
It's the title bitchess! !!

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1 Stability of Numerical Schemes

We have the general scheme

$$\begin{aligned}U^{n+1} &= Q(t_n)U^n + \Delta t F^n \\U^0 &= g\end{aligned}$$

where $U^n \in \mathbb{R}^d$.

1.1 Duhamel's Principle

We are given the following discrete Duhamel's Principle:

$$U^n = S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu, \quad (1)$$

where $t_n = n\Delta t$, and

$$\begin{aligned}S_h(t, t) &= I, \quad t \in \mathbb{R} \\S_h(t_{n+1}, t_\mu) &= Q(t_n)S_h(t_n, t_\mu).\end{aligned}$$

We begin by showing that (1) holds by induction.

Base Case: $n = 0$

$$\begin{aligned}U^0 &= g \\&= S_h(0, 0)g + \Delta t 0 \\&= S_h(0, 0)g + \Delta t \sum_{\nu=0}^{-1} S_h(0, t_{\nu+1})F^\nu\end{aligned}$$

which fits the Duhamel. Now we assume that the discrete Duhamel's Principle fits the general scheme at step n , and we want to show that this implies that it fits for step $n + 1$.

$$\begin{aligned}U^{n+1} &= Q(t_n)U_n + F^n \\&= Q(t_n)(S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu) + F^n \quad \text{by induction assumption} \\&= Q(t_n)S_h(t_n, 0)g + \Delta t Q(t_n) \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu + S_h(t_{n+1}, t_{n+1})F^n \\&= S_h(t_{n+1}, 0)g + \Delta t \sum_{\nu=0}^n S_h(t_{n+1}, t_{\nu+1})F^\nu\end{aligned}$$

which fits Duhamel.

1.2 Bound in the h -norm

We now wish to show that

$$\|S_h(t_{\nu+1}, t_\nu)\|_h \leq K e^{a\Delta t} \implies \|U^n\|_h \leq K(e^{at_n}\|g\|_h + \int_0^{t_n} e^{a(t_n-s)} ds \max_{0 \leq \nu \leq n-1} \|F^\nu\|_h)$$

Taking $\|\cdot\|_h$ of both sides of (1), then by Cauchy-Schwartz inequality, we have

$$\begin{aligned} \|U^n\|_h &= \|S_h(t_n, 0)g + \Delta t \sum_{\nu=0}^{n-1} S_h(t_n, t_{\nu+1})F^\nu\|_h \\ &\leq \|S_h(t_n, 0)\|_h \|g\|_h + \|\Delta t\|_h \sum_{\nu=0}^{n-1} \|S_h(t_n, t_{\nu+1})\|_h \|F^\nu\|_h \\ &\leq K e^{at_n} \|g\|_h + \|\Delta t\|_h \sum_{\nu=0}^{n-1} \|S_h(t_n, t_{\nu+1})\|_h \|F^\nu\|_h \\ &\leq K e^{at_n} \|g\|_h + \Delta t \sum_{\nu=0}^{n-1} K e^{a(t_n-t_{\nu+1})} \|F^\nu\|_h \end{aligned}$$

we notice that $\Delta t \sum_{\nu=0}^{n-1} K e^{a(t_n-t_{\nu+1})}$ is a right Remman sum of a strictly decreasing function, thus

$$\begin{aligned} &\leq K e^{at_n} \|g\|_h + K \int_0^{t_n} e^{a(t_n-s)} ds \|F^\nu\|_h \\ &\leq K(e^{at_n}\|g\|_h + \int_0^{t_n} e^{a(t_n-s)} ds \max_{0 \leq \nu \leq n-1} \|F^\nu\|_h) \end{aligned}$$

IS a POSITIVE??

1.3 a Value

If $a = \Delta t^{-1/2}$, then we would have $\|S_h(t_{\nu+1}, t_\nu)\|_h \leq K e^{n\sqrt{\Delta t}}$. Plugging this into our second inequality, we obtain,

$$\|U^n\|_h \leq K(e^{n\sqrt{t_n}}\|g\|_h + \int_0^{t_n} e^{n\sqrt{t_n-s}} ds \max_{0 \leq \nu \leq n-1} \|F^\nu\|_h)$$

2 Linerization

The shallow water equation can be written in quasilinear form as

$$u_t + f'(u)u_x = 0$$

where $u = (h, hv)^T$,

$$f'(u) = \begin{pmatrix} 0 & 1 \\ -(\frac{u_2}{u_1})^2 + gu_1 & 2(\frac{u_2}{u_1}) \end{pmatrix}$$

and

$$q(x, 0) = \begin{pmatrix} H + \epsilon e^{-(x-L/2)^2/w^2} \\ 0 \end{pmatrix}$$

Now, to make this system linear we must simply pick a constant state (h_0, v_0) which is consistent with the boundary and initial conditions. We choose (h_0, v_0) at $x = 0$ and $t = 0$. We can now compute (h_0, v_0) using the initial condition. We get $(h_0, v_0) \approx (1, 0)$. Thus we have

$$f'(u) = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 9.61 & 0 \end{pmatrix}$$

We can see easily $f'(u)$ has eigenvalues $\lambda_{1,2} = \pm\sqrt{9.61} = \pm 3.1$ which are real. It also has a full set of eigenvectors i.e. $(1, 3.1)^T$ and $(-1, 3.1)$. This together confirm that the linear problem is hyperbolic. We also know that the wave speeds are the eigenvalues, so we have wave speeds $\pm\sqrt{9.61}$

2.1 b

We have the PDE,

$$u_t + f'(u)u_x = 0$$

which we have shown can be written as

$$\begin{aligned} u_t + VDV^{-1}u_x &= 0 \\ \implies V^{-1}u_t + DV^{-1}u_x &= 0 \end{aligned}$$

where

$$V = \begin{pmatrix} 1 & -1 \\ 3.1 & 3.1 \end{pmatrix}, D = \begin{pmatrix} 3.1 & 0 \\ 0 & -3.1 \end{pmatrix}, \text{ and } V^{-1} = \begin{pmatrix} -0.5 & 0.1613 \\ 0.5 & 0.1613 \end{pmatrix}$$

and initial condition

$$q(x, 0) = \begin{pmatrix} H + \epsilon e^{-(x-L/2)^2/w^2} \\ 0 \end{pmatrix}$$

Now, defining a new variable $r = V^{-1}q$ we have the decoupled system of equations

$$r_t + Dr_x = 0$$

and

as