Numerical Solutions of Differential Equations : HW1

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Introduction

This is the introduction bitches!

1 Conservation laws

We are given the following Euler equations in $u = (\rho, \rho v, E)$,

$$\rho_t + (\rho v)_x = 0 \quad \text{(Conservation of mass)} \tag{1}$$
$$(\rho v)_t + (\rho v^2 + p)_x = 0 \quad \text{(Conservation of momentum)} \tag{2}$$
$$E_t + (v(E + p))_x = 0 \quad \text{(Conservation of energy)} \tag{3}$$

and the variable change $u = (\rho, \rho v, E) = (u_1, u_2, u_3)$. We then have

$$f(u) = (f_1, f_2, f_3)^T = (u_2, u_2^2/u_1 + p(u_1), u_2(u_3 + p(u_1))/u_1)$$

Using the u coordinates, we can write equations (1), (2), (3) as

$$u_t + f(u)_x = 0$$

From the chain rule, we can rewrite our equation as

$$u_t + \frac{\partial f}{\partial u} u_x = 0 \tag{4}$$

We now wish to find the eigenvalues and eigenvectors of the matrix

$$\frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{pmatrix}$$

Using simple partial differentiation, we calculate

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 0 & 1 & 0 \\ a & b & 0 \\ c & d & e \end{pmatrix}$$

with $a = -(\frac{u_2}{u_1})^2 + p'(u_1)$,
$b = 2(\frac{u_2}{u_1})$,
$c = \frac{u_2 p'(u_1)}{u_1} - \frac{u_2(u_3 + p(u_1))}{u_1^2}$,
$d = \frac{u_3 + p(u_1)}{u_1}$,
and $e = \frac{u_2}{u_1}$

To get the eigenvalues, we look at the roots of the characteristic polynomial of this matrix

$$\lambda^3 - e\lambda^2 - b\lambda^2 + be\lambda - a\lambda + ae = 0$$

which gives us the three eigenvalues

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(b - \sqrt{4a + b^2}) \\
\lambda_2 &= \frac{1}{2}(b + \sqrt{4a + b^2}) \\
\lambda_3 &= e
\end{aligned}$$

Plugging in the actual values of a, b, and e, we get

$$\begin{aligned}
\lambda_1 &= \frac{u_2}{u_1} - \sqrt{p'(u_1)} \\
\lambda_2 &= \frac{u_2}{u_1} + \sqrt{p'(u_1)} \\
\lambda_3 &= \frac{u_2}{u_1}
\end{aligned}$$

After computing, we get the corresponding eigenvectors

$$\begin{pmatrix} 1 \\ \frac{u_2}{u_1} - \sqrt{p'(u_1)} \\ \frac{u2\sqrt{p'(u_1)} - (u_3 + p(u_2)))}{u_1} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \frac{u_2}{u_1} + \sqrt{p'(u_1)} \\ \frac{u2\sqrt{p'(u_1)} + (u_3 + p(u_2)))}{u_1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

1.1 Linearisation

We use formal linearisation, so we first define

$$u' = \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} U_1 + \delta v_1 \\ U_2 + \delta v_2 \\ U_3 + \delta v_3 \end{pmatrix}$$

where $U_{1,2,3}$ is a solution of the (1) and δ is assumed to be small. Inserting u' into (1), we get

$$\begin{aligned}
u'_t + \frac{\partial f}{\partial u'} u'_x = 0 \rightarrow \\
(u'_1)_t + (u'_2)_x = 0 \\
(-(\frac{u'_2}{u'_1})^2 + p'(u'_1))(u_1)_x + 2(\frac{u'_2}{u'_1})(u'_2)_x = 0 \\
(\frac{u'_2 p'(u_1)}{u_1} - \frac{u'_2(u'_3 + p(u'_1))}{u'^2_1})(u'_1)_x + (\frac{u'_3 + p(u'_1)}{u'_1})(u'_2)_x + \frac{u'_2}{u'_1}(u'_3) = 0
\end{aligned}$$

We linearise each equation separately

$$\begin{aligned}
(-(\frac{u'_2}{u'_1})^2 + p'(u'_1))(u_1)_x + 2(\frac{u'_2}{u'_1})(u'_2)_x = 0 \rightarrow \\
-u'^2_2(u_1)_x + u'^2_1 p'(u'_1)(u_1)_x + 2(u'_2 u'_1)(u'_2)_x = 0 \rightarrow \\
U_2^2(U_1)_x + U_2U_1(U_2)_x + p'(u'_1)U_1^2 + 2p'(u'_1)\delta v_1 U_1 + U_2^2(\delta v_1)_x + \\
+\delta v_2(U_1(U_2)_x + U_1U_2(U_2)_x + 2U_2(U_1)_x) + U_2U_1(\delta v_2)_x = 0
\end{aligned}$$

1.2 Hyperbolicity

We know that an equation of this form is hyperbolic if $\frac{\partial f}{\partial u}$ has real eigenvalues and linearly independent eigenvectors. It can be seen easily that the eigenvalues are real as long as $p'(u_1) \geq 0$. WHAT ABOUT EIGEN VECTORS? can p' = 0??

2 Heat Equation

2.1 Flux vector

2.2 Q(t)

We are now interested in computing :

$$Q(t) = \int_0^1 \int_0^1 q(x, y, t) dx dy$$

Using the fundamental theorem of calculus, we have :

$$Q(t) = Q(0) + \int_0^t Q'(\tau) d\tau$$

The initial condition gives :

$$Q(0) = \int_0^1 \int_0^1 q(x, y, 0) dx dy = \int_0^1 \int_0^1 0 \, dx dy = 0$$

We also know that (using the definition of Q) :

$$Q'(t) = \frac{d}{dt}(\int_0^1 \int_0^1 q(x, y, t) dx dy) = \int_0^1 \int_0^1 q_t(x, y, t) dx dy$$

We are now going to use the PDE to substitute q_t. This yields :

$$Q'(t) = \int_0^1 \int_0^1 q_t(x, y, t) dx dy = \int_0^1 \int_0^1 \nabla \cdot (\nabla q(x, y, t)) dx dy + \int_0^1 \int_0^1 S(x, y, t) dx dy$$

With divergence theorem, and if S means the boundary of the domain and $\mathbf{n}$ the outward unit normal to the boundary, we have :

$$Q'(t) = \oint_S \nabla q(x, y, t) \cdot \mathbf{n} \, ds + \int_0^1 \int_0^1 S(x, y, t) dx dy$$

Because the given boundary condition given is :

$$\nabla q(x, y, t) \cdot \mathbf{n} = 0$$

We finally have :

$$Q'(t) = \int_0^1 \int_0^1 S(x, y, t) dx dy$$

This yields an expression for Q as a function of t.

$$Q(t) = \int_0^t \int_0^1 \int_0^1 S(x, y, \tau) dx dy d\tau$$

Because S is a given function, this expression can always be computed as a function of t.

3 Discretization and implementation

blabla yo!

4 Numerical results

blabla bitches

5 Refinements

blabla