Numerical Solutions of Differential Equations : HW1

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Introduction

This is the introduction bitches!

1 Conservation laws

We are given the following Euler equations in $u = (\rho, \rho v, E)$,

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \quad \text{(Conservation of mass)} \\ (\rho v)_t + (\rho v^2 + p)_x &= 0 \quad \text{(Conservation of momentum)} \\ E_t + (v(E+p))_x &= 0 \quad \text{(Conservation of energy)} \end{aligned}$$

and the variable change $u = (\rho, \rho v, E) = (u_1, u_2, u_3)$. We then have

$$f(u) = (f_1, f_2, f_3)^T = (u_2, u_2^2/u_1 + p(u_1), u_2(u_3 + p(u_1))/u_1)$$

We now wish to find the eigenvalues and eigenvectors of the matrix

$$\frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{pmatrix}$$

Using simple partial differentiation, we calculate

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 0 & 1 & 0 \\ a & b & 0 \\ c & d & e \end{pmatrix}$$

with $a = -(\frac{u_2}{u_1})^2 + p'(u_1)$,
$b = 2(\frac{u_2}{u_1})$,
$c = \frac{u_2 p'(u_1)}{u_1} - \frac{u_2(u_3 + p(u_1))}{u_1^2}$,
$d = \frac{u_3 + p(u_1)}{u_1}$,
and $e = \frac{u_2 u_3}{u_1}$

To get the eigenvalues, we look at the roots of the characteristic polynomial of this matrix

$$\lambda^3 - e\lambda^2 - b\lambda^2 + be\lambda - a\lambda + ae = 0$$

which gives us the three eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(b - \sqrt{4a + b^2}) \\ \lambda_2 &= \frac{1}{2}(b + \sqrt{4a + b^2}) \\ \lambda_3 &= e \end{aligned}$$

Plugging in the actual values of a, b, and e, we get

$$\lambda_1 = \frac{u_2}{u_1}\sqrt{p'(u_1)}$$
$$\lambda_2 = \frac{u_2}{u_1}\sqrt{p'(u_1)}$$
$$\lambda_3 = \frac{u_2 u_3}{u_1}$$

2 Heat Equation

2.1 Flux vector

2.2 Q(t)

We are now interested in computing :

$$Q(t) = \int_0^1 \int_0^1 q(x, y, t) dxdy$$

Using the fundamental theorem of calculus, we have :

$$Q(t) = Q(0) + \int_0^t Q'(\tau) d\tau$$

The initial condition gives :

$$Q(0) = \int_0^1 \int_0^1 q(x, y, 0) dxdy = \int_0^1 \int_0^1 0 \, dxdy = 0$$

We also know that (using the definition of Q) :

$$Q'(t) = \frac{d}{dt}(\int_0^1 \int_0^1 q(x, y, t) dxdy) = \int_0^1 \int_0^1 q_t(x, y, t) dxdy$$

We are now going to use the PDE to substitute q_t. This yields :

$$Q'(t) = \int_0^1 \int_0^1 q_t(x, y, t) dxdy = \int_0^1 \int_0^1 \nabla \cdot (\nabla q(x, y, t)) dxdy + \int_0^1 \int_0^1 S(x, y, t) dxdy$$

With divergence theorem, and if S means the boundary of the domain and $\mathbf{n}$ the outward unit normal to the boundary, we have :

$$Q'(t) = \oint_S \nabla q(x, y, t) \cdot \mathbf{n} \, ds + \int_0^1 \int_0^1 S(x, y, t) dxdy$$

Because the given boundary condition given is :

$$\nabla q(x, y, t) \cdot \mathbf{n} = 0$$

We finally have :

$$Q'(t) = \int_0^1 \int_0^1 S(x, y, t) dxdy$$

This yields an expression for Q as a function of t.

$$Q(t) = \int_0^t \int_0^1 \int_0^1 S(x, y, \tau) dxdyd\tau$$

Because S is a given function, this expression can always be computed as a function of t.

3 Discretization and implementation

blabla yo!

4 Numerical results

blabla bitches

5 Refinements

blabla