It's the title bitchess! !!

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1 Stability of Numerical Schemes

We have the general scheme

$$\begin{aligned} U^{n+1} &= Q(t_n)U^n + \Delta t F^n \\ U^0 &= g \end{aligned}$$

where $U^n \in \mathbb{R}^d$.

1.1 Duhamel's Principle

We are given the following discrete Duhamel's Principle:

$$U^n = S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu, \tag{1}$$

where $t_n = n\Delta t$, and

$$\begin{aligned} S_h(t,t) &= I, \quad t \in \mathbb{R} \\ S_h(t_{n+1,t_\mu}) &= Q(t_n)S_h(t_n, t_\mu). \end{aligned}$$

We begin by showing that (1) holds by induction.

Base Case: $n = 0$

$$\begin{aligned} U^0 &= g \\ &= S_h(0,0)g + \Delta t 0 \\ &= S_h(0,0)g + \Delta t \sum_{\nu=0}^{-1} S_h(0, t_{\nu+1})F^\nu \end{aligned}$$

which fits the Duhamel. Now we assume that the discrete Duhamel's Principle fits the general scheme at step n, and we want to show that this implies that it fits for step $n+1$.

$$\begin{aligned} U^{n+1} &= Q(t_n)U_n + F^n \\ &= Q(t_n)(S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu) + F^n \quad \text{by induction assumption} \\ &= Q(t_n)S_h(t_n, 0)g + \Delta t Q(t_n) \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu + S_h(t_{n+1}, t_{n+1})F^n \\ &= S_h(t_{n+1}, 0)g + \Delta t \sum_{\nu=0}^{n} S_h(t_{n+1}, t_{\nu+1})F^\nu \end{aligned}$$

which fits Duhamel.

1.2 Bound in the h-norm

We now wish to show that

$$||S_h(t_{\nu+1}, t_\nu)||_h \leq Ke^{a\Delta t} \implies ||U^n||_h \leq K(e^{at_n}||g||_h + \int_0^{t_n} e^{a(t_n-s)}ds \max_{0\leq\nu\leq n-1} ||F^\nu||_h$$

Taking $||\cdot||_h$ of both sides of (1), then by Cauchy-Schwartz inequality, we have

$$\begin{aligned}
||U^n||_h &= ||S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu||_h \\
&\leq ||S_h(t_n, 0)||_h||g||_h + ||\Delta t||_h \sum_{\nu=0}^{n-1} ||S_h(t_n, t_{\nu+1})||_h||F^\nu||_h \\
&\leq Ke^{at_n}||g||_h + ||\Delta t||_h \sum_{\nu=0}^{n-1} ||S_h(t_n, t_{\nu+1})||_h||F^\nu||_h \\
&\leq Ke^{at_n}||g||_h + \Delta t \sum_{\nu=0}^{n-1} Ke^{a(t_n-t_{\nu+1})}||F^\nu||_h
\end{aligned}$$

we notice that $\Delta t \sum_{\nu=0}^{n-1} Ke^{a(t_n-t_{\nu+1})}$ is a right Remman sum of a strictly decreasing function, thus

$$\begin{aligned}
&\leq Ke^{at_n}||g||_h + K\int_0^{t_n} e^{a(t_n-s)}ds||F^\nu||_h \\
&\leq K(e^{at_n}||g||_h + \int_0^{t_n} e^{a(t_n-s)}ds \max_{0\leq\nu\leq n-1} ||F^\nu||_h)
\end{aligned}$$

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1.3 a Value

If $a = \Delta t^{-1/2}$, then we would have $||S_h(t_{\nu+1}, t_\nu)||_h \leq Ke^{n\sqrt{\Delta t}}$. Plugging this into our second inequality, we obtain,

$$||U^n||_h \leq K(e^{n\sqrt{t_n}}||g||_h + \int_0^{t_n} e^{n\sqrt{t_n-s}}ds \max_{0\leq\nu\leq n-1} ||F^\nu||_h)$$

2 Linerization

The shallow water equation can be written in quasilinear form as

$$u_t + f'(u)u_x = 0$$

where $u = (h, hv)^T$,

$$f'(u) = \begin{pmatrix} 0 & 1 \\ -(\frac{u_2}{u_1})^2 + gu_1 & 2(\frac{u_2}{u_1}) \end{pmatrix}$$

and

$$q(x, 0) = \begin{pmatrix} H + \epsilon e^{-(x-L/2)^2/w^2} \\ 0 \end{pmatrix}$$

Now, to make this system linear we must simply pick a constant state (h_0, v_0) which is consistent with the boundary and initial conditions. We choose (h_0, v_0) at $x = 0$ and $t = 0$. We can now compute (h_0, v_0) using the initial condition. We get $(h_0, v_0) \approx (1, 0)$. Thus we have

$$f'(u) = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 9.61 & 0 \end{pmatrix}$$

We can see easily $f'(u)$ has eigenvalues $\lambda_{1,2} = \pm\sqrt{9.61} = \pm 3.1$ which are real. It also has a full set of eigenvectors i.e. $(1, 3.1)^T$ and $(-1, 3.1)$. This together confirm that the linear problem is hyperbolic. We also know that the wave speeds are the eigenvalues, so we have wave speeds $\pm\sqrt{9.61}$

2.1 b

We have the PDE,

$$u_t + f'(u)u_x = 0$$

which we have shown can be written as

$$\begin{aligned} u_t + VDV^{-1}u_x &= 0 \\ \implies V^{-1}u_t + DV^{-1}u_x &= 0 \end{aligned}$$

where

$$V = \begin{pmatrix} 1 & -1 \\ 3.1 & 3.1 \end{pmatrix}, D = \begin{pmatrix} 3.1 & 0 \\ 0 & -3.1 \end{pmatrix}, \text{and} V^{-1} = \begin{pmatrix} 0.5 & 0.1613 \\ -0.5 & 0.1613 \end{pmatrix}$$

and initial condition

$$q(x, 0) = \begin{pmatrix} H + \epsilon e^{-(x-L/2)^2/w^2} \\ 0 \end{pmatrix}$$

Now, defining a new variable $r = V^{-1}q$ we have the decoupled system of equations

$$r_t + Dr_x = 0$$

and

$$r(x, 0) = V^{-1}q(x, 0) = \frac{1}{2}\begin{pmatrix} H + \epsilon e^{-(x-L/2)^2/w^2} \\ -H - \epsilon e^{-(x-L/2)^2/w^2} \end{pmatrix}$$

From Lavengen, we know the solutions of this system are

$$\begin{aligned} r_1(x, t) = r_n(x + \lambda_1 t, 0) &= \frac{1}{2}(H + \epsilon e^{-(x+3.1t-L/2)^2/w^2}) \\ \text{and} \quad r_2(x, t) = r_2(x + \lambda_2 t, 0) &= \frac{1}{2}(-H - \epsilon e^{-(x-3.1t-L/2)^2/w^2}) \end{aligned}$$

Finally switching back to q, we get

$$\begin{aligned} q(x, t) = Vr(x, t) &= \frac{1}{2}\begin{pmatrix} H + \epsilon e^{-(x+3.1t-L/2)^2/w^2} + H + \epsilon e^{-(x-3.1t-L/2)^2/w^2} \\ 3.1(H + \epsilon e^{-(x+3.1t-L/2)^2/w^2} - H - \epsilon e^{-(x-3.1t-L/2)^2/w^2}) \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2H + \epsilon e^{-(x+3.1t-L/2)^2/w^2} + \epsilon e^{-(x-3.1t-L/2)^2/w^2} \\ 3.1(\epsilon e^{-(x+3.1t-L/2)^2/w^2} - \epsilon e^{-(x-3.1t-L/2)^2/w^2}) \end{pmatrix} \end{aligned}$$

Now, simply plugging in $t = 1$ we get

$$q(x, 1) = \begin{pmatrix} -H - \epsilon e^{-(x+3.1-L/2)^2/w^2} - 3.1(H + \epsilon e^{-(x-3.1-L/2)^2/w^2}) \\ H + \epsilon e^{-(x+3.1-L/2)^2/w^2} + 3.1(H + \epsilon e^{-(x-3.1-L/2)^2/w^2}) \end{pmatrix}$$