
It's the title bitchess! !!

THOMAS Garrett & WEICKER David

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1 Stability of Numerical Schemes

We have the general scheme

$$\begin{aligned}U^{n+1} &= Q(t_n)U^n + \Delta t F^n \\U^0 &= g\end{aligned}$$

where $U^n \in \mathbb{R}^d$.

1.1 Duhamel's Principle

We are given the following discrete Duhamel's Principle:

$$U^n = S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu, \quad (1)$$

where $t_n = n\Delta t$, and

$$\begin{aligned}S_h(t, t) &= I, \quad t \in \mathbb{R} \\S_h(t_{n+1}, t_\mu) &= Q(t_n)S_h(t_n, t_\mu).\end{aligned}$$

We begin by showing that (1) holds by induction.

Base Case: $n = 0$

$$\begin{aligned}U^0 &= S_h(0, 0)g + \Delta t \sum_{\nu=0}^{-1} S_h(0, t_{\nu+1})F^\nu \\&= 0\end{aligned}$$

which fits the general scheme. Now we assume that the (1) fits the general scheme at step n , and we want to show that this implies that it fits for step $n + 1$.

$$\begin{aligned}U^{n+1} &= S_h(t_{n+1}, 0)g + \Delta t \sum_{\nu=0}^n S_h(t_{n+1}, t_{\nu+1})F^\nu \\&= Q(t_n)S_h(t_n, 0)g + \Delta t Q(t_n) \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu + S_h(t_{n+1}, t_{n+1})F^n \\&= Q(t_n)(S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu) + F^n \\&= Q(t_n)U_n + F^n\end{aligned}$$

which fits our general scheme.

1.2 Bound in the h -norm

We now wish to show that

$$\|S_h(t_{\nu+1}, t_\nu)\|_h \leq K e^{ah} \implies \|U^n\|_h \leq K(e^{at_n} \|g\|_h + \int_0^{t_n} e^{a(t_n-s)} ds \max_{0 \leq \nu \leq n-1} \|F^\nu\|_h)$$

Taking $\|\cdot\|_h$ of both sides of (1), then by Cuchy-Schwartz inequality, we have

$$\begin{aligned} \|U^n\|_h &= \|S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu\|_h \\ &\leq \|S_h(t_n, 0)\|_h \|g\|_h + \|\Delta t\|_h \sum_{\nu=0}^{n-1} \|S_h(t_n, t_{\nu+1})\|_h \|F^\nu\|_h \\ &\leq K e^{at_n} \|g\|_h + \|\Delta t\|_h \sum_{\nu=0}^{n-1} \|S_h(t_n, t_{\nu+1})\|_h \|F^\nu\|_h \\ &\leq K e^{at_n} \|g\|_h + \Delta t \sum_{\nu=0}^{n-1} K e^{a(t_n-t_{\nu+1})} \|F^\nu\|_h \end{aligned}$$

we notice that $\Delta t \sum_{\nu=0}^{n-1} K e^{a(t_n-t_{\nu+1})}$ is a right Remman sum of a strictly decreasing function, thus

$$\begin{aligned} &\leq K e^{at_n} \|g\|_h + K \int_0^{t_n} e^{a(t_n-s)} ds \|F^\nu\|_h \\ &\leq K(e^{at_n} \|g\|_h + \int_0^{t_n} e^{a(t_n-s)} ds \max_{0 \leq \nu \leq n-1} \|F^\nu\|_h) \end{aligned}$$

IS A POSITIVE??