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Chapter 1

Results and discussion

In this chapter, we will present the different results and discuss them.

Expliquer ce qu'on va faire et tester

Mettre ici les trois "super mesh"

1.1 Multigrid

In this section, we will test the coarse part of the preconditioner : the multigrid solver. This will be done in two steps. First, we will verify a well known property of the multigrid solvers : the h-independent convergence. We will also compare the number and iterations needed while varying key parameters of the model. Those tests will be performed on various meshes. The second part will focus on the influence of hanging nodes on the numerical solution.

Let us before all present a type of numerical solution that can be obtained using the multigrid solver. Figure 1.1 shows an example of the numerical solution computed. We can see that even with $p = 1$, we have a good approximation. This is because the forcing term is not at all oscillatory.

1.1.1 H-independent convergence

Let us first verify that our geometric multigrid solver has the required property and that the same number of iterations is needed to obtain a given accuracy, however small the elements. We will use the model problem throughout this section with the same right hand side. For all the tests below, the domain will be : $\Omega = [-1; 1]^2$. We will solve :

$$\nabla^2 u = -\frac{\pi^2}{2}\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y) \qquad \text{on } \Omega \tag{1.1}$$

$$u = 0 \qquad \text{on } \Gamma \tag{1.2}$$

It is easy to see that for the given domain, we have an analytic solution :

$$u(x, y) = \cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y))$$

Let us now explain how we define the error. We will look at the absolute difference between the value of the approximation and the value of the analytic solution at the global nodes and take the maximum. Formally, we have that the error after iteration k, e_k is :

$$e_k = \max_i |u(x_i, y_i) - u_i^k|$$

Where u_i^k is the value of our approximation at the global node i after iteration k. Since $u_i^0 = 0$ for all i, it is clear that $e_0 = 1$.

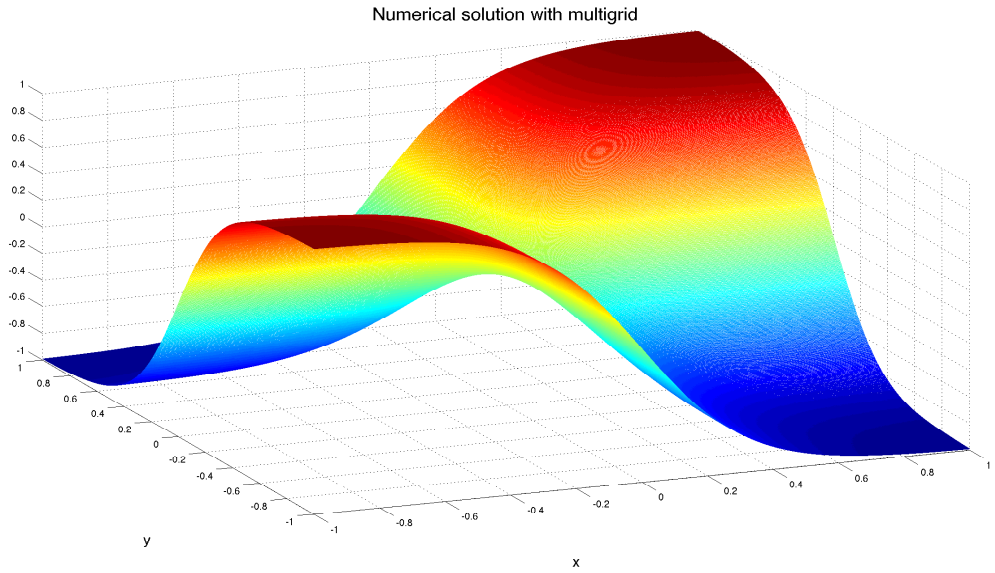


Figure 1.1: Numerical solution using the multigrid solver of $\nabla^2 u = f$ for $f = -2\tanh(3x)\tanh(3y)(18 - 9\tanh^2(3x) - 9\tanh^2(3y))$

Figure 1.2 shows the two "supra meshes" that we will refine during the tests. Some refinements will be uniform and some will be so that we have the presence of hanging nodes. We can note that even for the crooked mesh, the elements are not really distorted. It does not matter since we are only testing the h-independent convergence. Having a mesh with elements that are more distorted will only influence the accuracy of the approximation and not how the algorithm solve the linear system we want to solve.

Let us start with a simple V-cycle on the regular mesh (figure 1.2a) that we will refine uniformly. We will compare the errors when we increase the number of degrees of freedom and for different ν_1 and ν_2. The sum of the two smoothing parameters is chosen to be constant so that we have the same number of Jacobi iterations for all pairs ν_1 and ν_2. Here, we chose the sum to be equal to four.

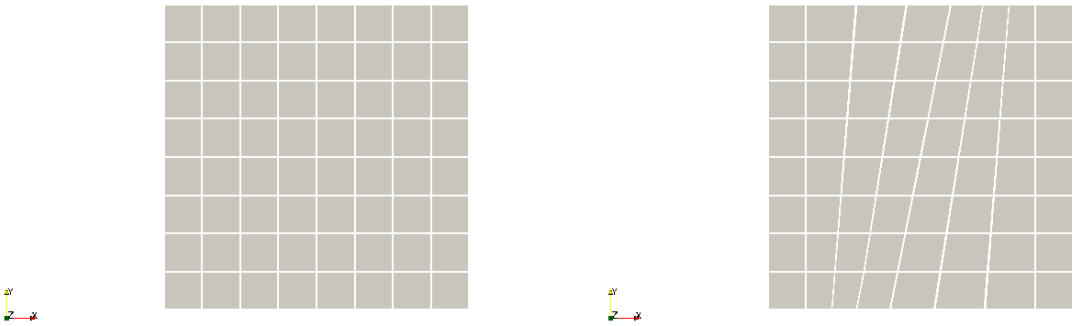
The results are shown in table 1.1. We can see that we indeed have an h-independent convergence of the solver. For every pair of the smoothing parameters, at least for the first few iterations, the error e_k is identical for all values of N. Except for the pair $\nu_1 = 4$ and $\nu_2 = 0$, each iteration roughly decrease the error by one decimal point.

We can also note that the values of the parameters influence the convergence. For this particular problem and this particular mesh, the values $\nu_1 = 0$ and $\nu_2 = 4$ seem to be the best as the error is smaller after the same number of iterations than for the other pairs. The less post smoothing iterations (ν_2) we do, the slower the convergence. This is true for $\nu_2 = 1$ where the error on the finest mesh after six iterations is a hundred times larger than on the same mesh after the same number of iterations for $\nu_2 = 4$, and it is clearer still for $\nu_2 = 0$ where the error is a thousand times larger after six iterations on the finest mesh.

We have to note that even tough the errors are identical for all meshes at first, we have a difference after a few iterations. This can be explained by the fact that our geometric multigrid algorithm actually solves a linear system whereas the error is measured as the difference between the analytic solution and the solution of the linear system. Thus, even if we solved the linear system exactly, we would still have an error and that error should decrease as the number of degrees of freedom increases. This is indeed what we observe here. For example, for the mesh with $N = 2.6\,10^5$, we can see that after six iterations, we have almost converged and that the error stays around $3.14\,10^{-6}$. Even if we did several more iterations, the error would not decrease significantly. That is because we have the solution of the linear system and the error is only due

N	$2.6\ 10^5$	$1.1\ 10^6$	$4.2\ 10^6$	$1.7\ 10^7$	$6.7\ 10^7$
	$\nu_1 = 2$	$\nu_2 = 2$			
e_1	3.56e-02	3.56e-02	3.56e-02	3.56e-02	3.56e-02
e_2	1.34e-03	1.34e-03	1.34e-03	1.34e-03	1.34e-03
e_3	5.42e-05	5.66e-05	5.72e-05	5.73e-05	5.73e-05
e_4	3.11e-06	2.90e-06	3.45e-06	3.59e-06	3.62e-06
e_5	3.30e-06	9.48e-07	3.62e-07	3.50e-07	3.85e-07
e_6	3.17e-06	8.14e-07	2.26e-07	8.26e-08	5.23e-08
	$\nu_1 = 3$	$\nu_2 = 1$			
e_1	3.70e-02	3.71e-02	3.71e-02	3.71e-02	3.71e-02
e_2	1.62e-03	1.63e-03	1.63e-03	1.63e-03	1.63e-03
e_3	1.04e-04	1.06e-04	1.07e-04	1.07e-04	1.07e-04
e_4	1.10e-05	1.22e-05	1.27e-05	1.29e-05	1.29e-05
e_5	4.47e-06	2.20e-06	1.96e-06	2.10e-06	2.13e-06
e_6	3.34e-06	1.00e-06	4.38e-07	3.53e-07	3.88e-07
	$\nu_1 = 1$	$\nu_2 = 3$			
e_1	3.57e-02	3.57e-02	3.57e-02	3.57e-02	3.57e-02
e_2	1.29e-03	1.29e-03	1.29e-03	1.29e-03	1.29e-03
e_3	4.66e-05	4.89e-05	4.95e-05	4.96e-05	4.97e-05
e_4	1.53e-06	1.53e-06	2.10e-06	2.24e-06	2.28e-06
e_5	3.08e-06	7.31e-07	1.43e-07	1.17e-07	1.51e-07
e_6	3.14e-06	7.83e-07	1.95e-07	4.80e-08	1.13e-08
	$\nu_1 = 4$	$\nu_2 = 0$			
e_1	4.55e-02	4.55e-02	4.55e-02	4.55e-02	4.55e-02
e_2	3.26e-03	3.26e-03	3.26e-03	3.26e-03	3.26e-03
e_3	4.64e-04	4.71e-04	4.71e-04	4.71e-04	4.71e-04
e_4	9.24e-05	9.60e-05	9.65e-05	9.67e-05	9.67e-05
e_5	1.74e-05	2.04e-05	2.12e-05	2.14e-05	2.14e-05
e_6	6.17e-06	3.86e-06	4.55e-06	4.74e-06	4.78e-06
	$\nu_1 = 0$	$\nu_2 = 4$			
e_1	3.58e-02	3.58e-02	3.58e-02	3.58e-02	3.58e-02
e_2	1.29e-03	1.29e-03	1.29e-03	1.29e-03	1.29e-03
e_3	4.53e-05	4.77e-05	4.82e-05	4.84e-05	4.84e-05
e_4	1.17e-06	1.31e-06	1.89e-06	2.03e-06	2.07e-06
e_5	3.03e-06	6.80e-07	9.14e-08	7.66e-08	1.12e-07
e_6	3.13e-06	7.76e-07	1.87e-07	4.04e-08	3.62e-09

Table 1.1: Errors after k iterations of a V-cycle (e_k) for the regular mesh uniformly refined to have N degrees of freedom and for different values of the parameters ν_1 and ν_2



(a) Regular mesh

(b) Crooked mesh

Figure 1.2: The two "supra meshes" that will be refined during the tests for the multigrid solver. We have one regular mesh (left) where the elements are squares and one crooked mesh (right) where the elements are slightly distorted.

N	$2.6\,10^5$	$1.1\,10^6$	$4.2\,10^6$	$1.7\,10^7$	$6.7\,10^7$
	$\nu_1 = 2$	$\nu_2 = 2$			
e_1	3.64e-02	3.64e-02	3.64e-02	3.64e-02	3.64e-02
e_2	2.56e-03	2.49e-03	2.46e-03	2.44e-03	2.43e-03
e_3	7.41e-04	6.33e-04	5.80e-04	5.54e-04	5.41e-04
e_4	6.15e-04	3.15e-04	1.62e-04	1.35e-04	1.28e-04
e_5	5.96e-04	3.01e-04	1.52e-04	7.67e-05	4.76e-05
e_6	5.91e-04	2.98e-04	1.50e-04	7.50e-05	3.77e-05

Table 1.2: Errors after k iterations of a V-cycle (e_k) for the crooked mesh uniformly refined to have N degrees of freedom and for $\nu_1 = 2$ and $\nu_2 = 2$

to the discretization. If we refine the mesh and go to $N = 6.7\,10^7$, then we can get smaller errors (of the order of 10^{-8}).

Let us now explore the results for the crooked mesh (figure 1.2b). Here also, we should expect an h-independent convergence. We only show the results for $\nu_1 = 2$ and $\nu_2 = 2$ but the same commentary applies for the other pairs. The results can be seen on table 1.2

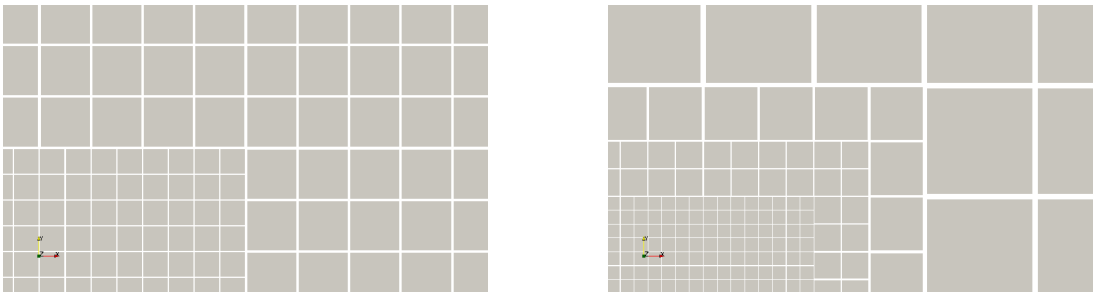
We can see that the for the first few iterations, the error e_k is independent of the mesh. However, the note we made earlier is much clearer here. Because the mesh is not regular, the effect of discretization are more important and therefore we will not reach the same accuracy than we did before. That is why after six iterations, the less refined grid ($N = 2.6\,10^5$) still has an error of $5.91\,10^{-4}$. More iterations will not have a great impact on the solution since the error is mostly due to the discretization.

1.1.2 Influence of hanging nodes

We will now investigate the influence of hanging nodes on our solution. We will present the results only for the regular mesh but the tests have been performed on both and the same conclusions apply to the crooked mesh.

We will compare the values of the error between one mesh with no hanging nodes, one where we only have refined the lower left part of the domain once, and one where we have a rapid transition between two parts of different refinement level (which will occur often in AMR).

Figure 1.3 presents the latter two meshes. For the mesh in figure 1.3b, we have refined thrice more in a certain region than in the adjacent one. Since we do not allow adjacent quadrants to



(a) Two levels

(b) Levels rapidly changing

Figure 1.3: Zoom on a certain part of the two meshes containing hanging nodes that will be used to test the multigrid solver. We have one mesh where we only have refined a part of the domain once more than the rest (left) and a mesh where we have refined a part thrice more than the rest which results in levels changing rapidly (right).

Meshes	No hanging nodes	Figure 1.3a	Figure 1.3b
N	$4.2\ 10^6$	$7.3\ 10^6$	$7.0\ 10^7$
e_1	3.56e-02	3.56e-02	3.56e-02
e_2	1.34e-03	1.34e-03	1.34e-03
e_3	5.72e-05	5.72e-05	5.74e-05
e_4	3.45e-06	3.47e-06	3.64e-06

Table 1.3: Errors after k iterations of a V-cycle (e_k) for a mesh without hanging nodes and the two meshes presented in figure 1.3. Each mesh has N degrees of freedom and the smoothing parameters were $\nu_1 = 2$ and $\nu_2 = 2$.

be more than one level apart, we obtain a "layer" where the levels of the quadrants is rapidly changing. In order to compare solutions, we made sure that the largest quadrants in all three meshes had the same size. This means that the meshes with hanging nodes have a lot more degrees of freedom.

Table 1.3 shows the results for the three different meshes using a V-cycle and with the smoothing parameters $\nu_1 = 2$ and $\nu_2 = 2$. Here again, we can see that the convergence is h-independent and that one iteration gives us roughly one more decimal. The presence of hanging nodes has no influence on the error we observe after a given number of iterations. The same result is observed with all pairs of the smoothing parameters and with other meshes.

This will be important in the next sections when we will use our multigrid solver as a preconditioner. Indeed, we will see that it is the coarse correction that allow for h-independent convergence.

1.2 Fine preconditioner

Let us now move on to the fine part of the preconditioner : the overlapping additive Schwarz preconditioner. We will test it by using the preconditioned conjugate gradients method described earlier but, for now, the preconditioner will only consist of the fine part (i.e. $P = P^f$).

As in the previous section, we will perform the tests in two parts : first, we will use meshes with elements that are distorted or not but with no hanging nodes. Then, we will see how the fine preconditioner performs in the presence of hanging nodes also for meshes that are distorted or not. Here, of course, we will use interpolations of higher degree. Typically, the tests will be performed for $p = 2, 4, 6, 8$.

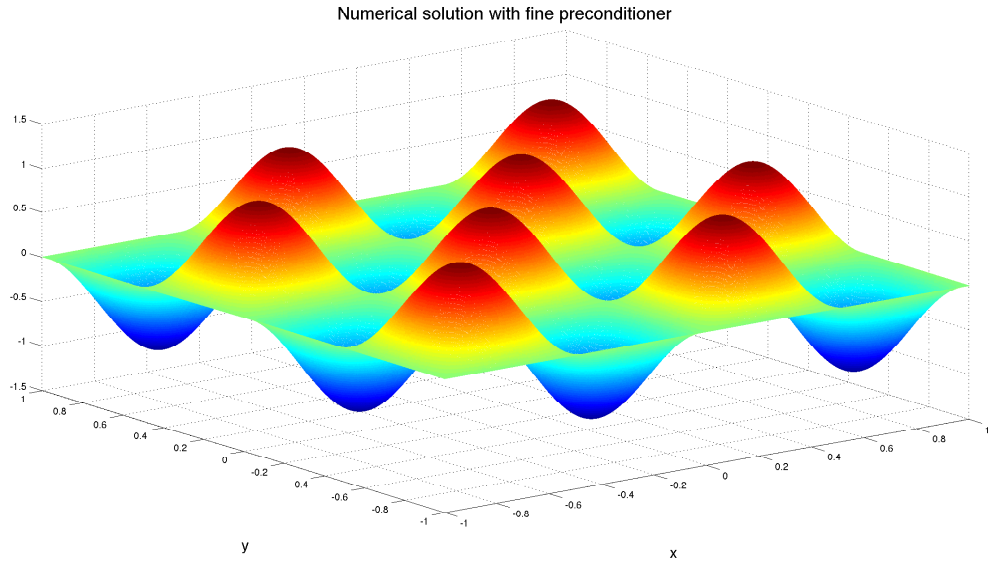


Figure 1.4: Numerical solution to problem 1.3 using an interpolation of order $p = 2$ and $1.0\,10^6$ degrees of freedom on a regular mesh with no hanging nodes.

1.2.1 No hanging nodes

Let us first present the problem we will use throughout this section. The forcing term will be chosen more oscillatory than in the previous part since we use interpolations of higher degree. As before, the domain is : $\Omega = [-1; 1]^2$ and Γ is the boundary. The problem is :

$$\nabla^2 u = -8 \sin(2\pi x) \sin(2\pi y) \qquad \text{on } \Omega \tag{1.3}$$

$$u = 0 \qquad \text{on } \Gamma \tag{1.4}$$

This problem has an analytic solution and it is easy to convince oneself that this solution is given by :

$$u(x, y) = \sin(2\pi x) \sin(2\pi y)$$

Figure 1.4 shows the numerical solution to the problem above for $p = 2$ and $1.0\,10^6$ degrees of freedom for a regular mesh. We can note that it is exactly the same number of degrees of freedom as if we had refined uniformly once more and used an interpolation of degree $p = 1$. Let us then compare how the two approximations perform. We solved the problem for $p = 1$ with our multigrid solver and the problem for $p = 2$ with the PCG and the fine preconditioner. Let us denote u_i^j as the value of the approximation for $p = j$ at node i. We have that :

$$e^1 = \max_i |u_i^1 - u(x_i, y_i)| = 5.02\,10^{-5}$$

$$e^2 = \max_i |u_i^2 - u(x_i, y_i)| = 1.01\,10^{-9}$$

We can see that with the same number of degrees of freedom, an approximation using $p = 2$ is much more accurate. This is because the solution is really smooth and is better approximated using a higher order interpolation than a bilinear interpolation on smaller quadrants. This is one example of the reasons we want to use higher order interpolations.

Number of quadrants	16^2	32^2	64^2	128^2	256^2
$p = 2$	$1.1\,10^3$	$4.2\,10^3$	$1.7\,10^4$	$6.6\,10^4$	$2.6\,10^5$
$p = 4$	$4.2\,10^3$	$1.7\,10^4$	$6.6\,10^4$	$2.6\,10^5$	$1.1\,10^6$
$p = 6$	$9.4\,10^3$	$3.7\,10^4$	$1.5\,10^5$	$5.9\,10^5$	$2.4\,10^6$
$p = 8$	$1.7\,10^4$	$6.6\,10^4$	$2.6\,10^5$	$1.1\,10^6$	$4.2\,10^6$

Table 1.4: Number of degrees of freedom for a regular mesh with different number of quadrants and for different degrees of interpolation.

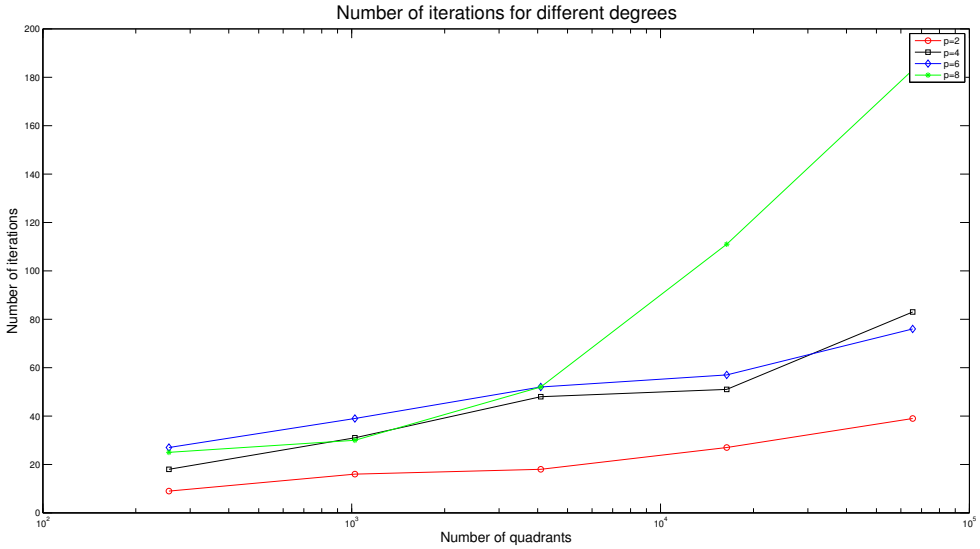


Figure 1.5: Number of iterations of PCG with only the fine preconditioner for different degree p of interpolation as a function of the number of quadrants in a regular mesh.

Regular mesh

Let us now move on to the comparison for different degrees of the number of iteration needed to reach a given accuracy as a function of the number of quadrants. We will take our regular mesh and uniformly refine it. Then, for $p = 2, 4, 6, 8$, we will see how many iterations are needed to reach a given error on the norm of the residual. Let us denote r_k the residual after iteration k of the preconditioned conjugate gradients. Of course, since our initial guess is zero, we have that $r_0 = b$ (since we are solving the linear system $Au = b$). For the following tests, our stopping criterion is given by :

$$\frac{||r_k||_2}{||r_0||_2} < 10^{-3}$$

Figure 1.5 shows the results. To put the data in perspective, we also have to show the number of degrees of freedom. Indeed, for a given number of quadrants, the higher degree the interpolation is, the more nodes we have. Table 1.4 contains the number of nodes for each mesh and for each degree p.

We can see that, even without the coarse preconditioner, we are solving the system in a small number of iterations compared to the number of degrees of freedom. For example, we only do about 80 iterations to solve the system with $2.4\,10^6$ degrees of freedom and with interpolations of degree $p = 6$.

We can also see that for every degree, the number of iterations increases when we refine the mesh. This is to be expected since the information from the boundaries has to go through more quadrant before propagate to the entire domain. Asymptotically, the number of iterations is

expected to double as the number of quadrants is multiplied by four (i.e. the mesh size is divided by two). We can see that it is not yet the case here.

A last remark we can make is that the number of iterations tends to increase when the degree of the interpolation increases. This is especially true for the finest mesh where we need 183 iterations for $p = 8$ where we only need 39 iterations for $p = 2$. This can be explained by the fact that the size of the overlap decreases when p grows. As mentioned in Ref here!!, this issue would be fixed if we imposed a constant overlap.

Meshes with distorted elements

Let us now move on to meshes that are not regular anymore. Let us remember that when we developed the fine preconditioner, we assumed that the elements were rectangular which allowed us to compute the analytic solution to the problem. This part explores the influence of having distorted elements on the number of iterations needed to obtain a given accuracy.

1.2.2 Influence of hanging nodes

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