k}{N} + 1 \pmod 2$ with $t_{k,j}$ a parameter. Similarly, we set $\phi(z^i) = (\phi(z_1^i), \ldots, \phi(z_n^i))$ and we define $\phi(\mathcal{CP}) = (\phi(z^i) + [0, 1]^n)_{1\leq i\leq m}$, the associated torus combinatorial cube packing.

In the remainder of this paper we do not use the above parameters but instead renumber them as $t_1, \ldots, t_N$. Without loss of generality, we will always assume that different coordinates have different parameters. Of course we can define combinatorial cube packings without using discrete cube packings. In the cube case, the relevant cubes are of the form $z + [0, 1]^n$ with $z_i = 0, 1$ or some parameter t. In the torus case, the relevant cubes are of the form $z + [0, 1]^n$ with $z_i = t$ or $t + 1$ and t a parameter. Two cubes $z^i + [0, 1]^n$ and $z^{i'} + [0, 1]^n$ are non-overlapping if there exists a coordinate j such that $z_j^i \equiv z_j^{i'} + 1 \pmod 2$. In the cube case this means that $z_j^i = 0$ or 1 and $z_j^{i'} = 1 - z_j^i$. In the torus case this means that z_j^i depends on the same parameter, say t, $z_j^i, z_j^{i'} = t$ or $t + 1$ and $z_j^i \neq z_j^{i'}$. A combinatorial cube packing is then a family of such cubes with any two of them being non-overlapping. Notions of tilings and extensibility are defined as well. Moreover, a discrete cube packing is extensible if and only if its associated combinatorial cube packing is extensible. Denote by $m(\mathcal{CP})$ the number of cubes of a combinatorial cube packing $\mathcal{CP}$ and by $N(\mathcal{CP})$ its number of parameters. Denote by $Comb^C(n)$, $Comb^T(n)$, the set of combinatorial cube packings of $[0, 2]^n$ and $\mathbb{R}^n/2\mathbb{Z}^n$ respectively.

Given two combinatorial cube packings $\mathcal{CP}$ and $\mathcal{CP}'$ (either on cube or torus), we say that $\mathcal{CP}'$ is a *subtype* of $\mathcal{CP}$ if after assigning the parameter of $\mathcal{CP}$ as 0, 1, or some parameter of $\mathcal{CP}'$, we get $\mathcal{CP}'$. So, necessarily $m(\mathcal{CP}') = m(\mathcal{CP})$ and $N(\mathcal{CP}') \leq N(\mathcal{CP})$ but the reverse implication is not true in general. A combinatorial cube packing is said to be *maximal* if it is not the subtype of any other combinatorial cube packing. Necessarily, a combinatorial cube packing $\mathcal{CP}$ is a subtype of at least one maximal combinatorial cube packing $\mathcal{CP}'$.

Given a combinatorial cube packing $\mathcal{CP}$, the number of discrete cube packings $\mathcal{CP}'$ such that $\phi(\mathcal{CP}') = \mathcal{CP}$ is denoted by $Nb(\mathcal{CP}, N)$. In the cube case we have $Nb(\mathcal{CP}, N) = (N-1)^{N(\mathcal{CP})}$.

The torus case is more complex but it is still possible to write explicit formulas: denote by $N_j(\mathcal{CP})$ the number of parameters which occur in the j-th coordinate of $\mathcal{CP}$. We then get

$$Nb(\mathcal{CP}, N) = \prod_{j=1}^{n} \prod_{k=1}^{N_j(\mathcal{CP})} (2N - 2(k-1)). \tag{3}$$

The asymptotic order of $Nb(\mathcal{CP}, N)$ is $(2N)^{N(\mathcal{CP})}$, which shows that $Nb(\mathcal{CP}, N) > 0$ for N large enough. More specifically, $N_j(\mathcal{CP}) \leq 2^n$ so $Nb(\mathcal{CP}, N) > 0$ if $N \geq 2^n$. Note that it is possible to have $\mathcal{CP}'$ a subtype of $\mathcal{CP}$ and $Nb(\mathcal{CP}', N) > Nb(\mathcal{CP}, N)$ for small enough N.

Denote by $f_N^T(n)$ the minimal number of cubes of non-extensible discrete torus cube packings $(z^i + [0, 1]^n)_{1\leq i\leq m}$ with $z^i \in \frac{1}{N}\mathbb{Z}^n$. Denote by $f_\infty^T(n)$ the minimal number of cubes of non-extensible combinatorial torus cube packings.

Proposition 2.1. *For $n \geq 1$ we have $\lim_{N\to\infty} f_N^T(n) = f_\infty^T(n)$.*

Proof. A discrete cube packing $\mathcal{CP}$ is extensible if and only if $\phi(\mathcal{CP})$ is extensible. Thus $f_\infty^T(n) \geq f_N^T(n)$. Take $\mathcal{CP}$ a non-extensible combinatorial torus cube packing with the minimal number of cubes. By the formula (3) there exists N_0 such that for $N > N_0$ we have $Nb(\mathcal{CP}, N) > 0$. The discrete cube packings $\mathcal{CP}'$ with $\phi(\mathcal{CP}') = \mathcal{CP}$ are non-extensible. So, we have $\lim_{N\to\infty} f_N^T(n) = f_\infty^T(n)$. □

In the cube case we have for $N \geq 2$ the equality $f_N^C(n) = 1$.

Two combinatorial cube packings $\mathcal{CP}$ and $\mathcal{CP}'$ are said to be equivalent if after a renumbering of the coordinates, parameters and cubes of $\mathcal{CP}$ one gets $\mathcal{CP}'$. The automorphism group of a combinatorial cube packing is the group of equivalences of $\mathcal{CP}$ preserving it. Testing equivalences and computing stabilizers can be done using the program `nauty` [10], which is a graph theory program for testing whether two graphs are isomorphic or not and computing the automorphism group. The method is to associate with a given combinatorial cube packing $\mathcal{CP}$ a graph $Gr(\mathcal{CP})$ which characterizes isomorphisms and automorphisms. The methods used to find such a graph $Gr(\mathcal{CP})$ are explained in the user manual of `nauty` and the corresponding programs are available from [4].

We now explain the sequential random cube packing. Given a discrete cube packing $\mathcal{CP} = (z^i + [0, 1]^n)_{1\leq i\leq m}$ denote by $Poss(\mathcal{CP})$ the set of cubes $z + [0, 1]^n$ with $z \in \frac{1}{N}\mathbb{Z}^n$ which do not overlap with $\mathcal{CP}$. Every possible cube $z + [0, 1]^n$ is selected with equal probability $\frac{1}{|Poss(\mathcal{CP})|}$. The sequential random cube packing process is thus a process that add cubes until the discrete cube packing is non-extensible or is a tiling.

Fix a combinatorial cube packing $\mathcal{CP}$, $N \geq 2^n$, and a discrete cube packing $\mathcal{CP}'$ such that $\phi(\mathcal{CP}') = \mathcal{CP}$. With any cube $w + [0, 1]^n \in Poss(\mathcal{CP}')$ we associate the combinatorial cube packing $\mathcal{CP}_w = \phi(\mathcal{CP}' \cup \{w + [0, 1]^n\})$. The set $Poss(\mathcal{CP}')$ is partitioned into classes $Cl_1, \dots, Cl_r$ with two cubes $w + [0, 1]^n$ and $w' + [0, 1]^n$ in the same class if $\mathcal{CP}_w = \mathcal{CP}_{w'}$. The combinatorial cube packing associated with Cl_i is denoted by $\mathcal{CP}_i$. The set $\{\mathcal{CP}_1, \dots, \mathcal{CP}_r\}$ of classes depends only on $\mathcal{CP}$. If we had chosen some $N \leq 2^n$, then some of the preceding $\mathcal{CP}_i$ might not have occurred. So, we have

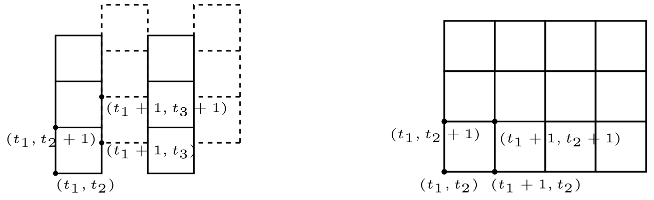
$$|Cl_i(N)| = \frac{Nb(\mathcal{CP}_i, N)}{Nb(\mathcal{CP}, N)}$$

and we can define the probability $p(\mathcal{CP}, \mathcal{CP}_i, N)$ of obtaining a discrete cube packing of combinatorial type $\mathcal{CP}_i$ from a discrete cube packing of combinatorial type $\mathcal{CP}$:

$$p(\mathcal{CP}, \mathcal{CP}_i, N) = \frac{|Cl_i(N)|}{|Cl_1(N)| + \cdots + |Cl_r(N)|} = \frac{Nb(\mathcal{CP}_i, N)}{Nb(\mathcal{CP}_1, N) + \cdots + Nb(\mathcal{CP}_r, N)}.$$

Given a combinatorial cube packing $\mathcal{CP}$ with m cubes, a *path* $p = \{\mathcal{CP}^0, \mathcal{CP}^1, \dots, \mathcal{CP}^m\}$ is a way of obtaining $\mathcal{CP}$ by adding one cube at a time starting from $\mathcal{CP}^0 = \emptyset$ and ending at $\mathcal{CP}^m = \mathcal{CP}$. The probability of obtaining $\mathcal{CP}$ along a path p is

$$p(\mathcal{CP}, p, N) = p(\mathcal{CP}^0, \mathcal{CP}^1, N) \times \cdots \times p(\mathcal{CP}^{m-1}, \mathcal{CP}^m, N).$$



A combinatorial cube tiling obtained with probability $\frac{1}{2}$.

A combinatorial cube tiling obtained with probability 0.

Fig. 1. Two two-dimensional torus combinatorial cube tilings.

The probability $p(\mathcal{CP}, N)$ of obtaining $\mathcal{CP}$ is the sum over all the paths p leading to $\mathcal{CP}$ of $p(\mathcal{CP}, p, N)$. The probabilities $p(\mathcal{CP}, p, N)$ and $p(\mathcal{CP}, N)$ are fractional functions of N, which implies that the limit values $p(\mathcal{CP}, \infty)$, $p(\mathcal{CP}, p, \infty)$ and $p(\mathcal{CP}, \mathcal{CP}', \infty)$ are well defined.

As N goes to ∞ we have the asymptotic behavior

$$|Cl_i(N)| \simeq (2N)^{nb_i}$$

with $nb_i = N(\mathcal{CP}_i) - N(\mathcal{CP})$ the number of new parameters in $\mathcal{CP}_i$ as compared with $\mathcal{CP}$. Clearly as N goes to ∞ only the classes with the largest nb_i have $p(\mathcal{CP}, \mathcal{CP}_i, \infty) > 0$. If Cl_i is such a class then we get

$$p(\mathcal{CP}, \mathcal{CP}_i, \infty) = \frac{1}{r'}$$

with r' the number of classes Cl_i having the largest nb_i and otherwise $p(\mathcal{CP}, \mathcal{CP}_i, \infty) = 0$. Analogously, for a path p leading to $\mathcal{CP}$ we can define $p(\mathcal{CP}, p, \infty)$ and $p(\mathcal{CP}, \infty)$. We say that a combinatorial cube packing $\mathcal{CP}$ is obtained with *strictly positive probability* if $p(\mathcal{CP}, \infty) > 0$, that is for at least one path p we have $p(\mathcal{CP}, p, \infty) > 0$. For a path $p = \{\mathcal{CP}^0, \mathcal{CP}^1, \dots, \mathcal{CP}^m\}$ we have $p(\mathcal{CP}, p, \infty) > 0$ if and only if every $\mathcal{CP}^i$ has $N(\mathcal{CP}^i)$ maximal among all possible extensions from $\mathcal{CP}^{i-1}$. This implies that each $\mathcal{CP}^i$ is maximal, i.e. is not the subtype of another type. As a consequence, we can define a sequential random cube packing process for combinatorial cube packing $\mathcal{CP}$ obtained with strictly positive probability and compute the probability $p(\mathcal{CP}, \infty)$.

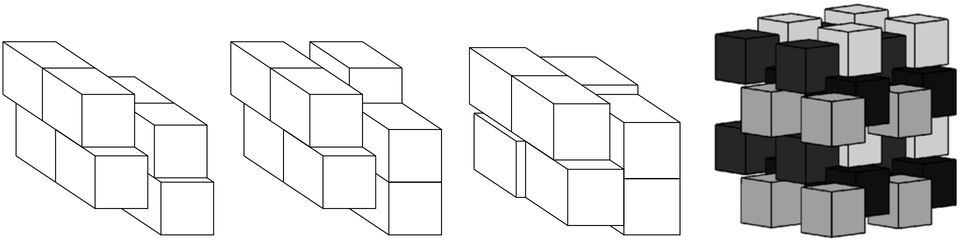
A combinatorial cube packing $\mathcal{CP}$ is said to have order $k = \mathrm{ord}(\mathcal{CP})$ if $p(\mathcal{CP}, N) = \frac{1}{(N-1)^k} f(N)$ with $\lim_{N\to\infty} f(N) \in \mathbb{R}_+^*$. A combinatorial cube packing is of order 0 if and only if it is obtained with strictly positive probability.

Let us denote by $M_N^C(n)$, $M_N^T(n)$ the random variables of the number of cubes of those non-extensible cube packings and by $E(M_N^C(n))$, $E(M_N^T(n))$ their expectation. From the preceding discussion we have

$$E(M_N^U(n)) = \sum_{\mathcal{CP} \in Comb^U(n)} p(\mathcal{CP}, N) m(\mathcal{CP}) \quad \text{with } U \in \{C, T\}.$$

Denote by $f_{>0,\infty}^T(n)$ the minimal number of cubes of non-extensible combinatorial torus cube packings obtained with strictly positive probability.

In dimension 2 (see Fig. 1), there are three combinatorial cube tilings. One of them is attained with probability 0; it is a subtype of the remaining two which are equivalent and attained with probability $\frac{1}{2}$. By applying the random cube packing process and doing reduction by isomorphism, one obtains the three-dimensional combinatorial cube packings obtained with strictly positive probability (see Fig. 2). The non-extensible cube packing shown in this figure already occurs in [9,5]. In dimension 4, the same enumeration method works (see Table 1) but for dimension 5 the enumeration is computationally too difficult.



7 parameters, probability $\frac{1}{3}$ | 7 parameters, probability $\frac{1}{3}$ | 6 parameters, probability $\frac{5}{18}$ | 6 parameters, probability $\frac{1}{18}$

Fig. 2. The three-dimensional combinatorial cube packings obtained with strictly positive probability; two laminations over two-dimensional cube tilings, the rod tiling and the smallest non-extensible cube packing.

3. Discrete random cube packings of the cube

We compute here the polynomials $C_k(n)$ occurring in Eq. (1) for $k \leq 6$. We compute the first three polynomials by an elementary method.

Lemma 3.1. *Put the cube $z^1 + [0, 1]^n$ in $[0, 2]^n$ and write*

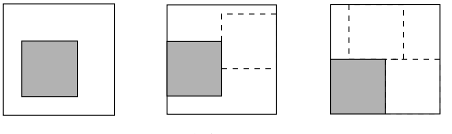
$$I = \{i : z_i^1 = 0 \text{ or } N\}$$

then do sequential random discrete cube packing.

(i) *The minimal number of cubes in the packing is $|I| + 1$.*
(ii) *The expected number of cubes in the packing is $|I| + 1 + O\left(\frac{1}{N+1}\right)$.*

Proof. Let us prove (i). If $|I| = 0$, then clearly one cannot insert any more cubes. We will assume $|I| > 1$ and do a reasoning by induction on $|I|$. If one puts in another cube $z^2 + [0, 1]^n$, there should exist an index $i \in I$ such that $|z_i^2 - z_i^1| = 1$. Take an index $j \neq i$ such that z_j^2 is 0 or 1. The set of possibilities for adding a subsequent cube is larger if $z_j \in \{0, 1\}$ than if $z_j \in \{\frac{1}{N}, \dots, \frac{N-1}{N}\}$. So, one can assume that for $j \neq i$, one has $0 < z_j^2 < 1$. This means that any cube $z + [0, 1]^n$ in subsequent insertion should satisfy $|z_i - z_i^2| = 1$, i.e. $z_i = z_i^1$. So, the sequential random cube packing can be done in one dimension less, starting with $z'^1 = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$. The induction hypothesis applies. Assertion (ii) follows easily by looking at the above process. For a given i the choice of z^2 with $0 < z_j^2 < 1$ for $j \neq i$ is the one with probability $1 - O\left(\frac{1}{N+1}\right)$. So, all neglected possibilities have probability $O\left(\frac{1}{N+1}\right)$ and with probability $1 - O\left(\frac{1}{N+1}\right)$ the number of cubes is the minimal possible. □

See below the two-dimensional possibilities:



$|I| = 0$ $\quad$ $|I| = 1$ $\quad$ $|I| = 2$

The random variable $M_N^C(n)$ is the number of cubes in the non-extensible cube packing obtained. $E(M_N^C(n))$ is the expected number of cubes and $E(M_N^C(n) \mid k)$ the expected number of cubes obtained by imposing the condition that the first cube $z^1 + [0, 1]^n$ has $|\{i : z_i^1 = 0 \text{ or } 1\}| = k$.

Theorem 3.2. *For any $n \geq 1$, we have*

$$E(M_N^C(n)) = 1 + \frac{2n}{N+1} + \frac{4n(n-1)}{(N+1)^2} + O\left(\frac{1}{N+1}\right)^3 \quad \text{as } N \to \infty.$$

Proof. If one chooses a vector z in $\{0, \ldots, N\}^n$ the probability that $|\{i : z_i^1 = 0 \text{ or } N\}| = k$ is $\left\{\frac{2}{N+1}\right\}^k \left\{\frac{N-1}{N+1}\right\}^{n-k} \binom{n}{k}$. Conditioning over $k \in \{0, 1, \ldots, n\}$, one obtains

$$E(M_N^C(n)) = \sum_{k=0}^{n} \left\{\frac{2}{N+1}\right\}^k \left\{\frac{N-1}{N+1}\right\}^{n-k} \binom{n}{k} E(M_N^C(n) \mid k). \tag{4}$$

So one gets

$$\begin{aligned} E(M_N^C(n)) &= \left(\frac{N-1}{N+1}\right)^n E(M_N^C(n) \mid 0) + n\frac{2}{N+1}\left(\frac{N-1}{N+1}\right)^{n-1} E(M_N^C(n) \mid 1) \\ &\quad + n(n-1)\frac{2}{(N+1)^2}\left(\frac{N-1}{N+1}\right)^{n-2} E(M_N^C(n) \mid 2) + O\left(\frac{1}{N+1}\right)^3. \end{aligned}$$

Clearly $E(M_N^C(n) \mid 0) = 1$ and $E(M_N^C(n) \mid 1) = 1 + E(M_N^C(n-1))$. By Lemma 3.1, $E(M_N^C(n) \mid 2) = 3 + O\left(\frac{1}{N+1}\right)$. Then, one has $E(M_N^C(n)) = 1 + O\left(\frac{1}{N+1}\right)$ and

$$\begin{aligned} E(M_N^C(n)) &= \left(1 - \frac{2}{N+1}\right)^n + \frac{2n}{N+1}\left(1 - \frac{2}{N+1}\right)^{n-1}\left(2 + O\left(\frac{1}{N+1}\right)\right) + O\left(\frac{1}{(N+1)^2}\right) \\ &= \left\{1 - \frac{2n}{N+1} + O\left(\frac{1}{(N+1)^2}\right)\right\} + \frac{4n}{N+1} + O\left(\frac{1}{(N+1)^2}\right) \\ &= 1 + \frac{2n}{N+1} + O\left(\frac{1}{(N+1)^2}\right). \end{aligned}$$

Inserting this expression into $E(M_N^C(n))$ and the formula (4) one gets the result. □

So, we get $C_0(n) = 1$, $C_1(n) = 2n$ and $C_2(n) = 4n(n-2)$. In order to compute $C_k(n)$ in general we use methods similar to the ones of Section 2. Given a cube $z + [0, 1]^n$ with $z_i \in \{0, \frac{1}{N}, \ldots, 1\}$ we define a face of the cube $[0, 1]^n$ in the following way: if $z_i = 0$ or 1 then we set $\psi(z_i) = 0$ or 1 whereas if $0 < z_i < 1$ we set $\psi(z_i) = t_i$ with t_i a parameter. When the parameters t_i of the vector $(\psi(z_1), \ldots, \psi(z_n))$ vary in $]0, 1[$ this vector describes a face of the cube $[0, 1]^n$, which we denote by $\psi(z)$. This construction was presented for the first time in [13,14].

If F and F' are two faces of $[0, 1]^n$, then we say that F is a subface of F' and write $F \subset F'$ if F is included in the closure of F'. A subcomplex of the hypercube $[0, 1]^n$ is a set of faces which contains all of its subfaces. If $\mathcal{CP}$ is a cube packing in $[0, 2]^n$, then the vectors z such that $z + [0, 1]^n$ is a cube which we can add to it are indexed by the faces of a subcomplex $[0, 1]^n$ with the dimension giving the exponent of $(N-1)^k$. The dimension of a complex is the highest dimension of its faces. Given a discrete cube packing $\mathcal{CP}$, we have seen in Section 2 that the size of $Poss(\mathcal{CP})$ depends only on the combinatorial type $\phi(\mathcal{CP})$. In the cube case which we consider in this section, $Poss(\mathcal{CP})$ itself depends only on the combinatorial type.

Theorem 3.3. *There exist polynomials $C_k(n)$ of n with $\deg C_k = k$ such that for any n and $N > 1$ one has*

$$E(M_N^C(n)) = \sum_{k=0}^{\infty} \frac{C_k(n)}{(N-1)^k}.$$

The polynomials $C_k(n)$ are given in Table 2.

Proof. The image $\psi(Poss(\mathcal{CP}))$ is a union of faces of $[0, 1]^n$, i.e. a subcomplex of the complex $[0, 1]^n$. Denote by $\dim(F)$ the dimension of a face F of the cube $[0, 1]^n$. Denote by $Poss(F)$ the set of vectors $z \in \{0, \frac{1}{N}, \dots, 1\}^n$ with $\psi(z) = F$. We have the formula

$$|Poss(F)| = (N-1)^{\dim(F)} \quad \text{and} \quad |Poss(\mathcal{CP})| = \sum_F (N-1)^{\dim(F)}.$$

The cubes whose corresponding faces in $[0, 1]^n$ have dimension $\dim(\psi(Poss(\mathcal{CP})))$ have the highest probability of being obtained. If one seeks the expansion of $E(M_N^C(n))$ up to order k and if $\mathcal{CP}$ is of order $\mathrm{ord}(\mathcal{CP})$ then we need to compute the faces of $\psi(Poss(\mathcal{CP}))$ of dimension at least $\dim(\psi(Poss(\mathcal{CP}))) - (k - \mathrm{ord}(\mathcal{CP}))$. The probabilities are then obtained in the following way:

$$p(F, N) = \frac{(N-1)^{\dim(F)}}{\sum_{F' \in \psi(Poss(\mathcal{CP})) \text{ with } \dim(F') \geq \dim(\psi(Poss(\mathcal{CP}))) - (k - \mathrm{ord}(\mathcal{CP}))} (N-1)^{\dim(F')}}. \tag{5}$$

The enumeration algorithm is then the following:

Input: Exponent k.
Output: List $\mathcal{L}$ of all inequivalent combinatorial types of non-extensible cube packings $\mathcal{CP}$ with order at most k and their probabilities $p(\mathcal{CP}, N)$ with an error of $O\left(\frac{1}{(N+1)^{k+1}}\right)$.

$\mathcal{T} \leftarrow \{\emptyset\}$.
$\mathcal{L} \leftarrow \emptyset$
while there is a $\mathcal{CP} \in \mathcal{T}$ **do**
  $\mathcal{T} \leftarrow \mathcal{T} \setminus \{\mathcal{CP}\}$
  $\psi(Poss(\mathcal{CP})) \leftarrow$ the complex of all possibilities of adding a cube to $\mathcal{CP}$
  $\mathcal{F} \leftarrow$ the faces of $\psi(Poss(\mathcal{CP}))$ of dimension at least
      $\dim(\psi(Poss(\mathcal{CP}))) - (k - \mathrm{ord}(\mathcal{CP}))$
  if $\mathcal{F} = \emptyset$ **then**
    if $\mathcal{CP}$ is equivalent to a $\mathcal{CP}'$ in $\mathcal{L}$ **then**
      $p(\mathcal{CP}', N) \leftarrow p(\mathcal{CP}', N) + p(\mathcal{CP}, N)$
    else
      $\mathcal{L} \leftarrow \mathcal{L} \cup \{\mathcal{CP}\}$
    end if
  else
    for $C \in \mathcal{F}$ **do**
      $\mathcal{CP}_{new} \leftarrow \mathcal{CP} \cup \{C\}$
      $p(\mathcal{CP}_{new}, N) \leftarrow p(\mathcal{CP}, N)p(C, N)$
      if $\mathcal{CP}_{new}$ is equivalent to a $\mathcal{CP}'$ in $\mathcal{T}$ **then**
        $p(\mathcal{CP}', N) \leftarrow p(\mathcal{CP}', N) + p(\mathcal{CP}_{new}, N)$
      else
        $\mathcal{T} \leftarrow \mathcal{T} \cup \{\mathcal{CP}_{new}\}$
      end if
    end for
  end if
end while

Let us prove that the coefficients $C_k(n)$ are polynomials in the dimension n. If C is the cube $[0, 1]^n$ then the number of faces of codimension l is $2^l \binom{n}{l}$, i.e. a polynomial in n of degree l. Suppose that a cube packing $\mathcal{CP} = (z^i + [0, 1]^n)_{1 \leq i \leq m}$ has $0 < z_j^i < 1$ for $n' \leq j \leq n$. Then all faces F of $\psi(Poss(\mathcal{CP}))$ of maximal dimension $d = \dim(\psi(Poss(\mathcal{CP})))$ have $0 < z_j < 1$ for $n' \leq j \leq n$ and $z \in F$. When one chooses a subface of F of dimension $d - l$, we have to choose some coordinates j to be equal to 0 or 1. Denote by l' the number of such coordinates j with $n' \leq j \leq n$. There are $2^{l'} \binom{n+1-n'}{l'}$ choices and they are all equivalent. There are still $l - l'$ choices to be made for $j \leq n' - 1$ but this number is finite

Table 2
The polynomials $C_k(n)$. $|Comb_k^C|$ is the number of types of combinatorial cube packings $\mathcal{CP}$ with $\text{ord}(\mathcal{CP}) \le k$.

k	$\lvert Comb_k^C\rvert$	$C_k(n)$
0	1	1
1	2	$2n$
2	3	$4n(n-2)$
3	7	$\frac{1}{3}\{28n^3 - 153n^2 + 149n\}$
4	18	$\frac{1}{2\cdot 3^2\cdot 5}\{2016n^4 - 21436n^3 + 58701n^2 - 40721n\}$
5	86	$\frac{1}{2^2 3^3\cdot 5\cdot 7}\{208724n^5 - 3516724n^4 + 18627854n^3 - 35643809n^2 + 20444915n\}$
6	1980	$\frac{1}{2^8 3^{11} 5^5 7^3 11^3\cdot 13\cdot 17}\{1929868729224214329703n^6 - 4692828379620116053 7385n^5$ $+3973790565954963301 71955n^4 - 144265997429108041377 0375n^3$ $+2205275555952621337847422n^2 - 111591132246678714324 1320n\}$

so in all cases the faces of $\psi(Poss(\mathcal{CP}))$ of dimension at least $d-l$ can be grouped in a finite number of classes with the size of the classes depending on n polynomially. Moreover, the number of classes of dimension d is finite so the term of higher order in the denominator of Eq. (5) is constant and the coefficients of the expansion of $p(F, N)$ are polynomial in n. $\square$

4. Combinatorial torus cube packings and lamination construction

Lemma 4.1. *Let $\mathcal{CP}$ be a non-extensible combinatorial torus cube packing.*
(i) *Every parameter t of $\mathcal{CP}$ which occurs as t also occurs as $t+1$.*
(ii) *Let $C_1, \dots, C_k$ be cubes of $\mathcal{CP}$ and C a cube which does not overlap with $\mathcal{CP}' = \mathcal{CP} - \{C_1, \dots, C_k\}$. The number of parameters of C which do not occur in $\mathcal{CP}'$ is at most $k-1$.*

Proof. (i) Suppose that a parameter t of $\mathcal{CP}$ occurs as t but not as $t+1$ in the coordinates of the cubes. Let $C = z + [0,1]^n$ be a cube having t in its j-th coordinate. If $C' = z' + [0,1]^n$ is a cube of $\mathcal{CP}$, then there exists a coordinate j' such that $z'_{j'} \equiv z_{j'} + 1 \pmod 2$. Necessarily $j' \ne j$ since $t+1$ does not occur, so $C + e_j$ does not overlap with C' as well and obviously $C + e_j$ does not overlap with C.
(ii) Let $C = z + [0,1]^n$ be a cube which does not overlap with the cubes of $\mathcal{CP}'$. Suppose that z has k coordinates $i_1 < \dots < i_k$ such that their parameters $t_1, \dots, t_k$ do not occur in $\mathcal{CP}'$. If $C_j = z^j + [0,1]^n$, then we fix $z_{i_j} \equiv z^j_{i_j} + 1 \pmod 2$ for $1 \le j \le k$ so that C does not overlap with $\mathcal{CP}$. This contradicts the fact that $\mathcal{CP}$ is extensible, so z has at most $k-1$ parameters which do not occur in $\mathcal{CP}'$. $\square$

Take two combinatorial torus cube packings $\mathcal{CP} = (z^i + [0,1]^n)_{1\le i\le m}$ and $\mathcal{CP}' = (z'^j + [0,1]^{n'})_{1\le j\le m'}$. Denote by $(z'^{i,j} + [0,1]^{n'})_{1\le j\le m'}$, with $1 \le i \le m$, m independent copies of $\mathcal{CP}'$; that is every parameter t'_k of z'^j is replaced by a parameter $t'_{i,k}$ in $z'^{i,j}$. One defines the combinatorial torus cube packing $\mathcal{CP} \ltimes \mathcal{CP}'$ by

$$(z^i, z'^{i,j}) + [0,1]^{n+n'} \quad \text{for } 1 \le i \le m \text{ and } 1 \le j \le m'.$$

Denote by $\mathcal{CP}_1$ the one-dimensional combinatorial packing formed by $(t + [0,1], t+1+[0,1])$. The combinatorial cube packings $\mathcal{CP}_1 \ltimes \mathcal{CP}_1$ and $\mathcal{CP}_1 \ltimes (\mathcal{CP}_1 \ltimes \mathcal{CP}_1)$ are the ones on the left of Figs. 1 and 2, respectively. Note that in general $\mathcal{CP} \ltimes \mathcal{CP}'$ is not isomorphic to $\mathcal{CP}' \ltimes \mathcal{CP}$.

Theorem 4.2. *Let $\mathcal{CP}$ and $\mathcal{CP}'$ be two combinatorial torus cube packings of dimension n and n', respectively.*

(i) *$m(\mathcal{CP} \ltimes \mathcal{CP}') = m(\mathcal{CP})m(\mathcal{CP}')$ and $N(\mathcal{CP} \ltimes \mathcal{CP}') = N(\mathcal{CP}) + m(\mathcal{CP})N(\mathcal{CP}')$.*
(ii) *$\mathcal{CP} \ltimes \mathcal{CP}'$ is extensible if and only if $\mathcal{CP}$ and $\mathcal{CP}'$ are extensible.*
(iii) *If $\mathcal{CP}$ and $\mathcal{CP}'$ are obtained with strictly positive probability and $\mathcal{CP}$ is non-extensible then $\mathcal{CP} \ltimes \mathcal{CP}'$ is attained with strictly positive probability.*
(iv) *One has $f^T_\infty(n+m) \le f^T_\infty(n) f^T_\infty(m)$ and $f^T_{>0,\infty}(n+m) \le f^T_{>0,\infty}(n) f^T_{>0,\infty}(m)$.*

Proof. Denote by $(z^i + [0,1]^n)_{1\le i\le m}$ and by $(z'^j + [0,1]^{n'})_{1\le j\le m'}$ the cubes of $\mathcal{CP}$ and $\mathcal{CP}'$ obtained in this order, i.e. first $z^1 + [0,1]^n$, then $z^2 + [0,1]^n$ and so on. Assertion (i) follows by simple counting.

If $\mathcal{CP}$ and $\mathcal{CP}'$, respectively, are extensible to $\mathcal{CP}\cup\{C\}$, $\mathcal{CP}'\cup\{C'\}$, then $\mathcal{CP}\ltimes\mathcal{CP}'$ is extensible to $(\mathcal{CP}\cup\{C\})\ltimes\mathcal{CP}'$ and $\mathcal{CP}\ltimes(\mathcal{CP}'\cup\{C'\})$, respectively, and so is extensible. Suppose now that $\mathcal{CP}$ and $\mathcal{CP}'$ are non-extensible and take a cube $z+[0,1]^{n+n'}$ with z expressed in terms of the parameters of $\mathcal{CP}\ltimes\mathcal{CP}'$. Then the cube $(z_1,\dots,z_n)+[0,1]^n$ overlaps with one cube of $\mathcal{CP}$, say $z^i+[0,1]^n$. Also $(z_{n+1},\dots,z_{n+n'})+[0,1]^{n'}$ overlaps with one cube of $\mathcal{CP}'$, say $z'^j+[0,1]^n$. So, $z+[0,1]^{n+n'}$ overlaps with the cube $(z^i,z'^{i,j})+[0,1]^{n+n'}$ and $\mathcal{CP}\ltimes\mathcal{CP}'$ is non-extensible, establishing (ii).

A priori there is no simple relation between $p(\mathcal{CP}\ltimes\mathcal{CP}',\infty)$ and $p(\mathcal{CP},\infty)$, $p(\mathcal{CP}',\infty)$. But we will prove that if $p(\mathcal{CP},\infty)>0$, $p(\mathcal{CP}',\infty)>0$ and $\mathcal{CP}$ is not extensible then $p(\mathcal{CP}\ltimes\mathcal{CP}',\infty)>0$. That is, to prove (iii) we have to provide one path, among the many possible, in the random sequential cube packing process for obtaining $\mathcal{CP}\ltimes\mathcal{CP}'$ with strictly positive probability from some corresponding paths of $\mathcal{CP}$ and $\mathcal{CP}'$. We first prove that we can obtain the cubes $((z^i,z'^{i,1})+[0,1]^{n+n'})_{1\le i\le m}$ with strictly positive probability in this order. Suppose that we add a cube $z+[0,1]^{n+n'}$ after the cubes $(z^{i'},z'^{i',1})+[0,1]^{n+n'}$ with $i'<i$. If we choose a coordinate $k\in\{n+1,\dots,n+n'\}$ such that $z_k=(z^{i'},z'^{i',1})_k+1$ for some $i'<i$ then we still have to choose a coordinate for all other cubes. This is because all parameters in $(z'^{i,1})_{1\le i\le m}$ are distinct. So, we do not gain anything in terms of dimension by choosing $k\in\{n+1,\dots,n+n'\}$ and the choice $(z^i,z'^{i,1})$ has the same or higher dimension. So, we can get the cubes $((z^i,z'^{i,1})+[0,1]^{n+n'})_{1\le i\le m}$ with strictly positive probability.

Suppose that we have the cubes $(z^i,z'^{i,j})+[0,1]^{n+n'}$ for $1\le i\le m$ and $1\le j\le m'_0$. We will prove by induction that we can add the cubes $((z^i,z'^{i,m'_0+1})+[0,1]^{n+n'})_{1\le i\le m}$. Denote by $n'_{m'_0}\le n'-1$ the dimension of choices in the combinatorial torus cube packing $(z'^j+[0,1]^{n'})_{1\le j\le m'_0}$.

Let $z+[0,1]^{n+n'}$ be a cube, which we want to add to the existing cube packing. Denote by $\mathcal{S}_z$ the set of i such that $z+[0,1]^{n+n'}$ does not overlap with $(z^i,z'^{i,j})+[0,1]^{n+n'}$ on a coordinate $k\le n$. The fact that $z+[0,1]^{n+n'}$ does not overlap with the cubes $(z^i,z'^{i,j})+[0,1]^{n+n'}$ fixes $n'-n'_{m'_0}$ coordinates of z. If $i\ne i'$ then the parameters in $z'^{i,j}$ and $z'^{i',j'}$ are different; this means that $(n'-n'_{m'_0})|\mathcal{S}_z|$ components of z are determined. Therefore, since $\mathcal{CP}$ is non-extensible, we can use Lemma 4.1.(ii) and so get the following estimate for the dimension D of choices:

$$\begin{aligned} D &\le \{n'-(n'-n'_{m'_0})|\mathcal{S}_z|\}+\{|\mathcal{S}_z|-1\}\\ &\le n'_{m'_0}-(n'-n'_{m'_0}-1)\{|\mathcal{S}_z|-1\}\\ &\le n'_{m'_0}. \end{aligned} \tag{6}$$

We conclude that we cannot do better in terms of dimension than adding the cubes $((z^i,z'^{i,m'_0+1})+[0,1]^{n+n'})_{1\le i\le m}$, which we do. So we have a path p with $p(\mathcal{CP}\ltimes\mathcal{CP}',p,\infty)>0$ which proves that $\mathcal{CP}\ltimes\mathcal{CP}'$ is obtained with strictly positive probability.

(iv) follows immediately from (iii) and (ii). □

There exist cube packings $\mathcal{CP}$, $\mathcal{CP}'$ obtained with strictly positive probability such that $p(\mathcal{CP}\ltimes\mathcal{CP}',\infty)>0$, which shows that the hypothesis $\mathcal{CP}$ non-extensible is necessary in (iii).

The third three-dimensional cube packing of Fig. 2, named the rod packing, has the cubes $(h^i+[0,1]^3)_{1\le i\le 8}$ with the following h^i:

$$\begin{array}{llll} h^1=(\ t_1,\ t_2,\ t_3) & h^5=(\ t_6+1,\ t_2+1,\ t_5+1)\\ h^2=(\ t_1+1,\ t_4,\ t_5) & h^6=(\ t_1,\ t_2,\ t_3+1)\\ h^3=(\ t_6,\ t_2+1,\ t_5+1) & h^7=(\ t_1+1,\ t_2,\ t_5+1)\\ h^4=(\ t_1+1,\ t_4+1,\ t_5) & h^8=(\ t_1,\ t_2+1,\ t_5) \end{array}$$

Taking eight $(n-3)$-dimensional combinatorial torus cube tilings $(w^{i,j})_{1\le j\le 2^{n-3}}$ with $1\le i\le 8$, one defines an n-dimensional *rod tiling* combinatorial cube packing

$$(z^i,w^{i,j})+[0,1]^n \quad \text{for } 1\le i\le 8 \text{ and } 1\le j\le 2^{n-3}.$$

Theorem 4.3. *The probability of obtaining a rod tiling is*

$$p_1^{15} \times q_{n-3}^8$$

where q_n is the probability of obtaining an n-dimensional cube tiling and p_1^{15} is a rational function of n.

Proof. Up to equivalence, one can assume that in the random cube packing process, one puts

$$z^1 = (h^1, w^{1,1}) = (t_1, t_2, t_3, \ldots) \quad \text{and} \quad z^2 = (h^2, w^{2,1}) = (t_1 + 1, t_4, t_5, \ldots).$$

Then there are $n(n-1)$ possible choices for the next cube; $2(n-1)$ of them are respecting the lamination. So, there are $(n-2)(n-1)$ choices which do not respect the lamination and their probability is $p_1^3 = \frac{n-2}{n}$. Without loss of generality, we can assume that one has

$$z^3 = (h^3, w^{3,1}) = (t_6, t_2 + 1, t_5 + 1, \ldots).$$

In the next five stages we add cubes with $n-3$ new parameters each. We have more than one type to consider under equivalence and we need to determine the total number of possibilities in order to compute the probabilities.

For the cube $z^4 + [0, 1]^n$ we should have three integers i_1, i_2, i_3 such that $z^4_{i_j} \equiv z^j_{i_j} + 1 \pmod 2$. Necessarily, the i_j are all distinct, which gives $n(n-1)(n-2)$ possibilities. There are exactly six possibilities with $i_j \leq 3$. One of them corresponds to the non-extensible cube packing of Fig. 2 on the first three coordinates for which the five others have a non-zero probability of being extended to the rod tiling. When computing later probabilities, we used the automorphism group of the existing configuration and gathered the possibilities of extension into orbits. At the fourth stage, the five possibilities split into two orbits:

(1) O_1^4: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 4\}$ with $p_1^4 = p_1^3 \frac{3}{n(n-1)(n-2)}$,
(2) O_2^4: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 7\}$ with $p_2^4 = p_1^3 \frac{2}{n(n-1)(n-2)}$; write $\Delta_2^4 = 3(n-3)(n-4)+3(n-3)+4$, the number of possibilities of adding a cube to the packing $((h^i, w^{i,1}) + [0, 1]^n)_{i\in\{1,2,3,7\}}$.

When adding a fifth cube one finds the following cases up to equivalence:

(1) O_1^5: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 4, 5\}$ with $p_1^5 = p_1^4 \frac{2}{2(n-1)(n-2)}$,
(2) O_2^5: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 4, 7\}$ with $p_2^5 = p_1^4 \frac{2}{2(n-1)(n-2)} + p_2^4 \frac{3}{\Delta_2^4}$,
(3) O_3^5: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 7, 8\}$ with $p_3^5 = p_2^4 \frac{1}{\Delta_2^4}$.

When adding a sixth cube one finds the following cases up to equivalence:

(1) O_1^6: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 4, 5, 6\}$ with $p_1^6 = p_1^5 \frac{1}{3(n-2)}$,
(2) O_2^6: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 4, 5, 7\}$ with $p_2^6 = p_1^5 \frac{2}{3(n-2)} + p_2^5 \frac{2}{n(n-2)}$,
(3) O_3^6: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 4, 7, 8\}$ with $p_3^6 = p_2^5 \frac{1}{n(n-2)} + p_3^5 \frac{3}{3(n-2)}$.

When adding a seventh cube one finds the following cases up to equivalence:

(1) O_1^7: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$ with $p_1^7 = p_1^6 + p_2^6 \frac{1}{n-1}$,
(2) O_2^7: $(h^i, w^{i,1})$ for $i \in \{1, 2, 3, 4, 5, 7, 8\}$ with $p_2^7 = p_2^6 \frac{1}{n-1} + p_3^6 \frac{2}{2(n-2)}$.

The combinatorial cube packing of eight cubes $((h^i, w^{i,1}) + [0, 1]^n)_{1\leq i\leq 8}$ is then obtained with probability $p_1^8 = p_1^7 + p_2^7 \frac{1}{n-2}$.

Then we add cubes in dimension $n-4$ following in fact the construction of Theorem 4.2. The parameters t_3, t_4 and t_6 appear only twice in the cube packing for the rods, which contain six cubes in total. So, when one adds cubes we have $8(n-3)$ choices respecting the cube packing, i.e. of the form $z^9 = (h^i, w^{i,2})$ with $w_j^{i,2} \equiv w_j^{i,1} \pmod 2$ for some $1 \leq j \leq n-3$. We also have $3(n-3)(n-4)$ choices not respecting the rod tiling structure, i.e. of the form $z^9 = (k^i, w)$ with k^i being one of h^i for $1 \leq i \leq 3$ with t_3, t_4 or t_6 replaced by another parameter. But after adding a cube $(h^i, w^{i,2}) + [0, 1]^n$

with h^i containing t_3, t_4 or t_6 this phenomenon cannot occur. Below a *type* T_r^h of probability p_r^h is a packing formed by the eight vectors $(h^i, w^{i,1})_{1\le i\le 8}$ and $h-8$ vectors of the form $(h^i, w^{i,2})$ amongst which r of the parameters t_3, t_4 or t_6 do not occur. Note that there may be several non-equivalent cube packings with the same type but this is not important since they have the same numbers of possibilities.

Adding the 9th cube one gets:

(1) $T_3^9, p_1^9 = p_1^8\frac{2(n-3)}{8(n-3)+3(n-3)(n-4)}$,
(2) $T_2^9, p_2^9 = p_1^8\frac{6(n-3)}{8(n-3)+3(n-3)(n-4)}$.

Adding the 10th cube one gets:

(1) $T_3^{10}, p_1^{10} = p_1^9\frac{n-3}{7(n-3)+3(n-3)(n-4)}$,
(2) $T_2^{10}, p_2^{10} = p_1^9\frac{6(n-3)}{7(n-3)+3(n-3)(n-4)} + p_2^9\frac{3(n-3)}{7(n-3)+2(n-3)(n-4)}$,
(3) $T_1^{10}, p_3^{10} = p_2^9\frac{4(n-3)}{7(n-3)+2(n-3)(n-4)}$.

Adding the 11th cube one gets:

(1) $T_2^{11}, p_1^{11} = p_1^{10}\frac{6(n-3)}{6(n-3)+3(n-3)(n-4)} + p_2^{10}\frac{2(n-3)}{6(n-3)+2(n-3)(n-4)}$,
(2) $T_1^{11}, p_2^{11} = p_2^{10}\frac{4(n-3)}{6(n-3)+2(n-3)(n-4)} + p_3^{10}\frac{4(n-3)}{6(n-3)+(n-3)(n-4)}$,
(3) $T_0^{11}, p_3^{11} = p_3^{10}\frac{2(n-3)}{6(n-3)+2(n-3)(n-4)}$.

Adding the 12th cube one gets:

(1) $T_2^{12}, p_1^{12} = p_1^{11}\frac{n-3}{5(n-3)+2(n-3)(n-4)}$,
(2) $T_1^{12}, p_2^{12} = p_1^{11}\frac{4(n-3)}{5(n-3)+2(n-3)(n-4)} + p_2^{11}\frac{3(n-3)}{5(n-3)+2(n-3)(n-4)}$,
(3) $T_0^{12}, p_3^{12} = p_2^{11}\frac{2(n-3)}{5(n-3)+2(n-3)(n-4)} + p_3^{11}\frac{5(n-3)}{5(n-3)+2(n-3)(n-4)}$.

Adding the 13th cube one gets:

(1) $T_1^{13}, p_1^{13} = p_1^{12}\frac{4(n-3)}{4(n-3)+2(n-3)(n-4)} + p_2^{12}\frac{2(n-3)}{4(n-3)+(n-3)(n-4)}$,
(2) $T_0^{13}, p_2^{13} = p_2^{12}\frac{2(n-3)}{4(n-3)+(n-3)(n-4)} + p_3^{12}\frac{4(n-3)}{4(n-3)}$.

Adding the 14th cube one gets:

(1) $T_1^{14}, p_1^{14} = p_1^{13}\frac{(n-3)}{3(n-3)+(n-3)(n-4)}$,
(2) $T_0^{14}, p_2^{13} = p_1^{13}\frac{2(n-3)}{3(n-3)+(n-3)(n-4)} + p_2^{13}\frac{3(n-3)}{3(n-3)}$.

Adding the 15th cube one gets:

(1) $T_0^{15}, p_1^{15} = p_1^{14}\frac{2(n-3)}{2(n-3)+(n-3)(n-4)} + p_2^{14}$.

After that if we add a cube $z+[0,1]^n$, then necessarily z is of the form (h^i, w). So, we have eight different $(n-3)$-dimensional cube packing problems show up and the probability is $p_1^{15}q_{n-3}^8$. □

A combinatorial torus cube packing $\mathcal{CP}$ is called *laminated* if there exist a coordinate j and a parameter t such that for every cube $z+[0,1]^n$ of $\mathcal{CP}$ we have $z_j \equiv t \pmod 1$.

Theorem 4.4. (i) *The probability of obtaining a laminated combinatorial cube packing is* $\frac{2}{n}$.
(ii) *For any* $n \ge 1$, *one has* $E(M_\infty^T(n)) \le 2^n(1-\frac{2}{n}) + \frac{4}{n}E(M_\infty^T(n-1))$.
(iii) *For any* $n \ge 3$, $\frac{1}{2^n}E(M_\infty^T(n)) \le 1 - \frac{2^n}{n!}\frac{1}{24}$.

Proof. Up to equivalence, we can assume that after the first two steps of the process, we have

$$z^1 = (t_1, \dots, t_n) \quad \text{and} \quad z^2 = (t_1 + 1, t_{n+1}, \dots, t_{2n-1}).$$

So, we consider lamination on the first coordinate. We then consider all possible cubes that can be added. Those cubes should have one coordinate differing by 1 from other vectors. This makes $n(n-1)$ possibilities. If a vector respects the lamination on the first coordinate then its first coordinate should be equal to t_1 or $t_1 + 1$. This makes $2(n-1)$ possibilities. So, the probability of having a family of cube respecting a lamination at the third step is $\frac{2}{n}$. But one sees easily that in all further steps, the choices breaking the lamination have a dimension strictly lower than the one respecting the lamination, so they do not occur and we get (i).

By separating between laminated and non-laminated combinatorial torus cube packings, bounding the number of cubes of non-laminated combinatorial torus cube packings by 2^n one obtains

$$E(M_\infty^T(n)) \le \left(1 - \frac{2}{n}\right) \times 2^n + \frac{2}{n}(E(M_\infty^T(n-1)) + E(M_\infty^T(n-1))),$$

which is (ii). (iii) follows by induction starting from $\frac{1}{8}E(M_\infty^T(3)) = \frac{35}{36}$ (see Table 1). $\square$

5. Properties of non-extensible cube packings

Theorem 5.1. *If a combinatorial torus cube packing has at least* $2^n - 3$ *cubes, then it is extensible.*

Proof. Our proof closely follows [5] but is different from it. Take $\mathcal{CP}'$ a combinatorial torus cube packing with $2^n - \alpha$ cubes, $\alpha \le 3$. Take N such that $Nb(\mathcal{CP}, N) > 0$ and $\mathcal{CP}$ a discrete cube packing with $\phi(\mathcal{CP}) = \mathcal{CP}'$. If $\mathcal{CP}$ is extensible then $\mathcal{CP}'$ is extensible as well.

We select $\delta \in \mathbb{R}$ and denote by I_j the interval $[\delta + \frac{j}{2}, \delta + \frac{j+1}{2}[$ for $0 \le j \le 3$. Denote by $n_{j,k}$ the number of cubes whose k-th coordinate modulo 2 belongs to I_j.

All cubes of $\mathcal{CP}$ whose k-th coordinate belongs to I_j, I_{j+1} form after removal of their k-th coordinate a cube packing of dimension $n-1$, which we denote by $\mathcal{CP}_{j,k}$. We write $n_{j,k} + n_{j+1,k} = 2^{n-1} - d_{j,k}$ and obtain the equations

$$d_{0,k} - d_{1,k} + d_{2,k} - d_{3,k} = 0 \quad \text{and} \quad \sum_{j=0}^{3} d_{j,k} = 2\alpha.$$

We can then write the vector $d_k = (d_{0,k}, d_{1,k}, d_{2,k}, d_{3,k})$ in the following way:

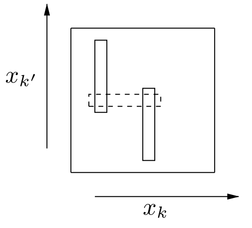
$$d_k = c_1(1, 1, 0, 0) + c_2(0, 1, 1, 0) + c_3(0, 0, 1, 1) + c_4(1, 0, 0, 1)$$
$$\text{with } \sum_{j=1}^{4} c_j = \alpha$$

and $c_i \in \mathbb{Z}^+$. This implies $d_{j,k} = c_j + c_{j+1} \le \sum c_j = \alpha$. This means that the $(n-1)$-dimensional cube packing $\mathcal{CP}_{j,k}$ has at least $2^{n-1} - 3$ cubes, so by an induction argument, we conclude that $\mathcal{CP}_{j,k}$ is extensible.

Suppose now that the k-th coordinates of the cubes in $\mathcal{CP}$ have values $0 < \delta_1 < \delta_2 < \cdots < \delta_M < 2$. So, the set of points in the complement of $\mathcal{CP}$ whose k-th coordinate belongs to the interval $[\delta_i, \delta_{i+1}[$ with $\delta_{M+1} = \delta_1 + 2$ can be filled by translates of the parallelepiped $Paral_k(\alpha) = [0, 1[^{k-1} \times [0, \alpha[\times [0, 1[^{n-k}$.

Note that as δ varies, the vector d_k varies as well. Suppose that for some i, we have the k-th layer $[\delta_i, \delta_{i+1}[$ being full and $[\delta_{i-1}, \delta_i[$ containing x translates with $x \le 3$ of the parallelepiped $Paral_k(\delta_{i+1} - \delta_i)$. Then if one selects another coordinate k', all parallelepipeds $Paral_{k'}(\delta'_{i'+1} - \delta'_{i'})$ filling the hole delimited by the parallelepiped $Paral_k(\delta_{i+1} - \delta_i)$ will have the same position in the k-th coordinate. This means that they will form x cubes and that the cube packing is extensible. This argument solves the case $\alpha = 1$, because up to symmetry $d_k = (0, 1, 1, 0)$.

If $\alpha = 2$, then the case of a vector of coordinate d_k being equal by symmetry to $(0, 2, 2, 0)$ or $(0, 1, 2, 1)$ is also solved because we have seen that a full layer implies that we can fill the hole. We have the remaining case $(1, 1, 1, 1)$. If the hole of this cube packing cannot be filled, then we have a structure of this form:



Selecting another coordinate k', we get that the two parallelepipeds $z + Paral_k(\delta_{i+1} - \delta_i)$ and $z' + Paral_k(\delta_{i'+1} - \delta_{i'})$ have $z_l = z'_l$ for $l \neq k, k'$. This is impossible if $n \geq 4$. So, if $d_k = (1, 1, 1, 1)$ for some k and δ, then the hole can be filled.

If $\alpha = 3$, and d_k, up to symmetry, is equal to $(0, 3, 3, 0)$ or $(0, 2, 3, 1)$, then we have a full layer and so we can fill the hole. If the vector $d_k = (2, 1, 1, 2)$ occurs, then by the same argument as for $(1, 1, 1, 1)$ we can fill the hole. □

Proposition 5.2. (i) *Non-extensible combinatorial torus cube packings of dimension n have at least $n+1$ cubes.*

(ii) *If $\mathcal{CP}$ is a combinatorial torus cube tiling, then in a coordinate j a parameter t occurs the same number of times as t and $t + 1$.*

Proof. (i) Suppose that a combinatorial torus cube packing $\mathcal{CP}$ has $m \leq n$ cubes $(z^i + [0, 1]^n)_{1\leq i\leq m}$. By fixing $z_i = z_i^i + 1$ for $i = 1, 2, \dots, m$ we get that the cube $z + [0, 1]^n$ does not overlap with $\mathcal{CP}$.

(ii) Without loss of generality, we can assume that a given parameter t occurs only in one coordinate k as t and $t+1$. The cubes occurring in the layer $[t, t+1]$, $[t+1, t+2]$ on j-th coordinate are the ones with $x_j = t, t + 1$; we denote by V_t and V_{t+1} their volume. Now if we interchange t and $t + 1$ we still obtain a tiling, so $V_t \leq V_{t+1}$ and $V_{t+1} \leq V_t$. So, $V_t = V_{t+1}$ and the number of cubes with $x_j = t$ is equal to the number of cubes with $x_j = t + 1$. □

Take a combinatorial torus cube packing $\mathcal{CP}$ obtained with strictly positive probability. Let us choose a path p to obtain $\mathcal{CP}$. Denote by $N_{k,p}(\mathcal{CP})$ the number of cubes obtained with k new parameters along the path p.

Proposition 5.3. *Let $\mathcal{CP}$ be a non-extensible combinatorial torus cube packing, p a path with $p(\mathcal{CP}, p, \infty) > 0$.*

(i) *$N_{n,p}(\mathcal{CP}) = 1$ and $N_{n-1,p}(\mathcal{CP}) = 1$.*
(ii) *$N_{n-2,p}(\mathcal{CP}) \leq 2$ and $N_{n-2,p}(\mathcal{CP}) = 2$ if and only if $\mathcal{CP}$ is laminated.*
(iii) *One has $N_{k,p}(\mathcal{CP}) \geq 1$ for $0 \leq k \leq n$.*
(iv) *$N(\mathcal{CP}) = \sum_{k=0}^{n} kN_{k,p}(\mathcal{CP}) \geq \frac{n(n+1)}{2}$.*
(v) *If $N(\mathcal{CP}) = \frac{n(n+1)}{2}$ then $N_{k,p}(\mathcal{CP}) = 1$ for $k \geq 1$.*

Proof. The first cube $z^1 + [0, 1]^n$ has n new parameters, but the second cube $z^2 + [0, 1]^n$ should not overlap with the first one so it has $n - 1$ parameters and $N_{n,p}(\mathcal{CP}) = 1$. Without loss of generality, we can assume that $z^1 = (t_1, \dots, t_n)$ and $z^2 = (t_1 + 1, t_{n+1}, \dots, t_{2n-1})$. When adding the third cube $z^3 + [0, 1]^n$, we have to set up two coordinates depending on the parameters t_i, $i \leq 2n - 1$; thus $N_{n-1,p}(\mathcal{CP}) = 1$.

If $z_1^3 = t_1$ or t_1+1 then we have a laminated cube packing; we can add a cube with $n-2$ parameters and $N_{n-2,p}(\mathcal{CP}) = 2$. Otherwise, we do not have a laminated cube packing; three coordinates of z^4 need to be expressed in terms of preceding cubes and thus $N_{n-2,p}(\mathcal{CP}) = 1$.

(iii) The proof is by induction; suppose one has put $m' = \sum_{l=k}^{n} N_{l,p}(\mathcal{CP})$ cubes. Then the cube $z^{m'} + [0, 1]^n$ has k new parameters $t'_1, \dots, t'_k$ in coordinates $i_1, \dots, i_k$. The cube $C = z + [0, 1]^n$ with $z_{i_1} = t'_1 + 1$ and $z_i = z_i^{m'}$ for $i \notin \{i_1, \dots, i_k\}$ has $k - 1$ free coordinates $\{i_2, \dots, i_k\}$ and thus $k - 1$ new parameters. So, $N_{k-1,p}(\mathcal{CP}) \geq 1$.

(iv) and (v) are elementary. □

Conjecture 5.4. *Let $\mathcal{CP}$ be a combinatorial torus cube packing and p a path with $p(\mathcal{CP}, p, \infty) > 0$.*

(i) *For all $k \geq 1$ one has $\sum_{k=0}^{l} N_{n-k,p}(\mathcal{CP}) \leq 2^l$.*

(ii) *$N(\mathcal{CP}) \leq 2^n - 1$; if $N(\mathcal{CP}) = 2^n - 1$, then $\mathcal{CP}$ is obtained via a lamination construction.*

A *perfect matching* of a graph G is a set $\mathcal{M}$ of edges such that every vertex of G belongs to exactly one edge of $\mathcal{M}$. A 1-*factorization* of a graph G is a set of perfect matchings which partitions the edge set of G. The graph K_4 has one 1-factorization; the graph K_6 has, up to isomorphism, exactly one 1-factorization with symmetry group Sym(5).

Proposition 5.5. *Let $\mathcal{CP}$ be a non-extensible combinatorial torus cube packing.*

(i) *If n is even then $\mathcal{CP}$ has at least $n + 2$ cubes.*

(ii) *If n is odd and $\mathcal{CP}$ has $n + 1$ cubes then $N(\mathcal{CP}) = \frac{n(n+1)}{2}$. Fix a coordinate j and a parameter t occurring in at least one cube. Then the number of cubes containing t, respectively $t + 1$ in coordinate j is exactly* 1.

(iii) *If n is odd then isomorphism classes of non-extensible combinatorial torus cube packings with $n+1$ cubes are in one to one correspondence with isomorphism classes of 1-factorizations of K_{n+1}.*

(iv) *If n is odd then the non-extensible combinatorial torus cube packings with $n+1$ cubes are obtained with strictly positive probability and $f^T_\infty(n) = f^T_{>0,\infty}(n) = n + 1$.*

Proof. We take a non-extensible cube packing $\mathcal{CP}$ with $n + 1$ cubes. Suppose that for a coordinate j we have two cubes $z^i + [0, 1]^n$ and $z^{i'} + [0, 1]^n$ with $z^i_j = z^{i'}_j = t$. If a vector z has $z_j = t + 1$, then $z + [0, 1]^n$ does not overlap with $z^i + [0, 1]^n$ and $z^{i'} + [0, 1]^n$. There are $n - 1$ remaining cubes with which $z + [0, 1]^n$ should not overlap but we have $n - 1$ remaining coordinates so it is possible to choose the coordinates of z so that $z + [0, 1]^n$ does not overlap with $\mathcal{CP}$. This is impossible; therefore parameters appear always at most once as t and at most once as $t + 1$ in a given coordinate.

By Lemma 4.1, every parameter t appears also as $t + 1$. So, every parameter t appears once as t and once as $t + 1$. This implies that we have an even number of cubes and so (i). Every coordinate has $\frac{n+1}{2}$ parameters, which gives $\frac{n(n+1)}{2}$ parameters and so (ii).

(iii) Assertion (ii) implies that any two cubes C_i and $C_{i'}$ of $\mathcal{CP}$ have exactly one coordinate on which they differ by 1. So, every coordinate corresponds to a perfect matching and the set of n coordinates to the 1-factorization.

(iv) Since parameters t appear only once as t and once as $t+1$, the dimension of choices after k cubes are put in is $n - k$ and one sees that such a cube packing is obtained with strictly positive probability. The existence of a 1-factorization of K_{2p} (see, for example, [1,6]) gives $f^T_\infty(n) \leq f^T_{>0,\infty}(n) \leq n + 1$. Combining this with Proposition 5.2.i, we have the result. □

Conjecture 5.6. *If n is even then there exist non-extensible combinatorial torus cube packings with $n + 2$ cubes and $\frac{n(n+1)}{2}$ parameters.*

In dimension 4 there is a unique cube packing (obtained with probability $\frac{1}{480}$) satisfying this conjecture:

$$\begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ t_5 & t_6 & t_7 & t_4 + 1 \\ t_1 + 1 & t_8 & t_7 + 1 & t_9 \\ t_5 + 1 & t_8 + 1 & t_3 + 1 & t_{10} \\ t_1 + 1 & t_6 + 1 & t_7 & t_{10} + 1 \\ t_5 & t_2 + 1 & t_7 + 1 & t_9 + 1 \end{pmatrix}.$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{14}+1 & t_{17}+1 & t_{19} \\ t_1+1 & t_6 & t_{10} & t_{13}+1 & t_{16}+1 & t_{19} \\ t_2 & t_5+1 & t_{11} & t_{13} & t_{18} & t_{20} \\ t_2+1 & t_7 & t_9+1 & t_{15} & t_{16} & t_{21} \\ t_3 & t_6+1 & t_{12} & t_{14} & t_{18}+1 & t_{21}+1 \\ t_3+1 & t_8 & t_{10}+1 & t_{15}+1 & t_{17} & t_{20}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{13}+1 & t_{17}+1 & t_{19}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{14}+1 & t_{16}+1 & t_{19}+1 \end{pmatrix}$$
$$21\ parameters, |Aut| = 4$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{13} & t_{17} & t_{21} \\ t_1+1 & t_6 & t_{10} & t_{14} & t_{18} & t_{21} \\ t_2 & t_5+1 & t_{10}+1 & t_{15} & t_{19} & t_{22} \\ t_2+1 & t_6+1 & t_9+1 & t_{16} & t_{20} & t_{22} \\ t_3 & t_7 & t_{11} & t_{13}+1 & t_{18}+1 & t_{22}+1 \\ t_3+1 & t_8 & t_{12} & t_{14}+1 & t_{17}+1 & t_{22}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{15}+1 & t_{20}+1 & t_{21}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{16}+1 & t_{19}+1 & t_{21}+1 \end{pmatrix}$$
$$22\ parameters, |Aut| = 64$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{13} & t_{17} & t_{21} \\ t_1+1 & t_6 & t_{10} & t_{14} & t_{18} & t_{21} \\ t_2 & t_5+1 & t_{10}+1 & t_{15} & t_{19} & t_{22} \\ t_2+1 & t_6+1 & t_9+1 & t_{16} & t_{20} & t_{22} \\ t_3 & t_7 & t_{11} & t_{13}+1 & t_{18}+1 & t_{22}+1 \\ t_3+1 & t_8 & t_{12} & t_{14}+1 & t_{17}+1 & t_{22}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{16}+1 & t_{19}+1 & t_{21}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{15}+1 & t_{20}+1 & t_{21}+1 \end{pmatrix}$$
$$22\ parameters, |Aut| = 64$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{13} & t_{17} & t_{21} \\ t_1+1 & t_6 & t_{10} & t_{14} & t_{18} & t_{22} \\ t_2 & t_5+1 & t_{10}+1 & t_{15} & t_{19} & t_{22} \\ t_2+1 & t_6+1 & t_9+1 & t_{16} & t_{20} & t_{21} \\ t_3 & t_7 & t_{11} & t_{13}+1 & t_{20}+1 & t_{22}+1 \\ t_3+1 & t_8 & t_{12} & t_{14}+1 & t_{19}+1 & t_{21}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{16}+1 & t_{17}+1 & t_{22}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{15}+1 & t_{18}+1 & t_{21}+1 \end{pmatrix}$$
$$22\ parameters, |Aut| = 16$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{13} & t_{17} & t_{21} \\ t_1+1 & t_6 & t_{10} & t_{14} & t_{18} & t_{22} \\ t_2 & t_5+1 & t_{10}+1 & t_{15} & t_{19} & t_{22} \\ t_2+1 & t_6+1 & t_9+1 & t_{16} & t_{20} & t_{21} \\ t_3 & t_7 & t_{11} & t_{13}+1 & t_{20}+1 & t_{22}+1 \\ t_3+1 & t_8 & t_{12} & t_{16}+1 & t_{17}+1 & t_{22}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{14}+1 & t_{19}+1 & t_{21}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{15}+1 & t_{18}+1 & t_{21}+1 \end{pmatrix}$$
$$22\ parameters, |Aut| = 16$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{13} & t_{17} & t_{21} \\ t_1+1 & t_6 & t_{10} & t_{14} & t_{18} & t_{21} \\ t_2 & t_5+1 & t_{11} & t_{15} & t_{18}+1 & t_{22} \\ t_2+1 & t_6+1 & t_9+1 & t_{16} & t_{19} & t_{22} \\ t_3 & t_7 & t_{10}+1 & t_{13}+1 & t_{20} & t_{22}+1 \\ t_3+1 & t_8 & t_{12} & t_{14}+1 & t_{17}+1 & t_{22}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{15}+1 & t_{19}+1 & t_{21}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{16}+1 & t_{20}+1 & t_{21}+1 \end{pmatrix}$$
$$22\ parameters, |Aut| = 16$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{13} & t_{17} & t_{21} \\ t_1+1 & t_6 & t_{10} & t_{14} & t_{18} & t_{22} \\ t_2 & t_5+1 & t_{11} & t_{15} & t_{19} & t_{22}+1 \\ t_2+1 & t_6+1 & t_9+1 & t_{16} & t_{20} & t_{21} \\ t_3 & t_7 & t_{10}+1 & t_{13}+1 & t_{20}+1 & t_{22} \\ t_3+1 & t_8 & t_{12} & t_{14}+1 & t_{19}+1 & t_{21}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{15}+1 & t_{18}+1 & t_{21}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{16}+1 & t_{17}+1 & t_{22}+1 \end{pmatrix}$$
$$22\ parameters, |Aut| = 32$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{13} & t_{17} & t_{21} \\ t_1+1 & t_6 & t_{10} & t_{14} & t_{18} & t_{21} \\ t_2 & t_5+1 & t_{11} & t_{15} & t_{18}+1 & t_{22} \\ t_2+1 & t_6+1 & t_9+1 & t_{16} & t_{19} & t_{22} \\ t_3 & t_7 & t_{10}+1 & t_{13}+1 & t_{20} & t_{22}+1 \\ t_3+1 & t_8 & t_{12} & t_{15}+1 & t_{19}+1 & t_{21}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{14}+1 & t_{17}+1 & t_{22}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{16}+1 & t_{20}+1 & t_{21}+1 \end{pmatrix}$$
$$22\ parameters, |Aut| = 8$$

$$\begin{pmatrix} t_1 & t_5 & t_9 & t_{13} & t_{17} & t_{21} \\ t_1+1 & t_6 & t_{10} & t_{14} & t_{18} & t_{22} \\ t_2 & t_5+1 & t_{11} & t_{15} & t_{19} & t_{22}+1 \\ t_2+1 & t_6+1 & t_9+1 & t_{16} & t_{20} & t_{21} \\ t_3 & t_7 & t_{10}+1 & t_{13}+1 & t_{20}+1 & t_{22} \\ t_3+1 & t_8 & t_{12} & t_{15}+1 & t_{18}+1 & t_{21}+1 \\ t_4 & t_7+1 & t_{12}+1 & t_{14}+1 & t_{19}+1 & t_{21}+1 \\ t_4+1 & t_8+1 & t_{11}+1 & t_{16}+1 & t_{17}+1 & t_{22}+1 \end{pmatrix}$$
$$22\ parameters, |Aut| = 16$$

Fig. 3. The non-extensible six-dimensional combinatorial cube packings with eight cubes and at least 21 parameters.

Proposition 5.7. (i) *There are nine isomorphism types of non-extensible combinatorial torus cube packings in dimension* 6 *with eight cubes and at least* 21 *parameters (see Fig. 3); they are not obtained with strictly positive probability.*

(ii) $8 = f^T_\infty(6) < f^T_{>0,\infty}(6)$.

Table 3
List of permutations describing combinatorial torus cube tilings with $\frac{n(n+1)}{2}$ parameters in dimensions 3, 5, 7, 9.

$n = 3$	(1, 2, 3)		
$n = 5$	(1, 2, 3, 4, 5)	(1, 2)(3, 5, 4)	(1, 4, 5, 3, 2)
$n = 7$	(1, 2, 3, 4, 5, 6, 7)	(1, 7)(2, 5, 4, 3, 6)	(1, 6, 2, 5, 4, 3, 7)
	(1, 7)(2, 5, 6)(3, 4)	(1, 6, 2, 3, 7)(4, 5)	(1, 7)(2, 3, 4, 5, 6)
	(1, 3, 7)(2, 6)(4, 5)	(1, 3, 7)(2, 5, 4, 6)	(1, 5, 4, 3, 2, 6, 7)
$n = 9$	(1, 2, 3, 4, 5, 6, 7, 8, 9)	(1, 6, 7, 4, 3, 5, 9)(2, 8)	(1, 5, 6, 7, 4, 3, 9)(2, 8)
	(1, 5, 9)(2, 8)(3, 6, 7, 4)	(1, 9)(2, 5, 4, 3, 6, 7, 8)	(1, 6, 7, 8, 2, 5, 4, 3, 9)
	(1, 5, 4, 3, 9)(2, 8)(6, 7)	(1, 9)(2, 5, 6, 7, 8)(3, 4)	(1, 9)(2, 3, 6, 5, 4, 7, 8)
	(1, 6, 5, 4, 7, 8, 2, 3, 9)	(1, 9)(2, 3, 6, 7, 8)(4, 5)	(1, 6, 7, 8, 2, 3, 9)(4, 5)
	(1, 9)(2, 3, 4, 7, 6, 5, 8)	(1, 9)(2, 3, 4, 5, 6, 7, 8)	(1, 9)(2, 3, 6)(4, 7, 8, 5)
	(1, 6, 2, 3, 9)(4, 7, 8, 5)	(1, 5, 4, 7, 8, 6, 2, 3, 9)	(1, 8, 3, 7, 6, 4, 9)(2, 5)
	(1, 7, 6, 4, 9)(2, 5)(3, 8)	(1, 7, 6, 9)(2, 8, 3, 4, 5)	(1, 4, 5, 2, 8, 3, 7, 6, 9)
	(1, 4, 9)(2, 5)(3, 7, 6, 8)	(1, 4, 8, 3, 7, 6, 9)(2, 5)	(1, 7, 6, 3, 4, 5, 2, 8, 9)
	(1, 4, 5, 2, 8, 9)(3, 7, 6)	(1, 4, 9)(2, 5)(3, 8, 7, 6)	(1, 3, 7, 6, 9)(2, 8, 4, 5)
	(1, 7, 6, 5, 4, 3, 2, 8, 9)	(1, 9)(2, 5, 8)(3, 4)(6, 7)	

Proof. (ii) follows immediately from (i). The enumeration problem in (i) is solved in the following way: instead of adding cube after cube like in the random cube packing process, we add coordinate after coordinate in all possible ways and reduce by isomorphism. The computation returns the listed combinatorial torus cube packings. Given a combinatorial cube packing $\mathcal{CP}$, in order to prove that $p(\mathcal{CP}, \infty) = 0$, we consider all $(8!)$ possible paths p and see that for all of them $p(\mathcal{CP}, p, \infty) = 0$. $\square$

Proposition 5.8. *If* $n = 3, 5, 7, 9$*, then there exist combinatorial torus cube tilings obtained with strictly positive probability and* $\frac{n(n+1)}{2}$ *parameters.*

Proof. If n is odd consider the matrix $H_n = m_{i,j}$ with all elements satisfying $m_{i+k,i} = m_{i-k,i} + 1$ for $1 \le k \le \frac{n-1}{2}$, the addition being modulo n. The matrix for $n = 5$ is

$$H_5 = \begin{pmatrix} t_1 & t_7 + 1 & t_{13} + 1 & t_{14} & t_{10} \\ t_6 & t_2 & t_8 + 1 & t_{14} + 1 & t_{15} \\ t_{11} & t_7 & t_3 & t_9 + 1 & t_{15} + 1 \\ t_{11} + 1 & t_{12} & t_8 & t_4 & t_{10} + 1 \\ t_6 + 1 & t_{12} + 1 & t_{13} & t_9 & t_5 \end{pmatrix}.$$

Then form the combinatorial cube packing with the cubes $(z^i + [0, 1]^n)_{1 \le i \le n}$ and z^i being the i-th row of H_n. It is easy to see that the number of parameters of cubes which we can add after z^i is $n - i$. So, those first n cubes are attained with the minimal number $\frac{n(n+1)}{2}$ of parameters and with strictly positive probability. If a cube $z + [0, 1]^n$ is non-overlapping with $z^i + [0, 1]^n$ for $i \le n$ then there exist $\sigma(i) \in \{1, \dots, n\}$ such that $z_{\sigma(i)} = z^i_{\sigma(i)} + 1$. If $i \ne i'$ then $\sigma(i) \ne \sigma(i')$, which proves that $\sigma \in \mathrm{Sym}(n)$. We also add the n cubes corresponding to the matrix $H_n + Id_n$. So, there are $n!$ possibilities for adding new cubes and we need to prove that we can select $2^n - 2n$ non-overlapping cubes amongst them.

The symmetry group of the n cubes $(z^i + [0, 1]^n)_{1 \le i \le n}$ is the dihedral group D_{2n} with $2n$ elements. It acts on $\mathrm{Sym}(n)$ by conjugation and so we simply need to list the relevant set of inequivalent permutations in order to describe the corresponding cube packings. See Table 3 for the permutations found for $n = 3, 5, 7, 9$. $\square$

The cube packing of above theorem was obtained for $n = 5$ by a random method, i.e., adding a cube whenever possible by choosing at random. Then the packings for $n = 7$ and 9 were built using the matrix H_n and consideration of all possibilities invariant under the dihedral group D_{2n}, by computer. But for $n = 11$ this method does not work. It would be interesting to know in which dimensions n combinatorial torus cube tilings with $\frac{n(n+1)}{2}$ parameters do exist.

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