Ungapped bricklayer lattice

a and b are unit vectors of an arbitrary bricklayers lattice of squares

we want to see if they can be combined to make a larger bravais squares whose volume is k

```
a = UnitVector[1]
```

```
{1, 0}
```

```
b = {0, 1}
```

```
{0, 1}
```

```
FindInstance[{eqnx, eqny, eqnz} /. {k → 34}, {n1, n2, n3, n4}, Integers]
```

```
{{n1 → -5, n2 → 3, n3 → -3, n4 → -5}}
```

```
b = {1 / 2, 1}
```

$\left\{\frac{1}{2}, 1\right\}$

```
FindInstance[{eqnx, eqny, eqnz} /. {k → 18}, {n1, n2, n3, n4}, Integers]
```

```
{{n1 → -3, n2 → 3, n3 → -3, n4 → -3}}
```

```
b = {1 / 3, 1}
```

$\left\{\frac{1}{3}, 1\right\}$

```
FindInstance[{eqnx, eqny, eqnz} /. {k → 18}, {n1, n2, n3, n4}, Integers]
```

```
{{n1 → -3, n2 → 3, n3 → -3, n4 → -3}}
```

```
b = {1 / d, 1}
```

$\left\{\frac{1}{d}, 1\right\}$

```
FindInstance[{eqnx, eqny, eqnz} /. {k → 60}, {d, n1, n2, n3, n4}, Integers]
```

```
{}
```

```
b = {c / d, 1}
```

$\left\{\frac{c}{d}, 1\right\}$

```
FindInstance[{eqnx, eqny, eqnz, d > 0, c < d, c > 0} /. {k → 64, b → c / d},
  {c, d, n1, n2, n3, n4}, Integers]
```

```
$Aborted
```

```
{0, 1}
```

```
Clear[a]
```

```
FindInstance[{eqnx, eqny, eqnz} /. {k → 18}, {n1, n2, n3, n4}, Integers]
{{n1 → -3, n2 → 3, n3 → -3, n4 → -3}}
eqnx := (Norm[n1 a + n2 b]^2 == k)
eqny := (Norm[n3 a + n4 b]^2 == k)
eqnz := ((n1 n4 - n2 n3) == k)
```

Here are the sum of square solutions

```
In[79]:= Sort[DeleteDuplicates[Flatten[Table[i^2 + j^2, {i, 1, 8}, {j, 1, 8}]]]]
Out[79]= {2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, 32, 34, 37, 40, 41, 45,
  50, 52, 53, 58, 61, 65, 68, 72, 73, 74, 80, 85, 89, 98, 100, 113, 128}
```

Gapped bricklayer lattice

- **The constraints needed to satisfy the small lattice and large lattice commensurate**

n1, n2, n3, n4 must be integers
a and b are the lattice vectors of the lattice of squares

The two lattice vectors of the torus are n1 a + n2 b and n3 a + n4 b. These must be orthogonal:

```
eqnzg := (Dot[n1 a + n2 b, n3 a + n4 b] == 0)
```

and equal length

```
eqnxyg := (Norm[n1 a + n2 b] == Norm[n3 a + n4 b])
eqnxys := (Norm[n1 a + n2 b]^2 == Norm[n3 a + n4 b]^2)
```

The unit vectors are a, along the row of bricks

```
a = UnitVector[1]
{1, 0}
```

and b from one row to the next

The shift is c and the gap is d - 1

```
b = {c, d}
{c, d}
b = {m / (n - 1), 1 + 1 / (n - 1)}
```

$$\left\{\frac{m}{-1+n},\ 1+\frac{1}{-1+n}\right\}$$

```
In[105]:= eqnn := (n == Abs[n1 n4 - n2 n3])
```

- **N = 6**

- **N = 11**
- **N = 14**
- **N = 21 no luck**
- **N = 22 a bricklayer with 1x2 bricks**

A bricklayer where the brick is a 1x2 block

```
nss = {n1 → 4, n2 → 1, n3 → -3, n4 → 2}
{n1 → 4, n2 → 1, n3 → -3, n4 → 2}
n1 n4 - n2 n3 /. nss
11
cdss = Solve[{eqnxyg, eqnzg} /. nss, {c, d}, Reals]
```

$$\left\{\left\{c \to \frac{2}{5}, d \to -\frac{11}{5}\right\}, \left\{c \to \frac{2}{5}, d \to \frac{11}{5}\right\}\right\}$$

Show that the torus does have N = 11 of the 1x2 bricks

```
Abs[Cross[{n1, n2, 0} , {n3, n4, 0}][[3]]] /. nss
11
```

Compute the density

```
22 / Norm[n1 a + n2 b]^2 /. cdss /. nss
```

$$\left\{\frac{10}{11}, \frac{10}{11}\right\}$$

```
22 / Norm[n1 a + n2 b]^2 /. cdss /. nss // N
{0.909091, 0.909091}
```

- **N = 27 a standard gapped bricklayer**

General analysis

- **The known optimal bricklayer solutions for N<28**

```
In[92]:= FindInstance[{eqnzg, eqnxys, eqnn} /. n → 6, {n1, n2, n3, n4, m}, Integers]
Out[92]= {{n1 → -2, n2 → 1, n3 → -2, n4 → -2, m → -2}}
In[94]:= Reduce[{eqnzg, eqnxys, eqnn} /. {n → 6, n1 → 2, n2 → -1, n3 → 2, n4 → 2, m → -2}]
Out[94]= True
Abs[Cross[{n1, n2, 0} , {n3, n4, 0}][[3]]] /.
  {c → 3 / 5, d → 6 / 5, n1 → 0, n2 → 2, n3 → 3, n4 → -1}
```

```
FindInstance[{eqnzg, eqnxys, eqnn} /. n → 11, {n1, n2, n3, n4, m}, Integers]
```

```
{{n1 → 3, n2 → 1, n3 → -2, n4 → 3, m → 3}}
```

```
FindInstance[{eqnzg, eqnxys, eqnn} /. n → 14, {n1, n2, n3, n4, m}, Integers]
```

```
{{n1 → -4, n2 → -3, n3 → 2, n4 → -2, m → -8}}
```

```
FindInstance[{eqnzg, eqnxys, eqnn, n2 == n3} /. {n → 14, m → -8},
 {n1, n2, n3, n4}, Integers, RandomSeed → 36]
```

```
{}
```

```
FindInstance[{eqnzg, eqnxys, eqnn} /. n → 27, {n1, n2, n3, n4, m}, Integers]
```

```
{{n1 → -2, n2 → -5, n3 → 5, n4 → -1, m → -5}}
```

```
In[110]:= Reduce[{eqnzg, eqnxys, eqnn} /. {n → 27, n1 → 2, n2 → 5, n3 → -5, n4 → 1, m → -5}]
```

```
Out[110]= True
```

- **Other examples that are not optimal or don' t exists**

▪ The full list of sums of two squares and gapped bricklayers less than 101

```
In[90]:= sts = Sort[DeleteDuplicates[Flatten[Table[i^2 + j^2, {i, 1, 10}, {j, 1, 10}]]]]
```

```
Out[90]= {2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, 32, 34, 37, 40, 41, 45, 50, 52,
 53, 58, 61, 65, 68, 72, 73, 74, 80, 82, 85, 89, 90, 97, 98, 100, 101, 104,
 106, 109, 113, 116, 117, 125, 128, 130, 136, 145, 149, 162, 164, 181, 200}
```

```
In[88]:= gb = Table[{i,
    FindInstance[{eqnzg, eqnxys, eqnn} /. n → i, {n1, n2, n3, n4, m}, Integers]
   },
   {i, 2, 100}];
```

```
In[89]:= TableForm[gb]
```

Out[89]//TableForm=

2	n1 → -2 n2 → 0 n3 → -1 n4 → -1 m → -1
3	n1 → -1 n2 → 1 n3 → -2 n4 → -1 m → -1
4	
5	
6	n1 → -2 n2 → 1 n3 → -2 n4 → -2 m → -2
7	
8	
9	
10	
11	n1 → 3 n2 → 1 n3 → -2 n4 → 3 m → 3
12	
13	
14	n1 → -4 n2 → -3 n3 → 2 n4 → -2 m → -8
15	
16	
17	
18	n1 → -2 n2 → -4 n3 → 4 n4 → -1 m → -4

19	
20	
21	
22	
23	
24	
25	
26	$n1\to -2$ $n2\to -4$ $n3\to 5$ $n4\to -3$ $m\to 7$
27	$n1\to -2$ $n2\to -5$ $n3\to 5$ $n4\to -1$ $m\to -5$
28	
29	
30	$n1\to 0$ $n2\to -5$ $n3\to 6$ $n4\to -2$ $m\to 12$
31	
32	
33	
34	
35	$n1\to -5$ $n2\to -5$ $n3\to 4$ $n4\to -3$ $m\to -13$
36	
37	
38	$n1\to -2$ $n2\to -6$ $n3\to 6$ $n4\to -1$ $m\to -6$
39	
40	
41	
42	$n1\to -3$ $n2\to -5$ $n3\to 6$ $n4\to -4$ $m\to 9$
43	
44	
45	
46	
47	
48	
49	
50	
51	$n1\to -2$ $n2\to -7$ $n3\to 7$ $n4\to -1$ $m\to -7$
52	
53	
54	$n1\to -13$ $n2\to -7$ $n3\to 4$ $n4\to -2$ $m\to -83$
55	
56	
57	
58	
59	$n1\to -15$ $n2\to -7$ $n3\to 2$ $n4\to -3$ $m\to -99$
60	
61	
62	$n1\to -4$ $n2\to -6$ $n3\to 7$ $n4\to -5$ $m\to 11$
63	
64	
65	
66	$n1\to -2$ $n2\to -8$ $n3\to 8$ $n4\to -1$ $m\to -8$
67	

```
68
69
70
71
72
73
74    n1 → -14  n2 → -8  n3 → 4  n4 → -3  m → -100
75    n1 → -8  n2 → -7  n3 → 5  n4 → -5  m → -31
76
77
78
79
80
81
82
83    n1 → -2  n2 → -9  n3 → 9  n4 → -1  m → -9
84
85
86    n1 → -7  n2 → -9  n3 → 8  n4 → -2  m → -47
87
88
89
90    n1 → -10  n2 → -8  n3 → 5  n4 → -5  m → -55
91
92
93
94
95
96
97
98    n1 → -11  n2 → -9  n3 → 6  n4 → -4  m → -75
99
100
```

■ Check Chris 6 - 28 - 2011 notes

In[123]:=
```
rr1 := -(n1 n3) / (n2 n4) + (n2 n3 + n1 n4)^2 / (2 n2 n4)^2
```

In[124]:=
```
Simplify[rr1 /. (n1 n4 - n2 n3) → n]
```

Out[124]= $\frac{(n2\ n3 - n1\ n4)^2}{4\ n2^2\ n4^2}$

In[125]:=
```
Reduce[rr1 == n^2 / (2 n2 n4)^2 /. n → n1 n4 - n2 n3]
```

Out[125]= True

In[128]:=
```
rr2 := -(n1^2 - n3^2) / (n2^2 - n4^2) + (n1 n2 - n3 n4)^2 / (n2^2 - n4^2)^2
```

In[129]:= `Simplify[rr2]`

Out[129]= $$\frac{(n2\ n3 - n1\ n4)^2}{\left(n2^2 - n4^2\right)^2}$$

In[130]:= `cc1 := (n1 n2 - n3 n4) / (n2^2 - n4^2)^2`

In[131]:= `cc2 := (n2 n3 + n1 n4) / (2 n2 n4)`

`Simplify[rr1 - (rr1 - rr2 + Abs[cc1 - cc2])^2 / (4 (cc1 - cc2)^2)] /.`

Out[132]= $$-\frac{1}{16\ n2^2\ n4^2}\left(16\ n1\ n2\ n3\ n4 - 4\ (n2\ n3 + n1\ n4)^2 + \left((n2\ n3 - n1\ n4)^2\ \left(n2^4 - 6\ n2^2\ n4^2 + n4^4\right) + 4\ n2^2\ n4^2\ \left(n2^2 - n4^2\right)^2\ \mathrm{Abs}\left[-\frac{n2\ n3 + n1\ n4}{2\ n2\ n4} + \frac{n1\ n2 - n3\ n4}{\left(n2^2 - n4^2\right)^2}\right]\right)^2 \Big/ \left(n2^5\ n3 + n1\ n2^4\ n4 - 2\ n2^3\ n3\ n4^2 + n1\ n4^5 - 2\ n1\ n2^2\ n4\ \left(1 + n4^2\right) + n2\ n3\ n4^2\ \left(2 + n4^2\right)\right)^2\right)$$