Square packing

Gapped bricklayer solutions

Derive the gapped brick layer equations

The two vectors of the larger square are

```
Vec1 = n1 {1, 0} + n2 {c, d};
Vec2 = n3 {1, 0} + n4 {c, d};
```

Get the equations that must be satisfied

```
PeriodicEqu = FullSimplify[{Vec1 . Vec2 == 0, Vec1 . Vec1 == Vec2 . Vec2}]
```

$$\left\{d^2\, n2\, n4 + (n1 + c\, n2)\, (n3 + c\, n4) == 0,\ d^2\, n2^2 + (n1 + c\, n2)^2 == d^2\, n4^2 + (n3 + c\, n4)^2\right\}$$

```
FullSimplify[Solve[PeriodicEqu, {c, d}],
 {Element[n1, Integers], Element[n2, Integers], Element[n3, Integers], Element[n4, Integers]}]
```

$$\left\{\left\{d \to -\frac{\sqrt{n4^3}\ \mathrm{Abs}[n2\, n3 - n1\, n4]}{n4^{3/2}\left(n2^2 + n4^2\right)},\ c \to -\frac{n1\, n2 + n3\, n4}{n2^2 + n4^2}\right\},\right.$$

$$\left.\left\{d \to \frac{\sqrt{n4^3}\ \mathrm{Abs}[n2\, n3 - n1\, n4]}{n4^{3/2}\left(n2^2 + n4^2\right)},\ c \to -\frac{n1\, n2 + n3\, n4}{n2^2 + n4^2}\right\}\right\}$$

$$\mathrm{gap}[n1\_, n2\_, n3\_, n4\_] := \frac{n2\, n3 - n1\, n4}{\left(n2^2 + n4^2\right)}$$

$$\mathrm{shift}[n1\_, n2\_, n3\_, n4\_] := -\frac{n1\, n2 + n3\, n4}{n2^2 + n4^2}$$

```
number[n1_, n2_, n3_, n4_] := n2 n3 - n1 n4

density[n1_, n2_, n3_, n4_] := 1 / gap[n1, n2, n3, n4]

BestDensity[nn_, st_] := (
  lastd = 0; where = 1;
  For[i = 1, i < Length[st],
   If[st[[i]][[5]] == nn && st[[i]][[8]] >= lastd, (lastd = st[[i]][[8]]; where = i;)], i++];
  {where, st[[where]]}
 )
```

Old equations for gapped brick layer solutions (incorrect)

Find some special solutions

Special gapped bricklayer solutions

```
eqn = FullSimplify[
  {number[n1, n2, n3, n4] == nn, gap[n1, n2, n3, n4] == nn / (nn - 1)}, {Element[n1, Integers],
   Element[n2, Integers], Element[n3, Integers], Element[n4, Integers]}]
```

$$\left\{\text{n2 n3} == \text{n1 n4} + \text{nn},\ \frac{\text{n2 n3} - \text{n1 n4}}{\text{n2}^2 + \text{n4}^2} == \frac{\text{nn}}{-1 + \text{nn}}\right\}$$

```
solution = FullSimplify[Solve[eqn, {n1, n2}]]
```

$$\left\{\left\{\text{n1} \to -\frac{\text{nn} + \text{n3}\sqrt{-1 - \text{n4}^2 + \text{nn}}}{\text{n4}},\ \text{n2} \to -\sqrt{-1 - \text{n4}^2 + \text{nn}}\right\},\right.$$
$$\left.\left\{\text{n1} \to \frac{-\text{nn} + \text{n3}\sqrt{-1 - \text{n4}^2 + \text{nn}}}{\text{n4}},\ \text{n2} \to \sqrt{-1 - \text{n4}^2 + \text{nn}}\right\}\right\}$$

These solutions require $-1 - \text{n4}^2 + \text{nn}$ is a square integer which implies that **nn-1 must be a sum of two squares.** The packing density is (n-1)/n. If we take n+1 squares, which is now known to be a density 1 packing and remove a square, we get a packing density of n/(n+1) which is slightly better than the gapped bricklayer of this form. This is actually obvious enough from the equation for the gap, which is n/(n2^2+n4^2).

```
FullSimplify[shift[n1, n2, n3, n4] /. solution[[1]]]
```

$$\frac{\text{n3} - \text{n3 nn} - \text{nn}\sqrt{-1 - \text{n4}^2 + \text{nn}}}{\text{n4}\ (-1 + \text{nn})}$$

Here are two special solutions of this form:

```
FullSimplify[Solve[(n3 - n3 nn - nn Sqrt[-1 - n4^2 + nn])/(n4 (-1 + nn)) == 1, n3]]
```

$$\left\{\left\{\text{n3} \to \frac{\text{n4} - \text{n4 nn} - \text{nn}\sqrt{-1 - \text{n4}^2 + \text{nn}}}{-1 + \text{nn}}\right\}\right\}$$

```
FullSimplify[Solve[(n3 - n3 nn - nn Sqrt[-1 - n4^2 + nn])/(n4 (-1 + nn)) == m / (nn - 1), n3]]
```

$$\left\{\left\{\text{n3} \to \frac{\text{m n4} + \text{nn}\sqrt{-1 - \text{n4}^2 + \text{nn}}}{1 - \text{nn}}\right\}\right\}$$

Density 1, zero shift solutions

```
eqn = FullSimplify[{shift[n1, n2, n3, n4] == 0, gap[n1, n2, n3, n4] == 1}]
```

$$\left\{\frac{\text{n1 n2} + \text{n3 n4}}{\text{n2}^2 + \text{n4}^2} == 0,\ \frac{\text{n2 n3} - \text{n1 n4}}{\text{n2}^2 + \text{n4}^2} == 1\right\}$$

```
solution = FullSimplify[Solve[eqn, {n2, n1}]]
```

$\{\{\text{n1} \to -\text{n4},\ \text{n2} \to \text{n3}\}\}$

```
FullSimplify[number[n1, n2, n3, n4] /. solution[[1]]]
```

$\text{n3}^2 + \text{n4}^2$

All zero gap solutions look like this, however:

```
eqn = FullSimplify[{gap[n1, n2, n3, n4] == 1}]
```

$$\left\{\frac{n2\ n3 - n1\ n4}{n2^2 + n4^2} == 1\right\}$$

```
solution = FullSimplify[Solve[eqn, {n1}]]
```

$$\left\{\left\{n1 \to -\frac{n2^2 - n2\ n3 + n4^2}{n4}\right\}\right\}$$

```
FullSimplify[number[n1, n2, n3, n4] /. solution[[1]]]
```

$$n2^2 + n4^2$$

```
FullSimplify[shift[n1, n2, n3, n4] /. solution[[1]]]
```

$$\frac{n2 - n3}{n4}$$

Start Finding numerical solutions

Enumerating "all" solutions

```
solutiontable = Flatten[Table[If[b^2 + d^2 > 0 && number[a, b, c, d] > 1 && gap[a, b, c, d] ≥ 1,
      {a, b, c, d, number[a, b, c, d], shift[a, b, c, d], gap[a, b, c, d], density[a, b, c, d]},
      {a, b, c, d, 0, 0, 0, 0}], {a, -10, 10}, {b, -10, 10}, {c, -10, 10}, {d, -10, 10}], 3];
stable2 = Sort[solutiontable, #1[[5]] < #2[[5]] &];
```

```
Table[BestDensity[nn, stable2], {nn, 5, 18}]
```

$$\left\{\{169173, \{-9, -2, -7, -1, 5, -5, 1, 1\}\}, \left\{169493, \left\{-10, -2, -8, -1, 6, -\frac{28}{5}, \frac{6}{5}, \frac{5}{6}\right\}\right\},\right.$$
$$\left\{169709, \left\{-9, -2, -8, -1, 7, -\frac{26}{5}, \frac{7}{5}, \frac{5}{7}\right\}\right\}, \{170086, \{-10, -2, 6, 2, 8, -4, 1, 1\}\},$$
$$\left\{170385, \left\{-10, -3, -3, 0, 9, -\frac{10}{3}, 1, 1\right\}\right\}, \{170805, \{-10, -3, 0, 1, 10, -3, 1, 1\}\},$$
$$\left\{170981, \left\{-10, -3, -7, -1, 11, -\frac{37}{10}, \frac{11}{10}, \frac{10}{11}\right\}\right\}, \left\{171369, \left\{-9, -3, -7, -1, 12, -\frac{17}{5}, \frac{6}{5}, \frac{5}{6}\right\}\right\},$$
$$\{171582, \{-9, -2, 7, 3, 13, -3, 1, 1\}\}, \left\{171940, \left\{-10, -3, 2, 2, 14, -\frac{34}{13}, \frac{14}{13}, \frac{13}{14}\right\}\right\},$$
$$\left\{172208, \left\{-9, -3, 1, 2, 15, -\frac{29}{13}, \frac{15}{13}, \frac{13}{15}\right\}\right\}, \left\{172625, \left\{-10, -4, -4, 0, 16, -\frac{5}{2}, 1, 1\right\}\right\},$$
$$\left.\{172833, \{-9, -4, -2, 1, 17, -2, 1, 1\}\}, \left\{173334, \left\{-10, -3, 4, 3, 18, -\frac{7}{3}, 1, 1\right\}\right\}\right\}$$