

The Diameter and Chemical Distance of Random Clusters

Don Blair and Jon Machta

Department of Physics, University of Massachusetts, Amherst, MA 01003-3720

We report numerical results for the fractal dimension of the diameter (the “longest shortest path” along bonds) and the chemical distance of 2D and 3D Potts clusters for $q = 1, 2, 3, 4$. We find that the fractal dimension of the diameter and of the chemical distance of Potts clusters are equal to within numerical error, and we suggest a possible relationship between D_{min} and the dynamical exponent, z .

I. BACKGROUND

The Potts Model, initially introduced as a generalization of the 2-state Ising Model to q possible spin states, can in fact be mapped onto the Random Cluster Model for all $q \geq 0$, with $q \rightarrow 1$ corresponding to the Percolation Model, and $q \rightarrow 2$ corresponding to the Ising Model. The Potts Model has found application in an impressively diverse range of applications, including conformal field theory, percolation theory, knot theory, quantum groups, mathematical biology, and complex networks. Although easy to formulate, the model exhibits rich phase behavior, and its study has yielded many significant insights into critical phenomena in statistical physics.

An important geometric property of Potts clusters that has proved very useful in describing transport and diffusion processes in random media is the “chemical distance”, l – the length of the “chemical” or shortest path between two randomly chosen sites on a cluster. The average chemical distance on critical Potts clusters has been shown to scale as $\bar{l} \propto r^{d_{min}}$ at criticality, where r is the Euclidean distance between the endpoints of the chemical path l . Attempts to establish a relationship between d_{min} and other known critical exponents have, as yet, proved inconclusive [refs]. For the $q \rightarrow 1$ (Percolation) case, much work has already been done to determine d_{min} numerically [3, 4] and an exact solution has been found using results from conformal field theory [5].

In this paper we generalize previous studies of d_{min} for the 2D, $q = 1$ Potts Model by reporting results for the $q = 2, 3$ and 4 for both Potts Models in both 2D and 3D. We also study the critical scaling of a related quantity: the diameter, D , defined as the longest of all the shortest paths between points on a cluster. (An illustration of both D and l on a Potts cluster is shown in Figure [A]). We show that D also exhibits scaling behavior at criticality: $\bar{D} \propto r^{D_{min}}$; and that, significantly, $d_{min} = D_{min}$ to within the error of our numerical results.

We also propose a possible relationship between both D_{min} d_{min} and the dynamical exponent, z .

II. METHODS

A. Potts Model Simulation

- Swendsen Wang.
- Random Number Generator.
- Correlation time.
- Time to reach equilibrium.

B. Determining the chemical distance and the diameter

- Chemical distance.
- Naive method for diameter.
- Our method for the diameter.

III. RESULTS AND DISCUSSION

A. Results for 2D Potts Model

- Correlation time and time to reach equilibrium.
- Correspondence between d_{min} and D_{min} for $L \leq 128$.
- Results for d_{min} for $L > 128$.

q	1	2	3	4
d_{min}	1.127(3)	1.0911(2)	1.060(1)	1.023(7)
D_{min}	1.129(2)	1.09(1)	1.059(2)	1.025(2)

TABLE I: **Results for 2D Potts Model.** Scaling exponent for the chemical distance (d_{min}) and for the diameter (D_{min}) for the 2D Potts model with various values of q , with system size $L=4, 8, 16, 32, 48, 64, 96, 128$.

B. Results for 3D Potts Model

- Correlation time and time to reach equilibrium.
- Correspondence between d_{min} and D_{min} for $L \leq 96$.
- Results for d_{min} for $L > 96$.

q	1	2	3	4
d_{min}	1.127(3)	1.0911(2)	1.060(1)	1.023(7)
D_{min}	1.129(2)	1.09(1)	1.059(2)	1.025(2)

TABLE II: **Results for 3D Potts Model.** Scaling exponent for the chemical distance (d_{min}) and for the diameter (D_{min}) for the 3D Potts model with various values of q , with system size $L=4, 8, 16, 32, 48, 64, 96, 128$.

C. Postulated Relationship between D_{min} and z

Diffusion exponent, dynamic exponent, argument for relationship.

Acknowledgments

This work was supported by NSF grant ()

-
- [1] R. B. Potts, Proc. Cambridge Philos. Soc. **48**, 106 (1952).
 - [2] C. M. Fortuin and P. M. Kasteleyn, Physica **57**, 536 (1972).
 - [3] P. Grassberger, Math. Biosci. **63**, 157 (1983).
 - [4] H. J. Herrmann and H. E. Stanley, J. Phys. A: Math. Gen. **21**, L829 (1988).
 - [5] R. M. Ziff, J. Phys. A: Math. Gen. **32**, L457 (1999).
 - [6] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1992).
 - [7] H. J. Herrmann, D. C. Hong, and H. E. Stanley, J. Phys. A: Math. Gen. **17**, L261 (1984).
 - [8] R. H. Swendsen and J.-S. Wang, Phys. Rev. Lett. **57**, 2607 (1986).
 - [9] M. E. J. Newman and G. T. Barkema, *Monte Carlo Methods in Statistical Physics* (Oxford, Oxford, 1999).
 - [10] N. Metropolis, M. Rosenbluth, A. Teller, and E. Teller, J.Chem.Phys. **21**, 1087 (1953).
 - [11] G. N. Frederickson, SIAM J. Comput. **16**, 1004 (1987), ISSN 0097-5397.