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Optimal packings of up to five equal circles on a square flat torus

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Abstract We prove that a certain lattice arrangement of five equal circles on a square flat torus (the quotient of the plane by the lattice generated by two unit perpendicular vectors) is the densest possible arrangement. Our proof uses techniques from Rigidity Theory and Topological Graph Theory. We also apply these techniques to the cases of one to four equal circles on a square flat torus and prove that the known densest arrangements are the only locally maximally dense arrangements. Additionally, we establish the existence of a locally maximally dense lattice arrangement of $n = a^2 + b^2$ ($a > b > 0$ and $\gcd(a, b) = 1$) equal circles on a square flat torus.

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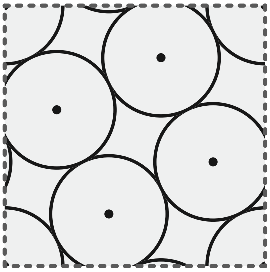
1 Introduction and techniques

In this article we consider a problem from discrete geometry in the area of equal circle packing. We prove that the following arrangement of five equal circles in a square flat torus, up to reflected and translated variants, is the densest possible arrangement.



This settles a conjecture made in the theses by Dickinson (1994) and Melissen (1997) where dense arrangements of small numbers of circles on a square flat torus are presented and explored.

In Melissen's thesis, which explores packings and coverings of various domains, he proves that specific packings of one to four equal circles on the square torus are the densest possible. He utilizes a technique similar to the one commonly used in proving the optimality of an arrangement of equal circles packed into a unit square. This technique centers on knowing a candidate for the densest arrangement of circles in the square which establishes a lower bound on the diameter of circles in the densest arrangement. One then cleverly uses this diameter to partition the square into regions with an appropriate diameter to prove the global optimality of the candidate arrangement. The approach employed in this article is fundamentally different. Here we prove the optimality of this arrangement of five circles on the square flat torus using techniques from the disparate fields of Rigidity Theory and Topological Graph Theory.

An optimal packing of n equal circles on a torus is naturally associated to an embedded graph by regarding the circle centers as vertices and the pairs of tangent circles as the edges connecting the vertices. We can view this graph as a type of tensegrity framework (an embedded graph with additional structure) on a torus. The results of Connelly then imply a lower bound on the number of circle–circle tangencies or equivalently the number of edges in the associated graph in any optimal arrangement. We also find an upper bound on the number of edges which allows us to make a finite list of the combinatorially distinct graphs with n vertices and a number of edges between the lower and upper bound. Using techniques from Topological Graph Theory we can enumerate all the possible embeddings of the combinatorially distinct graphs onto a topological torus. Finally, we use the geometry of a specific torus to eliminate those graphs which cannot be associated to any optimal circle packing. After demonstrating that the remaining graph or graphs are the packing graph associated to an optimally dense arrangement, we obtain a complete list of all locally and globally optimal arrangements. In the case of five circles, there is exactly one locally optimal arrangement. This technique does not rely on the circles being equal and can be adapted to any particular flat torus.

Besides the work of Dickinson and Melissen, packings on flat tori have been studied by other authors. In Heppes (1999), Heppes uses techniques similar to Melissen to determine the optimally dense arrangements of one to four equal circles in any rectangular flat torus. Przeworski (2006) explores packings of two equal circles on a torus, where the torus is not fixed, but is constrained to contain a closed geodesic of length one. He determines the densest packing in this situation. Lubachevsky et al. (1996) explore packings with large numbers (50–10,000) of equal circles packed on a square torus (among other domains). They used a billiards algorithm to discover their arrangements and they discuss large scale patterns as there is little hope of proving optimality. For similar explorations from a physics point of view, see the paper by Donev et al. (2004). Articles Reztsov and Sloan (1997) and Gensane and Ryckelynck (2008) explore optimal packings of squares on the square flat torus.

In Sect. 2, we review basic terms and definitions. Section 3 recalls results from Rigidity Theory which delineate the properties of packing graphs associated to optimally dense arrangements of equal circles. The power of these results is used in Sect. 4 to demonstrate an alternate proof of the optimality of arrangements of one to four circles presented in Dickinson (1994) and Melissen (1997). In Sect. 5, we discuss the enumeration of all the two-cell embeddings of graphs onto a torus with the properties determined by the results from Sect. 3. Finally, in Sect. 6, we present a tool that eliminates many of the embedded graphs from being the packing graph of an optimally dense arrangement and prove the optimality of the arrangement of five equal circles presented at the start of this section. These results also prove that there are no locally maximally dense arrangements of one to five circles on the square torus other than the globally maximally dense ones. Also in Sect. 6, we establish the local maximality of a lattice arrangement of $n = a^2 + b^2$ ($a > b > 0$ and $\gcd(a, b) = 1$) circles on the square flat torus.

2 Definitions and basic notions

In this section we review some terminology and recall some basic facts about circle packings. The quotient of the Euclidean plane by a lattice generated by two independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is called a flat torus. A fundamental domain of a flat torus is the set of points in the Euclidean plane, $\{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 | 0 \leq t_1, t_2 < 1\}$. When a basis for the lattice consists of two unit perpendicular vectors, the quotient torus is called the *square flat torus*. (In the literature this is sometimes denoted $\mathbb{R}^2/\mathbb{Z}^2$ or $[0, 1)^2$.) The *standard basis* for the square flat torus is the one where $\mathbf{v}_1 = \langle 1, 0\rangle$ and $\mathbf{v}_2 = \langle 0, 1\rangle$. For a point p, a lift of it is another point q in the Euclidean plane that is equivalent to p.

For a given flat torus, an arrangement of equal circles forms a *packing* on the torus if the interiors of the circles are disjoint. The density of a packing is the ratio of the area of the circles to the area of the flat torus. We define two packings with the same number of circles to be ϵ-*close* if there is a one-to-one correspondence between the circles, so that corresponding circles have centers that are all within a distance of ϵ of each other. We define a packing $\mathcal{P}$ to be optimal or *locally maximally dense* if there exists an $\epsilon > 0$ so that all ϵ-close packings of equal circles have a packing density no larger than that of $\mathcal{P}$. A packing $\mathcal{Q}$ is *globally maximally dense* if it is the densest possible packing. Rather than searching directly for the globally maximally dense packings, our techniques allow us to determine all the locally maximally dense arrangements of a fixed number of circles on a given flat torus. This allows us to easily determine the globally maximally dense packing(s).

The main structure that allows us to form a list of all the locally maximally dense packings for a fixed number of circles on a flat torus is the graph of a packing. Given a packing $\mathcal{P}$ on a flat torus, the *graph associated to* $\mathcal{P}$, denoted $G_{\mathcal{P}}$, has geometric vertices and edges defined as follows. (This is also sometimes called a kissing graph.) The center of each circle in the packing is associated to a vertex (with a location in the torus) of $G_{\mathcal{P}}$ and two vertices of $G_{\mathcal{P}}$ are connected with an edge (with a length on the torus) if and only if the corresponding circles are tangent to each other. Thus each packing of equal circles on a flat torus is naturally associated to an embedding of a graph on a flat torus where all the edges are equal in length. The next section will illuminate some of the properties of packing graphs associated to locally maximally dense packings on a flat torus.

For n circles on a flat torus we can find an upper bound for the diameter of the circles (or common edge length of the associated packing graph) by using the L. Fejes Tóth–Thue Theorem (Fejes Tòth 1964; Thue 1977) that states that the densest packing of equal circles in the Euclidean plane is uniquely achieved by the triangular close packing, where each circle is tangent to six others. The triangular close packing has density $\frac{\pi}{\sqrt{12}}$ and the packing density on the torus cannot exceed this bound. In the case of the square flat torus we have the following result.

Proposition 2.1 (Diameter upper bound) *The common diameter of a packing of n equal circles on a square flat torus may not exceed* $\frac{2}{\sqrt{n\sqrt{12}}}$.

3 Results from Rigidity Theory

This section reviews results from Rigidity Theory and is included for the convenience of the reader. The presentation here is the specialization of a much broader theory to the cases needed for the study of circle packings on a flat torus. For more details about the broader theory see Connelly (1988) and the references it contains.

What is the minimum number of edges that the graph associated to a circle packing must contain in order for the packing to be locally maximally dense? Connelly has answered this in Connelly (1988, 1990, 2008). The answer comes from studying tensegrity frameworks and determining when such a framework is rigid.

A tensegrity framework is essentially a graph embedded in a Riemannian manifold with some additional structure. To be a tensegrity framework, each edge in an embedded graph must be designated as a strut, bar, or cable and each vertex must be designated as either variable or fixed. Each edge in the graph that is designated as a strut is not allowed to decrease in length as the location of its endpoint vertices change. This is in contrast to cables (bars) where the length is not allowed to increase (change) as its endpoints change location. A *strut tensegrity framework* is an embedded graph in which all edges are designated as struts and all vertices are variable. The variable vertices (and incident edges) in a tensegrity framework can be (possibly) moved using a flex of a framework. Intuitively, a flex is motion of the vertices of an embedded graph that respects the distance constraints and fixes all fixed vertices.

To each circle packing we associate a strut tensegrity framework on a torus by regarding all edges in its packing graph as struts and all vertices as variable. This is appropriate because as we move the vertices of a circle packing graph to try and improve the density, we want the length of the edges to either increase (or remain unchanged), in order to possibly increase (or maintain) the density.

If the location of the vertices of a strut tensegrity framework with n vertices are denoted $p_1, p_2, \ldots, p_n$, then a *flex* of the framework is a collection of n continuous functions $\{p_i(t)|i = 1, \ldots, n\}$ from the interval $[0, 1]$ to the flat torus, where

1. $p_i(0) = p_i$ for all i, and
2. for each pair (i, j) where $\overline{p_i p_j}$ is a strut in the framework, $|p_i(t) - p_j(t)|$ is not a decreasing function on $[0, 1]$.

If each $p_i(t)$ is obtained by restricting the same rigid motion of the torus to each p_i then we say the motion is a *trivial flex*. A strut tensegrity framework is considered *rigid* if the only flex of the framework is a trivial flex.

In the context of strut tensegrity frameworks, there is another notion of rigidity that is easily checked and, in our case, is equivalent to the definition of rigidity given above. An *infinitesimal flex* of a strut tensegrity framework is a collection of n vectors $\{\mathbf{p}'_i \mid i = 1, \ldots, n\}$ where for each pair (i, j) where $\overline{p_i p_j}$ is a strut

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \geq 0. \tag{1}$$

In the context of a flat torus, the only *trivial infinitesimal flex* of a strut tensegrity framework is the one where, for a fixed non-zero vector $\mathbf{v}$, $\mathbf{p}'_i = \mathbf{v}$ for all i. (That is, p'_i is the time zero derivative of a family of rigid motions at p_i.) A strut tensegrity framework is considered *infinitesimally rigid* if the only infinitesimal flex of the framework is a trivial infinitesimal flex.

In Connelly (1988), Connelly proves that a strut tensegrity framework is rigid if and only if it is infinitesimally rigid. Combining the Rigidity Theory results with the ideas of circle packing, we have the following.

Proposition 3.1 *If the strut tensegrity framework associated to a circle packing $\mathcal{P}$ is infinitesimally rigid or rigid, then the circle packing $\mathcal{P}$ is locally maximally dense.*

We are now in a position to state the main result from Connelly (1988) (specialized to the context of flat tori) which is almost a converse of Proposition 3.1.

Theorem 3.2 (Connelly) *Let $\mathcal{P}$ be a packing that is locally maximally dense on a flat torus. Then there is a sub-packing $\mathcal{Q}$ of $\mathcal{P}$ such that the associated strut tensegrity framework is infinitesimally rigid and the circles not in $\mathcal{Q}$ (possibly an empty set) are prevented from increasing their radius, but are free to move (as in Fig. 1).*

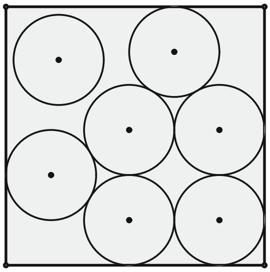


Fig. 1 The globally maximally dense arrangement of seven circles (due to Schaer 1965) in a hard-boundary square contains a free circle

There exist locally maximally dense packings of equal circles which contain an individual circle that is free to move (i.e. not held fixed by its neighbors), but the common diameter of all the circles cannot increase. For example, this occurs in the globally maximally dense arrangement of seven circles packed into a hard-boundary square (see Fig. 1). We call circles which are free to move *free circles* (or *floaters* or *rattlers*). If we remove any free circles from a locally maximally dense arrangement, then we obtain a locally maximally dense packing for fewer circles in the flat torus. Conversely, if there is room for another circle (free or not) in a globally maximally dense packing of n circles, then adding this circle gives us an arrangement realizing the globally maximal density (there may be several arrangements with this maximal density) for $n+1$ circles.

In this article, we will determine all the locally maximally dense arrangements for one to five circles without free circles. Therefore, for the remainder of this article, we assume that all of our graphs are connected. It turns out that none of these arrangements admit free circles, so we will have created an exhaustive list of locally maximally dense packings.

Now we observe that we can find a lower bound on the number of edges (and their arrangement) incident to a vertex in the packing graph associated to a locally maximally dense packing with no free circles.

Proposition 3.3 *Let $\mathcal{P}$ be a locally maximally dense packing of circles with no free circles. Then no circle in $\mathcal{P}$ has its points of tangency contained in a closed semi-circle. In particular, every circle is tangent to at least three circles.*

Proof If there were such a circle in a locally maximally dense packing, then we would have a non-trivial infinitesimal flex of the associated strut tensegrity framework contradicting Theorem 3.2. The infinitesimal flex would be given by assigning a non-zero vector to the vertex (corresponding to the circle) that points directly away from the closed semi-circle and the zero vector to remaining vertices. □

Finally, we answer the question raised at the start of this section. Connelly proves in Connelly (1990, 2008) a lower bound on the number of edges that a packing graph must contain in order for the associated packing to be locally maximally dense.

Proposition 3.4 (Minimum Edges) *Let $\mathcal{P}$ be a locally maximally dense packing of n circles on a flat torus with no free circles. Then the packing graph associated to $\mathcal{P}$ contains at least $2n-1$ edges.*

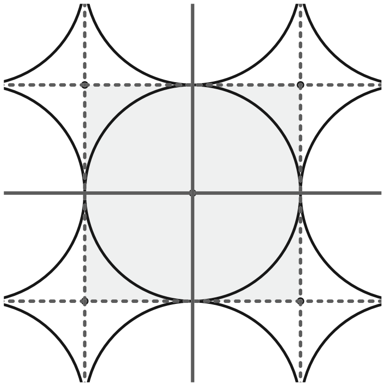


Fig. 2 The only locally maximally dense packing of one circle on the square torus and therefore the globally maximally dense packing. The fundamental region is *shaded* and the packing graph is shown

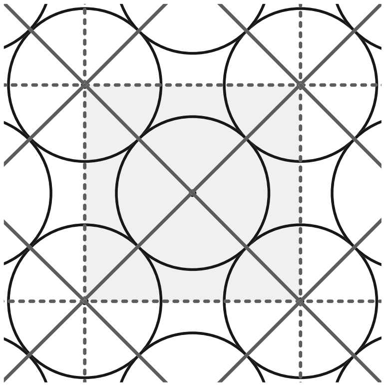


Fig. 3 The only locally maximally dense packing of two circles on the square torus and therefore the globally maximally dense packing. The fundamental region is *shaded* and the packing graph is shown

This lower bound is powerful and allows us to determine all the locally maximally dense packings of one to four circles on the square torus.

4 Packings of one to four circles

In this section, the results from Sect. 3 are applied to prove that arrangements of one to four equal circles on a square torus are globally maximally dense. While Melissen (1997) has already established the global maximality of these arrangements, this illustrates the power of the Rigidity Theory approach and also proves that there are no other locally maximally dense arrangements for one to four circles (which has not been established before).

The only locally maximally dense packing of one circle with diameter 1 is the one shown in Fig. 2. It is also the only packing where a circle is tangent to itself. Therefore we turn our attention to packings of two circles.

Proposition 4.1 *The only locally maximally dense packing of two circles on the square torus* (*up to translated variants*) *is the packing pictured in Fig.* 3. *The common diameter is* $\frac{\sqrt{2}}{2}$.

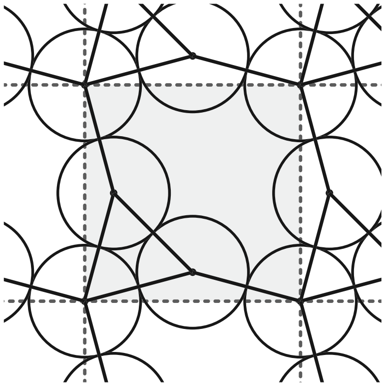


Fig. 4 The only locally maximally dense packing of three circles on the square torus and therefore the globally maximally dense packing. The fundamental region is *shaded* and the packing graph is shown

Proof Using a translation we may fix the first circle at the origin (always the lower left hand corner of the fundamental domain) and by the Minimum Edges Proposition 3.4 we must place the second circle in a location where it is tangent to the first circle in at least three ways. If we place this circle anywhere except at the center of this fundamental domain, then a maximum of two tangencies are formed. We can use Inequalities (1) and Proposition 3.1 to show that this packing is locally maximally dense. $\square$

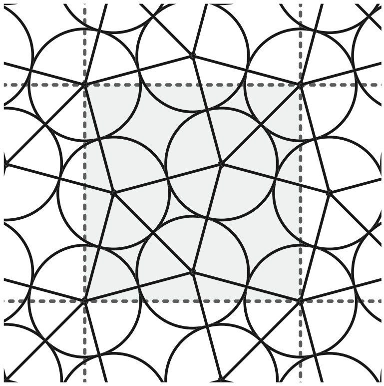
Notice that if there is a pair of circles that are tangent to each other in at least three different ways in a square torus then the above argument implies that the diameter of the circles must be $\frac{\sqrt{2}}{2}$ and the diameter upper bound (Proposition 2.1) implies that $n < 3$. Therefore we observe that if an equal circle packing on the square torus contains a pair of circles that are tangent to each other in three (or more) different ways, then the packing contains one or two circles.

Proposition 4.2 *The only locally maximally dense packing of three circles on the square torus* (*up to reflected and translated variants*) *is the packing pictured in Fig.* 4. *The common diameter is* $\frac{\sqrt{6}-\sqrt{2}}{2}$.

Proof By the Minimum Edges Proposition 3.4 we must place three circles in locations where they are tangent to each other in least five ways. By our previous observation each pair of circles can be tangent in at most two different ways, further as there are only three different circles, there must be one circle that is tangent to the other two circles in two different ways. Translating this circle to the origin we may see that the other two circles must be on the perpendicular bisectors of the lines segments connecting the origin and $(0, 1)$ and the origin and $(1, 0)$. These two circles must be tangent to each other and this leads to the arrangement given in Fig. 4. We can use Inequalities (1) and Proposition 3.1 to show that this packing is locally maximally dense. $\square$

Having established the only locally maximally dense packing of three circles on the square torus, the globally maximally dense packing for four circle easily follows.

Fig. 5 The only locally maximally dense packing of four circles on the square torus and therefore the globally maximally dense packing. The fundamental region is *shaded* and the packing graph is shown



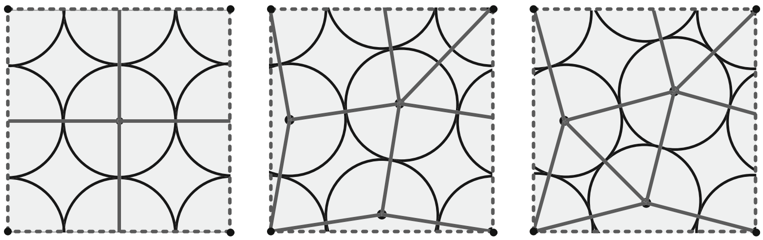


Fig. 6 One parameter family of packings of four circles. The diameters increase until the only locally maximally dense arrangement is achieved

Proposition 4.3 *The only locally maximally dense packing of four circles on the square torus (up to reflected and translated variants) is the packing pictured in Fig. 5. The common diameter is* $\frac{\sqrt{6}-\sqrt{2}}{2}$.

Proof Observe that in the globally maximally dense packing for three circles a fourth circle can be located at $(\frac{\sqrt{3}}{1+\sqrt{3}}, \frac{\sqrt{3}}{1+\sqrt{3}})$. The resulting packing must be globally maximally dense. Demonstrating that there are no other locally maximally dense arrangements requires a little more work. The Minimum Edges Proposition 3.4 implies that there must be at least seven tangencies. As there are only four circles and hence $\binom{4}{2} = 6$ possible single tangencies, there must be at least one pair of circles tangent to each other in two different ways (doubly tangent). One of these circles can be translated to the origin and we can use symmetries so that the other lies on the perpendicular bisector of the line segment between the origin and $(1, 0)$. This forces the location of the remaining two circles and leads to a one-parameter family (see Fig. 6) of packings whose diameters increase until the packing becomes the one pictured in Fig. 5. Therefore there is only one locally maximally dense packing of four circles in the square torus. □

This proof leads to an observation about packings containing doubly tangent circles. If two circles are doubly tangent we know that in the Euclidean plane the second circle must be tangent to the first and a lift of the first. As the minimal distance from a circle to a lift of itself is one in the case of a square torus, the triangle inequality implies that the common diameter of the circles is $\frac{1}{2}$ or larger. However, the diameter upper bound (Proposition 2.1) is greater than $\frac{1}{2}$ only if $1 \leq n \leq 4$, so the equal circle packing contains at most four circles.

Collecting the results from the Rigidity Theory section and the observations from the discussion about the packings of one to four circles and translating them into the language of the packing graphs yields the following proposition. This proposition will help us list all of the possible packing graphs of locally maximally dense arrangements of five circles on the torus using the techniques of Topological Graph Theory.

Proposition 4.4 *Given a locally maximally dense packing, $\mathcal{P}$, of $n > 4$ equal circles (without any free circles) on a torus, the packing graph $G_{\mathcal{P}}$ satisfies the following conditions.*

1. *It contains at least $2n - 1$ and at most $3n$ edges.*
2. *It contains no loops and no multi-edges.*
3. *Every vertex is connected to at least three and at most six others.*

5 Results from Topological Graph Theory

In this section, we discuss and review previously known techniques for determining all the two-cell embeddings of a combinatorial graph onto a topological torus. A graph embedded in a torus is a two-cell embedding if each region determined by the graph on the torus is homeomorphic to an open disk. For full details see the book by White (2001). One powerful tool for enumerating all the two-cell embeddings of a graph on surfaces is Edmonds' permutation technique (Edmonds 1960) which we outline below.

Let G be a connected (simple) graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. The set of neighbors of $i \in V(G)$ is given by $N(i) = \{j \in V(G) \mid ij \in E(G)\}$. A *rotation at i* is a cyclic permutation $\rho_i : N(i) \to N(i)$, and a choice of rotation at each vertex is called a *rotation scheme on G*. Each rotation scheme on G determines a two-cell embedding of G onto some orientable surface (the genus is determined by the rotation scheme) and there is a one-to-one correspondence between oriented and labeled embeddings of G and rotation schemes for G (see White 2001, Sec. 6.6). The labeling of the graph is inherited from the numbering of the vertex set and determined by each rotation scheme. Each rotation scheme also determines the orientation of the embedding of G on the surface. The proof of this correspondence is given by the algorithm below that computes the number of faces determined by the embedding given by a rotation scheme, ρ, on G. This algorithm takes as input a rotation scheme, ρ and a list, L, of the oriented edges of G (each edge of G corresponds to two oriented edges) that are initially unmarked.

Face Tracing Algorithm

faceCount $\leftarrow 0$
for e **in** unmarked edges of L **do**
  mark e
  nextEdge $\leftarrow$ empty
  tempEdge $\leftarrow e$

while $e \neq$ nextEdge **do**
  nextEdge $\leftarrow j\rho_j(i)$ (where tempEdge $= ij$)
  mark nextEdge
  tempEdge $\leftarrow$ nextEdge
 faceCount $\leftarrow$ faceCount $+1$
output faceCount

This uses the fact that for each oriented edge ij the next edge (as determined by the rotation scheme ρ) in that face is $j\rho_j(i)$. Therefore the while loop in the above algorithm cycles through the edges of the face containing the edge e until it returns to e. Oriented edges are used because each edge belongs to two faces with a different orientation, so once all oriented edges are marked the algorithm terminates and all the faces determined by the rotation scheme have been traced. The number of vertices, unoriented edges and the number of faces along with the Euler formula determine the genus of the surface determined by ρ.

The number of rotation schemes for a given graph is $\prod_{i=1}^{n}(|N(i)|-1)!$. Searching all of these rotation schemes for torus (genus 1) embeddings using the algorithm above determines all of the possible oriented labeled two-cell torus embeddings of a graph G. However many of these are essentially the same: some are merely relabelings of others, some are identical except for the choice of orientation or both. As circle packing graphs do not come with natural labels or orientations, we are going to regard two rotation schemes, ρ and σ, on G as equivalent if there exists an α in the automorphism group of G such that $\alpha\rho_i\alpha^{-1} = \sigma_{\alpha(i)}$ for all i (the same graph with two different labelings are the same) or $\alpha\rho_i^{-1}\alpha^{-1} = \sigma_{\alpha(i)}$ for all i (the same graph with opposite orientations are the same). See Mull et al. (1988) for more details. This gives us an easily programable method for determining all the possible unlabeled, unoriented two-cell torus embeddings of a given combinatorial graph onto a topological torus.

Finally, we prove that only two-cell embeddings can be the packing graph associated to a locally maximally dense packing on a flat torus. So by enumerating all the two-cell embeddings of a graph we can begin to ask a reverse question: Which embedded graph corresponds to a locally maximally dense packing?

Proposition 5.1 *If a packing of circles on a flat torus is locally maximally dense without any free circles, then the associated packing graph is a two-cell embedding.*

Proof Suppose there was a locally maximally dense packing whose associated packing graph was not two-cell embedded on a flat torus. Consider the region determined by the graph that is not a two-cell. One component of the boundary of this region must have a vertex that is furthest into this region when lifted into the Euclidean plane. By assigning this vertex a non-zero vector pointing into the region and the zero vector to all other vertices, we obtain a non-trivial infinitesimal flex of the associated strut tensegrity framework, contradicting Theorem 3.2. □

6 Packings of five circles

In this section, we establish the main result. The previous section explained how we enumerate all embeddings of a combinatorial graph onto a topological torus. Now we

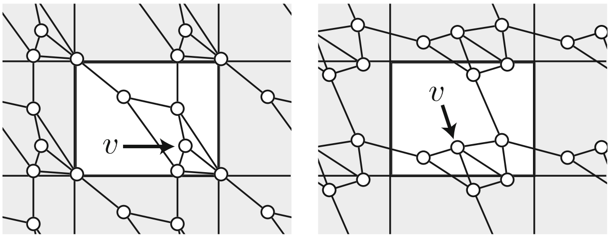


Fig. 7 Two embedding of K_5 minus an edge that cannot be the packing graph of any local maximally dense equal circle packing. The embedding on the *left* cannot be equilateral because then the vertex v would have at most 240° surrounding it. The embedding on the *right* is not the packing graph associated to a locally maximally dense packing because the vertex v would correspond to a circle whose tangencies are restricted to a closed semi-circle

impose the constraints from a particular torus to eliminate potential packing graphs on the torus as coming from optimal arrangements. In the case of five circles, this process eliminates all but one graph which we prove is the packing graph of a locally maximally dense arrangement. Additionally, in this section, we note that the same technique to prove the local optimality of five circles generalizes to a similar lattice arrangement of $n = a^2 + b^2$ ($a > b > 0$ and $\gcd(a, b) = 1$) circles on the square flat torus.

Our approach to the study of packings of n ($n \geq 5$) circles on the square torus requires us to answer the following questions.

1. For a fixed n, how many combinatorially distinct graphs satisfy the properties given in Proposition 4.4?
2. How many ways can those combinatorial graphs from Question 1 above be embedded on a flat torus?

Question 1 is readily answered in the case of five circles. Packing five circles onto a flat torus requires at least nine and at most ten tangencies. This means, combinatorially, that the packing graph of any locally maximally dense packing is isomorphic to the complete graph on five vertices, denoted K_5, or to K_5 with an edge removed.

Question 2 is answered using the ideas outlined in Sect. 5. We have determined that there are 462 labeled, oriented two-cell embeddings, which reduce to 6 unoriented, unlabeled two-cell embeddings of K_5 on a flat torus. Further, there are 206 labeled, oriented two-cell embeddings, which reduce to 14 unoriented, unlabeled two-cell embeddings of K_5 minus an edge on a flat torus. (The numbers for K_5 agree with those derived in Mull et al. (1988) and Gagarin et al. (2007). K_5 minus edge was not studied in either of these references.) This means that there are 20 potential packing graphs and one or more of them must correspond to locally maximally dense packings on the square torus. Ten of these 20 two-cell embeddings are pictured in Figs. 7, 10, and 11. The software by Kocay (2007) was invaluable in drawing these embeddings on the torus and calculating the automorphism groups of combinatorial graphs.

6.1 Eliminating potential packing graphs

For each potential packing graph on a torus we need to ask:

1. Could the embedded graph be the graph associated to a circle packing?
   and
2. If so, could that circle packing be locally maximally dense?

The first question asks if we can force the embedded graph to be equilateral with every angle greater than or equal to 60°. The second question asks if the embedded graph, when viewed as an equilateral strut tensegrity framework, is infinitesimally rigid. For many of the graphs, the answer to one or both of these questions is no and this allows us to eliminate many embedded graphs from corresponding to locally maximally dense packings. The following proposition uses these ideas.

Proposition 6.1 *If a graph embedded on a torus contains a vertex surrounded by any one of the following face patterns, then the embedded graph cannot be the graph associated to a locally maximally dense equal circle packing. The forbidden face patterns are* (1) *two triangles and a polygon,* (2) *three triangles and a polygon,* (3) *five triangles,* (4) *four triangles and a quadrilateral,* (5) *six polygons with at least one non-triangle,* (6) *a triangle, a quadrilateral and a polygon,* (7) *two triangles and two quadrilaterals,* (8) *three quadrilaterals, or* (9) *seven* (*or more*) *polygons.*

Proof Some of these face patterns can be eliminated by observing that an equal circle packing graph is equilateral and all angles are at least 60°. For example, two triangles and a quadrilateral cannot surround a vertex because then there would be at most 240° around that vertex. See left side of Fig. 7.

Sometimes, if the embedded graph is forced to be equilateral and corresponds to an equal circle packing, then at least one new edge would be forced. For example, if a vertex is surrounded by three quadrilaterals (rhombi) then as the angles in a rhombus in a equal circle packing graph are at most 120°, all three angle around the vertex must be 120°. Therefore each rhombus has a 60° angle and there is a pair of vertices (in each rhombus) forced to be the common edge length apart. However, in this case, the circles corresponding to those vertices must be tangent, but are not connected with an edge in the graph.

The other observation is that while certain embeddings could be equilateral, they cannot be associated to a locally maximally dense equal circle packing. For example, a vertex cannot be surrounded by three (equilateral) triangles and a polygon (with five or more sides) because the tangencies at that vertex would be contained in a closed semi-circle, violating Proposition 3.3. See the right side of Fig. 7. The other cases follow similarly. □

In the case of 5 circles, of the 20 potential packing graphs, we can eliminate 12 of them using this proposition. This leaves the ones pictured in Figs. 10 and 11. In order to help eliminate six of remaining eight embeddings we use the following proposition.

Proposition 6.2 *If the packing graph of a locally maximally dense packing of five circles contains a triangle, then lifts of the same vertex connected with a chain of three edges are a unit length apart.*

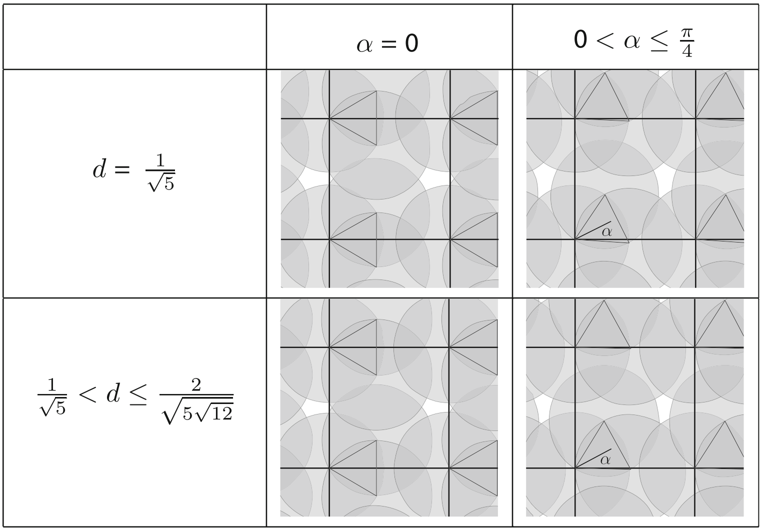


Fig. 8 Representative arrangements of the disks of exclusion on the square torus for various values of d and α. In each of the cases the *white area* outside of the disks of exclusion has diameter strictly less than d, the common diameter

Proof Suppose the packing graph associated to a locally maximally dense packing of five equal circles on the square torus contains a triangle. Using the symmetries of square torus, we may assume that one vertex of the triangle is at the origin in the standard fundamental domain and that line connecting the vertex at the origin to the midpoint of the opposite side of the triangle makes an angle α with the vector $\langle 1, 0\rangle$ and that $0 \leq \alpha \leq \frac{\pi}{4}$. Let the common edge length in the packing graph be d (i.e. the diameter of the circles in the packing). A *disk of exclusion for circle* **C** is a disk of *radius* d centered at C (see Fig. 8). By the definition of a circle packing, another circle cannot have its center located interior to another circle's disk of exclusion. Since we have three tangencies, by the Minimum Edges Proposition 3.4 we must place two more circles in the fundamental domain so their centers are outside or on the boundary of the disks of exclusion to create at least six additional tangencies.

Examining the case when d is between the maximum of $\frac{2}{\sqrt{5\sqrt{12}}}$ and $\frac{1}{\sqrt{5}}$ and $0 \leq \alpha \leq \frac{\pi}{4}$, we can show that the diameter of the region outside of the union of the three disks of exclusion is strictly less than d. (These are the unshaded regions in Fig. 8.) To see this, observe that for a fixed α, as the radius of the disks of exclusion increases above $\frac{1}{\sqrt{5}}$, the diameter of the region outside of the union of them decreases because the centers of the disks are fixed. Therefore we only need to examine the diameter of the region when $d = \frac{1}{\sqrt{5}}$ and α varies between 0 and $\frac{\pi}{4}$. From the graph of this upper bound on the diameter (see Fig. 9) we can see it is always less than $\frac{1}{\sqrt{5}}$. Therefore the diameter of this region is always less than d and is too small for two new circle centers. Hence, when the packing graph contains a triangle, the diameter of the circles in the packing must be less than $\frac{1}{\sqrt{5}}$.

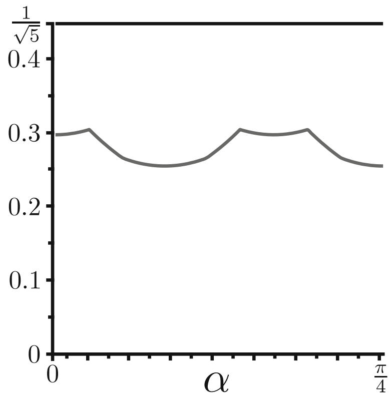


Fig. 9 A graph of the upper bound (for d between $\frac{2}{\sqrt{5\sqrt{12}}}$ and $\frac{1}{\sqrt{5}}$) on the diameter of the region outside of disks of exclusion pictured in Fig. 8 as a function of α

Two lifts of the same vertex in a packing graph (in the Euclidean plane) must differ by an element of the square lattice. Suppose two lifts are separated by a chain of three edges, then the smallest possible value for the diameter is the length of the lattice vector divided by 3. If the lattice vector has length $\sqrt{2}$ (or larger) then as $\frac{\sqrt{2}}{3} > \frac{1}{\sqrt{5}}$, this is impossible, so the lifts must be separated by a shorter lattice vector, that is, a lattice vector of length one. □

Notice that all the embeddings in Fig. 10 contain a triangle. Further observe that in each embedding there is a vertex, v, connected to six different lifts of v by chains of length 3. By Proposition 6.2 this vertex is a unit length from six of its lifts. This is impossible in a square lattice, where a vertex can have only four unit distance lifts. This eliminates these six embeddings and implies that the only possible packing graphs that lead to locally maximally dense packings on the square torus are the embeddings in Fig. 11.

However, the embedding in Fig. 11b is not the packing graph of a locally maximally dense arrangement on the square torus. An equilateral embedding of this type forces an edge between v and u. Therefore, the only remaining embedding (Fig. 11a) must be the packing graph associated to a locally maximally dense packing.

6.2 The locally maximally dense packing of five circles

Theorem 6.3 *The only locally maximally dense packing of five circles on the square torus (up to reflected and translated variants) is the packing pictured in Fig. 12. It has packing graph homotopic to the embedding in Fig.* 11a *and the common diameter of the circles is* $\frac{1}{\sqrt{5}}$.

Proof Using Inequalities (1) on all of the edges in the associated packing graph, we can prove that the packing pictured in Fig. 12 is a locally maximally dense equal circle packing.

It is possible that there could be another basis for the square torus in which embedding in Fig. 11a is realized as a locally maximally dense equal circle packing

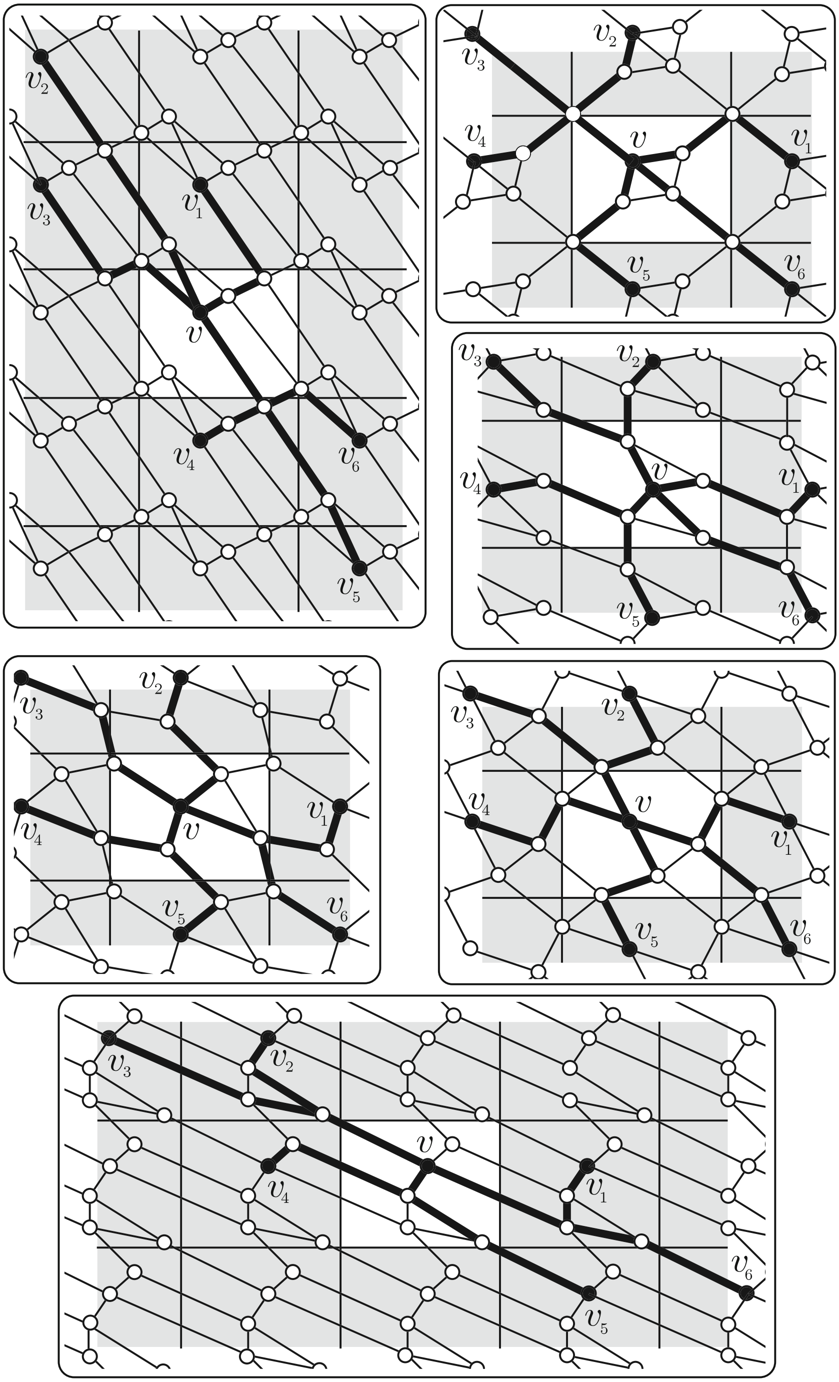
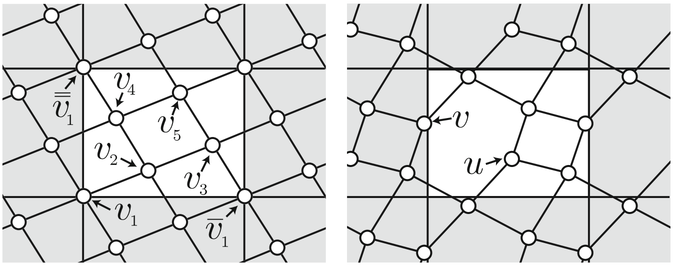


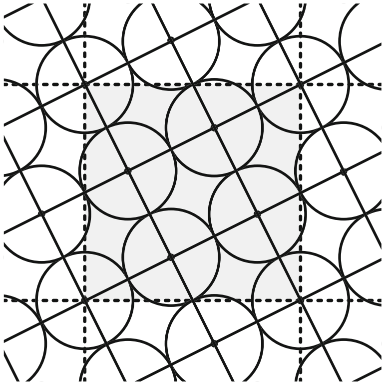
Fig. 10 Six of the 20 inequivalent embeddings of K_5 or K_5 minus edge on a flat torus. For each embedding a vertex, v, and chains of three edges connecting six of the lifts of v to v are highlighted in *boldface*. (In this figure the *unshaded region* is the fundamental domain)



(a) Packing graph of a locally maximally dense arrangement. **(b)** Not the packing graph of any packing.

Fig. 11 The remaining potential packing graphs after Propositions 6.1 and 6.2 are used to eliminate the 18 of the original 20 possibilities. The fundamental region is the *unshaded region* in each of these

Fig. 12 The only locally maximally dense packing of five circles on the square torus and therefore the globally maximally dense packing. The fundamental region is *shaded* and the packing graph is shown



graph. (That is, the edge chains in the embedding of Fig. 11a could be in different homotopy classes on the torus if a different basis for the square lattice is used, but the face arrangement and degree are the same.) However, by observing that if the embedding pictured in Fig. 11a is forced to be equilateral (with common edge length d), then the distance from v_1 to its lift $\overline{v}_1$ is at most $\sqrt{7}d$ because the largest the angle at v_3 in the chain of three edges from v_1 to $\overline{v}_1$ is 120°. Using the diameter upper bound (Proposition 2.1), we notice that $\sqrt{7}d < \sqrt{7}(\frac{2}{\sqrt{5\sqrt{12}}}) < \sqrt{2}$. As v_1 and $\overline{v}_1$ are connected with a lattice vector and the lattice vectors of the square lattice have lengths $\{1, \sqrt{2}, 2, \sqrt{5}, \ldots\}$, they must be connected with a length one lattice vector. The similar argument applies to v_1 and $\overline{\overline{v}}_1$. Hence any equilateral embedding with all angles larger than or equal to 60°, with the same face type as in Fig. 11a, on the square torus must result in the embedding pictured in Fig. 11a in the standard basis. An argument similar to the proof of Theorem 4 in Connelly (1982) shows that this locally maximally dense packing in this homotopy class is unique (up to a rigid motion). The infinitesimal rigidity of the strut tensegrity framework guarantees a non-zero stress on every strut (see Roth and Whiteley 1981, Theorem 5.2) which leads to an energy function with a unique maximum, so any other equilateral infinitesimally rigid strut tensegrity framework must be congruent to the packing graph associated to the packing in Fig. 12. □

The proof of the locally maximality in the first part of this proposition generalizes to the following.

Proposition 6.4 *If* $n = a^2 + b^2$ *with integers* $a > b > 0$ *and* $\gcd(a, b) = 1$ *then there exists a locally maximally dense equal circle packing of n circles on the square torus whose packing graph is the union of squares. The common diameter is* $\frac{\sqrt{n}}{n}$.

Proof Let the centers of the circles lie on a superlattice (of the square lattice of the torus) generated by the equi-length perpendicular vectors $\mathbf{u}_1 = \langle \frac{a}{a^2+b^2}, \frac{b}{a^2+b^2} \rangle$ and $\mathbf{u}_2 = \langle -\frac{b}{a^2+b^2}, \frac{a}{a^2+b^2} \rangle$. The condition on the greatest common divisor of a and b guarantees we can find appropriate chains of edges (containing all the centers of the circles) from the circle at $(0, 0)$ to a lift this circle in two different directions (one by following the centers at the integer multiples of $\mathbf{u}_1$ and the other by following integer multiples of $\mathbf{u}_2$). Using the $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ basis for the vectors in Inequality (1) greatly simplify the computation. We can show that these inequalities only have the zero solution and hence the arrangement is locally maximally dense. □

This supports the conjectured globally maximally dense packing of $10 = 3^2 + 1^2$ circles on a square torus in (Melissen, 1997, Fig. 2.12 l). A slight modification of this argument shows that the conjectured globally maximally dense packings of 12 and 15 circles on a square torus in (Melissen, 1997, Fig. 2.12 n and q) are locally maximally dense.

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