Project Assignment 1A

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SF2812 Applied Linear Optimization

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Problem description

The telecommunication company Nett provides capacity between cites in Europe. The cities are Stockholm, Berlin, London, Warsaw, Paris, Madrid and Rome. Some of the cities are connected in which traffic can be sent in both directions. The connections and the maximum traffic which can be sent between the cites respectively can be seen in Figure 1.

Nett wishes to provide 50 Gbit/s between Stockholm and Rome at the same time they provide 40 Gbit/s between London and Warsaw. They need help in routing the traffic because of the capacity limitations. Also, they would like to investigate if there is any slack in the network, potential to adding another traffic route and how to handle fluctuations in capacity.

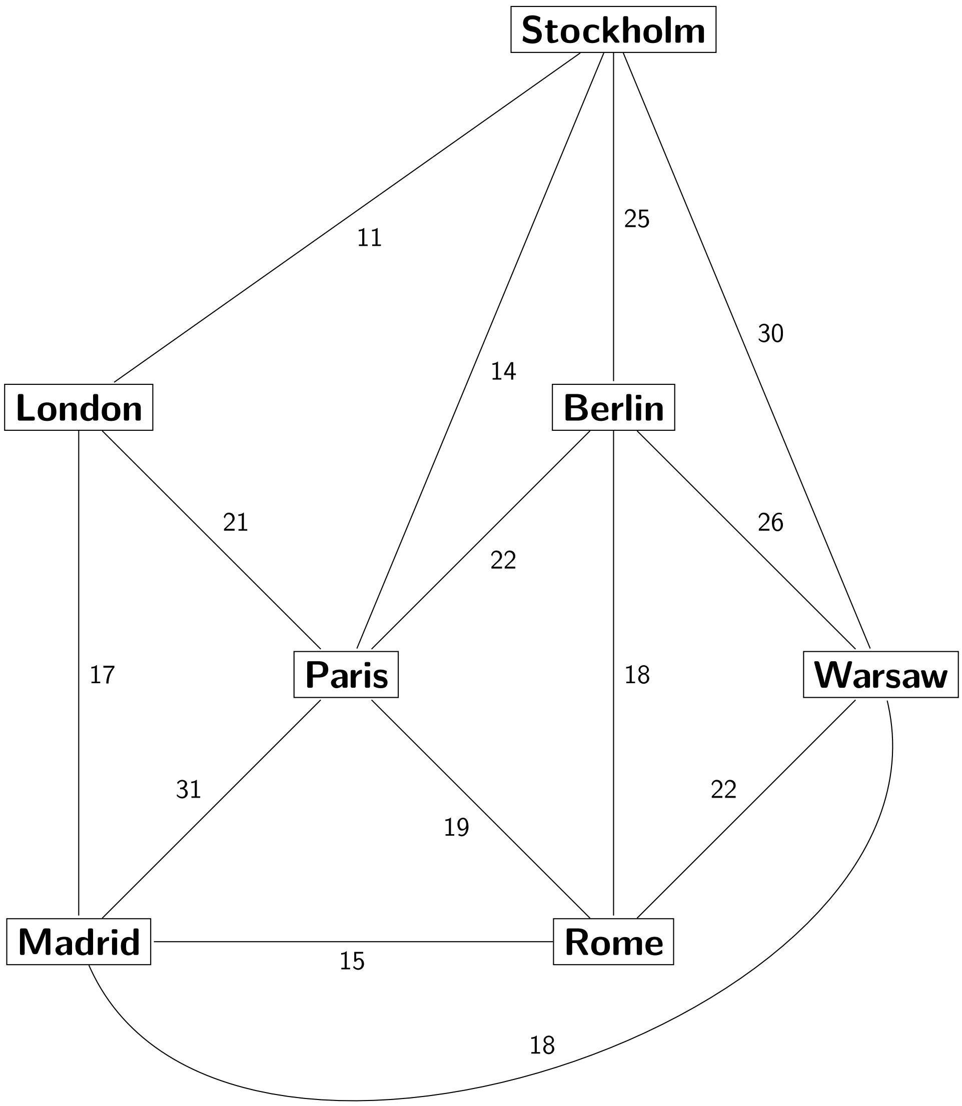

Figure 1. System modelled as a network where maximum capacity is marked out on the arcs.

1 Routing the traffic

In this section we investigate the best way to sent the traffic in the network. 50Gbit/s must be sent from Stockholm to Rome at the same time 40Gbit/s is sent from London to Warsaw without violating the capacity limitations. We would like to spread the traffic evenly. Therefore, we minimize the strain on any link in the network, i.e. traffic divided by the capacity.

1.1 Mathematical formulation

To solve this we treat the cities and links as a network and the traffic as a flow throughout the network. The cities are nodes connected by links which are arcs. Stockholm is a source node where 50Gbit/s comes in which will be distributed to the sink node Rome. The same goes for London and Warsaw respectively.

We start by introduction the set I :

$$I = \{Sto, Lon, Ber, War, Par, Rom, Mad\}$$

The set I make up for the network's nodes and an arc between node i and j is denoted as (i,j). The flows from Stockholm and London can be sent in the same link in either direction. However, the two flows must be treated separately. That being said, the 50Gbit/s that comes in from Stockholm is the same 50Gbit/s that arrives at Rome. We must keep track of the flows which is why we introduce another index k. The flow from Stockholm to Rome is denoted as $k = 1$ and the flow from London to Warsaw is $k = 2$.

For each arc in the network, there will be a corresponding variable $x_{i,j,k}$ which denotes the flow from node i to j of flow k. Solving for the variable x will provide the strategy for how to send the traffic between the cites. Though, there are two constraints which must be taken into account.

The first constraint deals with conservation of flow, i.e. the flow into a node is equal the flow out.

$$\sum_{j\in I} x_{i,j,1} - \sum_{j\in I} x_{j,i,1} = a_i \qquad \text{for } \forall\, i \in I$$

$$\sum_{j\in I} x_{i,j,2} - \sum_{j\in I} x_{j,i,2} = b_i \qquad \text{for } \forall\, i \in I$$

$$a_i = \begin{bmatrix} a_{Sto} \\ a_{Ber} \\ a_{War} \\ a_{Lon} \\ a_{Par} \\ a_{Mad} \\ a_{Rom} \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -50 \end{bmatrix} \text{ and } b_i = \begin{bmatrix} b_{Sto} \\ b_{Ber} \\ b_{War} \\ b_{Lon} \\ b_{Par} \\ b_{Mad} \\ b_{Rom} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -40 \\ 40 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The other constraint is that both types of traffic, in either direction, must not exceed the capacity in each link. If there is no direct connections between two cities, the capacity is zero and no traffic can be sent in this arc.

$$\sum_{k=1}^{2} x_{i,j,k} + \sum_{k=1}^{2} x_{j,i,k} \leq c_{i,j} \qquad \text{for } \forall\, i,j \in I$$

where $c_{i,j}$ is the maximum capacity in arc (i,j).

Finally, we define the objective function as the utility in a link. As stated before, utility is defined as the strain on any link in the network and we wish to minimize the maximum of this. Hence, we introduce the variable $maxUtility$ and put this into our objective function. We also set $maxUtility$ to less or equal to one since the traffic cannot exceed the capacity. A value of 1 is the highest value utility can take and means that the entire capacity in a link is used.

Taking all of the information into consideration, we can formulate an optimization problem mathematically.

$$
\begin{array}{lll}
\underset{x,maxUtility}{\text{minimize}} & maxUtility & \\
\text{subject to} & \sum_{j\in I} x_{i,j,1} - \sum_{j\in I} x_{j,i,1} = a_i & \forall\, i \in I \\
& \sum_{j\in I} x_{i,j,2} - \sum_{j\in I} x_{j,i,2} = b_i & \forall\, i \in I \\
& \sum_{k=1}^{2} x_{i,j,k} + \sum_{k=1}^{2} x_{j,i,k} \leq maxUtility * c_{i,j} & \forall\, i,j \in I \\
& maxUtility \leq 1 & \\
& x_{i,j,k} \geq 0 & \text{for } k = 1,2 \text{ and } \forall\, i,j \in I
\end{array}
$$

2 Model 2:

2.1 Problem description

The problem in the previous section is now to be extended such as to calculate the maximum available traffic between Berlin and Madrid under the same conditions as in the previous section, ie the capacity limits given in figure 1 as well as 50 Gbit/s of traffic Stockholm-Rome and 40 Gbit/s London-Warsaw.

2.2 Mathematical model

Once again we introduce the set I :

$$I = \{Sto, Lon, Ber, War, Par, Rom, Mad\}$$

representing the nodes of the system.

The flow from Stockholm to Rome is denoted $k = 1$ and the flow from London to Warsaw is $k = 2$. Thus, the added flow from Berlin to Rome is $k = 3$. The variable $x_{i,j,k}$ represents the flow of traffic type k from node i to node j.

As before, the first group of constraints treats conservation of flow:

$$\sum_{j\in I} x_{i,j,1} - \sum_{j\in I} x_{j,i,1} = a_i \quad i \in I$$

$$\sum_{j\in I} x_{i,j,2} - \sum_{j\in I} x_{j,i,2} = b_i \quad i \in I$$

$$\sum_{j\in I} x_{i,j,3} - \sum_{j\in I} x_{j,i,3} = qd_i \quad i \in I$$

Where a_i, b_i and d_i denotes the sources and sinks for each type of traffic and q is the total amount of traffic for $k = 3$. a_i and b_i is defined as previously

$$\text{and } d_i = \begin{bmatrix} d_{Sto} \\ d_{Ber} \\ d_{War} \\ d_{Lon} \\ d_{Par} \\ d_{Mad} \\ d_{Rom} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The transmitted data is not to exceed the capacity between each node, which may be modelled as a forth constraint:

$$\sum_{k=1}^{3} x_{i,j,k} + \sum_{k=1}^{3} x_{j,i,k} \leq c_{i,j} \quad i, j \in I$$

where $c_{i,j}$ is the capacity between node i and j. The objective function is the variable q, which is to be maximized.

The problem may thus be formulated as

$$\begin{array}{lll}
\max & q & \\
\text{s.t.} & \sum_{j \in I} x_{i,j,1} - \sum_{j \in I} x_{j,i,1} = a_i & i \in I \\
& \sum_{j \in I} x_{i,j,2} - \sum_{j \in I} x_{j,i,2} = b_i & i \in I \\
& \sum_{j \in I} x_{i,j,3} - \sum_{j \in I} x_{j,i,3} = q d_i & i \in I \\
& \sum_{k=1}^{3} x_{i,j,k} + \sum_{k=1}^{3} x_{j,i,k} \leq c_{i,j} & i, j \in I \\
& x_{i,j,k} \geq 0 & i, j \in I \quad k \in \{1, 2, 3\}
\end{array}$$

3 Model 3

3.1 Presentation of the problem

In this section, we are going to generalize the model a little bit. Up to this point, we have considered that each link has a known capacity and that this quantity is exact. This is, however, not true in general. We can expect the actual capacities to be close to the numbers given but we should take into account that they can fluctuate around those values. Our new model will thus have to deal with this uncertainty. Because the value of the capacities can be considered as an unknown parameter, we will use stochastic programming.

We also know that the 50 Gbit/s-demand can be rerouted after the actual values are known but the 40 Gbit/s-demand must be determined on beforehand (and thus not cannot be rerouted after knowing the actual capacities).

3.2 Mathematical model

Just as in the previous models, we introduce the set I :

$$I = \{Sto, Lon, Ber, War, Par, Rom, Mad\}$$

The main difference with previous models is the presence of uncertainty. As usual, we will handle this by discretizing this uncertainty into different scenarios s with an associated probability p_s. Each scenario corresponds to a complete set of known capacities. We can for example say that the second scenario is the one where the capacities are exactly those given. The scenarios and the corresponding probabilities must be chosen beforehand and have an influence on the solution. We can intuitively say that if the scenarios do not vary much with respect to the mean, the solution will not vary much either. On the other hand, if at least one scenario is very different from the mean values given, we can expect a quite different solution. In this section, we assume that the set of scenarios are given. The results section presents the set of scenarios chosen by our group but it can easily be changed.

Let us assume that we have S scenarios and define the parameters :

$c_{i,j,s}$	capacity in scenario s between cities i and j $i, j \in I$ and $s = 1, ...S$

For every scenario, we also introduce the variable $maxUtility_s$, the maximum utility in scenario s. It is clear that those variables must lie between 0 and 1.

Our goal is to minimize the expected value of the maximum utility. Since scenario s has probabily p_s, we have that our objective function is :

$$\sum_{s=1}^{S} p_s \, maxUtility_s$$

Let us also introduce the variables x, depicting the 40 Gbit/s-flow. Because this flow must be determined on beforehand, it cannot depend on the scenarios.

$x_{i,j}$	flow in Gbit/s going from i to j for the 40 Gbit/s-flow $i, j \in I$

All x are positive. Similarly, let us define the variables y for the 50 Gbit/s-flow. This can be rerouted and thus we could potentially have different flows for different scenarios.

$y_{i,j,s}$	flow in Gbit/s going from i to j for the 50 Gbit/s-flow and scenario s $i, j \in I$ and $s = 1, ..., S$

Here also, all y are positive. Just as previously, the first constraint to think about is the conservation of flow. If we think about London as a source and Warsaw as a sink, we have, for the 40 Gbit/s-flow :

$$\sum_{j \in I} x_{i,j} - \sum_{j \in I} x_{j,i} = b_i = \begin{bmatrix} b_{Sto} \\ b_{Ber} \\ b_{War} \\ b_{Lon} \\ b_{Par} \\ b_{Mad} \\ b_{Rom} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -40 \\ 40 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This must hold for every scenario since we cannot reroute this traffic. Concerning the 50 Gbit/s-flow, the conservation equation is given by :

$$\sum_{j \in I} y_{i,j,s} - \sum_{j \in I} y_{j,i,s} = a_i = \begin{bmatrix} a_{Sto} \\ a_{Ber} \\ a_{War} \\ a_{Lon} \\ a_{Par} \\ a_{Mad} \\ a_{Rom} \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -50 \end{bmatrix}$$

The other constraint is that the amount of flow cannot exceed the capacity. As stated above, we have a given capacity for each scenario. Because we know that the maximum utility is not larger than 1 and that we are minimizing a weighted sum of the maximum utilities, we have the following constraint :

$$x_{i,j} + x_{j,i} + y_{i,j,s} + y_{j,i,s} \leq maxUtility_s \, c_{i,j,s}$$

This equation must hold for every $i, j \in I$ and every scenario $s = 1, ..., S$ because we do not want to find ourselves in a position where we cannot route the traffic.

We now have everything we need and can state the full mathematical model.

$$\begin{aligned}
\min_{x,y,maxUtility} \quad & \sum_{s=1}^{S} p_s \, maxUtility_s & \\
s.t. \quad & maxUtility_s \leq 1 & s = 1, ..., S \\
& \sum_{j \in I} x_{i,j} - \sum_{j \in I} x_{j,i} = b_i & i \in I \\
& \sum_{j \in I} y_{i,j,s} - \sum_{j \in I} y_{j,i,s} = a_i & i \in I, s = 1, ..., S \\
& x_{i,j} + x_{j,i} + y_{i,j,s} + y_{j,i,s} \leq maxUtility_s \, c_{i,j,s} & i, j \in I, s = 1, ..., S \\
& x_{i,j} \geq 0 & i, j \in I \\
& y_{i,j,s} \geq 0 & i, j \in I, s = 1, ..., S
\end{aligned}$$

We can note that we do not need to add the positivity constraint for $maxUtility_s$ because the fourth constraint already demands it (because x,y and c are all positive).

We can also note that if the problem does not have a feasible region, it means that, at least in one of the scenario, it is not possible to route the traffic.

4 Results

4.1 Results model 1

The solution to routing the traffic in the network when minimizing the maximum utility in any link can be seen in Figure 2. The figure shows how much of each type of traffic that is to be sent in each arc. The blue numbers, $k = 1$, denotes the flow from Stockholm to Rome and the red numbers, $k = 2$, denotes the flow from London to Warsaw.

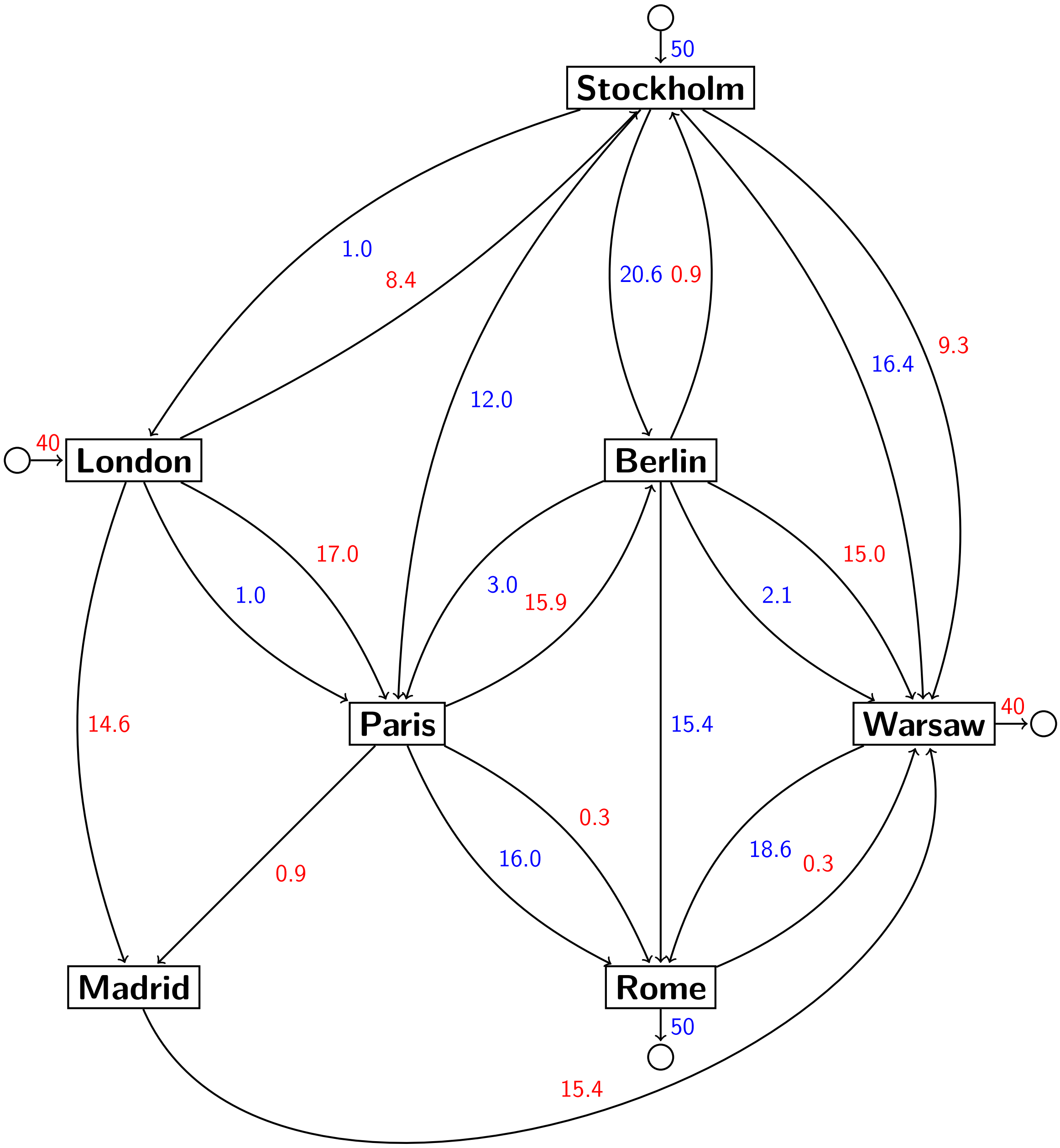

Figure 2. Result of routing for each type of traffic rounded to the nearest decimal.

Addition to the solution of the variable $x_{i,j,k}$, we also solved for the variable $maxUtility$. The minimum value of utility when routing traffic is

$$maxUtility = 0.857$$

To determine whether there is any slack in the network, let us investigate the utility in each link.

Table 1: Utility for each link

Link	Utility
Sto – Lon	85.7%
Sto – Par	85.7%
Sto – Ber	85.7%
Sto – War	85.7%
Lon – Par	85.7%
Lon – Mad	85.7%
Par – Ber	85.7%
Par – Mad	2.8%
Par – Rom	85.7%
Ber – War	65.9%
Ber – Rom	85.7%
War – Rom	85.7%
Mad – War	85.7%
Mad – Rom	0%

For almost all of the links, the utility is 85.7%, which means the traffic is spread evenly. However, there are three links in which there is slack; arcs (Par, Mad), (Ber, War) and (Mad, Rom). The connection between Madrid and Rome is not used at all and could be taken away without changing the routing of traffic.

On the other hand, let us now investigate in which links Nett should buy extra capacity in order to create more slack, i.e. decrease the maximum utility. One might think that Nett should invest in all links where the utility is 85.7%. Though, this is not true because there are some links where the strain is more severe. In other words, there are some key links in the network which limit the minimum value of the utility.

Along with the solution to the variables in the network problem, we also got *marginal values* for all the capacities in the network respectively. Consider the third constraint in the optimization problem, which controls that the traffic does not exceed the capacity. If we were to move all the variables to the left hand side and keep the constants on the right hand side, the marginal values denote the amount that the objective function change if the right hand side were increased by 1.0. In our case, we simply look at which capacities to increase in order to generate the highest decrease in utility.

There are 5 links, with the same marginal value, which we recommend Nett to invest in extra capacity. The links are

$$(Sto, Lon), (Sto, Par), (Ber, Par), (War, Mad) \text{ and } (War, Rom)$$

4.2 Results model 2

The maximum available capacity between Berlin and Madrid was found by maximizing the objective function under the constraints in the mathemathical model section above. The resulting optimal solution was calculated using GAMS to be $q = 15$ Gbit/s. Table 2 below presents the resulting system of traffic flow between each node separated into each type of traffic.

Table 2: The traffic flow for each type of traffic ($k = 1, 2, 3$) for each node

	$k=1$	$k=2$	$k=3$
Sto – Par	14.0		
Sto – Ber	14.0	3.0	
Sto – War	22.0	8.0	
Lon – Sto		11.0	
Lon – Par		12.0	
Lon – Mad		17.0	
Par – Ber		11.0	11.0
Par – Mad		1.0	
Par – Rom	14.0		
Ber – War		14.0	
Ber – Rom	14.0		
War – Rom	22.0		
Mad – Par			11.0
Mad – War		18.0	
Mad – Rom			4.0
Rom – Ber			4.0

4.3 Results model 3

This section presents the results for the third model (stochastic programming). It sometimes refers to the mathematical model available in the modelisation section.

The first thing to do is to define the different scenarios. Our group chose three different scenarios, called low, medium and high. The low scenario is when all the actual capacities are below the mean by a certain factor p. The medium scenario is when the real values are actually the one given. And the high scenario is when the values are above the mean by a factor p. So if we reuse the notation defined above :

$$c_{i,j,s} = \begin{cases} (1-p)c_{i,j} & \text{if } s = 1 \\ c_{i,j} & \text{if } s = 2 \\ (1+p)c_{i,j} & \text{if } s = 3 \end{cases}$$

We also need to assign probability to each scenario. To preserve the fact that the mean value should be $c_{i,j}$, we have that $p_1 = p_3$. We chose :

$$p_s = \begin{cases} 0.25 & \text{if } s = 1 \\ 0.5 & \text{if } s = 2 \\ 0.25 & \text{if } s = 3 \end{cases}$$

With those scenarios and a chosen factor of $p = 0.1$, we get the following expected utility :

$$\sum_{s=1}^{3} p_s \, maxUtility_s = 0.8615$$

It is a little bit over the utility obtained when we have certainty over the capacities. It is to be expected. Having complete information will yield better results. We also present below the utilities for each scenario :

	$s = 1$	$s = 2$	$s = 3$
$maxUtility_s$	0.952	0.857	0.779

We can see that, the better the scenario, the better the utility. That is quite intuitive. We can also note that if $s = 2$ (the capacities are the same as in the deterministic model), then it is possible to route the traffic to find the optimal value found with that model.

Let us finally look at the proposed traffic. The 40 Gbit/s-flow is given in the table below. In our model, this corresponds to the variables $x_{i,j}$.

i\j	Sto	Lon	Par	Ber	Mad	Rom
Sto		9.286				
Par		16.286				
Ber	2.143		15.000			
War	7.143			17.143	15.429	0.286
Mad		14.429	1.000			
Rom			0.286			

Note that this traffic is fixed and cannot depend on the scenarios. We can also check that we have indeed 40Gbit/s leaving Warsaw and 40 Gbit/s entering London. The conservation of flow is also respected.

Let us compare this solution to the one obtained for the deterministic model. We can see that it does not change much.

Let us also look at the 50Gbit/s-flow. In our model, this is the variable y. It can depend on the scenarios so we will have a different flow for each scenario. The table below gives the results.

	low	medium	high
Lon - Sto	0.143	0.143	0.143
Par - Sto	12	12	12
Par - Lon		0.143	
Par - Ber	3.857	3.857	3.857
Par - Mad			0.143
Ber - Sto	19.286	19.286	19.286
War - Sto	18.571	18.571	18.571
Mad - Lon	0.143		0.143
Rom - Par	15.857	16	16
Rom - Ber	15.429	15.429	15.429
Rom - War	18.571	18.571	18.571
Rom - Mad	0.143		