Project Assignment 1A

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SF2812 Applied Linear Optimization

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Problem description

The telecommunication company Nett provides capacity between cites in Europe. The cities are Stockholm, Berlin, London, Warsaw, Paris, Madrid and Rome. Some of the cities are connected in which traffic can be sent in both directions. The connections and the maximum traffic which can be sent between the cites respectively can be seen in Figure 1.

Nett wishes to provide 50 Gbit/s between Stockholm and Rome at the same time they provide 40 Gbit/s between London and Warsaw. They need help in routing the traffic because of the capacity limitations. Also, they would like to investigate if there is any slack in the network, potential to adding another traffic route and how to handle fluctuations in capacity.

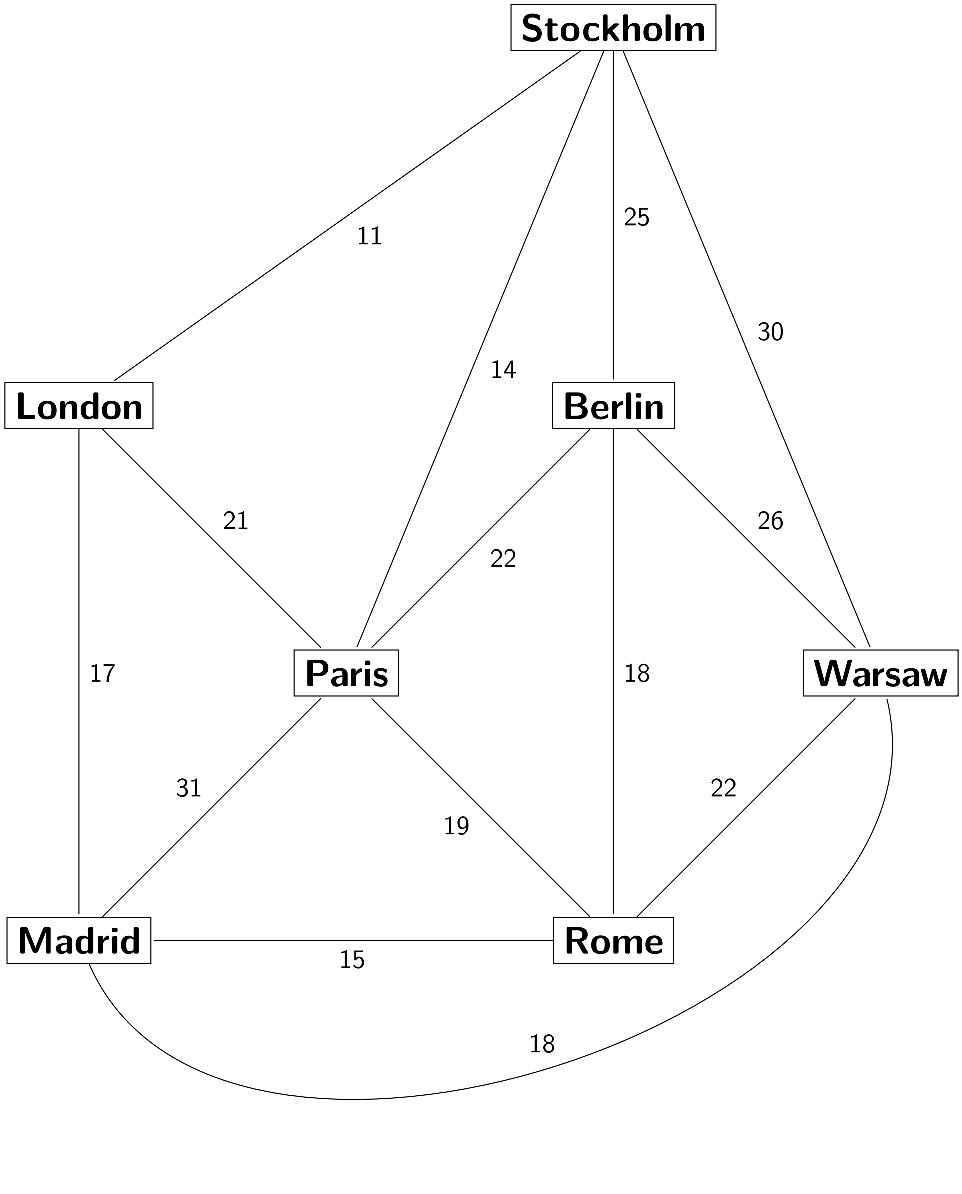


Figure 1. System modelled as a network where maximum capacity is marked out on the arcs.

1 Routing the traffic

In this section we investigate the best way to sent the traffic in the network. 50Gbit/s must be sent from Stockholm to Rome at the same time 40Gbit/s is sent from London to Warsaw without violating the capacity limitations. We would like to spread the traffic evenly. Therefore, we minimize the strain on any link in the network, i.e. traffic divided by the capacity.

1.1 Mathematical formulation

To solve this we treat the cities and links as a network and the traffic as a flow throughout the network. The cities are nodes connected by links which are arcs. Stockholm is a source node where 50Gbit/s comes in which will be distributed to the sink node Rome. The same goes for London and Warsaw respectively.

We start by introduction the set I :

$$I = \{Sto, Lon, Ber, War, Par, Rom, Mad\}$$

The set I make up for the network's node and an arc between node i and j is denoted as (i,j). The flows from Stockholm and London can be sent in the same link in either direction. However, the two flows must be treated independently. That being said, the 50Gbit/s that comes in from Stockholm is the same 50Gbit/s that arrives at Rome. We must keep track of the flows which is why we introduce another index k. The flow from Stockholm to Rome is $k = 1$ and the flow from London to Warsaw is $k = 2$.

For each arc in the network, there will be a corresponding variable $x_{i,j,k}$ which denotes the flow from node i to j of flow k. Solving for the variable x will provide the strategy for how to send the traffic between the cites. Though, there are two constraints which must be taken into account.

The first constraint deals with conservation of flow, i.e. the flow into a node is equal the flow out.

$$\sum_{j \in I} x_{i,j,k} - \sum_{j \in I} x_{j,i,k} = b_i$$

$$\text{where } b_i = \begin{bmatrix} b_{Sto} \\ b_{Ber} \\ b_{War} \\ b_{Lon} \\ b_{Par} \\ b_{Mad} \\ b_{Rom} \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ -40 \\ 40 \\ 0 \\ 0 \\ -50 \end{bmatrix}$$

The other constraint is that both types of traffic, in either direction, must not exceed the capacity in each link. If there is no direct connections between two cities, the capacity is zero and no traffic can be sent in this arc.

$$\sum_{k=1}^{2} x_{i,j,k} + \sum_{k=1}^{2} x_{j,i,k} \leq c_{i,j}$$

where $c_{i,j}$ is the maximum capacity in arc (i,j)

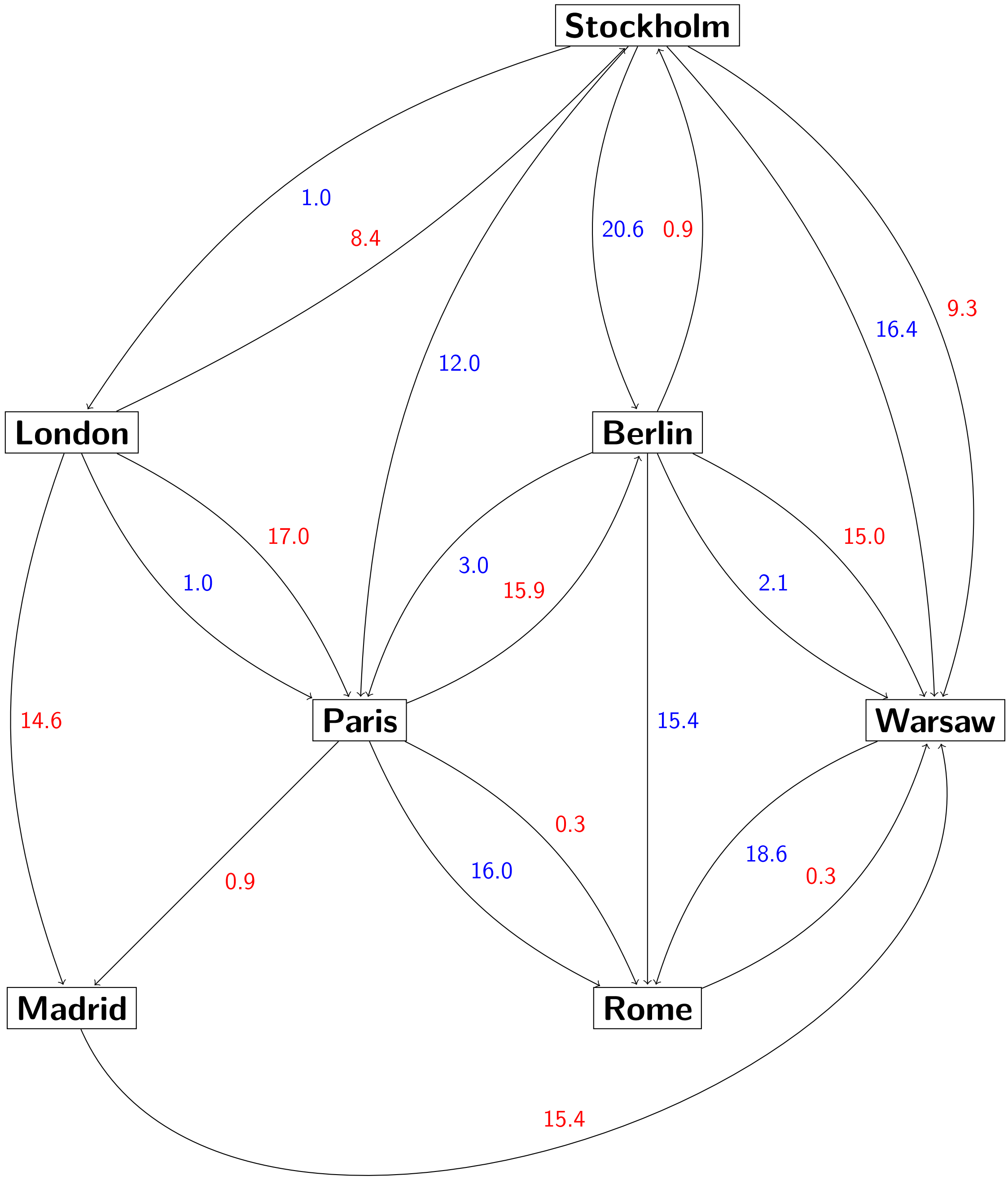
Finally, we define the objective function as the utility in a link.

$$\frac{\sum_{k=1}^{2} x_{i,j,k} + \sum_{k=1}^{2} x_{j,i,k}}{c_{i,j}}$$

Taking all the above into consideration, the mathematical formulation of the network flow optimization problem becomes:

$$\begin{array}{ll} \min & \frac{\sum_{k=1}^{2} x_{i,j,k} + \sum_{k=1}^{2} x_{j,i,k}}{c_{i,j}} \\ \text{s.t.} & \sum_{j \in I} x_{i,j,k} - \sum_{j \in I} x_{j,i,k} = b_i \\ & \sum_{k=1}^{2} x_{i,j,k} + \sum_{k=1}^{2} x_{j,i,k} \leq c_{i,j} \\ & x_{i,j,k} \geq 0 \quad \forall\, i, j \in I \end{array}$$

1.2 Result


Stockholm
London
Berlin
Paris
Warsaw
Madrid
Rome
1.0
8.4
12.0
20.6
0.9
16.4
9.3
1.0
17.0
3.0
15.9
2.1
15.0
14.6
15.4
0.3
0.9
16.0
18.6
0.3
15.4

2 Model 2

2.1 Problem description

The problem in the previous section is now to be extended such as to calculate the maximum available traffic between Berlin and Madrid under the same conditions as in the previous section, ie the capacity limits given in figure 1 as well as 50 Gbit/s of traffic Stockholm-Rome and 40 Gbit/s London-Warsaw.

2.2 Mathematical model

Once again we introduce the set I :

$$I = \{Sto, Lon, Ber, War, Par, Rom, Mad\}$$

representing the nodes of the system.

The flow from Stockholm to Rome is denoted $k = 1$ and the flow from London to Warsaw is $k = 2$. Thus, the added flow from Berlin to Rome is $k = 3$. The variable $x_{i,j,k}$ represents the flow of traffic type k from node i to node j.

As before, the first gropu of constraints treats conservation of flow:

$$\sum_{j \in I} x_{i,j,k} - \sum_{j \in I} x_{j,i,k} = a_i \quad k = 1$$

$$\sum_{j \in I} x_{i,j,k} - \sum_{j \in I} x_{j,i,k} = b_i \quad k = 2$$

$$\sum_{j \in I} x_{i,j,k} - \sum_{j \in I} x_{j,i,k} = qd_i \quad k = 3$$

(Change as above)

The transmitted data is not to exceed the capacity between each node, which may be modelled as a second constraint:

$$\sum_{k=1}^{3} x_{i,j,k} + \sum_{k=1}^{3} x_{j,i,k} \leq c_{i,j}$$

where $c_{i,j}$ is the capacity between node i and j.

The problem may thus be formulated as

$$\begin{array}{lll}
\max & q & \\
\text{s.t.} & \sum_{j \in I} x_{i,j,k} - \sum_{j \in I} x_{j,i,k} = a_i + b_i + qd_i & \\
& \sum_{k=1}^{3} x_{i,j,k} + \sum_{k=1}^{3} x_{j,i,k} \leq c_{i,j} & \forall\, i, j \in I \\
& x_{i,j,k} \geq 0 & \forall\, i, j \in I \quad k \in \{1, 2, 3\}
\end{array}$$

2.3 Results

Results

3 Model 3

3.1 Presentation of the problem

In this section, we are going to generalize the model a little bit. Up to this point, we have considered that each link has a known capacity and that this quantity is exact. This is, however, not true in general. We can expect the actual capacities to be close to the numbers given but we should take into account that they can fluctuate around those values. Our new model will thus have to deal with this uncertainty. Because the value of the capacities can be considered as an unknown parameter, we will use stochastic programming.

We also know that the 50 Gbit/s-demand can be rerouted after the actual values are known but the 40 Gbit/s-demand must be determined on beforehand (and thus not cannot be rerouted after knowing the actual capacities).

3.2 Mathematical model

Just as in the previous models, we introduce the set I :

$$I = \{Sto, Lon, Ber, War, Par, Rom, Mad\}$$

The main difference with previous models is the presence of uncertainty. As usual, we will handle this by discretizing this uncertainty into different scenarios s with an associated probability p_s. Each scenario corresponds to a complete set of known capacities. We can for example say that the second scenario is the one where the capacities are exactly those given. The scenarios and the corresponding probabilities must be chosen beforehand and have an influence on the solution. We can intuitively say that if the scenarios do not vary much with respect to the mean, the solution will not vary much either. On the other hand, if at least one scenario is very different from the mean values given, we can expect a quite different solution. In this section, we assume that the set of scenarios are given. The results section presents the set of scenarios chosen by our group but it can easily be changed.

Let us assume that we have S scenarios and define the parameters :

$c_{i,j,s}$	capacity in scenario s between cities i and j $i, j \in I$ and $s = 1, ...S$

For every scenario, we also introduce the variable $maxUtility_s$, the maximum utility in scenario s. It is clear that those variables must lie between 0 and 1.

Our goal is to minimize the expected value of the maximum utility. Since scenario s has probably p_s, we have that our objective function is :

$$\sum_{s=1}^{S} p_s \, maxUtility_s$$

Let us also introduce the variables x, depicting the 40 Gbit/s-flow. Because this flow must be determined on beforehand, it cannot depend on the scenarios.

$x_{i,j}$	flow in Gbit/s going from i to j for the 40 Gbit/s-flow $i, j \in I$

All x are positive. Similarly, let us define the variables y for the 50 Gbit/s-flow. This can be rerouted and thus we could potentially have different flows for different scenarios.

$y_{i,j,s}$	flow in Gbit/s going from i to j for the 50 Gbit/s-flow and scenario s $i, j \in I$ and $s = 1, ..., S$

Here also, all y are positive. Just as previously, the first constraint to think about is the conservation of flow. If we think about London as a source and Warsaw as a sink, we have, for the 40 Gbit/s-flow :

$$\sum_{j \in I} x_{i,j} - \sum_{j \in I} x_{j,i} = b_i = \begin{bmatrix} b_{Sto} \\ b_{Ber} \\ b_{War} \\ b_{Lon} \\ b_{Par} \\ b_{Mad} \\ b_{Rom} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -40 \\ 40 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This must hold for every scenario since we cannot reroute this traffic. Concerning the 50 Gbit/s-flow, the conservation equation is given by :

$$\sum_{j \in I} y_{i,j,s} - \sum_{j \in I} y_{j,i,s} = a_i = \begin{bmatrix} a_{Sto} \\ a_{Ber} \\ a_{War} \\ a_{Lon} \\ a_{Par} \\ a_{Mad} \\ a_{Rom} \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -50 \end{bmatrix}$$

The other constraint is that the amount of flow cannot exceed the capacity. As stated above, we have a given capacity for each scenario. Because we know that the maximum utility is not larger than 1 and that we are minimizing a weighted sum of the maximum utilities, we have the following constraint :

$$x_{i,j} + x_{j,i} + y_{i,j,s} + y_{j,i,s} \leq maxUtility_s \, c_{i,j,s}$$

This equation must hold for every $i, j \in I$ and every scenario $s = 1, ..., S$ because we do not want to find ourselves in a position where we cannot route the traffic.

We now have everything we need and can state the full mathematical model.

$$\begin{aligned}
\min_{x,y,maxUtility} \quad & \sum_{s=1}^{S} p_s \, maxUtility_s & \\
s.t. \quad & maxUtility_s \leq 1 & s = 1, ..., S \\
& \sum_{j \in I} x_{i,j} - \sum_{j \in I} x_{j,i} = b_i & i \in I \\
& \sum_{j \in I} y_{i,j,s} - \sum_{j \in I} y_{j,i,s} = a_i & i \in I, s = 1, ..., S \\
& x_{i,j} + x_{j,i} + y_{i,j,s} + y_{j,i,s} \leq maxUtility_s \, c_{i,j,s} & i, j \in I, s = 1, ..., S \\
& x_{i,j} \geq 0 & i, j \in I \\
& y_{i,j,s} \geq 0 & i, j \in I, s = 1, ..., S
\end{aligned}$$

We can note that we do not need to add the positivity constraint for $maxUtility_s$ because the fourth constraint already demands it (because x,y and c are all positive).

We can also note that if the problem does not have a feasible region, it means that, at least in one of the scenario, it is not possible to route the traffic.

4 Results

4.1 Results 1

result1

4.2 Results 2

results2

4.3 Results 3

This section presents the results for the third model (stochastic programming). It sometimes refers to the mathematical model available in the modelisation section.

The first thing to do is to define the different scenarios. Our group chose three different scenarios, called low, medium and high. The low scenario is when all the actual capacities are below the mean by a certain factor p. The medium scenario is when the real values are actually the one given. And the high scenario is when the values are above the mean by a factor p. So if we reuse the notation defined above :

$$c_{i,j,s} = \begin{cases} (1-p)c_{i,j} & \text{if } s = 1 \\ c_{i,j} & \text{if } s = 2 \\ (1+p)c_{i,j} & \text{if } s = 3 \end{cases}$$