It's the title bitchess! !!

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1 Stability of Numerical Schemes

We have the general scheme

$$\begin{aligned} U^{n+1} &= Q(t_n)U^n + \Delta t F^n \\ U^0 &= g \end{aligned}$$

where $U^n \in \mathbb{R}^d$.

1.1 Duhamel's Principle

We are given the following discrete Duhamel's Principle:

$$U^n = S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu, \tag{1}$$

where $t_n = n\Delta t$, and

$$\begin{aligned} S_h(t,t) &= I, \quad t \in \mathbb{R} \\ S_h(t_{n+1,t_\mu}) &= Q(t_n)S_h(t_n, t_\mu). \end{aligned}$$

We begin by showing that (1) holds by induction.

Base Case: $n = 0$

$$\begin{aligned} U^0 &= S_h(0,0)g + \Delta t \sum_{\nu=0}^{-1} S_h(0, t_{\nu+1})F^\nu \\ &= 0 \end{aligned}$$

which fits the general scheme. Now we assume that the (1) fits the general scheme at step n, and we want to show that this implies that it fits for step $n+1$.

$$\begin{aligned} U^{n+1} &= S_h(t_{n+1}, 0)g + \Delta t \sum_{\nu=0}^{n} S_h(t_{+1}n, t_{\nu+1})F^\nu \\ &= Q(t_n)S_h(t_n, 0)g + \Delta t Q(t_n) \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu + S_h(t_{n+1}, t_{n+1})F^n \\ &= Q(t_n)(S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu) + F^n \\ &= Q(t_n)U_n + F^n \end{aligned}$$

which fits our general scheme.

1.2 Bound in the h-norm

We now wish to show that

$$||S_h(t_{\nu+1},t_\nu)||_h \leq Ke^{ah} \implies ||U^n||_h \leq K(e^{at_n}||g||_h + \int_0^{t_n} e^{a(t_n-s)}ds \max_{0\leq\nu\leq n-1}||F^\nu||_h$$

Taking $||\cdot||_h$ of both sides of (1), then by Cauchy-Schwartz inequality, we have

$$\begin{aligned}
||U^n||_h &= ||S_h(t_n,0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n,t_{\nu+1})F^\nu||_h \\
&\leq ||S_h(t_n,0)||_h||g||_h + ||\Delta t||_h \sum_{\nu=0}^{n-1} ||S_h(t_n,t_{\nu+1})||_h||F^\nu||_h \\
&\leq Ke^{at_n}||g||_h + ||\Delta t||_h \sum_{\nu=0}^{n-1} ||S_h(t_n,t_{\nu+1})||_h||F^\nu||_h \\
&\leq Ke^{at_n}||g||_h + \Delta t \sum_{\nu=0}^{n-1} Ke^{a(t_n-t_{\nu+1})}||F^\nu||_h
\end{aligned}$$

we notice that $\Delta t \sum_{\nu=0}^{n-1} Ke^{a(t_n-t_{\nu+1})}$ is a right Remman sum of a strictly decreasing function, thus

$$\begin{aligned}
&\leq Ke^{at_n}||g||_h + K\int_0^{t_n} e^{a(t_n-s)}ds||F^\nu||_h \\
&\leq K(e^{at_n}||g||_h + \int_0^{t_n} e^{a(t_n-s)}ds \max_{0\leq\nu\leq n-1}||F^\nu||_h)
\end{aligned}$$

IS a POSITIVE??

1.3 a Value

If $a = h^{-1/2}$, then we would have $||S_h(t_{\nu+1},t_\nu)||_h \leq Ke^{\sqrt{h}}$. Plugging this into our second inequality, we obtain,

$$||U^n||_h \leq K(e^{\sqrt{t_n}}||g||_h + \int_0^{t_n} e^{\sqrt{t_n-s}}ds \max_{0\leq\nu\leq n-1}||F^\nu||_h)$$

I dont get the hint