A Theorem on Bilinear Forms

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Theorem 1. *Assume F is a bilinear form. Let A be the diagonal matrix representing F, and assume it has no 0's on it's diagonal. Then F is positive definite if and only if the diagonal contains only strictly positive elements.*

Proof. Let the diagonal elements of the matrix be $a_1, \ldots, a_n$ such that the matrix has the form

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_n \end{pmatrix} \tag{1}$$

First assume that $a_1, \ldots a_n > 0$, and let x be an arbitrary non-zero vector such that

$$x = \sum_{i=1}^{n} \lambda_i e_i \tag{2}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and are not all 0. Then we have that

$$F(x, x) = \sum_{i=1}^{n} \lambda_i^2 a_i \geq 0. \tag{3}$$

However, $F(x, x) = 0$ if and only if all $\lambda_i = 0$, so we can conclude that

$$F(x, x) > 0 \tag{4}$$

and therefore that F is positive definite.

Now assume that some $a_k < 0$, then choose $x = e_k$. Then,

$$F(x, x) = F(e_k, e_k) = a_k < 0 \tag{5}$$

so F is not positive definite.

$\square$